Exclusion Process and Growth Models

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Séminaire P. Flajolet, IHP, March 2012

Introduction

The statistical mechanics of a system at thermal equilibrium is encoded in the Boltzmann-Gibbs canonical law:

$$P_{\mathrm{eq}}(\mathcal{C}) = \frac{\mathrm{e}^{-E(\mathcal{C})/kT}}{Z}$$

the Partition Function Z being related to the Thermodynamic Free Energy F:

$$F = -kTLog\ Z$$

This provides us with a well-defined prescription to analyze systems *at equilibrium*:

(i) Observables are mean values w.r.t. the canonical measure.

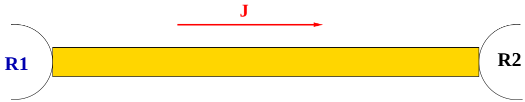
(ii) Statistical Mechanics predicts fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics.

Systems far from equilibrium

No fundamental theory is yet available.

- What are the relevant macroscopic parameters?
- Which functions describe the state of a system?
- Do Universal Laws exist? Can one define Universality Classes?
- Can one postulate a general form for the microscopic measure?
- What do the fluctuations look like ('non-gaussianity')?

Example: Stationary driven systems in contact with reservoirs.



Non-vanishing stationary current: a fingerprint for nonequilibrium

Rare Events and Large Deviations

Let $\epsilon_1, \dots, \epsilon_N$ be N independent binary variables, $\epsilon_k = \pm 1$, with probability p (resp. $q = 1 - p$). Their sum is denoted by $S_N = \sum_1^N \epsilon_k$.

- The Law of Large Numbers implies that $S_N/N \to p - q$ a.s.
- The Central Limit Theorem implies that $[S_N - N(p-q)]/\sqrt{N}$ converges towards a Gaussian Law.

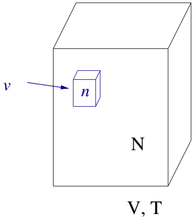
One can show that for $-1 < r < 1$, in the large N limit,

$$\Pr\left(\frac{S_N}{N} = r\right) \sim \mathrm{e}^{-N\,\Phi(r)}$$

where the positive function $\Phi(r)$ vanishes for $r = p - q$. The function $\Phi(r)$ is a Large Deviation Function: it encodes the probability of rare events.

$$\Phi(r) = \frac{1+r}{2}\ln\left(\frac{1+r}{2p}\right) + \frac{1-r}{2}\ln\left(\frac{1-r}{2q}\right)$$

Density fluctuations in a gas



Mean Density $\rho_0 = \frac{N}{V}$

In a volume v s. t. $1 \ll v \ll V$
$\langle \frac{n}{v} \rangle = \rho_0$

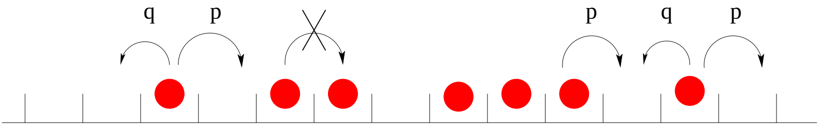
The probability of observing large fluctuations of density in v is given by

$$\mathrm{Pr}\left(\frac{n}{v} = \rho\right) \sim \mathrm{e}^{-v\,\Phi(\rho)}$$

with $\Phi(\rho) = f(\rho, T) - f(\rho_0, T) - (\rho - \rho_0)\frac{\partial f}{\partial \rho_0}$ where $f(\rho, T)$ is the *free energy per unit volume* in units of kT: the Thermodynamic Free Energy can be viewed as a Large Deviation Function.

Conversely, large deviation functions *may* play the role of potentials in non-equilibrium statistical mechanics.

ASEP



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

• EXCLUSION: Hard core-interaction; at most 1 particle per site.

• ASYMMETRIC: External driving; breaks detailed-balance (no microreversibility $\rightarrow$ current)

• PROCESS: Stochastic Markovian dynamics; no Hamiltonian

ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.
- Non-Hermitian Spin Chain: **ASEP is Integrable model solvable by Bethe Ansatz**

APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

Outline

1. Large deviations of the current in a closed ring
(S. Prolhac)

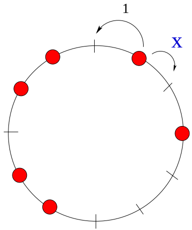
2. Fluctuations of the current in an open system
(A. Lazarescu)

3. Corner Dynamics in 2 and 3 dimensions
(P. Krapivsky, J. Olejarz and S. Redner)

1. Current Fluctuations

on a ring

Markov Equation for the ASEP on a ring



Master Equation for the Probability $P_t(x_1, \ldots, x_N)$ of being in configuration $1 \leq x_1 < \ldots < x_N \leq L$ at time t.

$$\begin{aligned}
\frac{\mathrm{d}P_t}{\mathrm{d}t} &= \sum_i{}' \left[P_t(x_1, \ldots, x_i - 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N)\right] \\
&+ x \sum_i{}' \left[P_t(x_1, \ldots, x_i + 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N)\right] \\
&= MP \,.
\end{aligned}$$

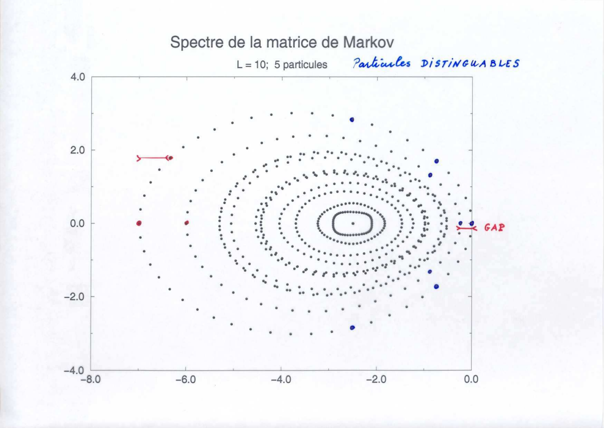
The sum being restricted to admissible configurations.

ASEP: SPECTRUM

Complex Eigenvalues $M\psi = E\psi$:

- Ground State : $E = 0$, $P = \Omega^{-1}$ (non-degenerate).
- Excited States : $\Re(E) < 0$ (Perron-Frobenius).

Excitations correspond to relaxation times.



Large Deviations of the Current

Statistics of the total current Y_t: total distance covered by all the N particles, hopping on a ring of size L, between time 0 and time t.

Let $P_t(\mathcal{C}, Y)$ be the joint probability of being at time t in configuration $\mathcal{C}$ with $Y_t = Y$. The time evolution of this joint probability can be deduced from the original Markov equation, by splitting the Markov operator

$$M = M_0 + M_+ + M_-$$

The Laplace transform of $P_t(\mathcal{C}, Y)$ with respect to Y, defined as $\hat{P}_t(\mathcal{C}, \mu) = \sum_Y \mathrm{e}^{\mu Y} P_t(\mathcal{C}, Y)$, satisfies a dynamical equation governed by the deformation of the Markov Matrix M, obtained by adding a jump-counting *fugacity* μ:

$$\frac{d\hat{P}_t}{dt} = M(\mu)\hat{P}_t$$

with

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

Cumulant generating function

In the long time limit, $t \to \infty$

$$\left\langle e^{\mu Y_t} \right\rangle \simeq e^{E(\mu) t}$$

where $E(\mu)$ is the eigenvalue of $M(\mu)$ with maximal real part.
Equivalently, $\Phi(j)$, the *large deviation function* of the current

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

is related to $E(\mu)$ by a Legendre transform

$$\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$$

The current statistics is reduced to an eigenvalue problem: This can be solved by Bethe Ansatz.

Deformed Markov Equation

The deformed Master Equation leads to the eigenvalue problem

$$E(\mu)\hat{P} = \sum_i{}' \left[e^{\mu} \hat{P}(x_1, \ldots, x_i - 1, \ldots, x_N; \mu) - \hat{P}(x_1, \ldots, x_i, \ldots x_N; \mu) \right]$$
$$+ x \sum_i{}' \left[e^{-\mu} \hat{P}(x_1, \ldots, x_i + 1, \ldots, x_N; \mu) - \hat{P}(x_1, \ldots, x_i, \ldots x_N; \mu) \right]$$

The sum being restricted to admissible configurations.

Integrability of ASEP: Bethe Ansatz

Eigenvector ψ of M written as a linear combination of plane waves, with pseudo-momenta given by $z_1, \dots z_N$:

$$\psi(x_1, \dots, x_N) = \sum_{\sigma \in \Sigma_N} \mathcal{A}_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

The Bethe Equations provide us with the quantification of the z_i's:

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x\mathrm{e}^{-\gamma} z_i z_j - (1+x) z_i + \mathrm{e}^{\gamma}}{x\mathrm{e}^{-\gamma} z_i z_j - (1+x) z_j + \mathrm{e}^{\gamma}}$$

The corresponding eigenvalues of $M(\gamma)$ are

$$E(\gamma; z_1, z_2 \dots z_N) = \mathrm{e}^{\gamma} \sum_{i=1}^{N} \frac{1}{z_i} + x\mathrm{e}^{-\gamma} \sum_{i=1}^{N} z_i - N(1+x)\,.$$

The Bethe equations do not decouple unless $x = 0$
(*This case was solved by B. Derrida and J. L. Lebowitz, 1998*).

The special case of TASEP

TASEP : $x = 0$

The Bethe Eigenvectors of M can be written as determinants.
The Bethe Equations reduce to with the quantification of the z_i's:

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{\mathrm{e}^{\gamma} - z_i}{\mathrm{e}^{\gamma} - z_j}$$

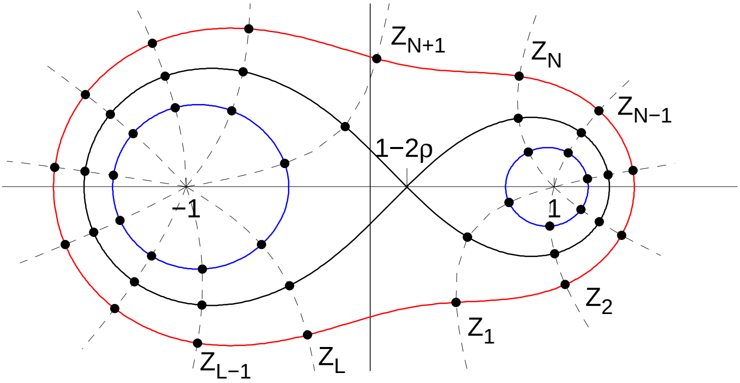
This leads to an effective single variable problem with a symmetric self-consistency condition:

$$z_i^{-L} (\mathrm{e}^{\gamma} - z_i)^N = (-1)^{N-1} \prod_{j=1}^{N} (\mathrm{e}^{\gamma} - z_j)$$

Note that the r.h.s. is a constant independent of i: DECOUPLING.

Labelling the roots of the Bethe Equations

Up to a change of variables, one can show that the the roots (for $x = 0$) belong to remarkable curves, **The Cassini Ovals**



Calculation of the GAP

The first excited state is solution of a transcendental equation. For a density ρ:

$$E_1 = \underbrace{-2\sqrt{\rho(1-\rho)}\frac{6.509189337\ldots}{L^{3/2}}}_{\text{RELAXATION}} \pm \underbrace{\frac{2i\pi(2\rho-1)}{L}}_{\text{OSCILLATIONS}} .$$

• Non-diffusive: Largest relaxation time $T \sim L^z$ with $z = 3/2$ *(D. Dhar, L.H. Gwa and H. Spohn, D. Kim, O. Golinelli and K. M.)*.

• Classification of higher excitations *(J. de Gier and F.H.L. Essler, 2006)*.

TASEP Current (Derrida Lebowitz 1998)

$E(\mu)$ is calculated by Bethe Ansatz to all orders in μ, thanks to the decoupling property of the Bethe equations.
The structure of the solution is given by a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\frac{1}{L}\sum_{k=1}^{\infty}\frac{[kL]!}{[kN]!\,[k(L-N)]!}\frac{B^k}{k}\,,$$

$$E = -\sum_{k=1}^{\infty}\frac{[kL-2]!}{[kN-1]!\,[k(L-N)-1]!}\frac{B^k}{k}\,.$$

Mean Total current:

$$J = \lim_{t\to\infty}\frac{\langle Y_t\rangle}{t} = \frac{N(L-N)}{L-1}$$

Diffusion Constant:

$$D = \lim_{t\to\infty}\frac{\langle Y_t^2\rangle - \langle Y_t\rangle^2}{t} = \frac{LN(L-N)}{(L-1)(2L-1)}\frac{C_{2L}^{2N}}{\left(C_L^N\right)^2}$$

Exact formula for the large deviation function.

Functional Bethe Ansatz for the General Case

After a change of variable, $y_i = \frac{1-e^{-\gamma} z_i}{1-x e^{-\gamma} z_i}$, the Bethe equations read

$$e^{L\gamma} \left(\frac{1-y_i}{1-xy_i} \right)^L = -\prod_{j=1}^{N} \frac{y_i - xy_j}{xy_i - y_j} \quad \text{for} \quad i = 1 \ldots N \,.$$

Let T be auxiliary variable playing a symmetric role w.r.t. all the y_i:

$$e^{L\gamma} \left(\frac{1-T}{1-xT} \right)^L = -\prod_{j=1}^{N} \frac{T - xy_j}{xT - y_j} \quad \text{for} \quad i = 1 \ldots N \,.$$

$$i.e.\, P(T) = e^{L\gamma}(1-T)^L \prod_{j=1}^{N} (xT - y_j) + (1-xT)^L \prod_{j=1}^{N} (T - xy_j) = 0.$$

But $P(y_i) = 0$ (Bethe Eqs.). Thus, $Q(T) = \prod_{i=1}^{N} (T - y_i)$ divides $P(T)$:

$$Q(T) \;\; \text{DIVIDES} \;\; e^{L\gamma}(1-T)^L Q(xT) + (1-xT)^L x^N Q(T/x).$$

Functional Bethe Ansatz

There exist two polynomials $Q(T)$ and $R(T)$ such that

$$\boxed{Q(T)R(T) = e^{L\gamma}(1-T)^L Q(xT) + x^N (1-xT)^L Q(T/x)}$$

where $Q(T)$ of degree N vanishes at the Bethe roots.
Functional Bethe Ansatz (Baxter's TQ equation): Restatement of the Bethe Ansatz as a purely algebraic problem. This equation is solved perturbatively w.r.t. γ.

Knowing $Q(T)$, we obtain an expansion of $E(\gamma)$. This provides the full statistics of the current and its large deviations.

Cumulants of the Current

- Mean Current: $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$

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$$D \sim 4\phi L\rho(1-\rho)\int_0^\infty du \frac{u^2}{\tanh \phi u} e^{-u^2}$$

when $L \to \infty$ and $x \to 1$ with *fixed value* of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

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- Third cumulant (Skewness):

$$\frac{E_3}{\phi(\rho(1-\rho))^{3/2}L^{5/2}} \simeq -\frac{4\pi}{3\sqrt{3}} +$$
$$12\int_0^{\infty} dudv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh \phi u \tanh \phi v}$$

$\to$ Non Gaussian fluctuations. TASEP limit for $\phi \to \infty$:

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

$$
\begin{aligned}
\frac{E_3}{6L^2} &= \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\,\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\,\frac{i^2+ij+j^2}{2}\,\frac{1+x^i}{1-x^i}\,\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N-i}C_L^{N-j}C_L^{N+i+j}}{(C_L^N)^3}\,\frac{i^2+ij+j^2}{2}\,\frac{1+x^i}{1-x^i}\,\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\,\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2 \\
&+ (1-x)\frac{N(L-N)}{4(L-1)(2L-1)}\,\frac{C_{2L}^{2N}}{(C_L^N)^2} \\
&- (1-x)\frac{N(L-N)}{6(L-1)(3L-1)}\,\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}
$$

The weakly symmetric case

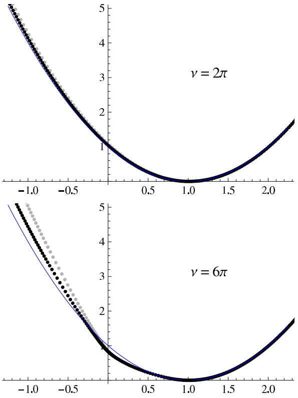
For large system sizes, $L \to \infty$, in the scaling limit $x = 1 - \frac{\nu}{L}$, the cumulant generating function is given by

$$E\left(\frac{\gamma}{L}\right) \simeq \frac{\rho(1-\rho)(\gamma^2+\gamma\nu)}{L} - \frac{\rho(1-\rho)\gamma^2\nu}{2L^2} + \frac{1}{L^2}\phi[\rho(1-\rho)(\gamma^2+\gamma\nu)]$$

$$\text{with} \quad \phi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$$

- The B_j's are Bernoulli Numbers.
- Leading order (in $1/L$): Gaussian fluctuations.
- Subleading (in $1/L^2$) : Non-Gaussian correction.
- Phase transition (*T. Bodineau and B. Derrida*) for $\nu \geq \nu_c = \frac{2\pi}{\sqrt{\rho(1-\rho)}}$

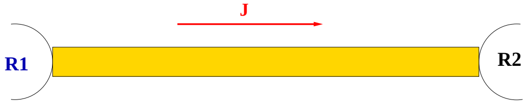
Behaviour of the large deviation function



2. Current Fluctuations in the open ASEP

The Current in the Open System

The fundamental paradigm



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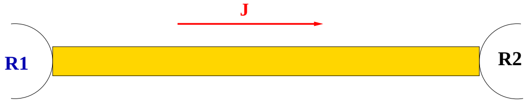
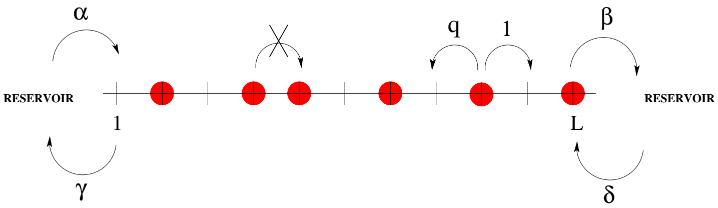


The asymmetric exclusion model with open boundaries



The Matrix Ansatz for TASEP (DEHP, 1993)

The stationary probability of a configuration $\mathcal{C}$ is given by

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle\alpha| \prod_{i=1}^{L} (\tau_i D + (1-\tau_i) E) |\beta\rangle \,.$$

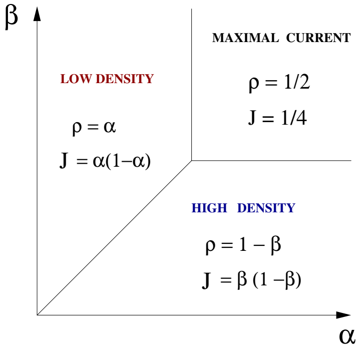
where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).
The normalization constant is $Z_L = \langle\alpha| (D+E)^L |\beta\rangle$

The operators D and E, the vectors $\langle\alpha|$ and $|\beta\rangle$ satisfy

$$\begin{aligned} D\,E &= D + E \\ D\,|\beta\rangle &= \frac{1}{\beta}\,|\beta\rangle \\ \langle\alpha|\,E &= \frac{1}{\alpha}\langle\alpha| \end{aligned}$$

This technique can be extended to PASEP by suitably deforming the quadratic algebra.

Phase Diagram (TASEP q=0)



This exact result is obtained using The Matrix Ansatz (DEHP, 1993).
Stationary probabilities are written as traces over a suitable quadratic algebra.

Large Deviations of the Current: Framework

Let N_t be the total (time-integrated) number of particles exchanged between the system and the left reservoir between 0 and t. When a particle enters or leaves the system at site 1:

$$N_t = N_t \pm 1$$

- Expectation value: $\lim_{t\to\infty} \frac{\langle N_t \rangle}{t} = J(q; \alpha, \beta, \gamma, \delta, L)$
- Variance: $\lim_{t\to\infty} \frac{\langle N_t^2 \rangle - \langle N_t \rangle^2}{t} = \Delta(q; \alpha, \beta, \gamma, \delta, L)$
- Cumulant Generating Function: $\langle \exp(\mu N_t) \rangle \simeq \exp(E(\mu) t)$

$$\boxed{E(\mu) = J\mu + \Delta \frac{\mu^2}{2} + E_3 \frac{\mu^3}{3!} + \dots}$$

The **Large-Deviation Function** $\Phi(j)$ of the total current

$$P\left(\frac{N_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

is the *Legendre transform* of the Cumulant Generating Function $E(\mu)$.

A 'simple' case

In the case $q = 0$ and $\alpha = \beta = 1$ a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k},$$

$$E = -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k}.$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $$E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368}\pi \sim -0.0090978\ldots$

Full Current Statistics of TASEP

For $q = 0$ and arbitrary (α, β), the parametric representation of $E(\mu)$ is

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha,\beta)\frac{B^k}{2k}$$
$$E = -\sum_{k=1}^{\infty} D_k(\alpha,\beta)\frac{B^k}{2k}$$

with

$$C_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}\,, \qquad a = \frac{1-\alpha}{\alpha}\,, \qquad b = \frac{1-\beta}{\beta}$$

Some explicit expressions

- Mean Current: (Same expression as in DEHP)

$$J = \frac{D_1(\alpha,\beta)}{C_1(\alpha,\beta)}$$

- Fluctuations:

$$\Delta = \frac{D_1\, C_2 - D_2\, C_1}{C_1^3}$$

- Saddle point analysis in the low density phase: $(\rho = \alpha)$

$$\begin{aligned}
E_1 &= \rho(1-\rho)\\
E_2 &= \rho(1-\rho)(1-2\rho)\\
E_3 &= \rho(1-\rho)(1-6\rho+6\rho^2)\\
E_4 &= \rho(1-\rho)(1-2\rho)(1-12\rho+12\rho^2)\\
E_5 &= \rho(1-\rho)(1-30\rho+150\rho^2-240\rho^3+120\rho^4)\ \ldots
\end{aligned}$$

Asymptotics in the TASEP Phase Diagram

In the limit $L \to \infty$ of systems of large size, we have

- Maximal Current phase $\alpha > 1/2$ and $\beta > 1/2$: Cumulants are independent from α and β

$$E_k \sim \pi(\pi L)^{k/2-3/2} \text{ for } \quad k \geq 2$$

- Low Density phase $\alpha < \min(\beta, 1/2)$:

$$E(\mu) = \frac{a}{a+1}\frac{\mathrm{e}^{\mu}-1}{\mathrm{e}^{\mu}+a}$$

By Legendre Transform, the current Large Deviation Function is

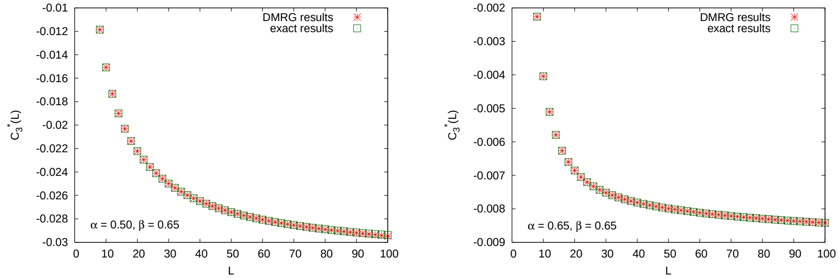
$$\Phi(j) = \alpha - r + r(1-r)\ln\left(\frac{1-\alpha}{\alpha}\frac{r}{1-r}\right)$$

where the current j is parametrized as $j = r(1-r)$.
Agrees with Macroscopic Fluctuation Theory (T. Bodineau and B. Derrida).

- Along the shock line $\alpha = \beta \leq 1/2$, fluctuations are enhanced

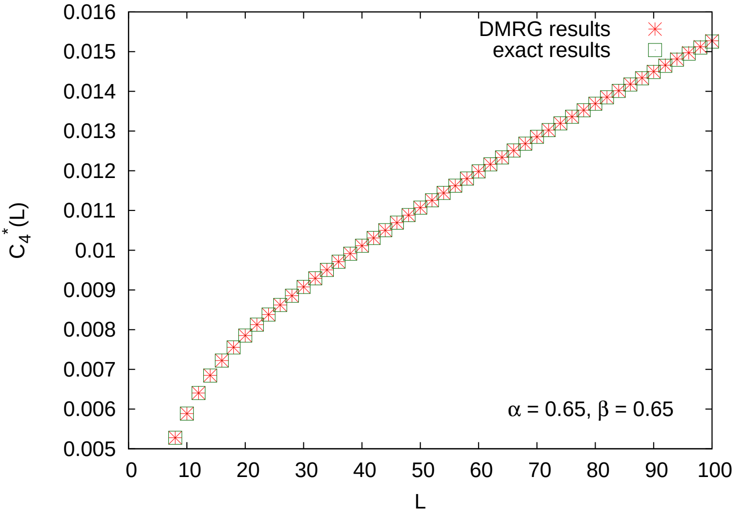
$$E_k \simeq \epsilon_k \alpha(1-\alpha)(1-2\alpha)^{k-1}L^{k-2} \text{ for } \; k \geq 2$$

DMRG Results (M. Gorissen, C. Vanderzande)

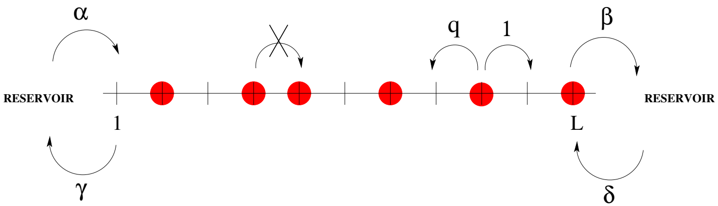


SKEWNESS

Fourth Cumulant (DMRG)



Current fluctuations in the general ASEP



The stationary probability of a configuration $\mathcal{C}$ is given by

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) | V \rangle .$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).

The operators D and E, the vectors $\langle W|$ and $|V\rangle$ now satisfy

$$\begin{aligned} D\,E - q E\,D &= D + E \\ (\beta\,D - \delta\,E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\,E - \gamma\,D) &= \langle W| \end{aligned}$$

Structure of the solution I

For arbitrary values of q and $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\mu = -\sum_{k=1}^{\infty} C_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(q; \alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

The coefficients C_k and D_k are given by contour integrals in the complex plane:

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

Structure of the solution II

The auxiliary function $W_B(z) = \sum_{k\geq 1} \phi_k(z) \frac{B^k}{k}$ solves a functional Bethe equation:

$$W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

- The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K(z_1, z_2)$$

$$\text{with kernel} \quad K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}$$

- The function $F(z)$ is given by

$$F(z) = \frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}$$

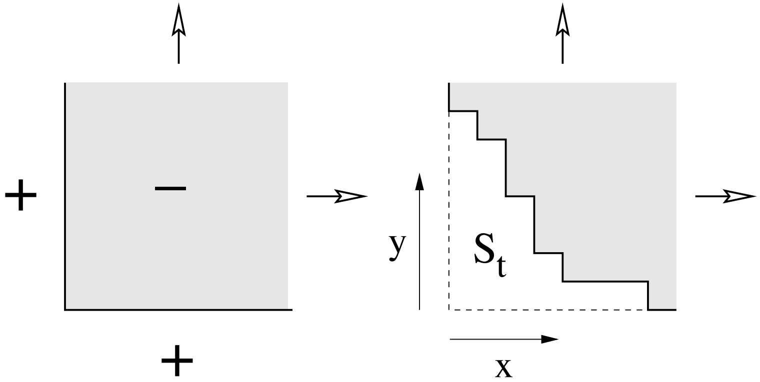
where $(x)_\infty = \prod_{k=0}^{\infty}(1 - q^k x)$ and $a_\pm$, $b_\pm$ are simple functions of the boundary rates.

Sketch of the Technique

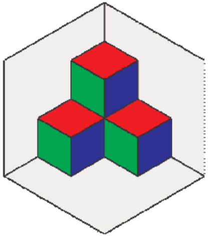
- The dominant eigenvector of the deformed operator $M(\mu)$ is formally expanded w. r. t. μ.
- The term of order k in this expansion can be expressed as the solution of a linear problem involving the original, non-deformed, matrix M.
- The knowledge of the dominant eigenvector at order k in μ enables us to calculate the eigenvalue $E(\mu)$ at order $k+1$.
- We construct of a Matrix Ansatz at each order k: This Ansatz involves $(2k+1)$ Tensor Products of the original quadratic algebra.
- This recursive structure is very closely related to the algebras used to solve multispecies exclusion process.
- The tensorial Matrix Ansatz leads to the parametric expression for $E(\mu)$

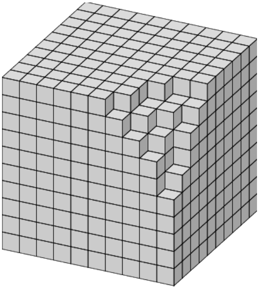
3. Shapes of Growth

A crystal growing on a corner in two dimensions

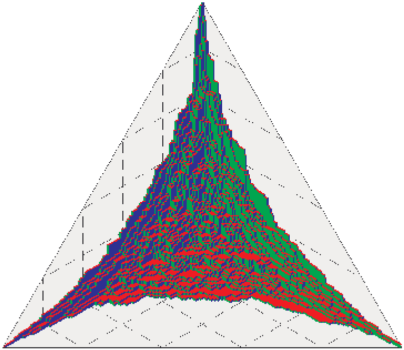


Corner Growth/Melting in three dimensions





Corner Growth in three dimensions



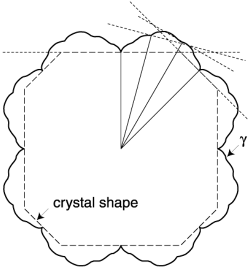
The Wulff Construction

What is the shape of a large crystal at thermodynamic equilibrium?
One has to minimize the surface free energy for a fixed total volume

The surface free energy σ depends on the local orientation of the crystal, given by the normal direction $\vec{n}$. The Wulff diagram is the surface (or curve)

$$\vec{n} \to \sigma(\vec{n})\vec{n}$$

The solution of the variational problem is purely geometrical:
La forme du cristal est la podaire (pedal) de la surface de Wulff



Limiting shapes of Young Tableaux

A Young Tableau of total size N encodes partitions of the integer N:

$$N = r_1 + r_2 + \ldots + r_p \qquad \text{with} \qquad r_1 \geq r_2 \geq \ldots \geq r_p$$

The generating function of the partition numbers was found by Euler and their asymptotics by Hardy and Ramanujan:

$$\sum_N p(N)\, x^N = \prod_k \frac{1}{1 - x^k} \qquad \text{with} \qquad p(N) \sim \exp\left(\pi \sqrt{2N/3}\right)$$

For Partitions of N with the uniform measure, Vershik (1996) found that the asymptotic shape of a Young Tableau of size N, $\ell \to \langle r_\ell \rangle_N$, becomes

$$\boxed{\exp(-x) + \exp(-y) = 1}$$

with the rescalings $\ell = x\, \frac{\sqrt{6N}}{\pi}$ and $r = y\, \frac{\sqrt{6N}}{\pi}$. *This limit shape can also be obtained by a Wulff construction by noting that the frontier of a tableau is a South-East directed walk and by enumerating such walks.* Fluctuation of this shape were studied by Borodin, Okounkov and Olshanki (2000).

Effects of the statistics

Another natural measure on Young Tableaux is the Plancherel measure

$$\mu(\lambda) = \frac{(f^{\lambda})^2}{N!}$$

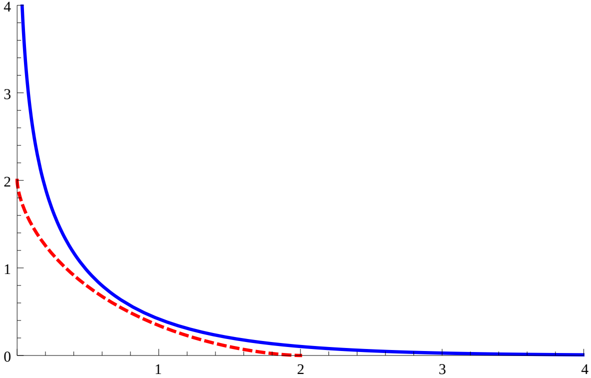
The limit shape of Young Tableaux, drawn with the Plancherel measure, was found by Logan and Shepp and by Vershik and Kerov (1977)

$$\begin{aligned} x &= y + 2\cos\theta \\ y &= \frac{2}{\pi}(\sin\theta - \theta\cos\theta) \end{aligned}$$

where the axis-coordinates have been scaled by $\sqrt{N}$ and $0 \leq \theta \leq \pi$.

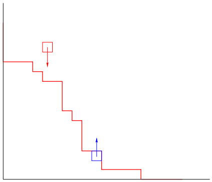
Note that $x_{max} = 2\sqrt{N}$: this is precisely the mean-length of the longest increasing subsequence extracted from N randomly ordered numbers (Ulam's problem, restated by Schensted).

Comparison: Plancherel versus uniform measure



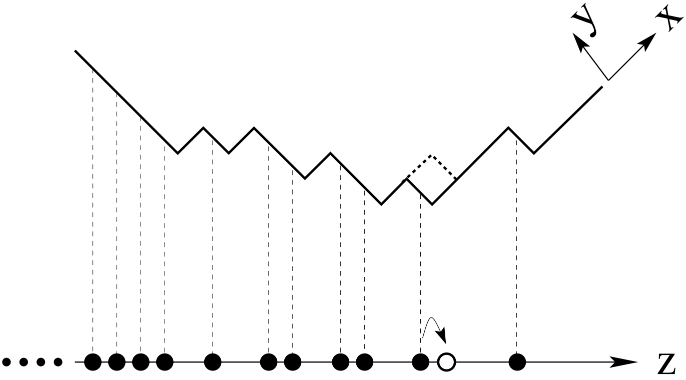
Far from equilibrium

We now focus on the non-equilibrium growth of a Young diagram: Evolution of a corner under Glauber dynamics at a vanishing low temperature.



Evaporation/Deposition events that respect the monotonicity constraint (no over-hangs). Different types of dynamics:
(i) Deposition and Evaporation with the same rate (Zero magnetic field).
(ii) Deposition with rate 1. No evaporation (Non-Zero magnetic field).
(iii) Deposition and Evaporation with different rates (Vanishingly small field). A negative field stabilizes the phase inside the corner and leads to a finite size equilibrium shape.

Mapping to a one-dimensional particle process



Evolution of the mean shape

Thanks to this mapping the shape of the crystal corresponds to the density profile in the particle language.
In the case of deposition only, H. Rost (1981) found the limiting shape of the crystal at time t from the TASEP:

$$\sqrt{x} + \sqrt{y} = \sqrt{t}$$

For evaporation and deposition occurring at the same rate (zero magnetic field), the limiting shape was calculated only recently:

$$\eta = \frac{1}{\sqrt{4\pi}} e^{-(\xi-\eta)^2} - \frac{\xi-\eta}{\sqrt{\pi}} \int_{\xi-\eta}^{\infty} d\zeta \, e^{-\zeta^2}$$

where $\xi = \frac{x}{\sqrt{4t}}$, and $\eta = \frac{y}{\sqrt{4t}}$. In particular, the diagonal $x = y$ crosses the interface at $\xi = \eta = (4\pi)^{-1/2}$ and therefore $x = y = \sqrt{t/\pi}$.

Statistics of the apex height

The height above the origin (i.e. the intersection of the diagonal $x = y$ with the interface) corresponds to the total current Q_t that has flown through the (0,1) bond in the exclusion process.
The statistics of Q_t was investigated by K. Johansson in the TASEP case. They found

$$Q_t = \frac{t}{4} + \frac{t^{1/3}}{2^{4/3}}\,\chi$$

where χ is a random variable distributed according to

$$\text{Prob}\,(\chi \leq s) = 1 - F_2(-s)$$

$$\text{with} \qquad F_2(s) = \exp\left(-\int_s^{\infty}(x-s)\,u(x)^2 dx\right)$$

$u(x)$ being the solution of Painlevé II equation $u'' = xu + 2u^3$, matching the Airy function at infinity. The Tracy-Widom function F_2 is the cumulative distribution of the maximal eigenvalue λ_{max} in a GUE.
The SEP case was studied recently by Derrida and Gerschenfeld.

Statistics of the Area

The area of the molten region can be written in terms of the displacements of the particles

$$S_t = \sum_{\text{all particles}} \text{displacements}$$

It is more convenient to use the representation of the area in terms of local Boolean variables $\tau_x(t) = 0, 1$ which indicate if site x is empty or occupied at time t:

$$S_t = \sum_{x=-\infty}^{+\infty} x[\tau_x(t) - \tau_x(0)]$$

We want to compute the statistics of S_t.

Some results

At all times, we have

$$\langle S_t \rangle = t$$

The variance of the area is given by

$$\lim_{t\to\infty} \frac{\langle S_t^2 \rangle - \langle S_t \rangle^2}{t^{3/2}} = \frac{4\sqrt{2}}{3\sqrt{\pi}} = 1.063846080\ldots$$

The probability that the quadrant is in the initial state at a large time t, i.e. the probability that $S_t = 0$:

$$\lim_{t\to\infty} \frac{\ln P_0(t)}{\sqrt{t}} = -\frac{1}{\sqrt{\pi}}\zeta\left(\frac{3}{2}\right) = -1.473874960\ldots$$

The large deviation property of the surface is given by

$$\text{Prob}\left(\frac{S_t}{t} = s\right) \sim \mathrm{e}^{-\sqrt{t}\phi(s)}$$

where $\phi(s) \sim (s-1)^2$ in the vicinity of $s = 1$.

Relation with Bethe Ansatz (Tracy and Widom)

The Green function $P_t^{(n)}(y_1, y_2, \ldots, y_n | x_1, x_2, \ldots, x_n)$ is known from Bethe Ansatz and is given by

$$\sum_\sigma \epsilon(\sigma) \oint \left[\prod_{k=1}^{n} \frac{dz_k}{2i\pi z_k} \mathrm{e}^{\mathrm{t}(\mathrm{z_k}+1/\mathrm{z_k}-2)} \mathrm{z}_{\sigma(\mathrm{k})}^{\mathrm{y_k}-\mathrm{x}_{\sigma(\mathrm{k})}} \right] \left[\prod_{k \langle l} \frac{z_{\sigma(k)} z_{\sigma(l)} + 1 - 2z_{\sigma(k)}}{z_k z_l + 1 - 2z_k} \right]$$

From this equation, one can calculate the values of the k-point correlations $\langle \tau_{y_1}(t) \ldots \tau_{y_k}(t) \rangle$.

For the SEP, this allows us to calculate the moments of the area.

This is not true for the general case (totally or partially asymmetric): open problem.

3d Plane Partitions with Uniform Measure

In a plane partition, the height satisfies

$$0 \leq h(k+1,l) \leq h(k,l) \quad \text{and} \quad 0 \leq h(k,l+1) \leq h(k,l)$$

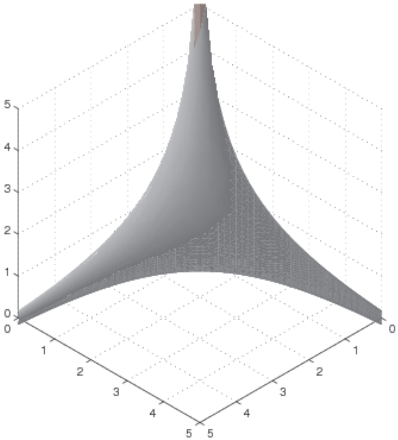
Consider plane partitions of total volume N, chosen randomly with uniform probability measure. The limiting shape of the corner crystal was found by R. Cerf and R. Kenyon, and also by A. Okounkov (ca 2000).

The limiting surface is $\left(\frac{\zeta(3)}{4}\right)^{-1/3} S_0$ (after rescaling the coordinates by $N^{1/3}$), S_0 being given by the parametric equations

$$S_0 = \{(f(A,B,C) - \ln A,\ f(A,B,C) - \ln B,\ f(A,B,C) - \ln C)\}$$

A, B, C are strictly positive and

$$f(A,B,C) = \frac{1}{4\pi^2} \int_{[0,2\pi]} \int_{[0,2\pi]} \ln \left| A + B\mathrm{e}^{\mathrm{i}\theta} + \mathrm{C}\mathrm{e}^{\mathrm{i}\phi} \right| d\theta d\phi$$


5
4
3
2
1
0
0
1
2
3
4
5
5
4
3
2
1
0

Growing Plane Partitions

What is the limit shape for a Plane Partition growing with Glauber dynamics with non-zero magnetic field?

We conjecture the following governing equation for the interface profile:

$$z_t = \frac{z_x}{z_x - 1}\,\frac{z_y}{z_y - 1}\left[1 - \frac{1}{z_x + z_y}\right]$$

- We seeked an equation of the form $z_t = F(z_x, z_y)$ involving only first derivatives.
- The correct 3d equation has to reduce to the 2d equation of motion on the boundaries $x = 0$ or $y = 0$. In 2 dimensions, the dynamics of an interface $\zeta(\eta)$ obeys

$$\zeta_t = \frac{\zeta_\eta}{\zeta_\eta - 1}$$

- The equation $z_t = F(z_x, z_y)$ must be invariant under exchange of any pair of coordinates:

$$F\left(\frac{1}{a}, \frac{1}{b}\right) = -\frac{1}{a} F\left(a, -\frac{a}{b}\right)$$

The method of characteristics

Any first order PDE $F(x, y, z, z_x, z_y) = 0$ can be shown to be equivalent to the following set of first order ODE's:

$$\begin{aligned}
x_t &= F_q \\
y_t &= F_r \\
z_t &= qF_q + rF_r \\
q_t &= -F_x - qF_z \\
r_t &= -F_y - rF_z
\end{aligned}$$

where $q = z_x$ and $r = z_y$.

In the case we are interested in, the function F does not depend on the variable z. The equations for the characteristics become Hamiltonian.

Solution of the PDE

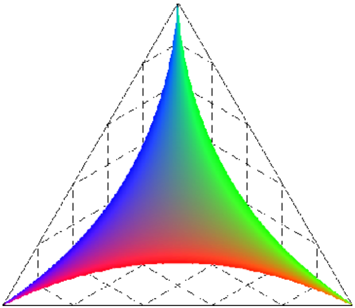
Using the method of characteristics, the conjectured equation can explicitly be solved:

$$\frac{x}{t} = A(q,r) \quad \frac{y}{t} = B(q,r) \quad \frac{z}{t} = C(q,r)$$

with

$$\begin{aligned} A &= \frac{r^2}{(r-1)(q-1)(q+r)} \left[\frac{1}{q-1} + \frac{1}{q+r} \right] \\ B &= \frac{q^2}{(r-1)(q-1)(q+r)} \left[\frac{1}{r-1} + \frac{1}{q+r} \right] \\ C &= \frac{q^2 r^2}{(r-1)(q-1)(q+r)} \left[\frac{1}{q-1} + \frac{1}{r-1} \right] \end{aligned}$$

Some numerical tests



This surface cuts the ground plane $z = 0$ along the curve $\sqrt{x} + \sqrt{y} = \sqrt{t}$. The intersection with the ray $x = y = z$ is predicted to be $\frac{t}{8}$. Numerical simulations give the speed 0.1261...

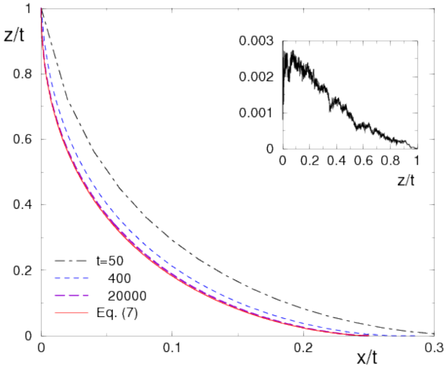
The volume can be calculated exactly and it grows as

$$V/t^3 = v = \frac{3\pi^2}{2^{11}} = 0.014457\ldots$$

The numerical measurement gives $v \approx 0.01472(3)$, within 1.8% of the prediction.

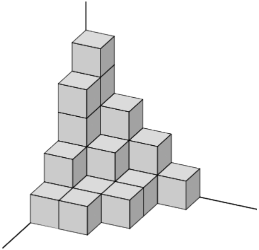
The intersection with the plane $x = y$ is given by

$$\frac{x}{t} = \frac{1}{2}\frac{z}{t} - \frac{3}{4}\left(\frac{z}{t}\right)^{2/3} + \frac{1}{4}$$

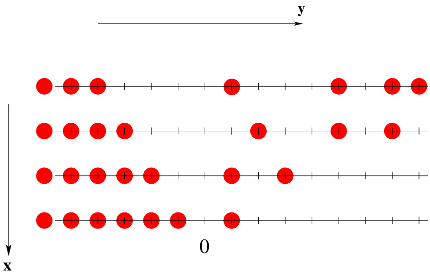


(Jason Olejarz, Sid Redner, P. K. and K. M: PRL **108** 016102 (2012))

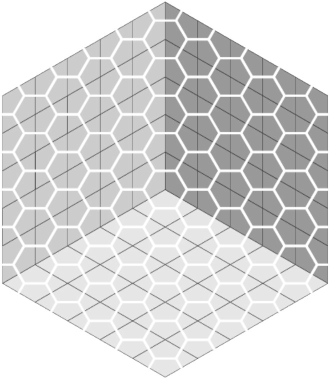
Equivalence with particle systems



Equivalent to family of coupled exclusion processes:



Dimers versus ABC model in the plane



More conjectures

- Diffusive Growth in 3d:

$$z_t = \frac{\left(1 - \frac{1}{z_x + z_y}\right)^2}{\left(1 - \frac{1}{z_x}\right)^2 \left(1 - \frac{1}{z_y}\right)^2} \left[\frac{z_{xx}}{z_x^2} + \frac{z_{yy}}{z_y^2} - \frac{z_{xy}}{z_x z_y} \right]$$

Higher Dimensions:

- **Ballistic:**

$$h_t = H = \prod_{1 \le i_1 < \ldots < i_p \le d-1} \left(1 - \frac{1}{h_{i_1} + \ldots + h_{i_p}}\right)^{(-1)^p}$$

- **Diffusive:**

$$\frac{h_t}{H^2} = \sum_{i=1}^{d-1} \frac{h_{ii}}{h_i^2} - \frac{2}{d-1} \sum_{i<j} \frac{h_{ij}}{h_i h_j}$$

A lot of work remains to be done to verify these conjectured equations and to extract from them other cogent tests.

Conclusions

Exact solutions of the asymmetric exclusion process are paradigms for the behaviour of systems far from equilibrium in low dimensions: Dynamical phase transitions, Non-Gibbsean measures, Large deviations, Fluctuations Theorems...

The large deviation functions (LDF) appear as the right generalization of the thermodynamic potentials: convex, optimized at the stationary state, and non-analytic features can be interpreted as phase transitions. Besides, the LDF's satisfy remarkable identities (Gallavotti-Cohen) valid far away from equilibrium. The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.

Interacting Particle Processes also represent Growth Models and can used to investigate properties of dimer tilings, Young tableaux and plane partitions.