# LYNDONTREE \& BINARY SEARCH TREE 

## LUCAS MERCIER

## \&

PHILIPPE CHASSAING INSTITUT ELIE CARTAN

GLOSSARYAlphabetn-letters long wordsLanguage$U$ is a factor of $W$$U$ is a Prefix of $W$$U$ is a suffix of $W$RotationNecklace, circular wordPrimitive word

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aabbaaababbaaabaaThe standard factorization of a Lyndon word is the first step in the construction of some basis of the free Lie algebra over A

Lyndon tree

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see also Pitted 1984

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w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}
$$



Lyudon tree \& BST
 * To use Jabbrour LD rewalk.

$$
w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}
$$



Lyudon tree \& BST
Thm Under $L_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ideas: * To find a BST somewhere * To use Jabbour LD resulte. $w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$
$\rightarrow$ Assume $n$ leaves of rype EO

Lyudon tree \& BST
Thm Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jabhour LD retulk. $w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$, Assume $n$ leaves of rype $\varepsilon 0 \rightarrow$ Jabbour: $n^{1-\eta(\alpha)}$
 of them ar level $\alpha \log n$

Lyudon tree \& BST
Thm Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jabhowr LD resulk. $w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$, Assume $n$ leaves of rype $\varepsilon 0 \rightarrow$ Jabbour: $n^{1-\eta(\alpha)}$
 of them ar level $\alpha \log n$ $n^{1-\eta_{2}^{(x)}}$ shmbes bchowing like i.i.d geometric $1 / 2$ the highest is $\left(1-\eta_{2}(\alpha)\right) \log _{2} n$ high

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$w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$, Assume $n$ leaves of type $\varepsilon 0 \rightarrow$ Jabbour: $\eta^{1-\eta(\alpha)}$
 of them at level $\alpha \log n$ $n^{1-n(\alpha)}$ shmbes behoving like i.i.d geometric $1 / 2$ the highest is $\left(1-\eta_{2}(\alpha)\right) \log _{2} n$ high $\downarrow$
contribution to $H_{n}$ :

$$
\left(\alpha+\frac{1-\eta_{2}(\alpha)}{\ln 2}\right) \times \log n
$$

Lyndon tree \& BST
Th Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ideas: ${ }^{\log }$ To find a BST somewhere * To use Jabhour LD result.
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 of them at level $\alpha \log n$ $n^{1-\eta_{2}^{(x)}}$ shmeer behoving like i.i.d geometric $1 / 2$ the highest is $\left(1-\eta_{2}(\alpha)\right) \log _{2} n$ high $\downarrow$

$$
\sup _{\alpha}\left\{\alpha+\frac{1-\eta_{2}(\alpha)}{\ln 2}\right\} \simeq 5,09 \ldots
$$

$$
\left(\alpha+\frac{1-\eta_{2}(\alpha)}{\ln 2}\right) \times \log n
$$

Lyudon tree \& BST
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Lyudon tree \& BST
Thm Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jabhour LD remalk. $w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$, Assume $n$ leaves of rype $\rightarrow 0 \rightarrow$ Jabb $l: n^{1-\eta\left(2^{(\alpha)}\right.}$
 of them recrel $\alpha \log n$

Lyudon tree \& BST
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Problem: ried factors $a^{k} b^{e}$


Lyudon tree \& BST
Thm Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jabtrour LD remults.
$w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$


Problem: ried factors $a^{k} b^{l}$ Solution: $a^{k_{i}} b^{\ell_{i}} a^{l_{i}} b^{\ell_{i}} \omega_{i}$, in which $w_{i} \infty^{\text {k }}$ i id linary words

Lyndon tree \& BST
Th Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jabhour LD result.
$w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$


Problem: Hied factors $a^{k} b^{l}$ Solution: $a^{k_{i}} b^{l_{i}} a^{l_{i}} b^{\ell_{i}} w_{i}$, in which $w_{i} \infty^{\text {k }}$ i id binary words Problem: Fer $w \in \mathcal{L}_{n}$, the $\#$ of factors is not easy to handle.

Lyndon tree \& BST
Th Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jobbrour LD results.
$w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$


Problem: Tied factors $a^{k} b^{e}$
Solution: $a^{k_{i}} b^{l_{i}} a^{k_{i}} b^{\ell_{i}} w_{i}$, in which $w_{i} \infty^{r^{r}}$ i id binary words
Problem: Fer $w \in \mathscr{L}_{n}$, the $\#$ of factors is not easy to handle.

Solution: Consider We the random $\infty^{\text {te }}$ binary word truncated after the $1^{\text {tr }}$ occurrence of $a^{l}$, then reversed (rob eLyndon).

Lyndon tree \& BST
Th Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere * To use Jabhour LD result.
$u=a^{5} b^{3} a^{3} b a b^{6} a^{4} b^{2} a^{2} b^{8}$


Problem: Fer $w \in \mathscr{L}_{n}$, the $\#$ of factors is not easy to handle.

Solution: Consider $W_{l}$ the random os te binary word truncated after the 1 tr occurrence of $a^{l}$, then reversed (robe Lyndon).

Assumption: $\left|W_{l}\right| \simeq 2^{l}$ and $H\left(\mathscr{L}\left(W_{l}\right)\right) \simeq \alpha l$

Lyndon tree \& BST
Th Under $U_{n}, \frac{M_{n}}{\log n} \xrightarrow{(P)} 5,09 \ldots$. Ibleas: ${ }^{\log }$ To find a BST somewhere $w=a^{5} b^{3} a^{3} b a b^{4} a^{4} b^{2} a^{2} b^{8}$


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$$
\Rightarrow H_{n} \simeq \alpha \log _{2} n
$$

Lyndon tree \& BST


* To ute Jabber LD remelt. $\omega=a^{8} b^{2} a^{2}=a_{a}^{4} a^{4} a^{8} \varepsilon^{2} a^{2} b^{8}$


Problem: Fer $w \in \mathscr{L}_{n}$, the $\#$ of faction is not easy i - handle.
Solution: Comider $W_{l}$ the random ob le binary word truncated after the $1^{\text {Et }}$ occurrence of $a^{l}$, then reversed (robe Landon).

Assumption: $\left|W_{l}\right| \simeq 2^{l}$ and $H\left(\mathcal{L}_{\left(w_{l}\right)}\right) \simeq \alpha l$

$$
\Rightarrow H_{n} \simeq \alpha \log _{2} n \Rightarrow \frac{\alpha}{\ln 2}=5,09 \ldots
$$

Lyndon tree \& Yule

$$
\begin{aligned}
& U^{(l)}=\left(U_{1}, U_{2}, \ldots . ., U_{T_{e}}\right) \rightarrow\left\{\begin{array}{l}
T_{e}=\inf \left\{k / U_{k}<2^{-l}\right\} \\
u_{i} i . i . d .
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \text { sequence of } 1 / U_{i} \text {, reversed. } \\
& a b c d e f g h i j k l m n o p
\end{aligned}
$$

Lyndon tree \& Yule

Lyndon tree \& Yule.


Lyudon tree \& Yule.


$$
\begin{array}{r}
-\log _{2} U_{x}>-\log _{2} U_{y}+1 \Rightarrow+1 \text { (@) } \\
Y_{x}=-\log _{2} U_{x} \sim \text { loi exponentielle } \\
\mathbb{E}\left[y_{x}\right]=\frac{1}{\ln ^{2}}
\end{array}
$$



Lyndon tree \& Yule.


Lyndon tree \& Yule


Lyndon tree \& Yule.


## Lyndon tree \& Yule.

!



Lyndon tree \& Yule.


Lyndon tree \& Yule.


Lyndon tree \& Yule.
depth of $m=5 \cdot+3-1+\sum\left\lfloor x_{i}\right\rfloor$
$\Pi_{m}^{0}$ : red point process l Auth endpoints $\Pi_{m}^{\circ}$ green point process


Lyndon tree \& Yule.

$$
\begin{aligned}
& \text { depth of } m=\left|\pi_{m}^{0}\right|+\left|\pi_{m}^{0}\right|+1+G\left(\pi_{m}^{*}\right) \quad \sum\left\lfloor x_{i j}=G\left(\pi_{m}^{0}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{n} \ln L_{\lambda n, n, \nu n}, G=\mu n \rightarrow \psi(\lambda, \mu, \nu) \\
& x_{4} \sup _{i \mu} \frac{1}{\sqrt{\ln 2}}\left(1+\mu+\nu+\frac{\psi(\lambda, \mu, \nu)}{\ln 2}\right)=5,09 \ldots
\end{aligned}
$$

