LYNDON TREE & BINARY SEARCH TREE

LUCAS MERCIER

g

PHILIPPE CHASSAING

INSTITUT ELIE CARTAN



Séminaire PHILIPPE FLAJOLET, 31 Janvier 2013

Alphabet

n-letters long words

🗌 Language

🗌 Uís a factor of W

🗌 Uis a Prefix of W

U uis a suffix of W

Rotation

🗌 Necklace, círcular word

Primitive word

Alphabet

 $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$

- n-letters long words
- 🗌 Language
- Ulis a factor of W
- 🗌 Uis a Prefix of W
- U is a Suffix of W
- Rotation
- Necklace, círcular word
- Primitive word

- \Box Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$
- \Box n-letters long words $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$
- 🗌 Language
- 🗌 Uís a factor of W
- 🗌 Uis a Prefix of W
- U is a suffix of W
- Rotation
- 🗌 Necklace, círcular word
- Primitive word

- Alphabet
- n-letters long words
- 🗌 Language
- Ulis a factor of W
- 🗌 Uis a Prefix of W
- Uís a Suffix of W
- Rotation
- Necklace, círcular word
- Primitive word

 $\mathcal{A} = \{a_1 < a_2 < \dots < a_k < \dots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$

- Alphabet
 n-letters long words
- 🗌 Language
- 🗆 Uís a factor of W
- 🗌 Uis a Prefix of W
- Uís a Suffix of W
- Rotation
- Necklace, círcular word
- Primitive word

 $egin{aligned} \mathcal{A} &= \{a_1 < a_2 < \cdots < a_k < \cdots\} \ &\omega &= \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n \ &\mathcal{A}^* &= \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots \ &\exists r, s \in \mathcal{A}^* \ ext{such that } w = rus \end{aligned}$

 $r = \emptyset$

Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ n-letters long words $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$ Language $\exists r, s \in \mathcal{A}^*$ such that w = rusUlis a factor of W 🗌 Uís a Prefix of W U is a Suffix of W Rotation 🗌 Necklace, círcular word Primitive word

Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ n-letters long words $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$ Language $\exists r, s \in \mathcal{A}^*$ such that w = rusUlis a factor of W $r = \emptyset$ 🗌 Uís a Prefix of W $s = \emptyset$ U is a Suffix of W Rotation 🗌 Necklace, círcular word Primitive word

Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ n-letters long words $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$ Language $\exists r, s \in \mathcal{A}^*$ such that w = rusUlis a factor of W $r = \emptyset$ 🗌 Uís a Prefix of W $s = \emptyset$ U is a Suffix of W Rotation $\omega = \omega_1 \omega_2 \dots \omega_n \to \tau \omega = \omega_2 \omega_3 \dots \omega_n \omega_1$ 🗌 Necklace, círcular word Primitive word

Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ n-letters long words $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$ Language $\exists r, s \in \mathcal{A}^*$ such that w = rusUlis a factor of W $r = \emptyset$ 🗌 Uís a Prefix of W $s = \emptyset$ U is a Suffix of W Rotation $\omega = \omega_1 \omega_2 \dots \omega_n \to \tau \omega = \omega_2 \omega_3 \dots \omega_n \omega_1$ $\langle \omega \rangle = \{ \tau^k \omega \mid k \in \mathbb{Z} \}$ Necklace, círcular word Primitive word

Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ n-letters long words $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$ Language $\exists r, s \in \mathcal{A}^*$ such that w = rusUlis a factor of W $r = \emptyset$ 🗌 Uís a Prefix of W $s = \emptyset$ U is a Suffix of W Rotation $\omega = \omega_1 \omega_2 \dots \omega_n \to \tau \omega = \omega_2 \omega_3 \dots \omega_n \omega_1$ $\langle \omega \rangle = \{ \tau^k \omega \mid k \in \mathbb{Z} \}$ Necklace, círcular word $\#\langle\omega\rangle = |\omega|$ Primitive word

Alphabet $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k < \cdots\}$ $\omega = \omega_1 \omega_2 \dots \omega_n, \ \omega_i \in \mathcal{A}, \ |\omega| = n, \ \omega \in \mathcal{A}^n$ n-letters long words $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \dots$ Language $\exists r, s \in \mathcal{A}^*$ such that w = rusUlis a factor of W $r = \emptyset$ 🗌 Uís a Prefix of W $s = \emptyset$ U is a Suffix of W Rotation $\omega = \omega_1 \omega_2 \dots \omega_n \to \tau \omega = \omega_2 \omega_3 \dots \omega_n \omega_1$ $\langle \omega \rangle = \{ \tau^k \omega \mid k \in \mathbb{Z} \}$ Necklace, círcular word $\#\langle\omega\rangle = |\omega|$ Primitive word

Lexicographic Order

Lexicographic Order

 $\mathbf{a} \prec \mathbf{b}$ if $\begin{cases}
\text{either } \exists \, p, \alpha, \beta \in \mathcal{A}^{\star}, a_i, a_j \in \mathcal{A} \text{ s.t. } \mathbf{i} < \mathbf{j}, \\
\mathbf{b} = p a_{\mathbf{j}} \beta, \\
\text{or } \mathbf{a} \text{ is a prefix of } \mathbf{b}
\end{cases}$

Lexicographic Order

 $\mathbf{a} \prec \mathbf{b}$ if $\begin{cases}
\text{either } \exists \, p, \alpha, \beta \in \mathcal{A}^{\star}, a_i, a_j \in \mathcal{A} \text{ s.t. } \mathbf{i} < \mathbf{j}, \\
\mathbf{b} = p a_{\mathbf{j}} \beta, \\
\text{or } \mathbf{a} \text{ is a prefix of } \mathbf{b}
\end{cases}$

w is a Lyndon word if w is primitive, and is the smallest word in its necklace

Lexicographic Order

 $\mathbf{a} \prec \mathbf{b}$ if $\begin{cases} \text{either } \exists \, p, \alpha, \beta \in \mathcal{A}^{\star}, a_i, a_j \in \mathcal{A} \text{ s.t. } i < j, \\ \mathbf{b} = p a_j \beta, \end{cases}$ or **a** is a prefix of **b**

w is a Lyndon word if w is primitive, and is the smallest word in its necklace

🗌 cbaa, baac, aacb, acba:

aacb is a Lyndon word,

Lexicographic Order

 $\mathbf{a} \prec \mathbf{b}$ if $\begin{cases}
\text{either } \exists \, p, \alpha, \beta \in \mathcal{A}^{\star}, a_i, a_j \in \mathcal{A} \text{ s.t. } i < j, \\
\mathbf{b} = p a_j \beta, \\
\text{or } \mathbf{a} \text{ is a prefix of } \mathbf{b}
\end{cases}$

w is a Lyndon word if w is primitive, and is the smallest word in its necklace

🗌 cbaa, baac, aacb, acba:

aacb is a Lyndon word,

🗌 aabaab, baac

are not

 \Box The standard right factor \lor of a word w is its smallest proper suffix.

- \Box The standard right factor \lor of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

-] The standard right factor v of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababbaaabaa

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababbaaabaa

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababbaaab<mark>aa</mark>

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababb<mark>aaabaa</mark>

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababbaaabaa

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababbaaabaa

-] The standard right factor \vee of a word w is its smallest proper suffix.
- \Box The related factorization w = uv is often called the standard factorization of w.

🗌 w=abaabbabaabb u=abaabbab v=aabb

 \square w=abaabbabaabb u'=ab v'=aabbabaabb v<v'

Theorem (Lyndon, 1954) Any word w may be written uniquely as a non-increasing product of Lyndon words (by iteration of a variant of the standard factorization).

aabbaaababbaaabaa

□ The standard factorization of a Lyndon word is the first step in the construction of some basis of the free Lie algebra over A

Lyndon tree

* If w= la is the standard decomposition of w, then the Lyndon tree of w, L (w), is given by L(w) = * Examples $\mathcal{L}(\alpha^{3}\beta^{4}) = \mathcal{L}(\alpha)$ L(a28")

Lyndon thee

+ If w= la is the standard decomposition of w, then the Lyndon tree of w, L (w), is given by L(2) L(w) = * Examples $\mathcal{L}(a^{3}b^{4}) = \mathbb{E}(a)$ $\mathbb{L}(a^{2}b^{2}) = a$ $\mathbb{E}(a^{2}b^{2})$

Lyndon thee

* If w= la is the standard decomposition of w, then the Lyndon tree of w, L (w), is given by L(2) L(w) = L(7) * Examples $\mathcal{L}(a^{3}b^{4}) = \mathbb{E}(a)$ $\mathbb{L}(a^{2}b^{2}) = \Box$ a L(a b3)

Lyndon thee

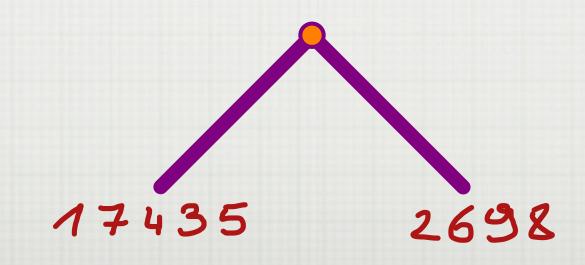
* If w= la is the standard decomposition of w, then the Lyndon tree of w, L (w), is given by L(w) = * Examples La) $\mathcal{L}(a^{3}b^{4}) =$ L(284) b L(ab2)

Lyndon tree

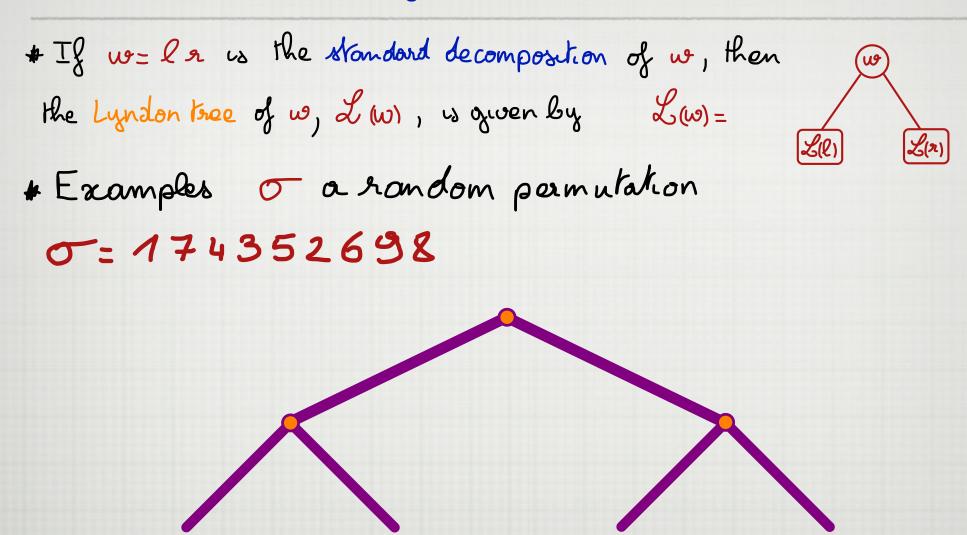
* If w= l x is the standard decomposition of w, then the Lyndon tree of w, L (w), is given by L(w) = * Examples L(a364 B(a) - B(a26")

Lyndon tree

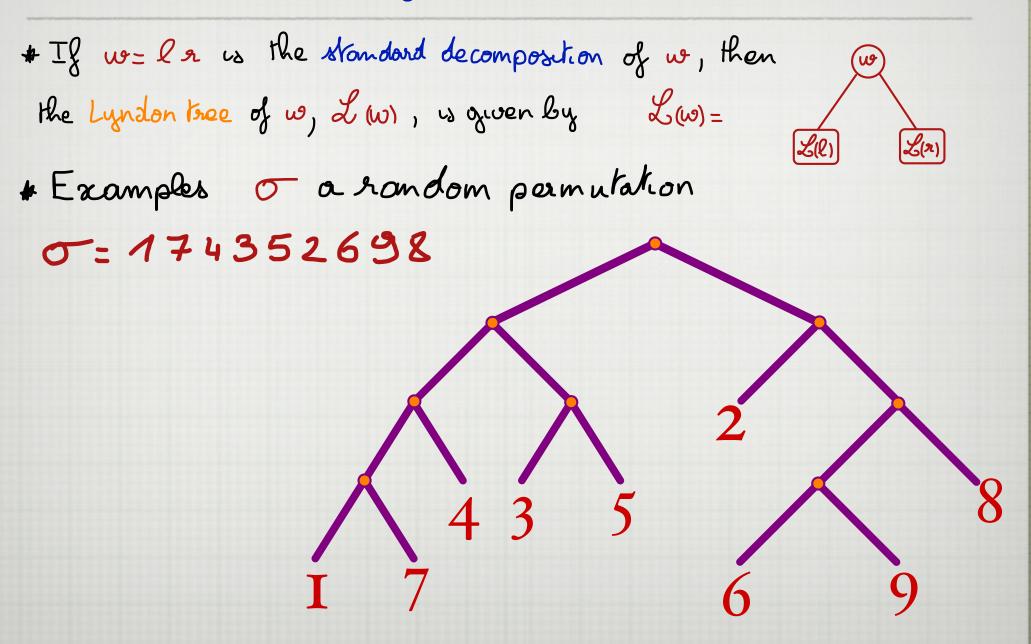
If w= la is the standard decomposition of w, then when the Lyndon tree of w, L(w), is given by L(w) =
Examples of a rondom permutation
0 = 174352698



Lyndon tree

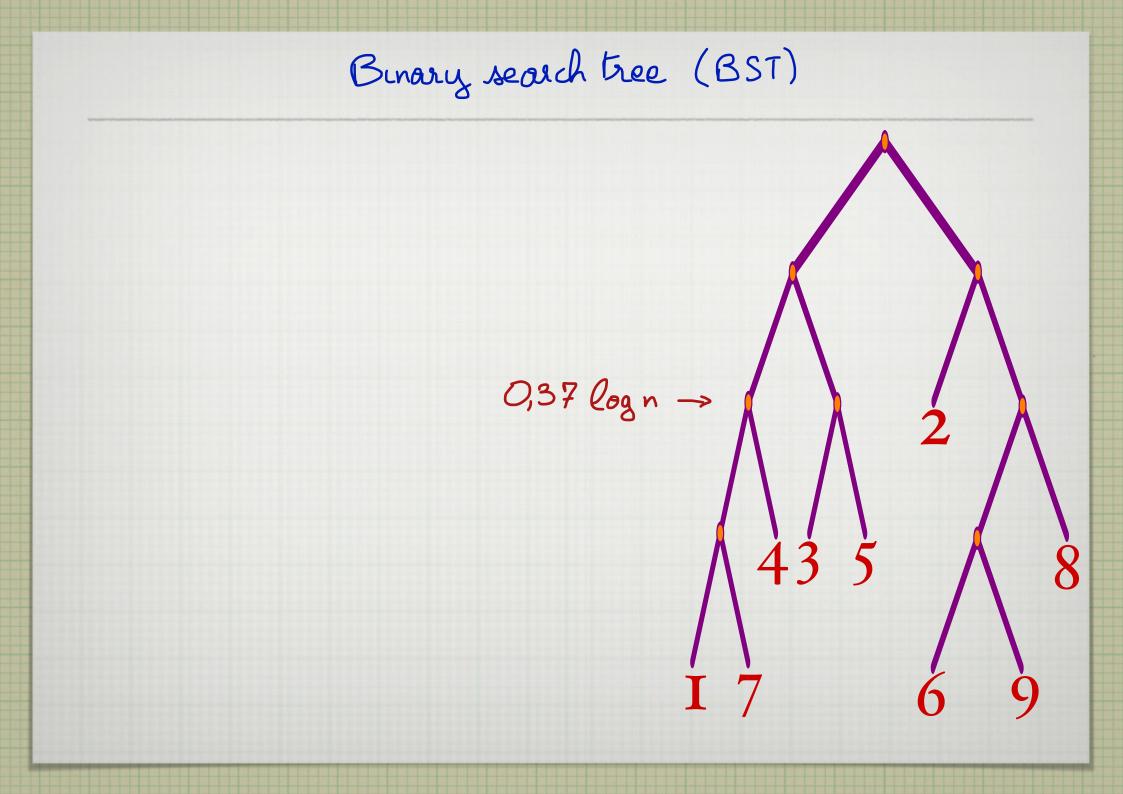


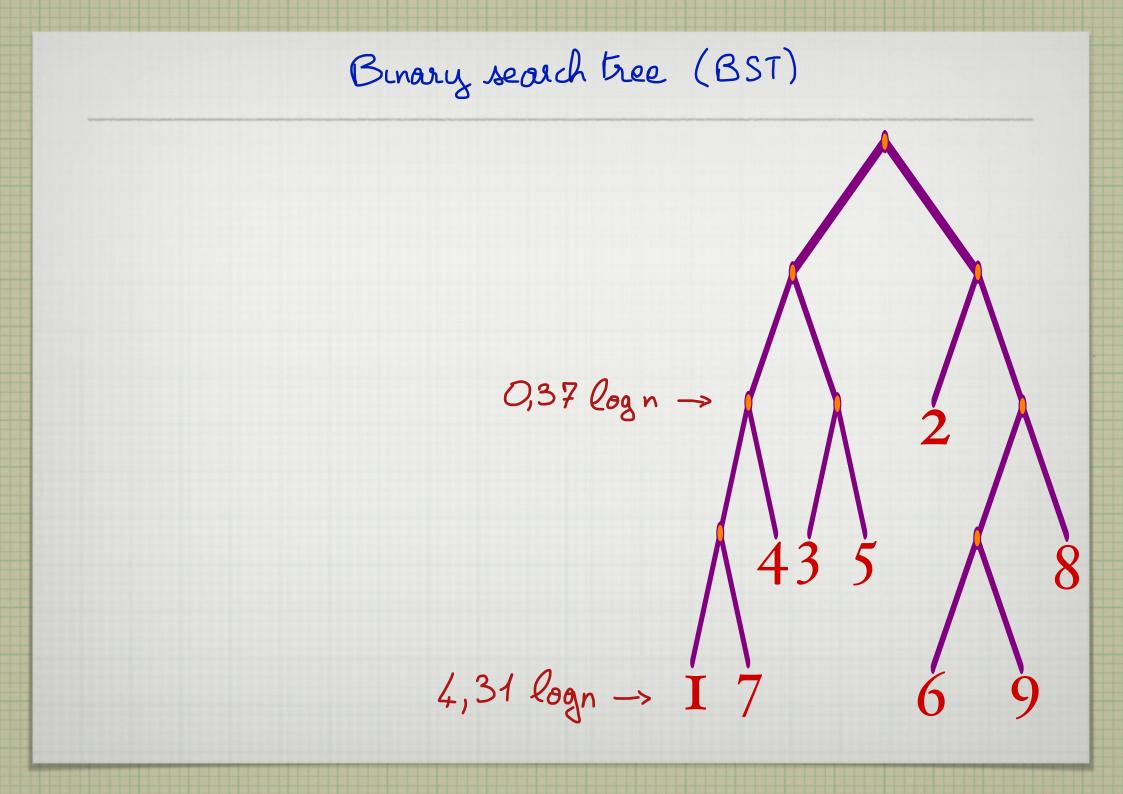
Lyndon thee



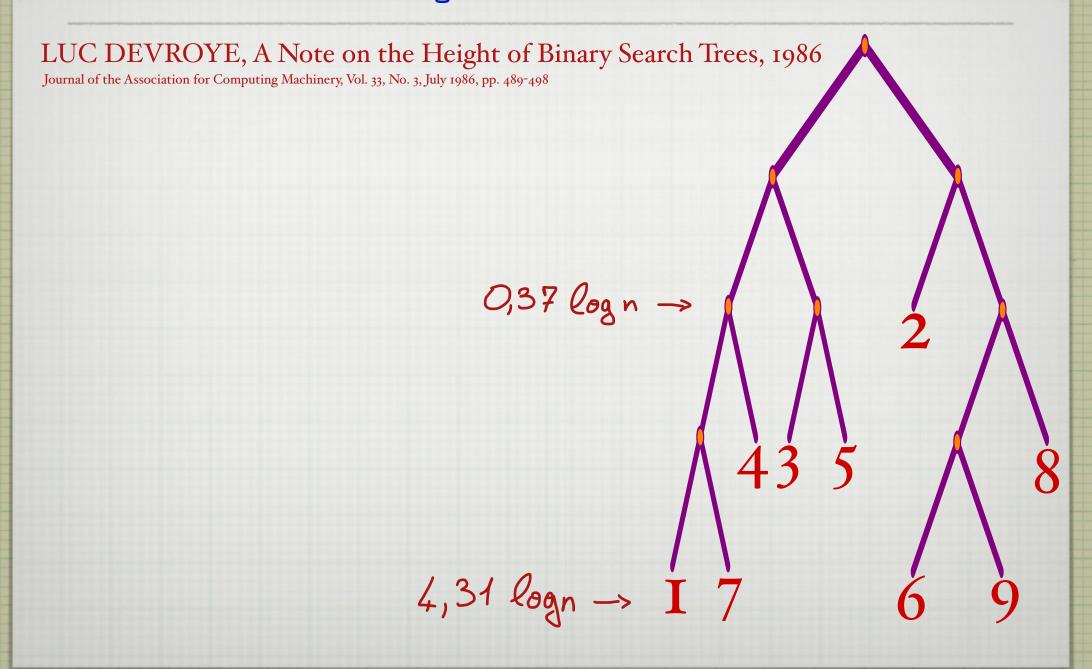
Lyndon tree

+ If w= lr is the standard decomposition of w, then the Lyndon tree of w, $\mathcal{L}(w)$, is given by $\mathcal{L}(w) =$ * Examples of a random permutation $\sigma = 174352698$ Binary search Tree (BST)





Binary search tree (BST)



Binary search tree (BST)

LUC DEVROYE, A Note on the Height of Binary Search Trees, 1986 Journal of the Association for Computing Machinery, Vol. 33, No. 3, July 1986, pp. 489-498 η : Cramer transform of the Poisson distribution of parameter λ c and c' are the rooks of n2(2)= 15 $\eta_{x}(x) = x \ln(\frac{x}{\lambda}) - x + \lambda$ 1C'= 0,37, log n see also Pittel 1984 (=4,31, logn ->

Binary search tree (BST)

LUC DEVROYE, A Note on the Height of Binary Search Trees, 1986

Journal of the Association for Computing Machinery, Vol. 33, No. 3, July 1986, pp. 489-498

B. CHAUVIN, M. DRMOTA, J. JABBOUR-HATTAB, The Profile of Binary Search Trees, 2001

Ann. Appl. Prob., 11:1042-1062.

0,37 log n -

2 logn + Vlegn Moro)

4,31 logn ->

Binary search tree (BST)

LUC DEVROYE, A Note on the Height of Binary Search Trees, 1986 Journal of the Association for Computing Machinery, Vol. 33, No. 3, July 1986, pp. 489-498 B. CHAUVIN, M. DRMOTA, J. JABBOUR-HATTAB, The Profile of Binary Search Trees, 2001 Ann. Appl. Prob., 11:1042-1062. J. JABBOUR-HATTAB, Martingales and large deviations for binary search trees, 2001 Random Structure and Algorithms, 19:112–127. 0,37 log n 2 (ogn + VBgn Moro) $\alpha \log n \longrightarrow \# \text{leaves at depth } \alpha \log n^2 \simeq n^{1-\eta_2(\kappa)}$ 4,31 logn $\rightarrow I 7$

Binary search tree (BST)

LUC DEVROYE, A Note on the Height of Binary Search Trees, 1986 Journal of the Association for Computing Machinery, Vol. 33, No. 3, July 1986, pp. 489-498 B. CHAUVIN, M. DRMOTA, J. JABBOUR-HATTAB, The Profile of Binary Search Trees, 2001 Ann. Appl. Prob., 11:1042-1062. J. JABBOUR-HATTAB, Martingales and large deviations for binary search trees, 2001 Random Structure and Algorithms, 19:112–127. 0,37 log n 2 logn Alogn → #fleaves at depth x logn} ~ n^{1-12(x)} 4,31 logn -> I

fine study of the top level see Matthew Roberts, 2010

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La

-					

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution In The Under Ln (dw), <u>Ill (d)</u> $\frac{1}{2}S_0 + \frac{1}{2}U_{[0,1]}$ Mondand Zohoorian

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La The Under Ln (dw), <u>Ill (d)</u> $\frac{1}{2}S_0 + \frac{1}{2}U_{[0,1]}$ Mondand Zohoorian 20te alphabet, Pouson Dindlet dist" - s Chassoning Zohoorian

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La The Under Un (dw), Ill (d) 1/2 So + 1/2 U[0,1] Mondand Zohoor an ste alphabet, Pouson Dindlet dist" -> Chassainez Zohoorian Idea the positions of the log_n long runs of "a" are uniform on [[1,n]], approximately they give the sizes of the factors (= subtrees)

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La The Under Un (dw), Ill (d) 1/2 So + 1/2 U[0,1] Mondand Zohoor an se alphabet, Pouson Dindlet dist" -> Chassaine Zohoorian Idea the positions of the log_n long runs of "a" are uniform on [[1,n]], approximately they give the sizes of the factors (= subtrees) Problems # # Ln and Ln unfriendly

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La The Under Un (dw), Ill idis 1/2 So + 1/2 U[0,1] Mondand Zohoor and se alphabet, Pouson Dindlet dist" -> Chassaine Zohoorian Idea the positions of the log_n long runs of "a" are uniform on [[1,n]], approximately they give the sizes of the factors (= subtrees) Problems # # Ln and Ln unfriendly * lexicographic order not easy to handle

Lyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La The Under Un (dw), Ill idi 1/2 So + 1/2 U[0,1] Mondand Zohoor an se alphabet, Pousson Dindlet dist" - s Chassaine Zohoorian Idea the positions of the log_n long runs of "a" are uniform on [[1,n]], approximately they give the sizes of the factors (= subtrees) Problems # # Ln and Ln unfriendly * lexicographic order not easy to handle * many red for the title of longest run to break the tie, one has to look at the suffix

Zyndon trees on the alphabet {a, b}

La set of n letters long Lyndon words, uniform probability distribution La The Under Un (dw), Ill idi 1/2 So + 1/2 U[0,1] Mondand Zohoor an se alphabet, Pouson Dindlet dist" -> Chassainy Zohoonan Idea the positions of the log_n long runs of "a" are uniform on [[1,n]], approximately they give the sizes of the factors (= subtrees) Problems # # In and Ln unfriendly * lexicographic order not easy to handle * many red for the title of longest run to break the tie, one has to look at the suffix $\mathcal{L}_{n}(k \text{ t.ed}) \simeq \sum_{\ell \in \mathcal{H}} e^{-2^{\alpha+\ell}} \frac{(2^{\alpha+\ell-1})^{k}}{k!} \text{ if } \{\log_{2} n\} \simeq \alpha$

The height Hn of the Lyndon tree

The Underly, $\frac{H_n}{\log n} \stackrel{(P)}{\longrightarrow} 5,09$.

The height Hn of the Lyndon tree

The Underly, $\frac{H_n}{\log n} \stackrel{(P)}{\longrightarrow} 5,09$.

Ideas # to find a BST somewhere

The height Hn of the Lyndon tree

The Underly, $\frac{H_n}{eog}$, $\frac{P}{5,09}$.

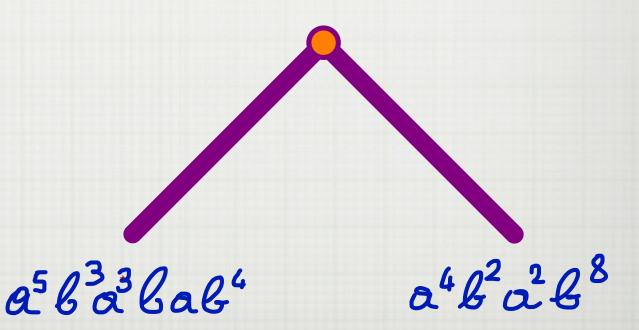
* to find a BST somewhere Ideas

* To use Jabbour LD results

The height Hn of the Lyndon tree

The Under Un, <u>Hn</u> (P), 5,09. Ideas * to find a BST somewhere to use Jabbour LD results

 $W = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$



The height Hn of the Lyndon tree

The Under Un, <u>Hn</u> (P), 5,09. Ideas * to find a BST somewhere to use Jabbour LD results

 $W = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$

a⁵b³a³bab⁴

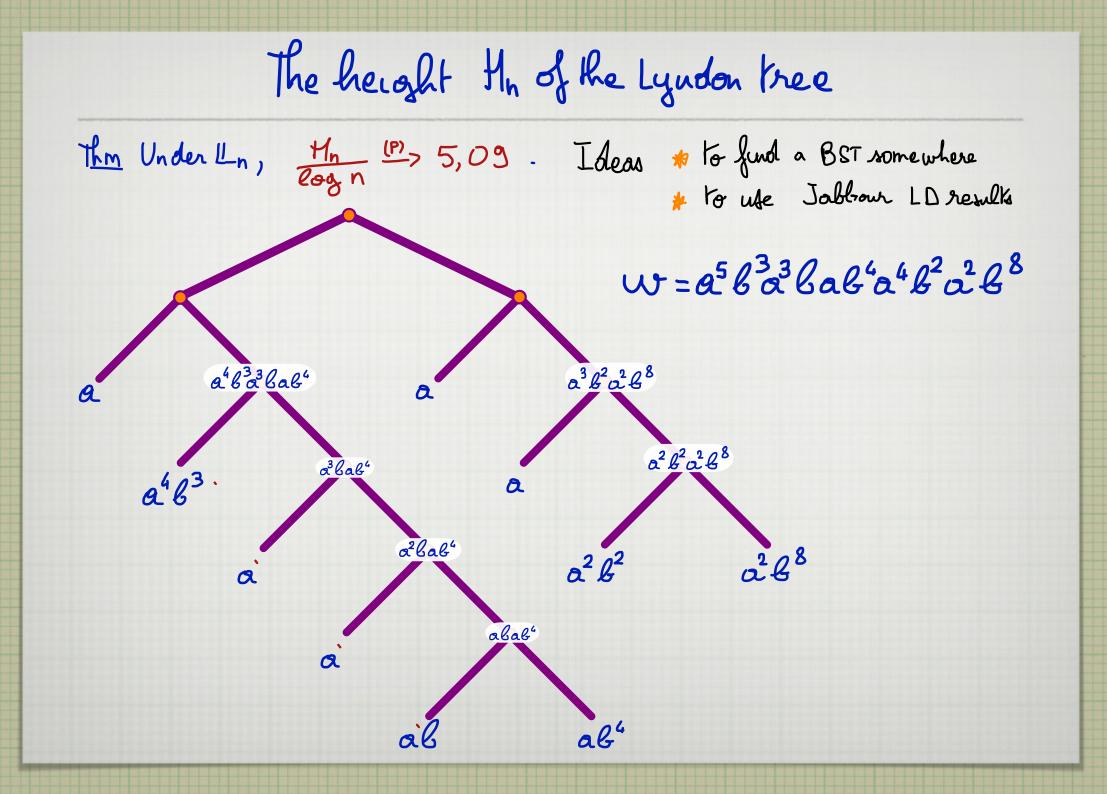
<u>A</u>

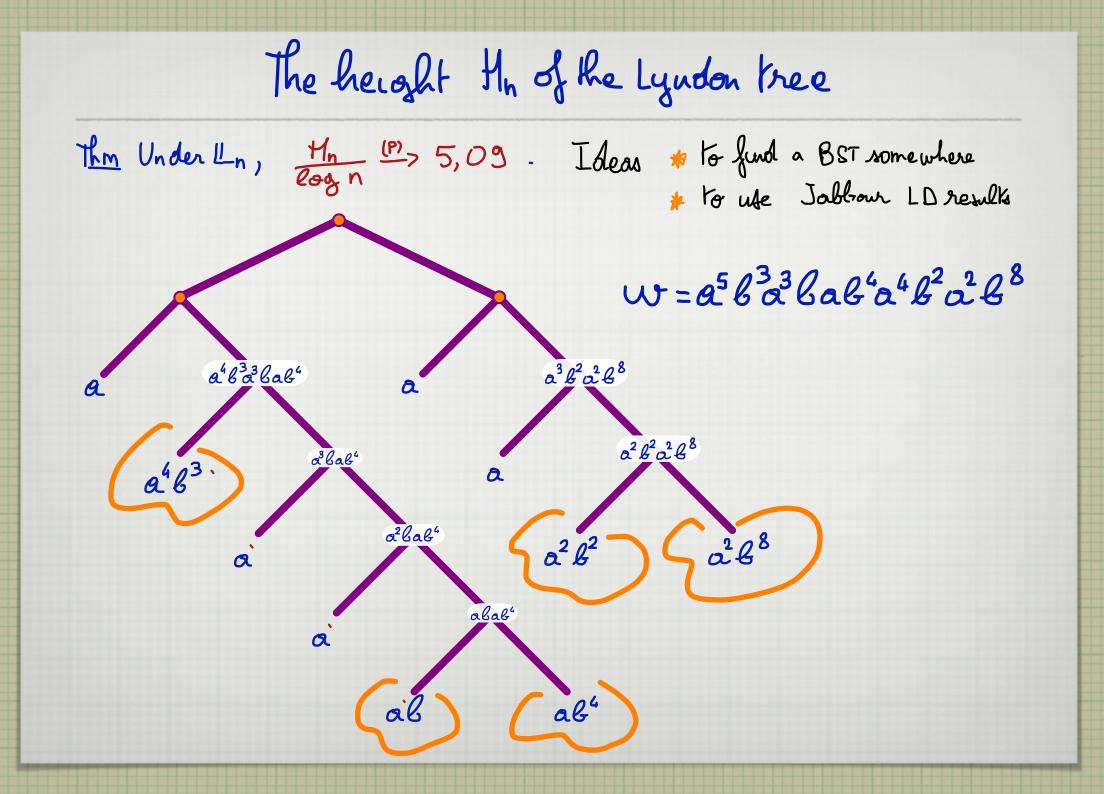
ababababé

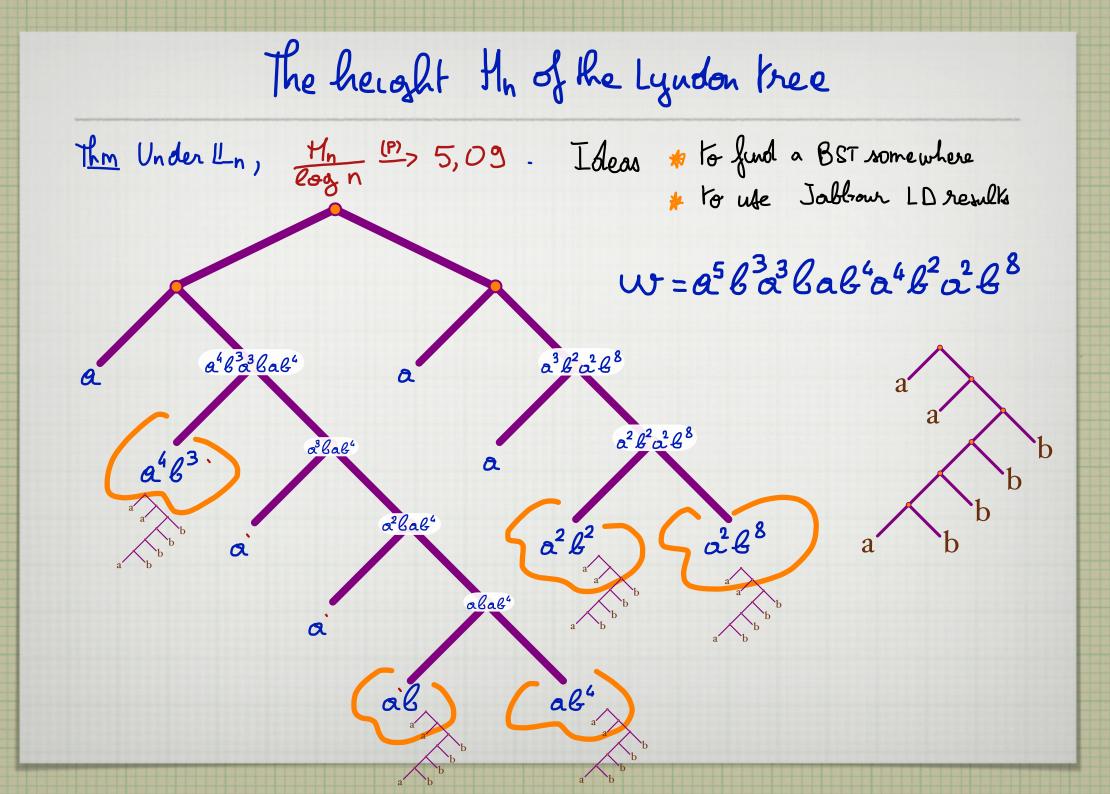
 $a^{3} h^{2} a^{2} h^{8}$

 $a^4 b^2 a^2 b^8$

A

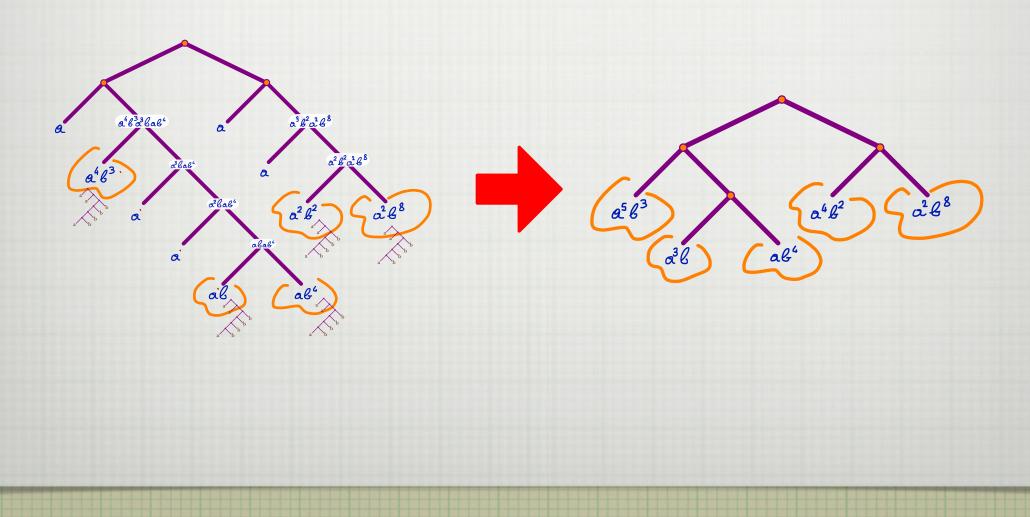






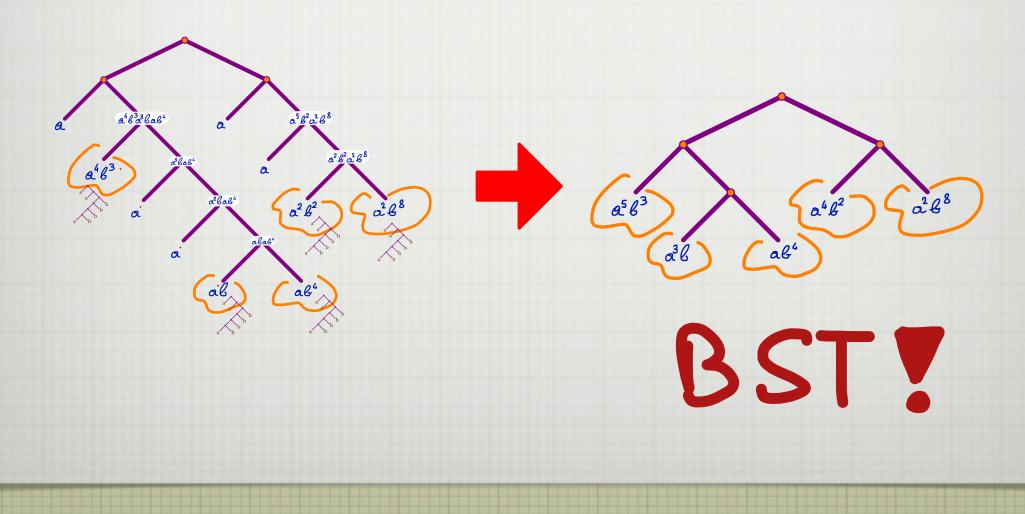
Lyndon tree & BST

The Under Ln, $\frac{H_n}{\log n} \stackrel{(P)}{\longrightarrow} 5,09$. Ideas * to find a BST somewhere * to use Jabbour LD results $W = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$



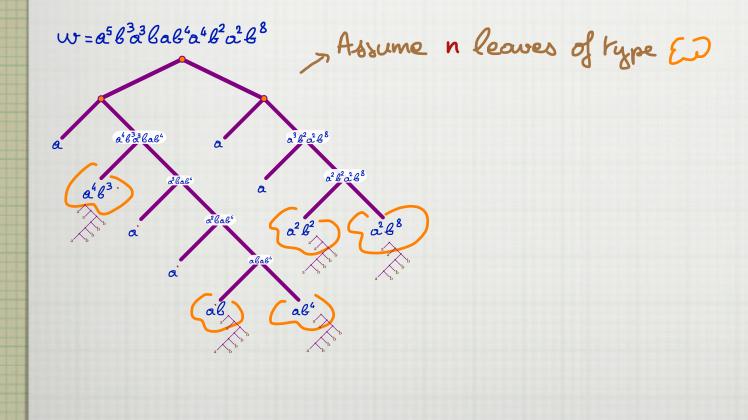
Lyndon tree & BST

The Under Ln, $\frac{H_n}{\log n} \stackrel{(P)}{\longrightarrow} 5,09$. Ideas * to find a BST somewhere * to use Jabbour LD results $W = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$



Lyndon tree & BST

Ideas * to find a BST somewhere * to use Jabbour LD results <u>Hn</u> (P) 5,09 log n Thm Under Un,



Lyndon tree & BST

to find a BST somewhere
* to use Jabbour LD results <u>Hn</u> (P) 5,09 log n Thm Under Un, Ideas

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$ Assume n leaves of type [-> Jabbour n¹-n^(a) of them at level alogn a362268 $a^2 b^2$

Lyndon tree & BST

<u>Hn</u> (P) 5,09 log n * to find a BST somewhere Thm Under Un, Ideas * to use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$ -> Jabbour n^{1-n(x)} , Assume n leaves of type ED of them at level alogn a⁴6³36a64 nt 100 shuls behaving a³b²a²6⁸ like i d'geometric 1/2 02 B2 allab4 the highest is all all shrubs (1-y(x)) log_n high

Lyndon tree & BST

<u>Hn</u> (P) 5,09 log n Thm Under Un, * to find a BST somewhere Ideas * To use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$ -> Jabbour n^{1-n(x)} , Assume n leaves of type [] of them at level alogn a a'b'a'bab' a $a^3 \beta^2 a^2 \theta^8$ nt 100 shuls behaving a alal' a b like i d'geometric 1/2 a288 the highest is all all (1-y_a) log_n high Contrubution to Hn: $\left(\alpha + \frac{1 - \eta(\alpha)}{\rho_{n-2}}\right) \times \log n$.

Lyndon tree & BST

<u>Hn</u> (P) 5,09. log n Thm Under Un, * to find a BST somewhere Ideas * to use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$ -> Jabbour n^{1-n(x)} , Assume n leaves of type En of them at level alogn a $a^{i} B^{3} \delta a B^{i}$ a $a^{3} B^{2} a^{2} B^{3}$ $a^{2} B^{2} a^{2} B^{3}$ nt nes behaving $a^{3}BaB^{4}$ a $a^{3}BaB^{4}$ $a^{2}B^{2}$ $a^{2}B^{3}$ $a^{2}B^{2}$ $a^{2}B^{3}$ $a^{2}B^$ a⁴b³. like i i d'geometric 1/2 the highest is all all (1-y_a) log_n high Contrubution to Hn: $\sup_{\alpha} \{\alpha + \frac{1 - \eta_{\alpha}}{n_{2}}\} \simeq 5,09$ $\left(\alpha + \frac{1 - \eta(\alpha)}{\rho_{2}}\right) \times \log n$.

Lyndon tree & BST # to find a BST somewhere
* to use Jabbour LD results <u>Hn</u> (P) 5,09 log n Thm Under Un, Ideas -> Jabban n^{1-n(x)} of them at kevel alogn $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$, Assume n leaves of type ED $a^3b^2a^2b^8$ $a^2 b^2$ al"

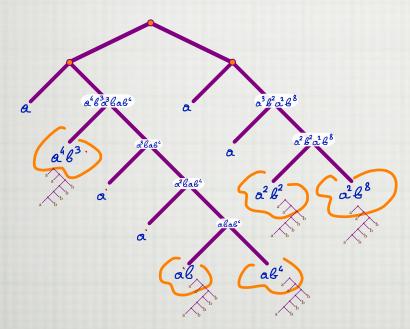
Lyndon tree & BST <u>Hn</u> (P) 5,09 log n * to find a BST somewhere Thm Under Un, Ideas * to use Jabbour LD results -> Jabber n^{1-n(x)} of them at kevel alogn $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$, Assume n leaves of type [] $a^3b^2a^2b^8$ a ba bab $a^2 b^2$ al"

Lyndon tree & BST

The Under Un, <u>Hn</u> (P), 5,09. Ideas * to find a BST somewhere log n * to use Jabbour LD results

Problem tred factors and

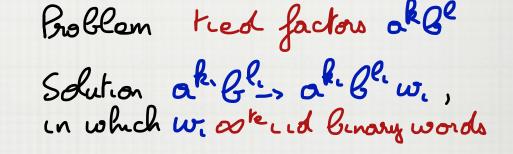
 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$

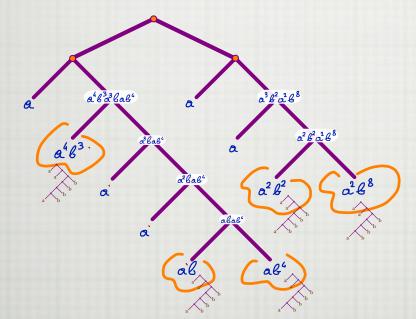


Lyndon tree & BST

<u>Hn</u> (P) 5,09 log n Ideas * to find a BST somewhere * to use Jabbour LD results Thm Under Un,

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$

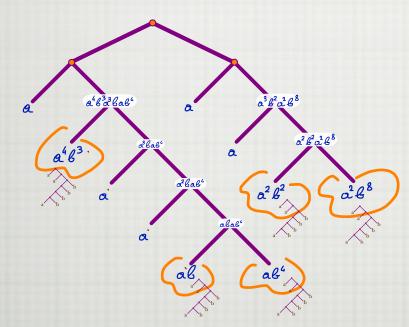




Lyndon tree & BST

<u>Hn</u> (P) 5,09 log n Thm Under Un, * to find a BST somewhere Ideas * to use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$

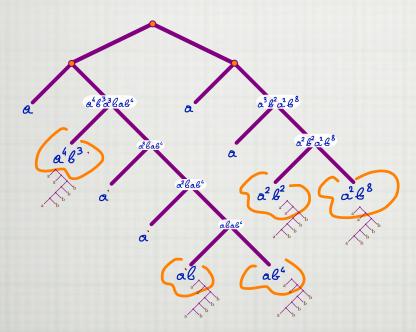


Problem tied factors at Be Solution ak Blis ak Bli w., in which wice keild binary words Problem For we Ln, the # of factors is not easy to handle

Lyndon tree & BST

The Under Un, <u>Hn</u> (P), 5,09. Ideas * to find a BST somewhere to use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$



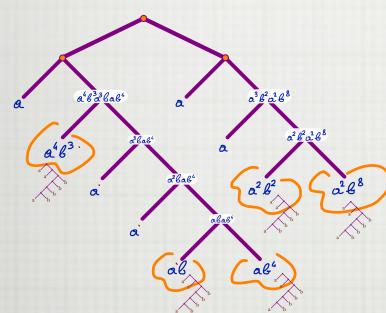
Problem tied factors at Be Solution ak Blis ak Bli w., in which wide kind binary words Problem For we Ln, the # of factors is not easy to handle Solution Consider We the rondom

aste burary word truncated after the 1st occurence of al, then reversed (ro be Lyndon)

Lyndon tree & BST

The Under Un, <u>Hn</u> (P), 5,09. Ideas * to find a BST somewhere log n * to use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$



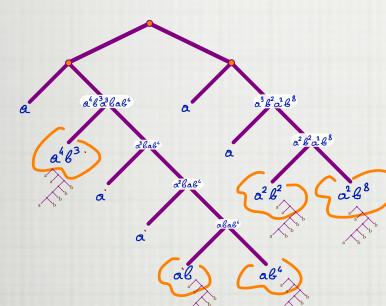
Problem For we Ln, the # of factors is not easy to handle Solution Consider We the rondom aste burary word truncated after the 1st occurence of a^e, then reversed (ro be Lyndon)

Assumption | We| ~ 2° and H(L(We)) ~ al

Lyndon tree & BST

The Under Un, <u>Hn</u> (P) > 5,09. Ideas * to find a BST somewhere log n * to use Jabbour LD results

 $w = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$



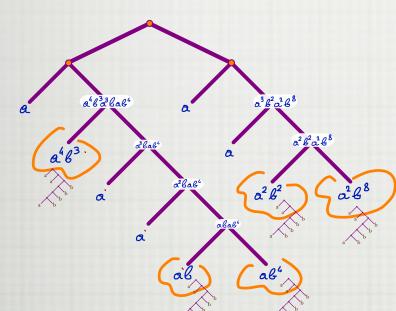
Problem For we Ln, the # of factors is not easy to handle Solution Consider We the rondom aste burary word truncated after the 1st occurence of a^e, then reversed (ro be Lyndon)

Assumption $|W_{\ell}| \simeq 2^{\ell}$ and $H(\mathcal{L}(W_{\ell})) \simeq \alpha \ell$ $\rightarrow H_{n} \simeq \alpha \log_{2} n$

Lyndon tree & BST

The Under Un, $\frac{H_n}{\log n} \xrightarrow{(P)} 5,09$. Ideas * to find a BST somewhere * to use Jabbour LD results

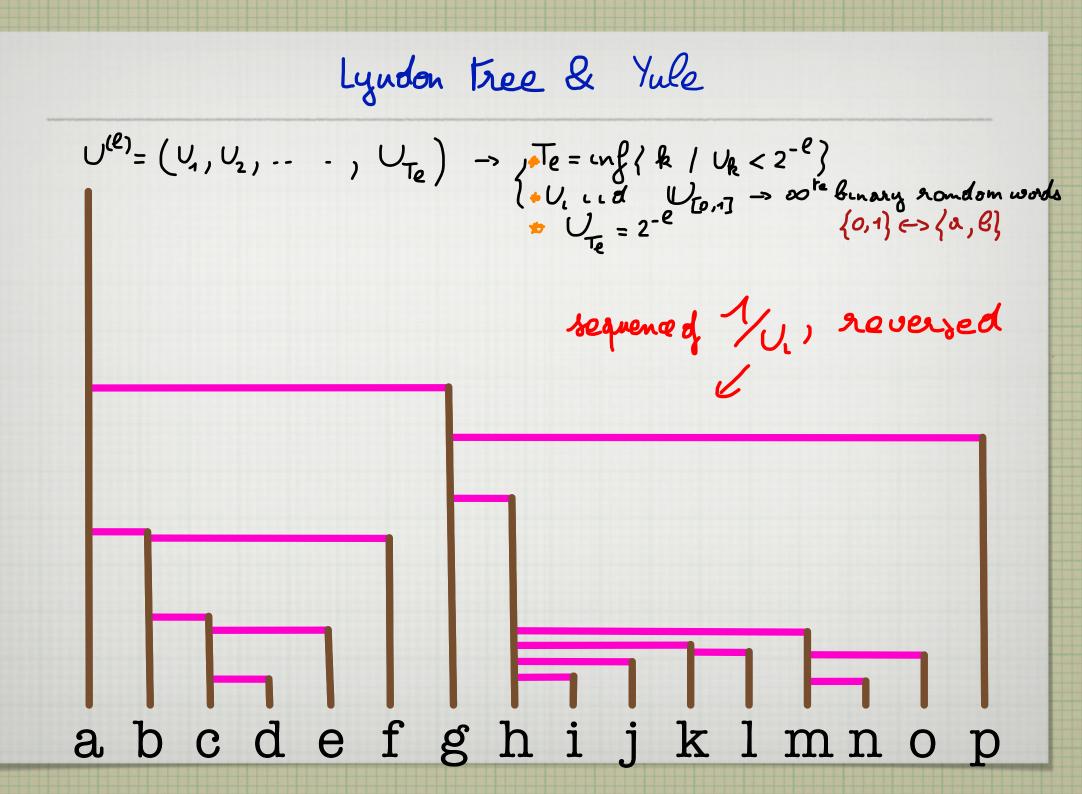
 $W = a^5 b^3 a^3 b a b^4 a^4 b^2 a^2 b^8$

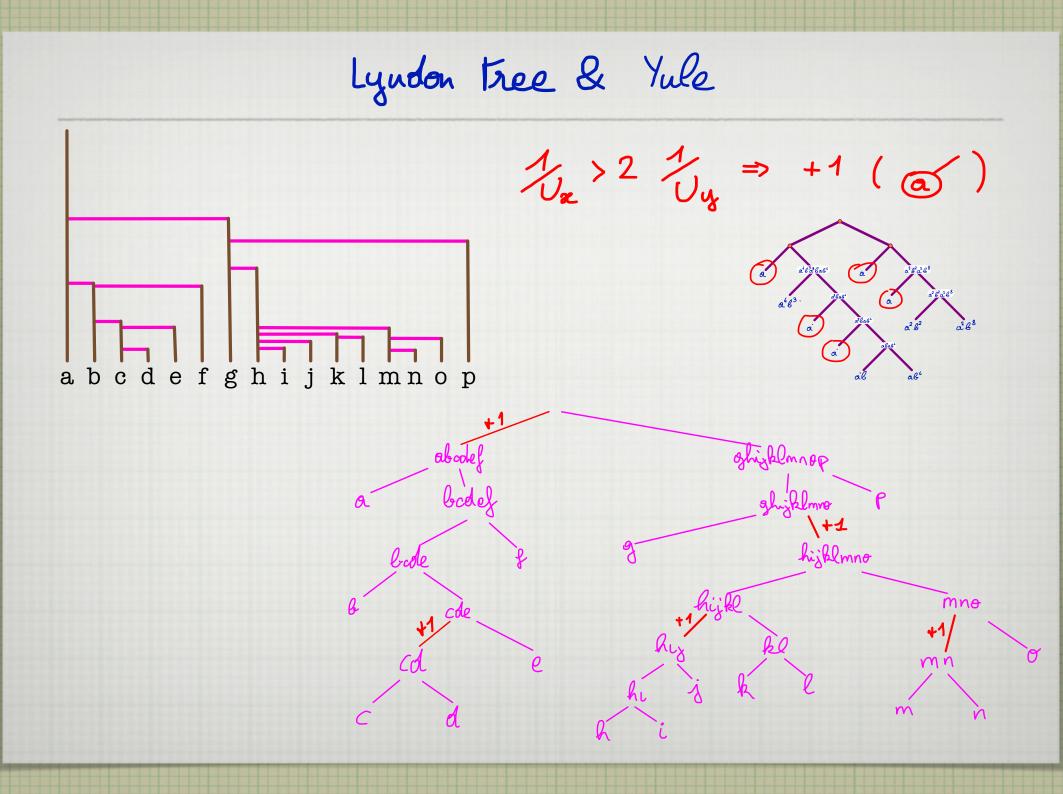


Problem For we Ln, the # of factors is not easy to handle Solution Consider We the rondom aste burary word truncated after the 1st occurence of al, then reversed (robe Lyndon)

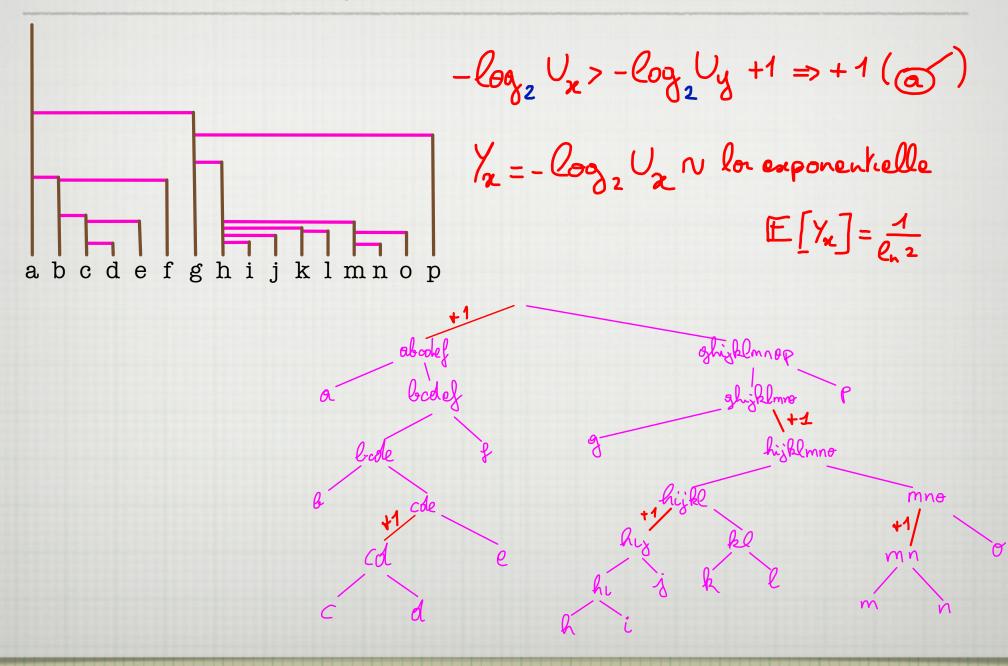
Assumption $|W_{\ell}| \simeq 2^{\ell}$ and $H(\mathcal{L}W_{\ell}) \simeq \alpha \ell$ $\Rightarrow H_{n} \simeq \alpha \log_{2} n \Rightarrow \frac{\alpha}{\ln 2} = 5,09$

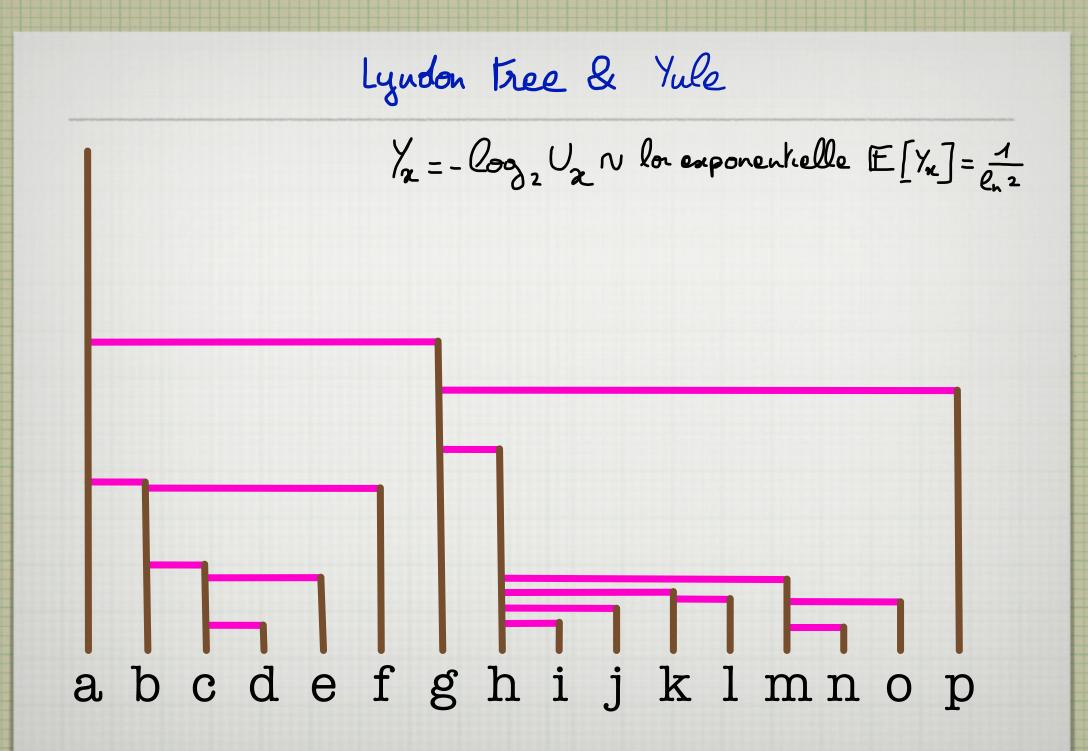
Lyndon tree & Yule $U^{(\ell)} = (U_1, U_2, \dots, U_{T_\ell}) \rightarrow \int_{T_\ell} T_\ell = \inf\{k \mid U_k < 2^{-\ell}\}$ $(U_1 \cup U_1 \cup U_1 \rightarrow \infty^{t_\ell} \text{ binary rondom words})$ $(U_1 \cup U_1 \cup U_1 \rightarrow \infty^{t_\ell} \text{ binary rondom words})$ $= U_{T_\ell} = 2^{-\ell} \qquad \{0, 1\} \in \{a, b\}$ sequenced 1/1, reversed abcdefghijklmnop

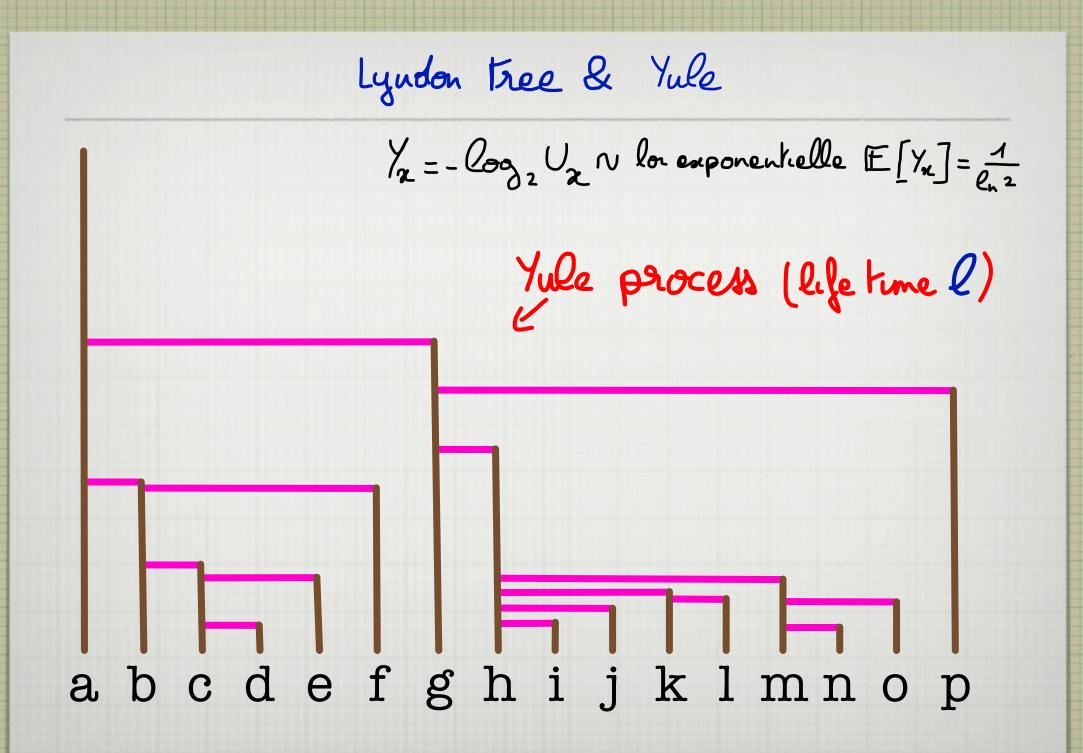


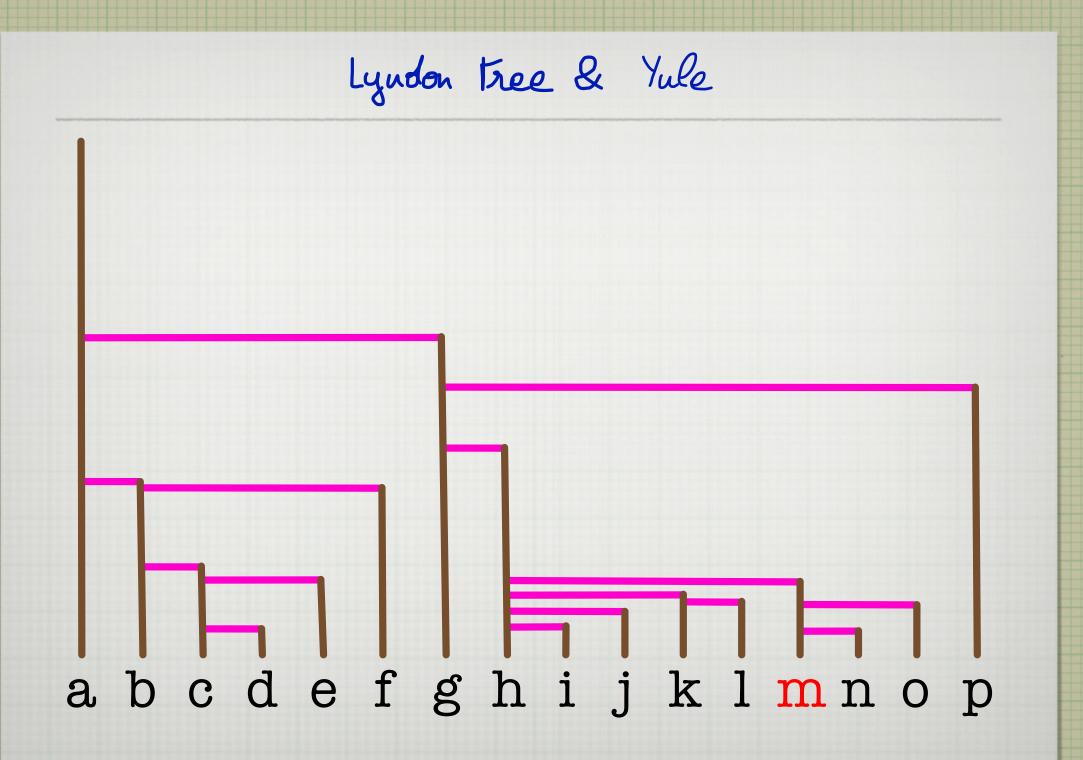


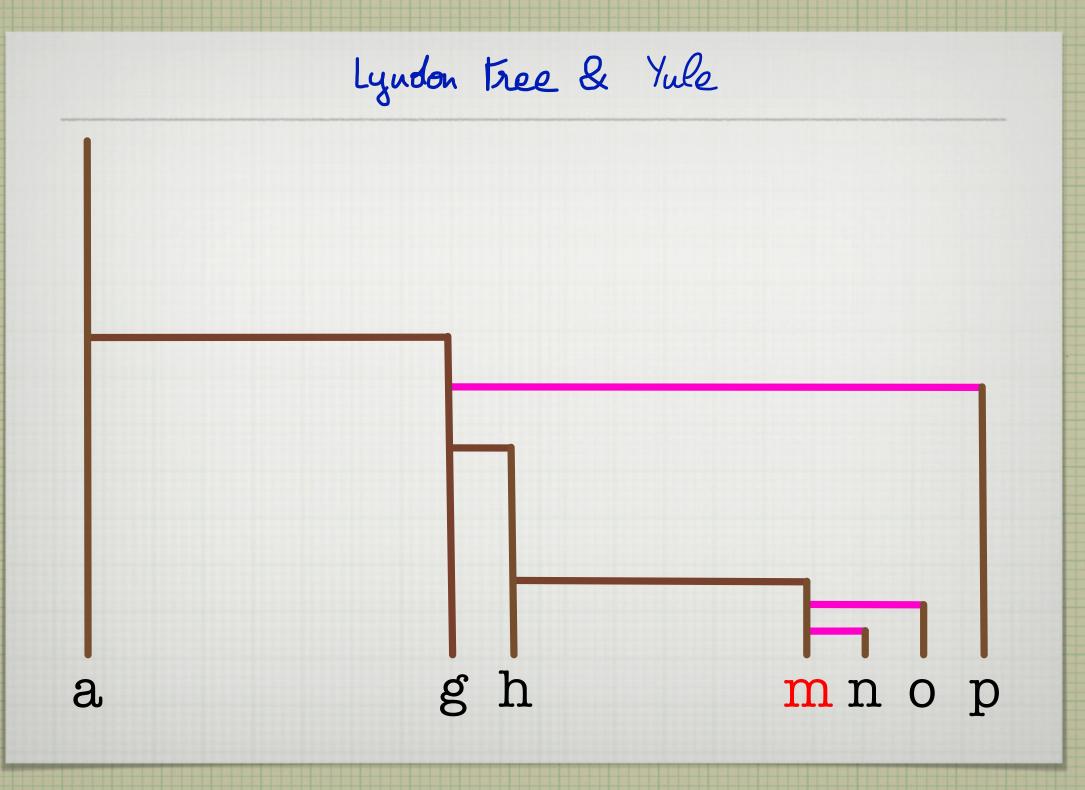
Lyndon tree & Yule

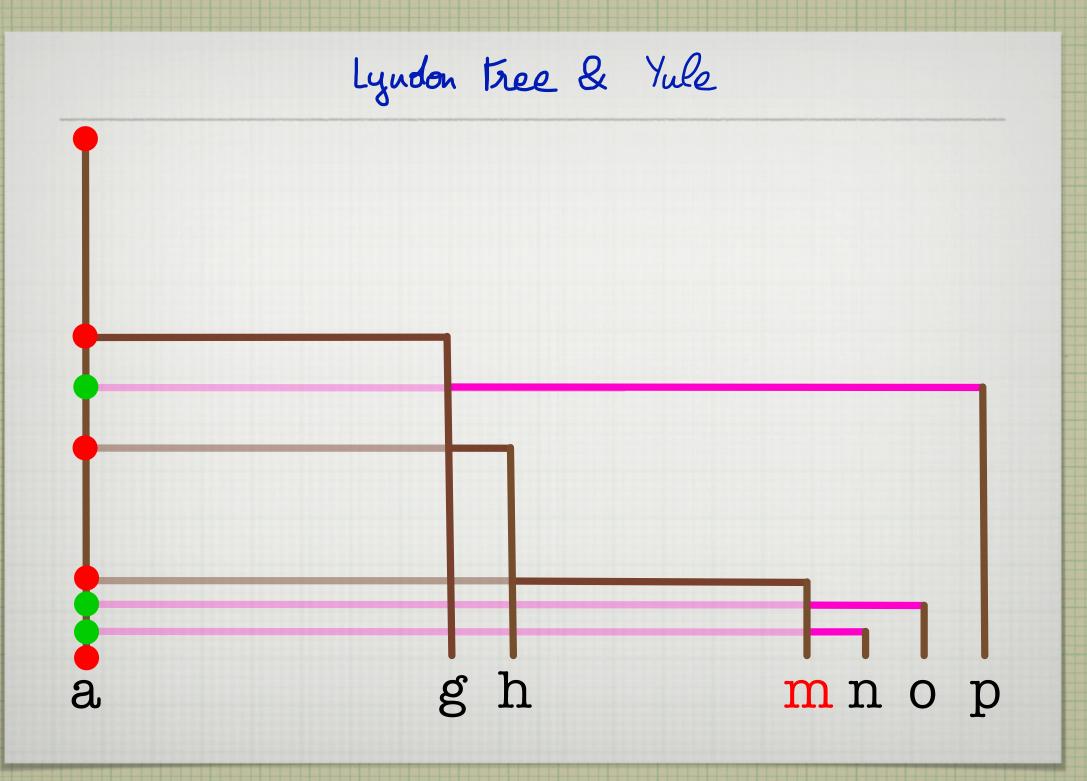


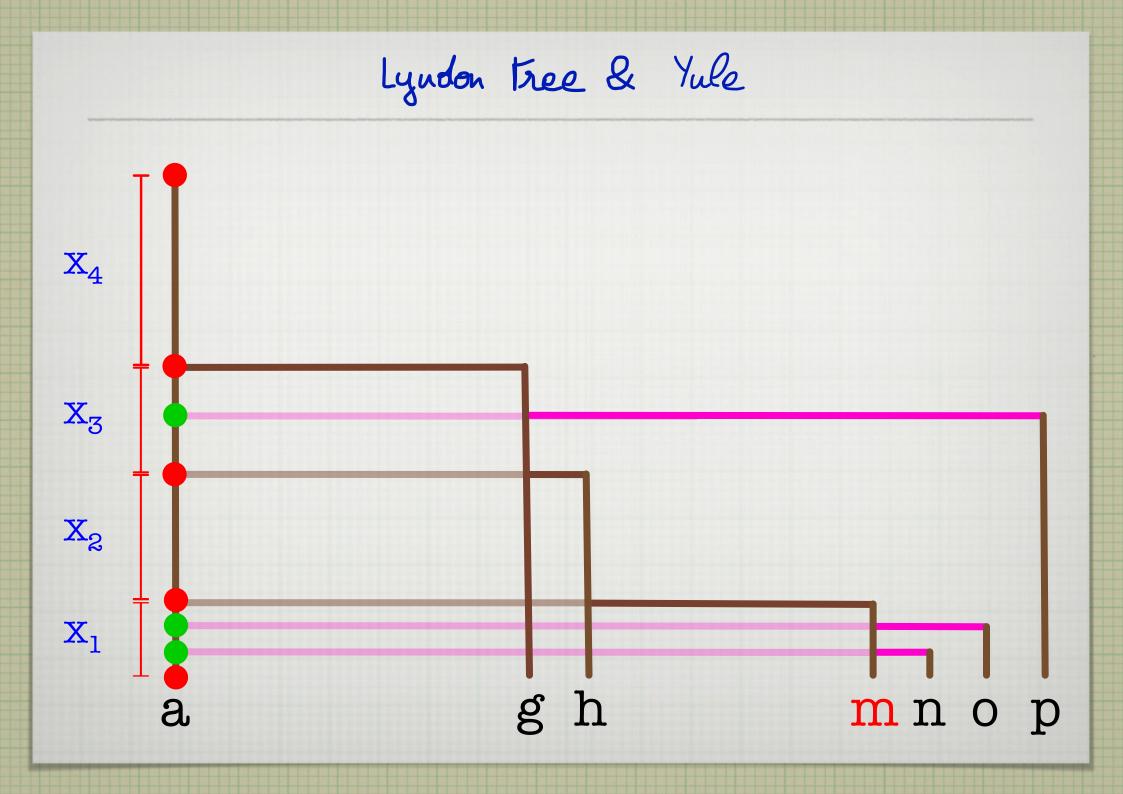












Lyndon tree & Yule $depth of m = 5 \cdot + 3 \cdot - 1 + \Sigma \lfloor x \rfloor$ X₄ X_3 X2 \mathbf{X}_1 g h mnop a

Lyndon tree & Yule

 $depholm = 5 + 3 - 1 + \Sigma [x]$ TIm red point process But the endpoints TTm green point process m is of type (n,k,A) if $\Sigma_{1}(x_{1}) = G(\Pi_{m})$ + | TT_m | = n # | TTm |= k E[# ofleaves (n, k, A)] X₄ ⋆ TIm ∈ A $= L l_{n,k,A}$ X₃ $= \frac{(e_{n2} e)^{n} z^{-\ell} (e_{n2} e)^{k} z^{-\ell}}{n!} (h_{n2} e)^{k} z^{-\ell}} (h_{n2} e)^{k} (A)$ X2 X₁ g h mnop

Lyndon tree & Yule

 $\operatorname{depth} o \int \mathbf{m} = |\Pi_{m}^{\bullet}| + |\Pi_{m}^{\bullet}| + 1 + G(\Pi_{m}^{\bullet}) \qquad \sum \mathcal{I}_{\mathcal{I}} = G(\Pi_{m}^{\bullet})$ $L_{l,n,k,A} = \mathbb{E}\left[\# \text{ of leaves } (n,k,A)\right] = \frac{(l_{n2} e)^{n} 2^{-l} (l_{n2} e)^{k} 2^{-l} (l_{n,e}(A))}{n!}$

 $\frac{1}{n}$ ln $L_{\lambda n, n, \nu n}, G=\mu n \rightarrow \Psi(\lambda, \mu, \nu)$

