Séminaire de Combinatoire Philippe Flajolet

Algèbres quadratiques, Robinson-Schensted, matrices à signes alternants et au delà ...

26 Mai 2011

IHP, Paris

XGV

LaBRI, Bordeaux

Heisenberg
operators
U, D

UD = DU + I

$$UD = DU + I$$

<u>Lemme</u> - Tout mot $w \in \{U, D\}^*$ s'écrit

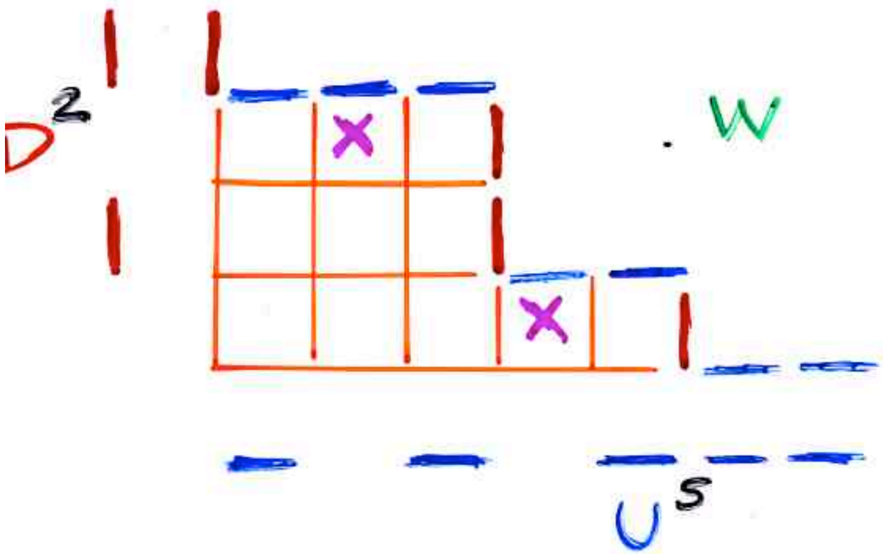
$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

notation $w \rightarrow F_w$ diagram Ferrers



<u>Prop</u>. $c_{i,j}(w)$ = nb de "placement" de k tours sur F

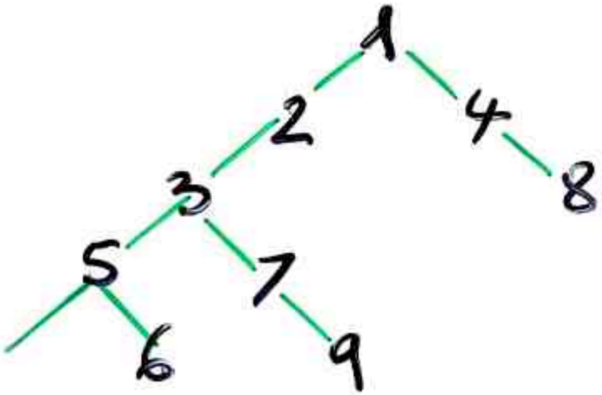
avec $i = |w|_D - k$

$j = |w|_U - k$

Prop. $w = (UD)^n$

$c_{k,k}(w) = S_{n+1,\,k+1}$ Stirling

nombre de partitions de $\{1, 2, \ldots, n+1\}$ en $(k+1)$ blocs

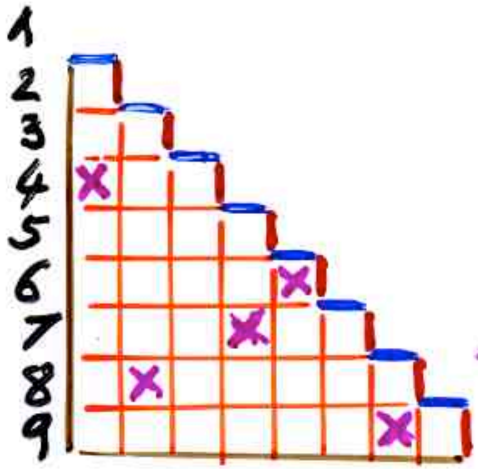


$\{1, 4, 8\}$

$\{2\}$

$\{3, 7, 9\}$

$\{5, 6\}$



K. Penson, I. Solomon
P. Blasiak, A. Horzela
G. Duchamps

P. Blasiak, P. Flajolet
2010

A produit par x

S $\frac{d}{dx}(\ )$

The cellular Ansatz

first part

$$UD = DU + I$$

$$UD \to DU \qquad UD \to I$$

$$UD = DU + Id$$

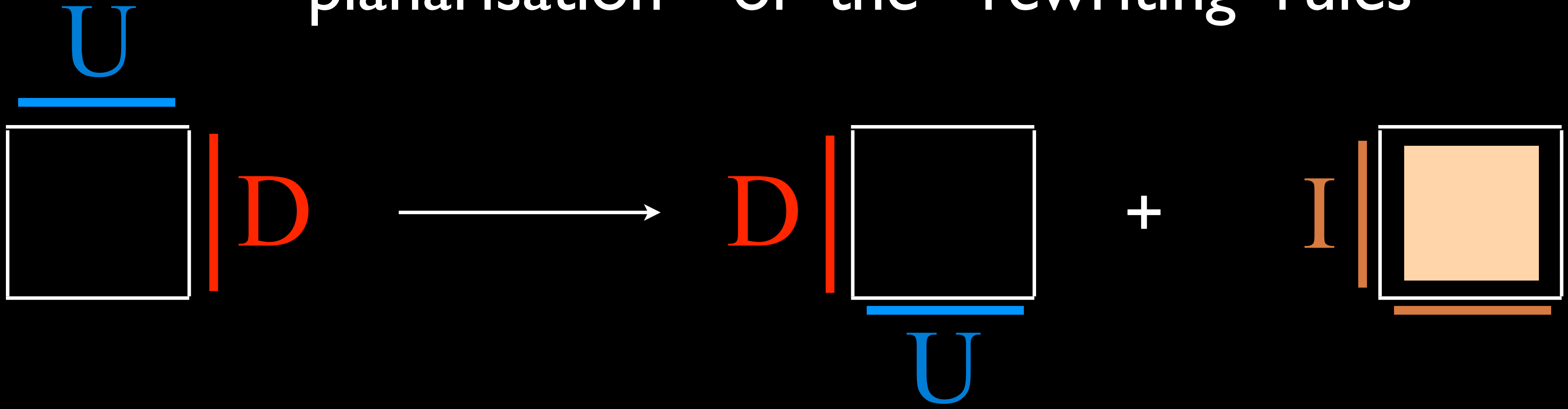
$$U^n D^n = ?$$

$$\begin{aligned}
UUUDDD &= UU(DU + Id)DD \\
&= UUDUDD + UUDD \\
&= UDUUDD + 2\,UUDD \\
&= DUUUDD + 3\,UUDD
\end{aligned}$$

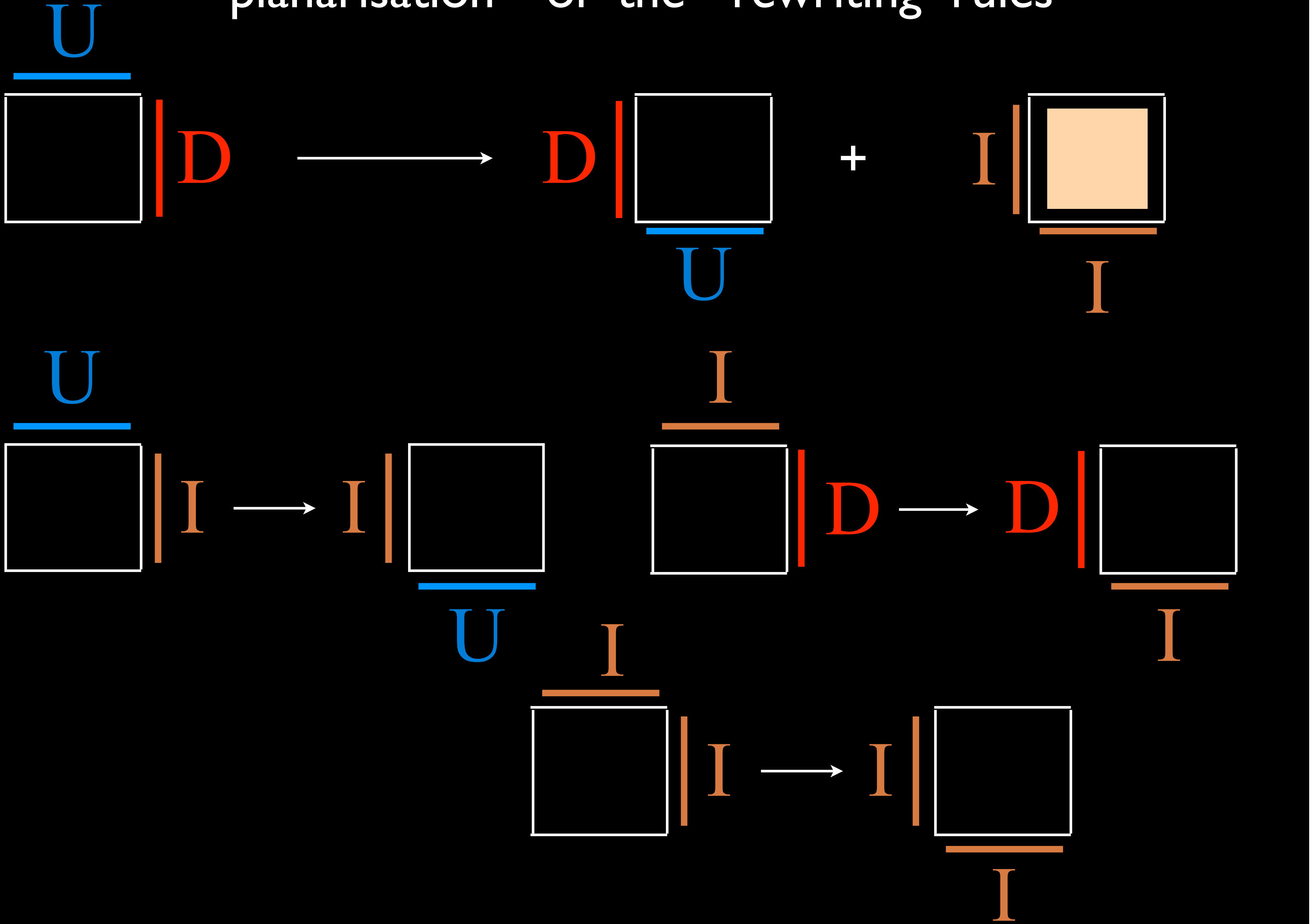
$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2\,UD \\
&= DUDU + DU + 2\,(DU + Id) \\
&= DDUU + 2\,DU \\
&= DDUU + 4\,DU + 2\,Id
\end{aligned}
$$

$$
\begin{aligned}
U^3 D^3 &= DU\,(DDUU + 4\,DU + 2\,Id) + \\
&\qquad 3\,(DDUU + 4\,DU + 2\,Id) \\
&= DD\,UDUU + DDUU \\
&\quad + 4\,(DDUU + DU) + 2\,DU \\
&\quad + 3\,DDUU + 12\,DU + 6\,Id \\
&= D^3 U^3 + 9\,D^2 U^2 + 18\,DU + 6\,Id
\end{aligned}
$$

“planarisation” of the “rewriting rules”



“planarisation” of the “rewriting rules”

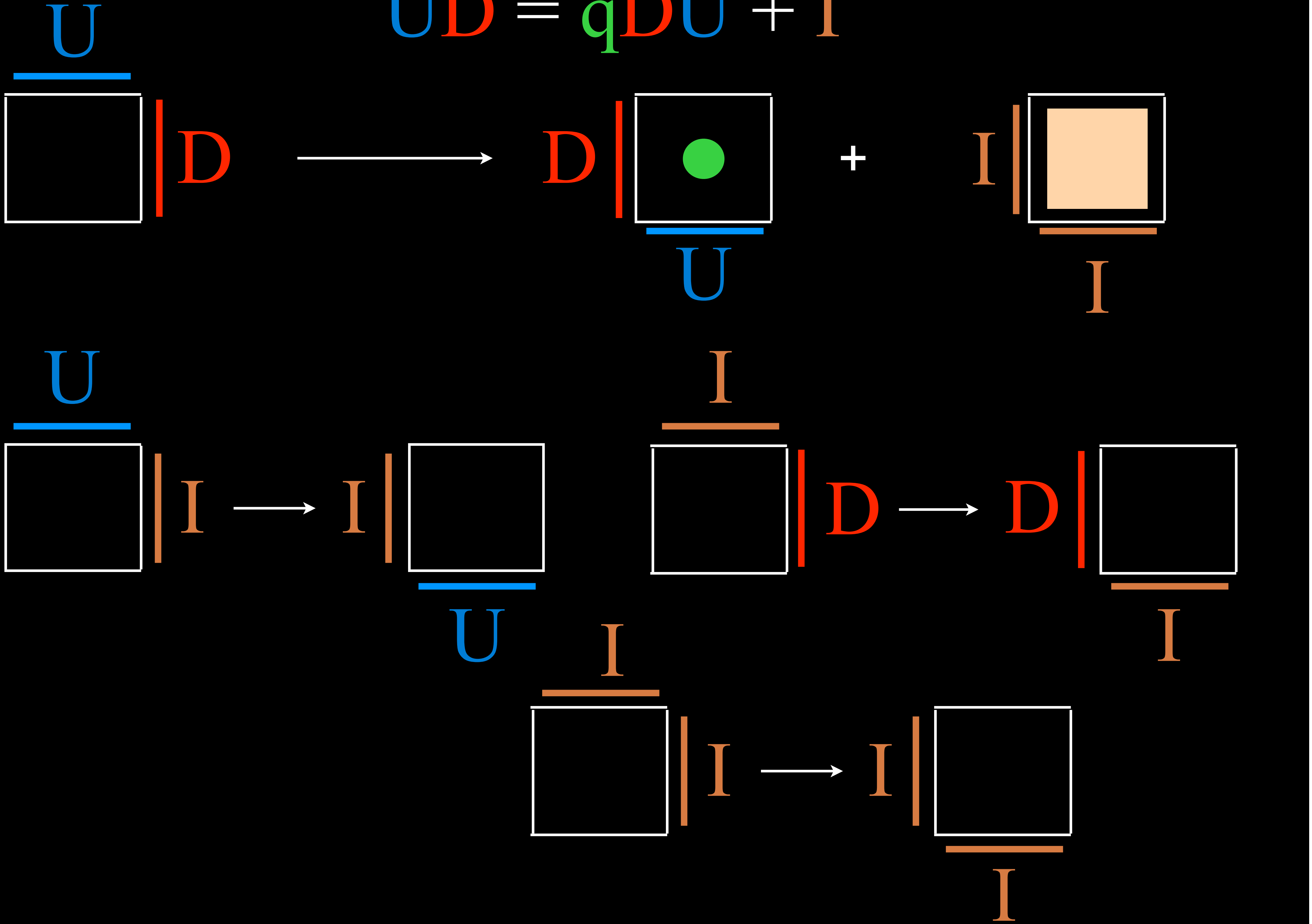


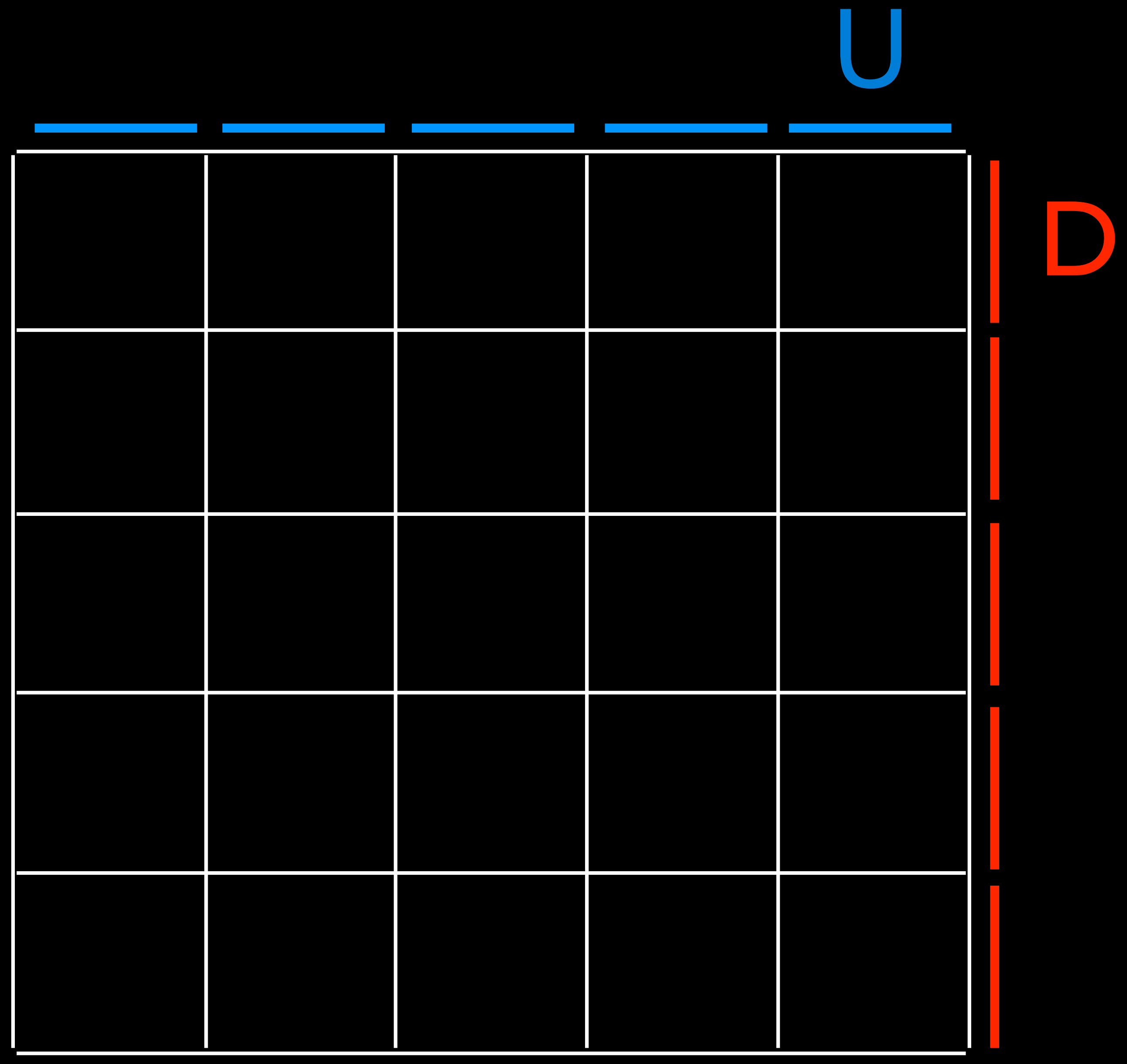
$$
\begin{cases}
UD = DU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$

$$
\begin{cases}
UD \rightarrow DU \qquad UD \rightarrow I_v I_h \\
U I_v \rightarrow I_v U \\
I_h D \rightarrow D I_h \\
I_h I_v \rightarrow I_v I_h
\end{cases}
$$

rewriting rules

$$UD = qDU + I$$

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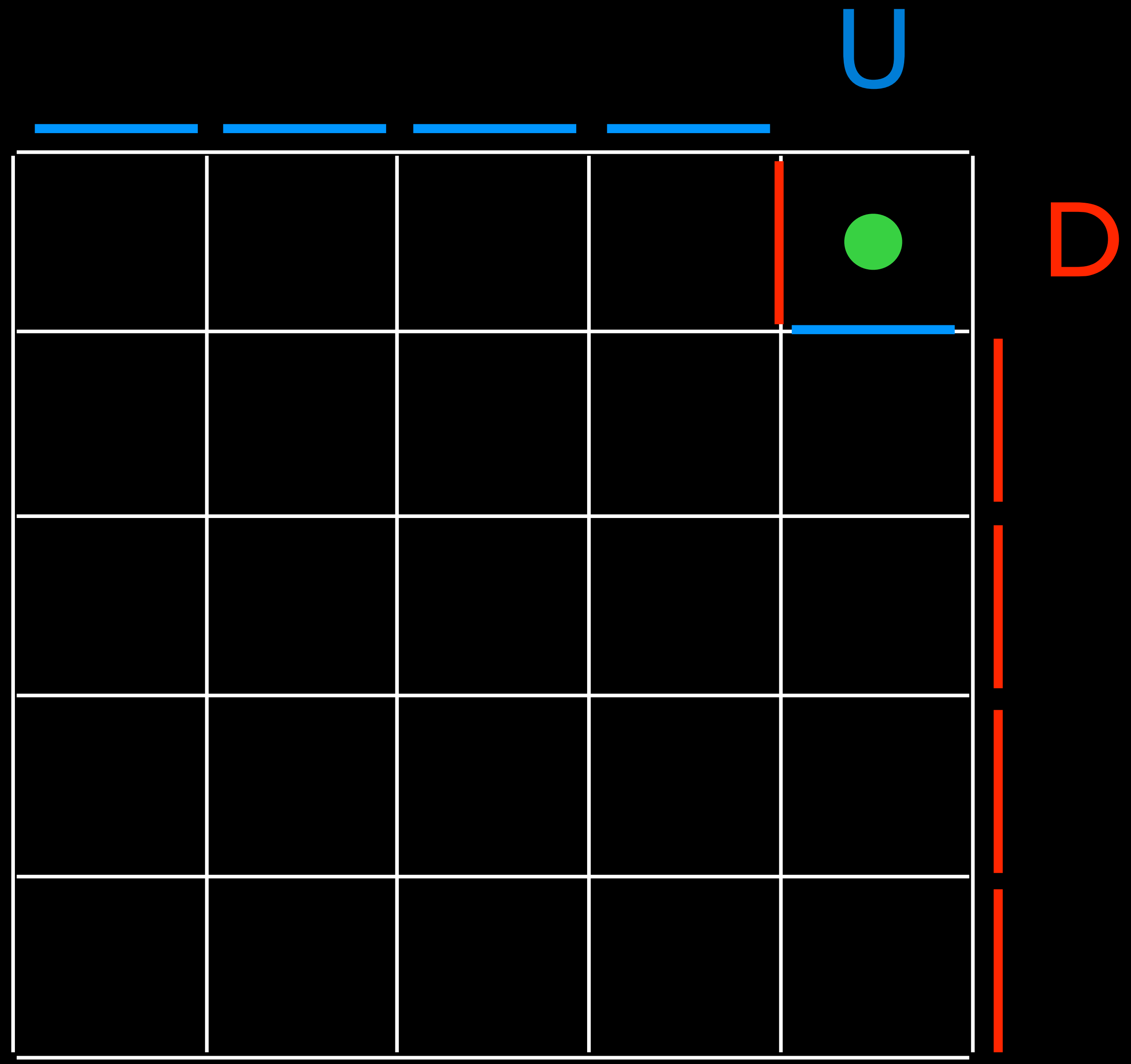
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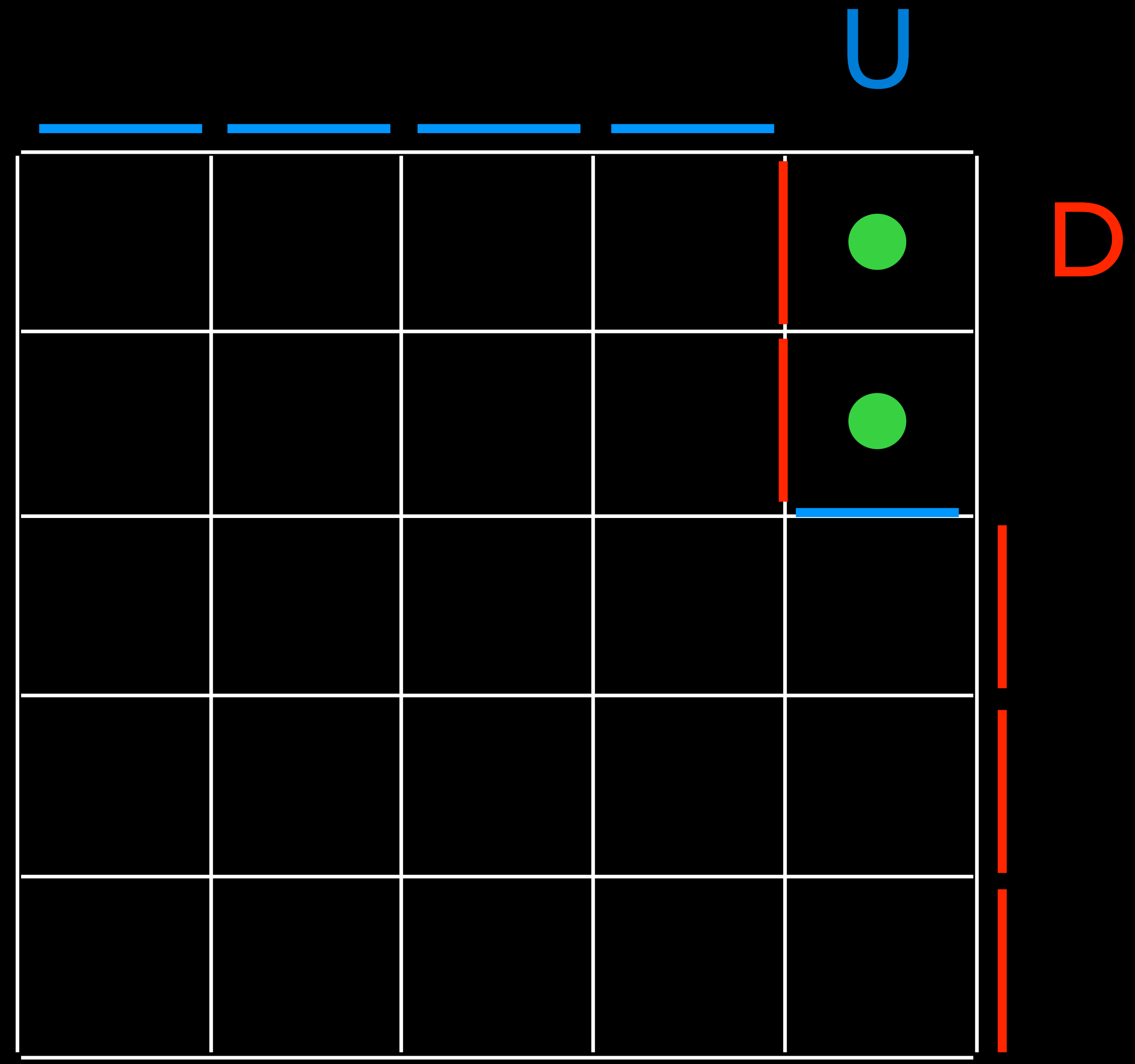
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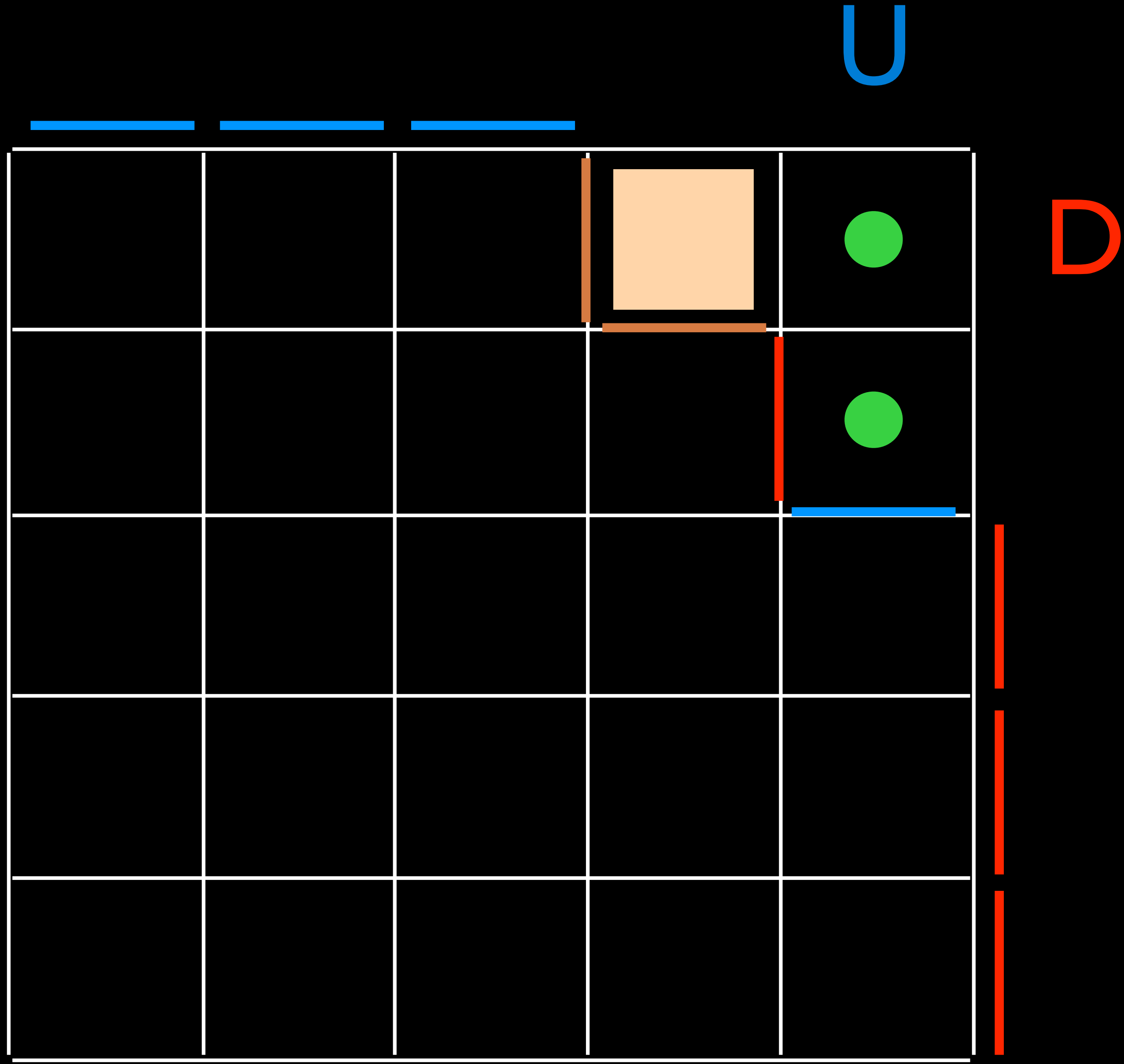

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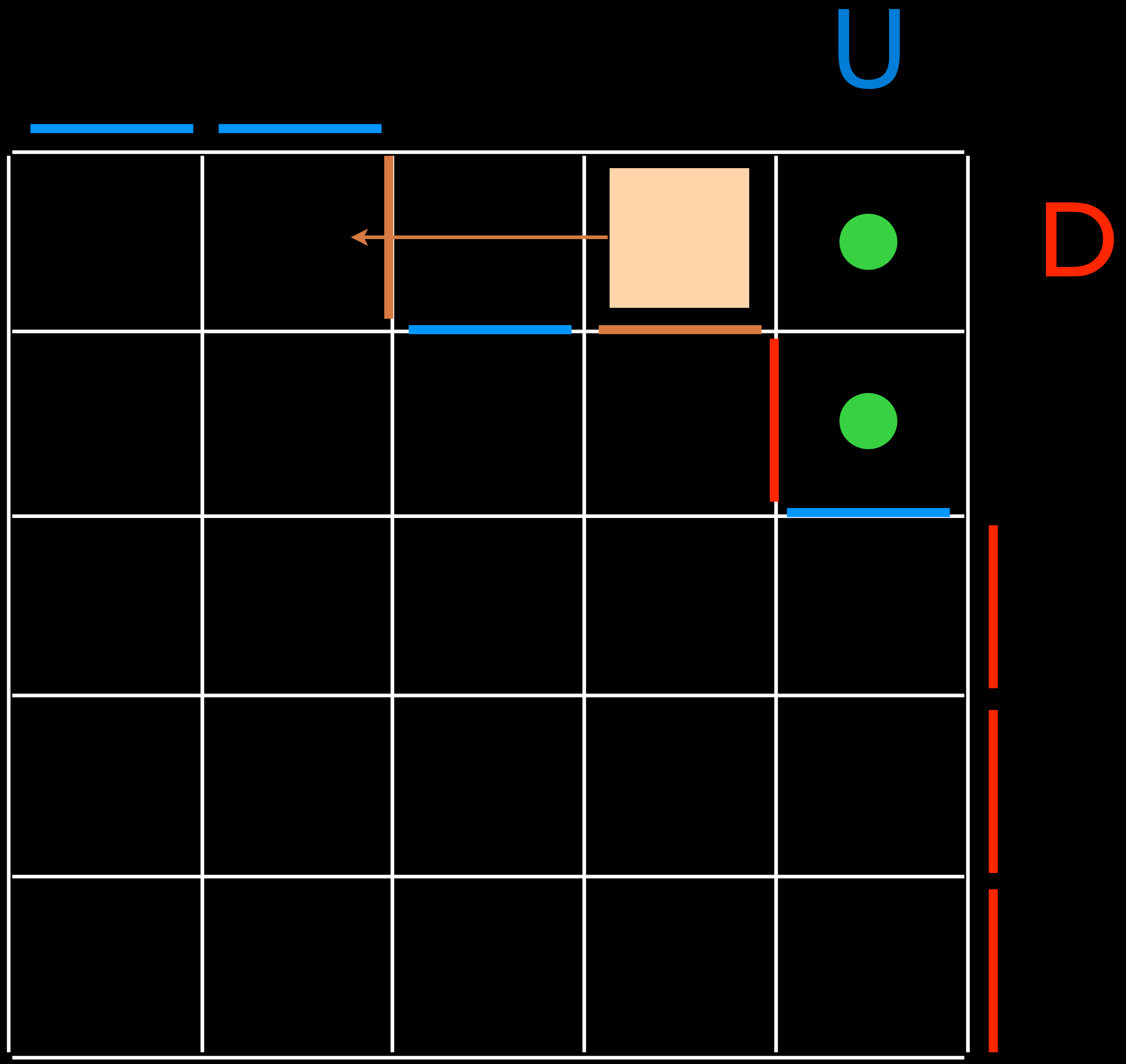
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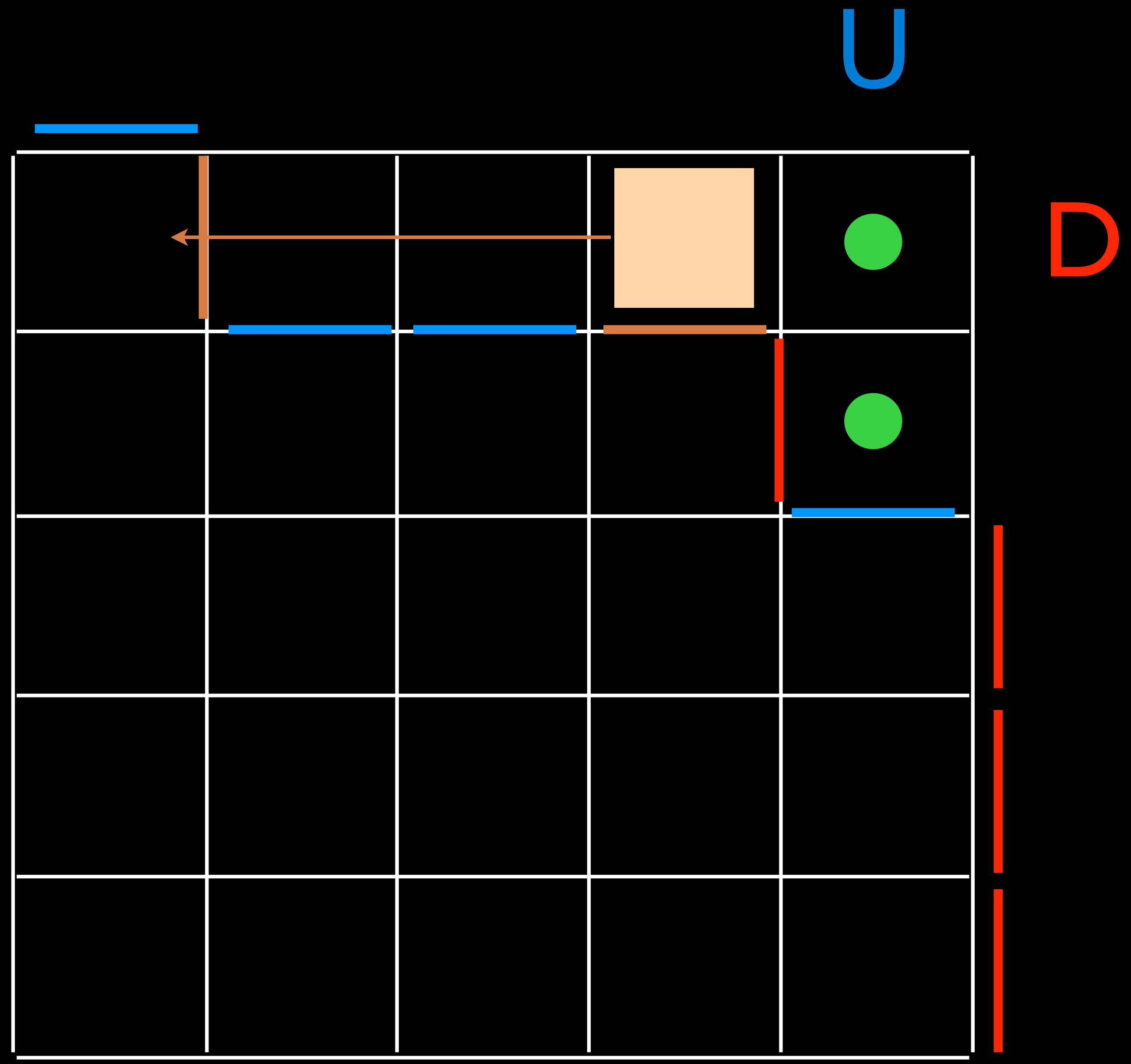
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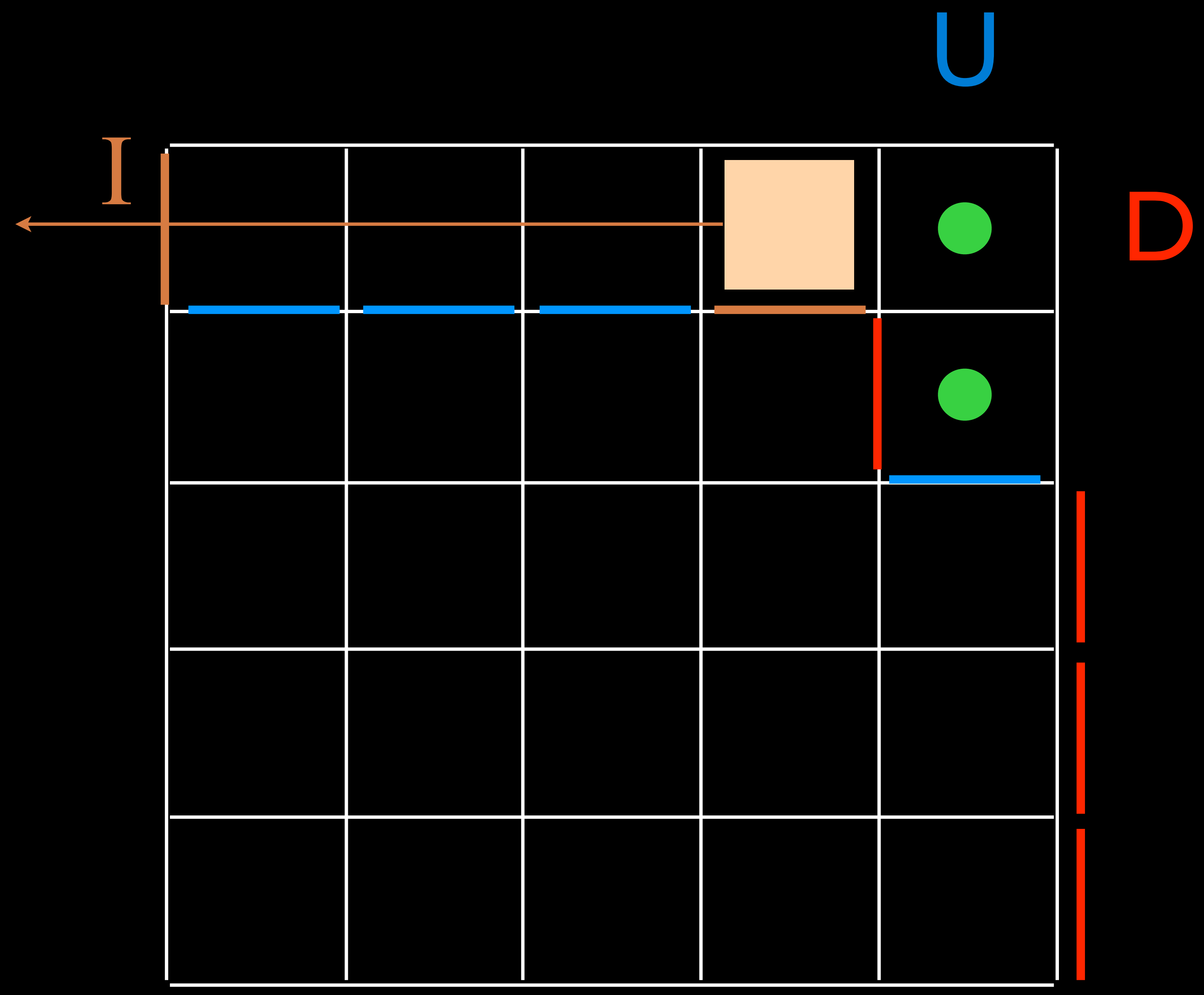

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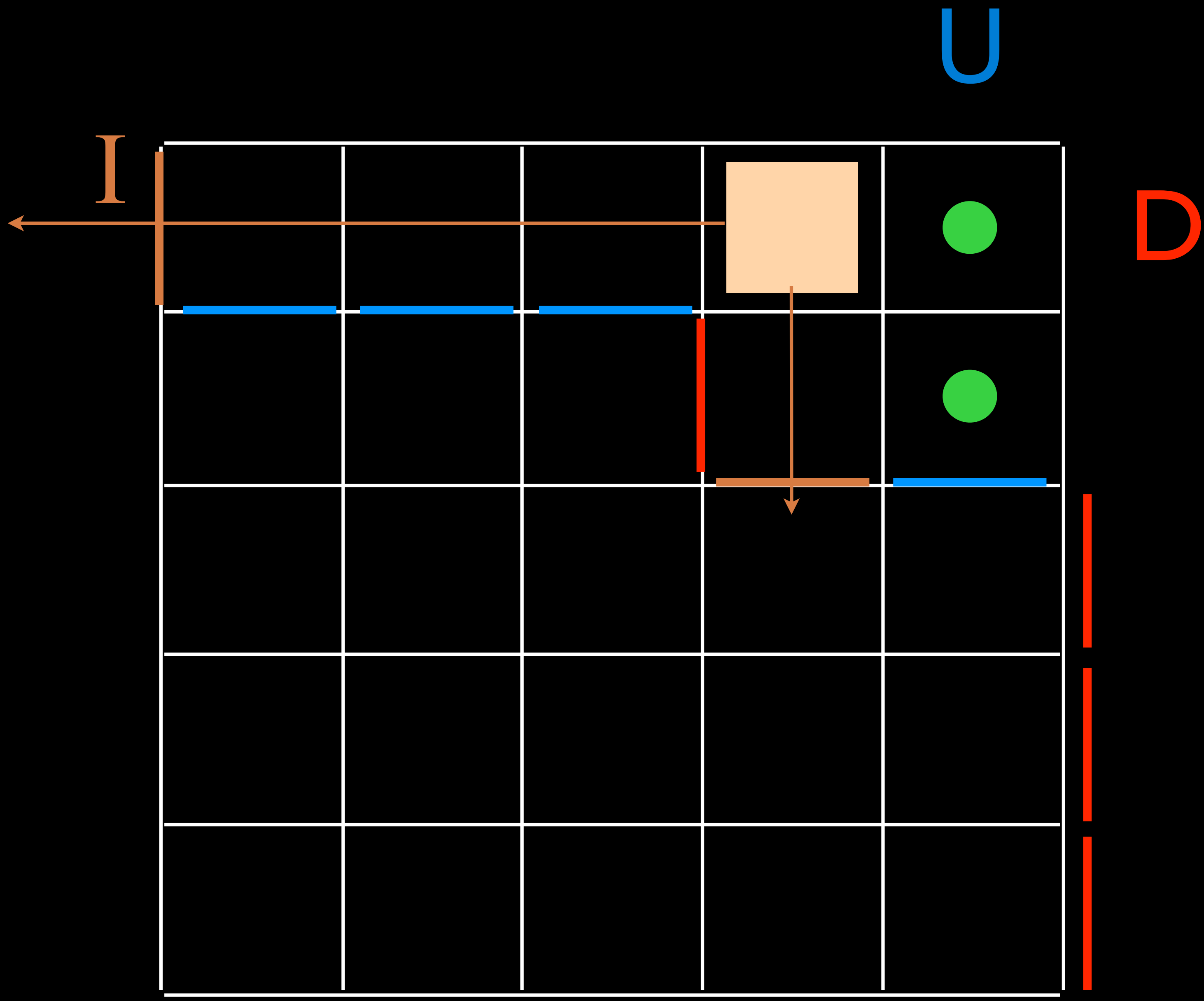
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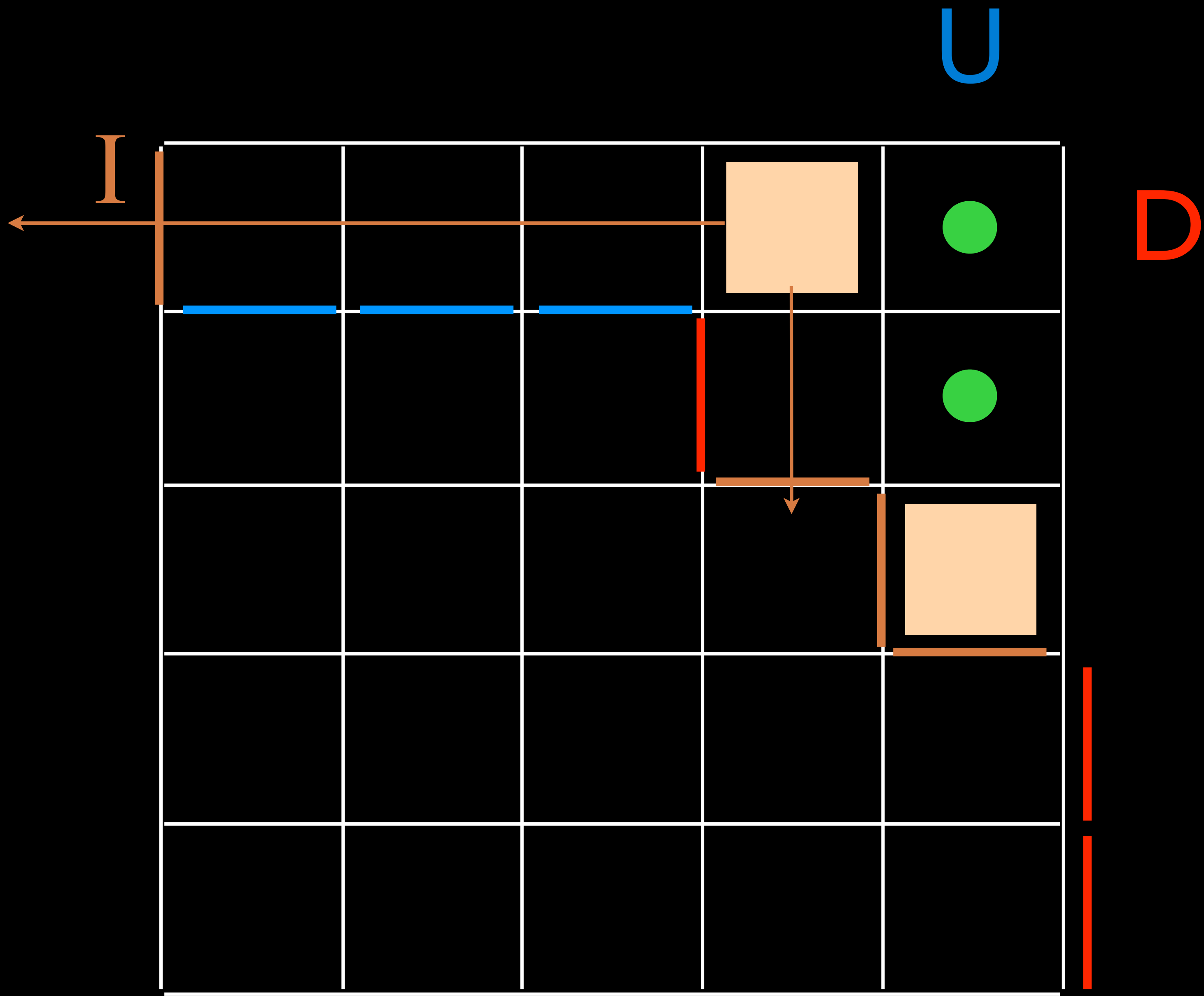
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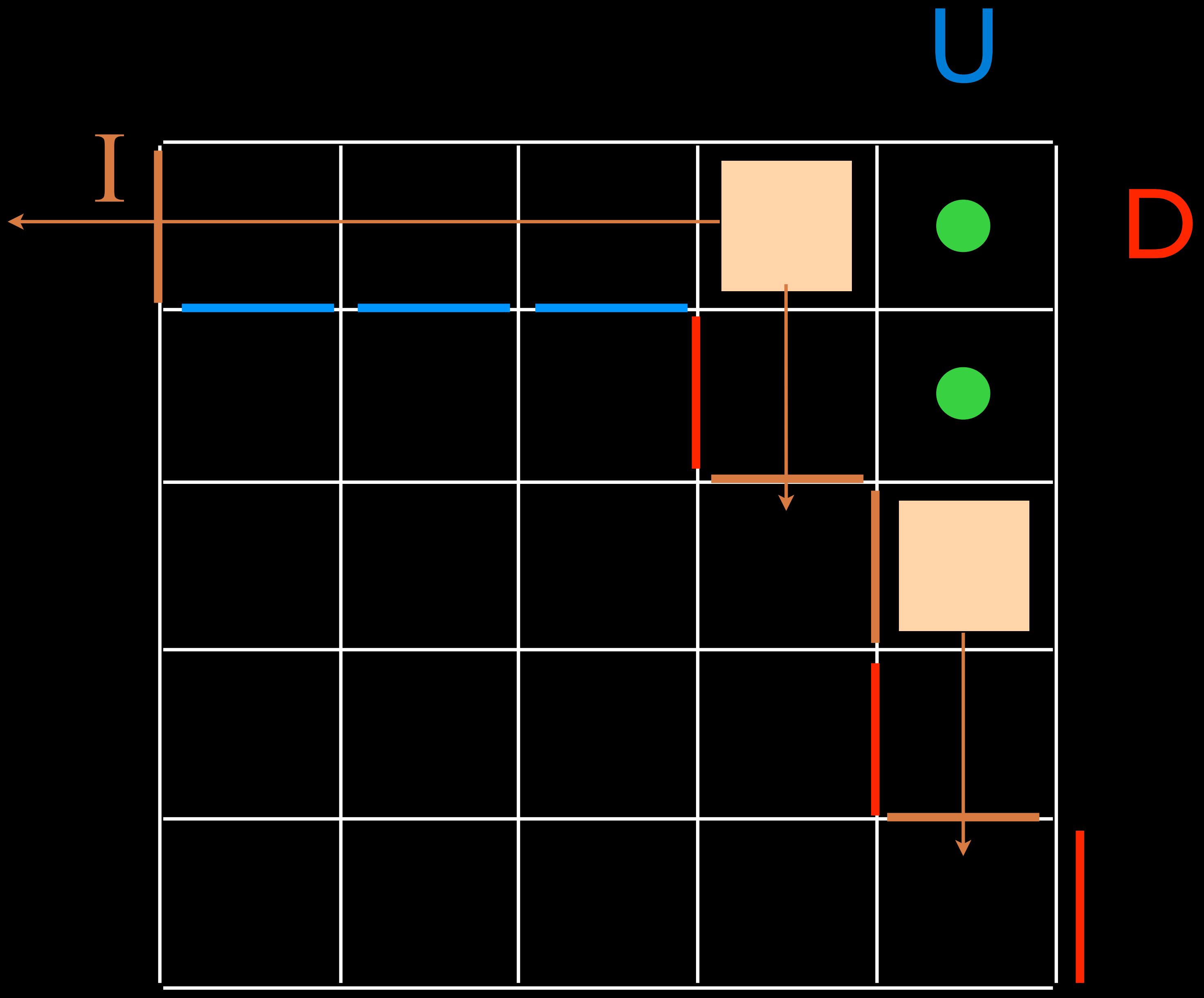
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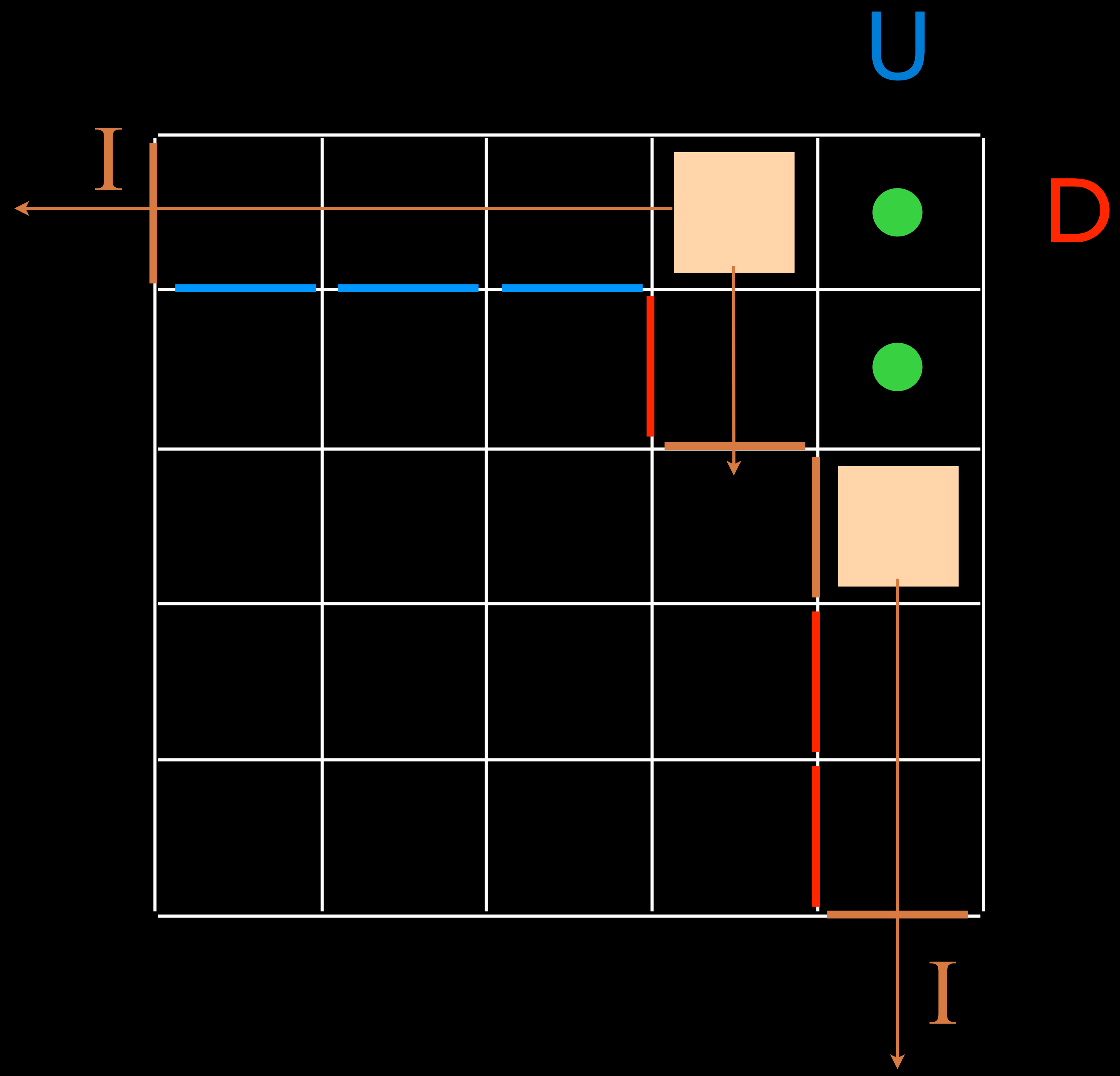
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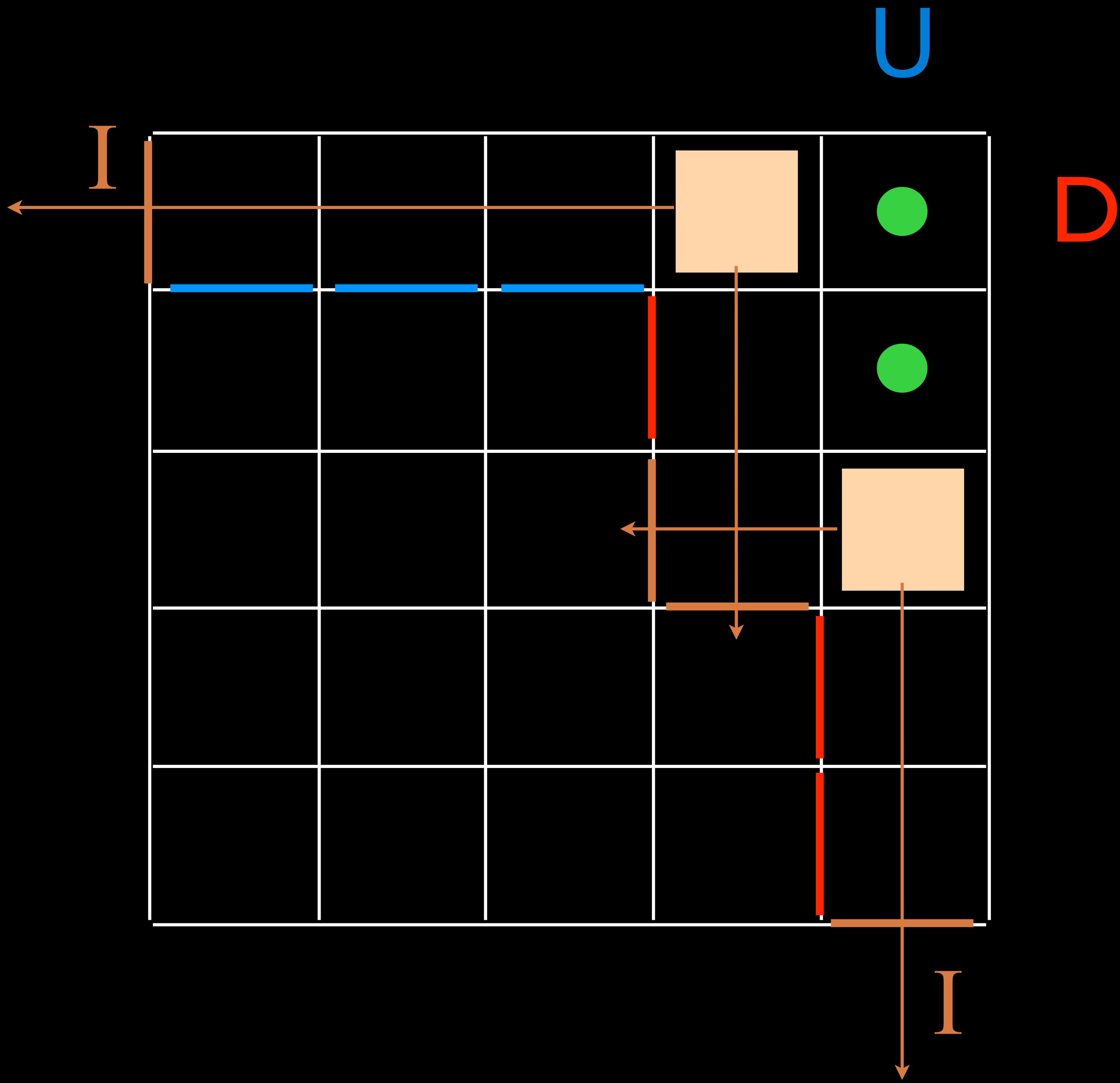
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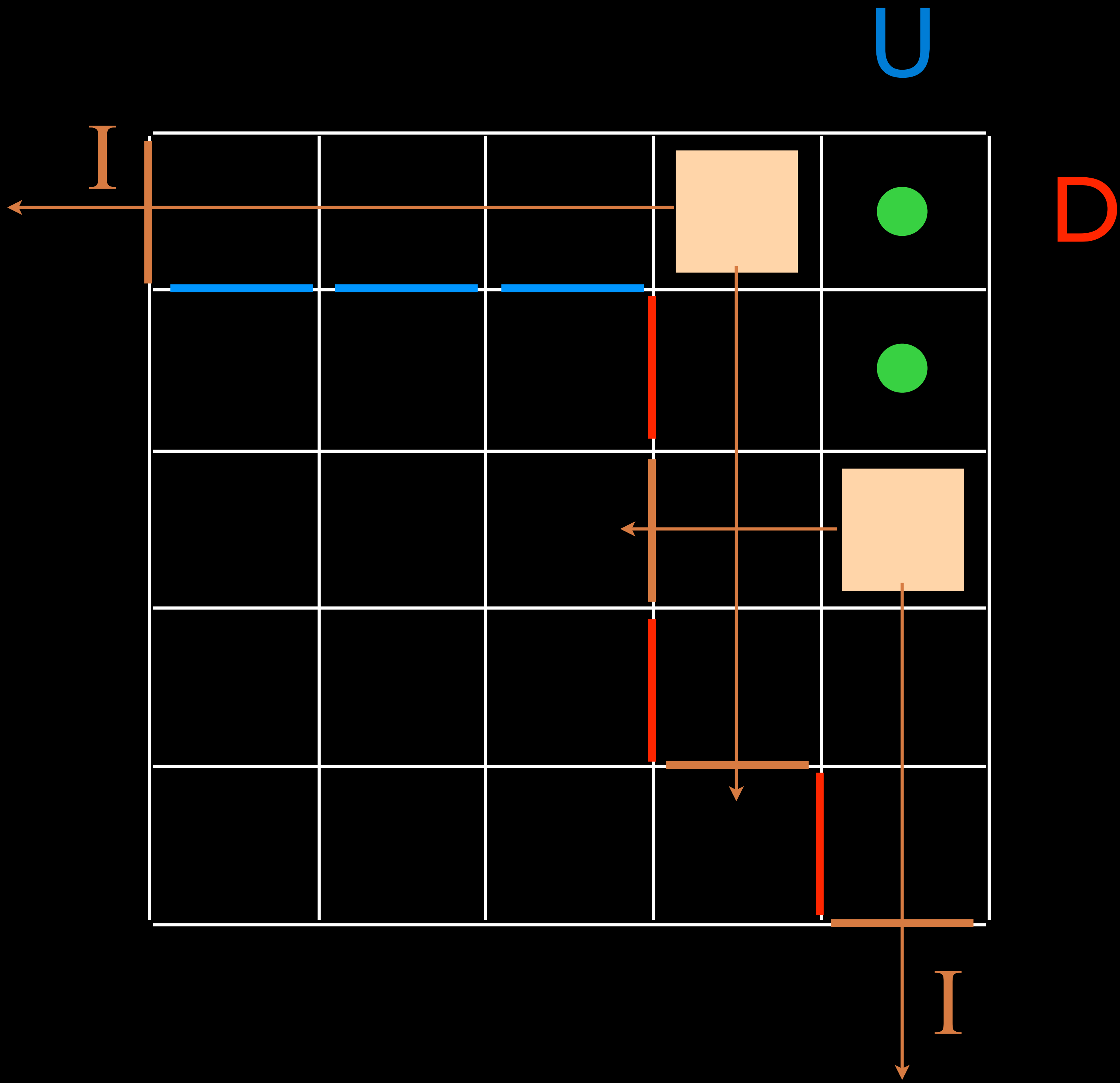
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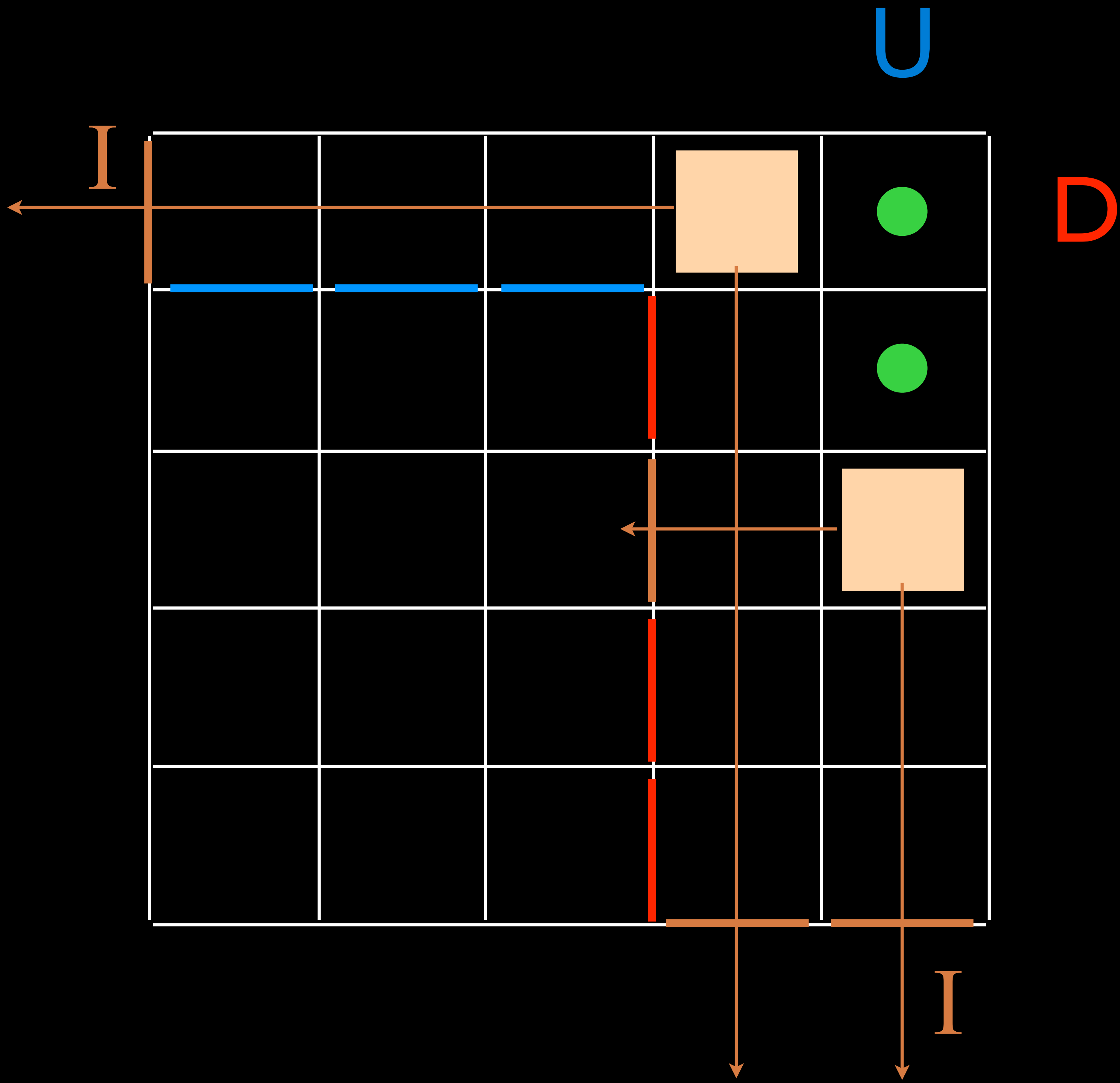

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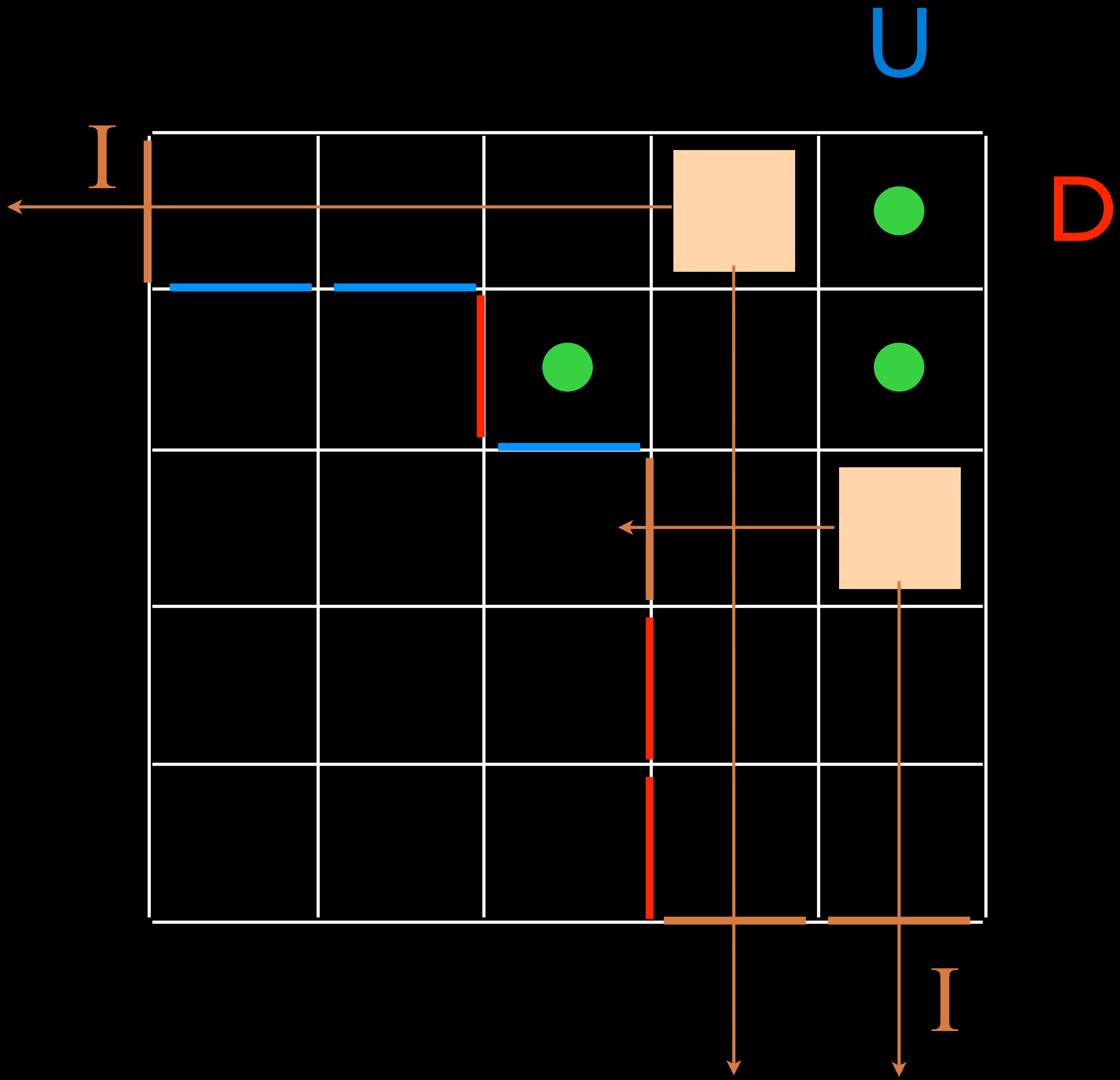
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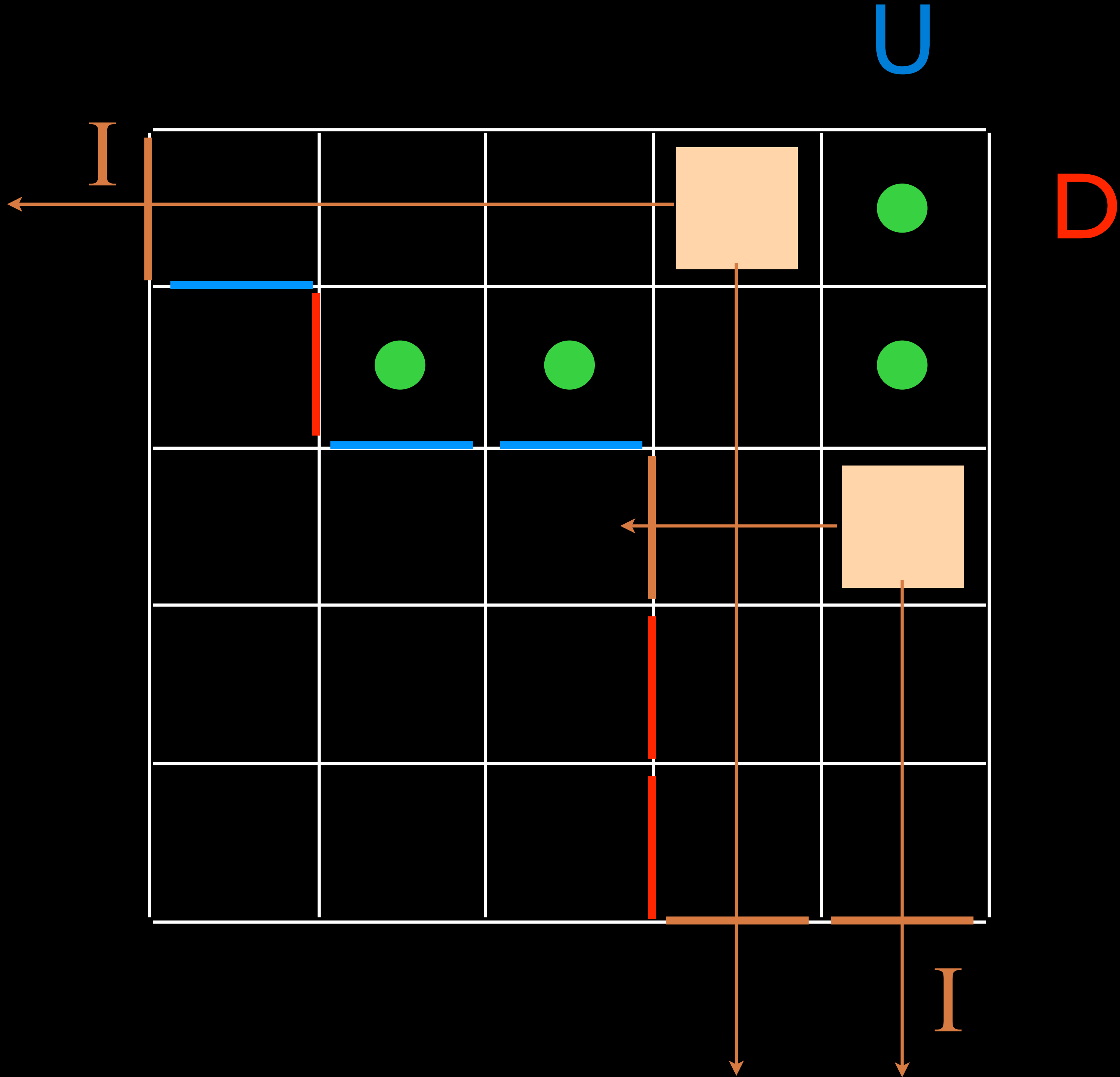

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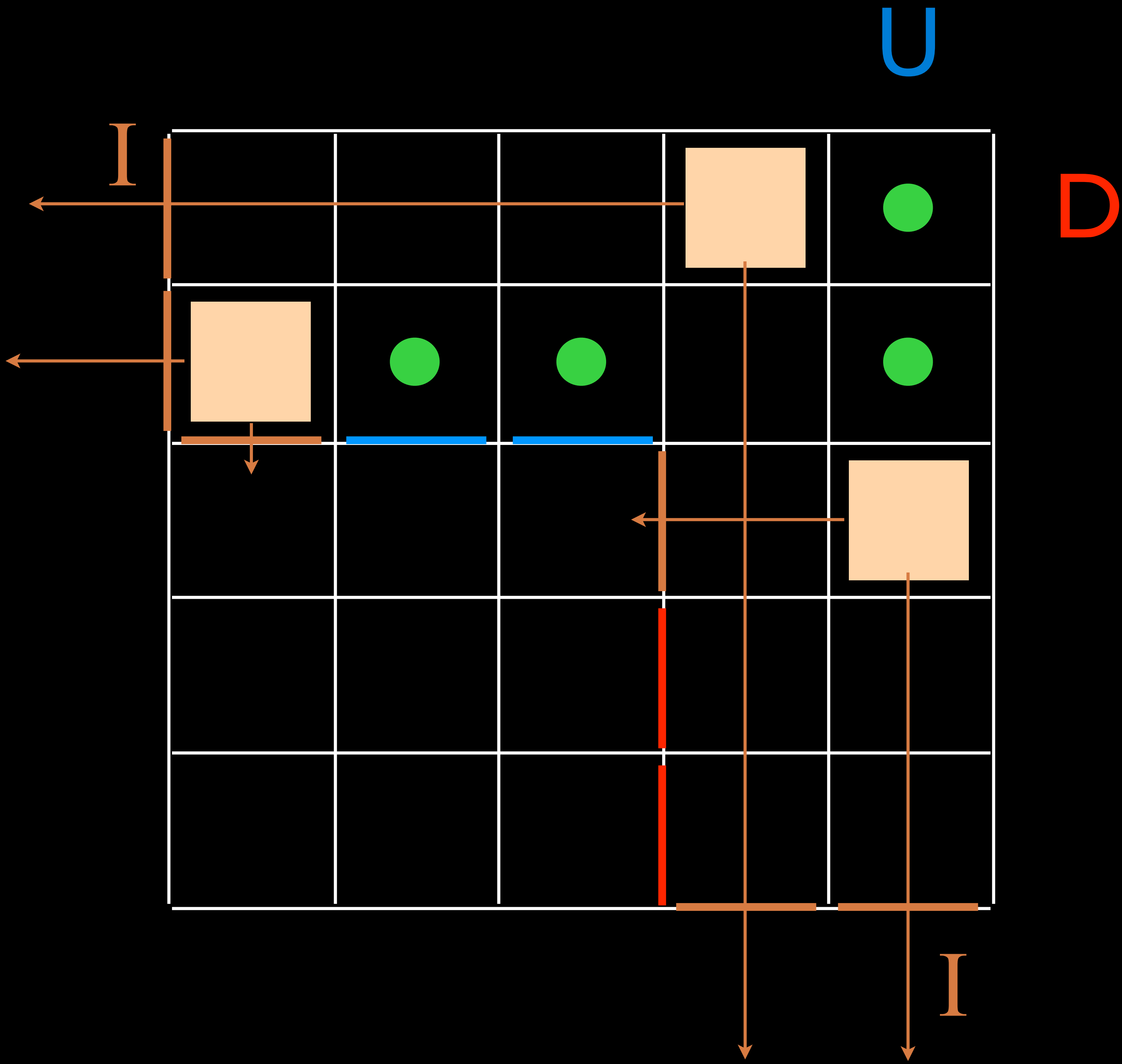
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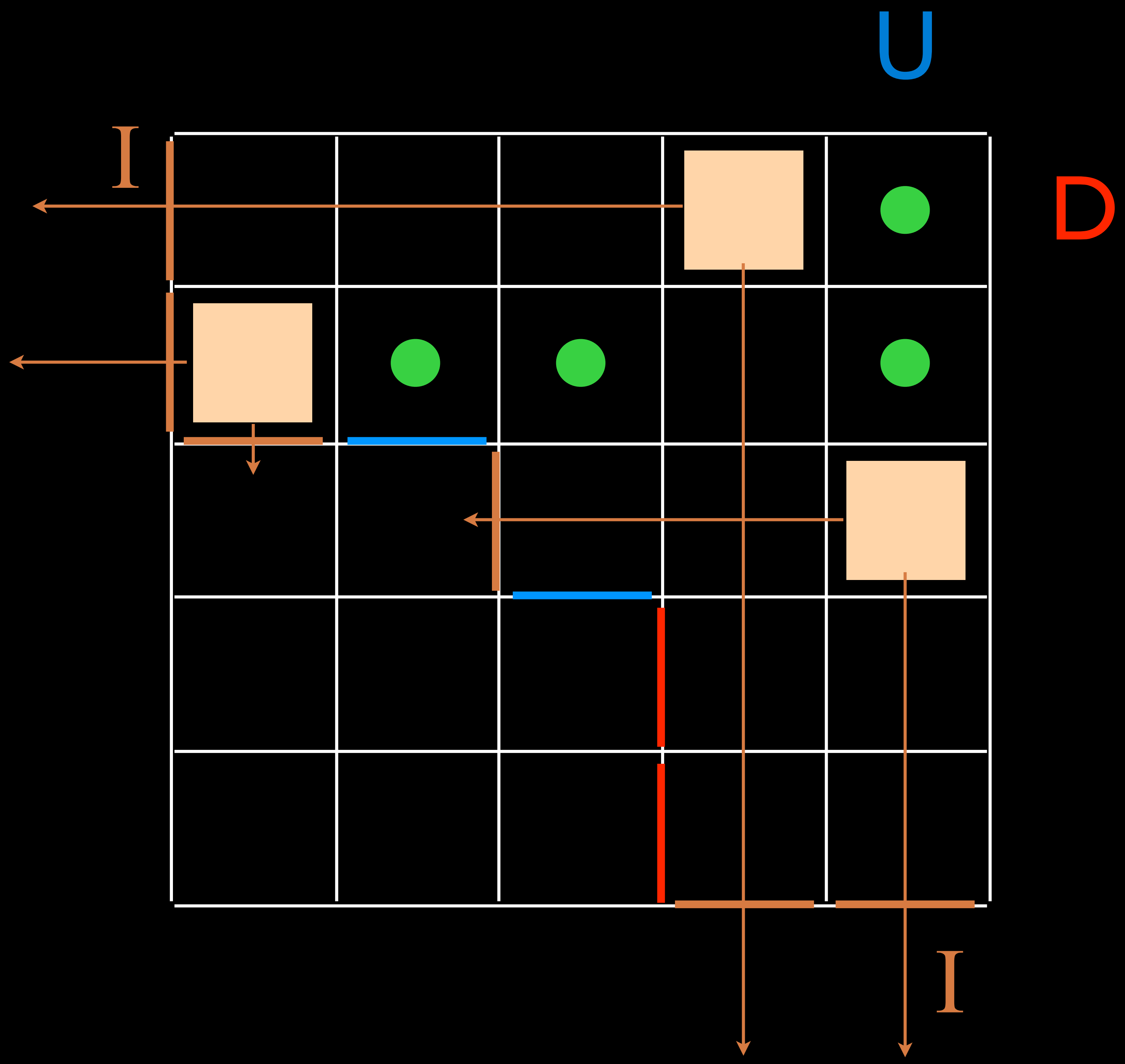
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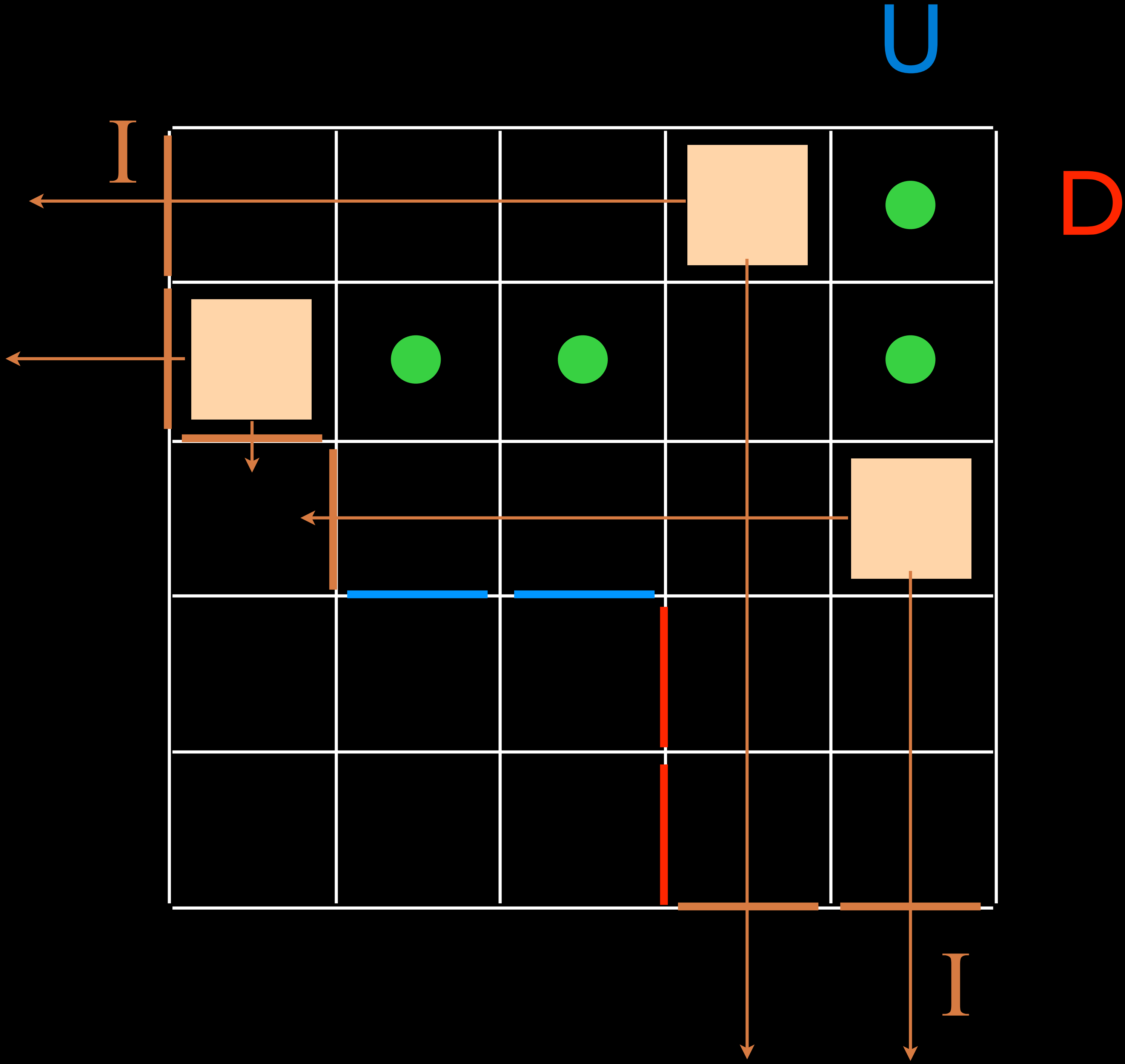
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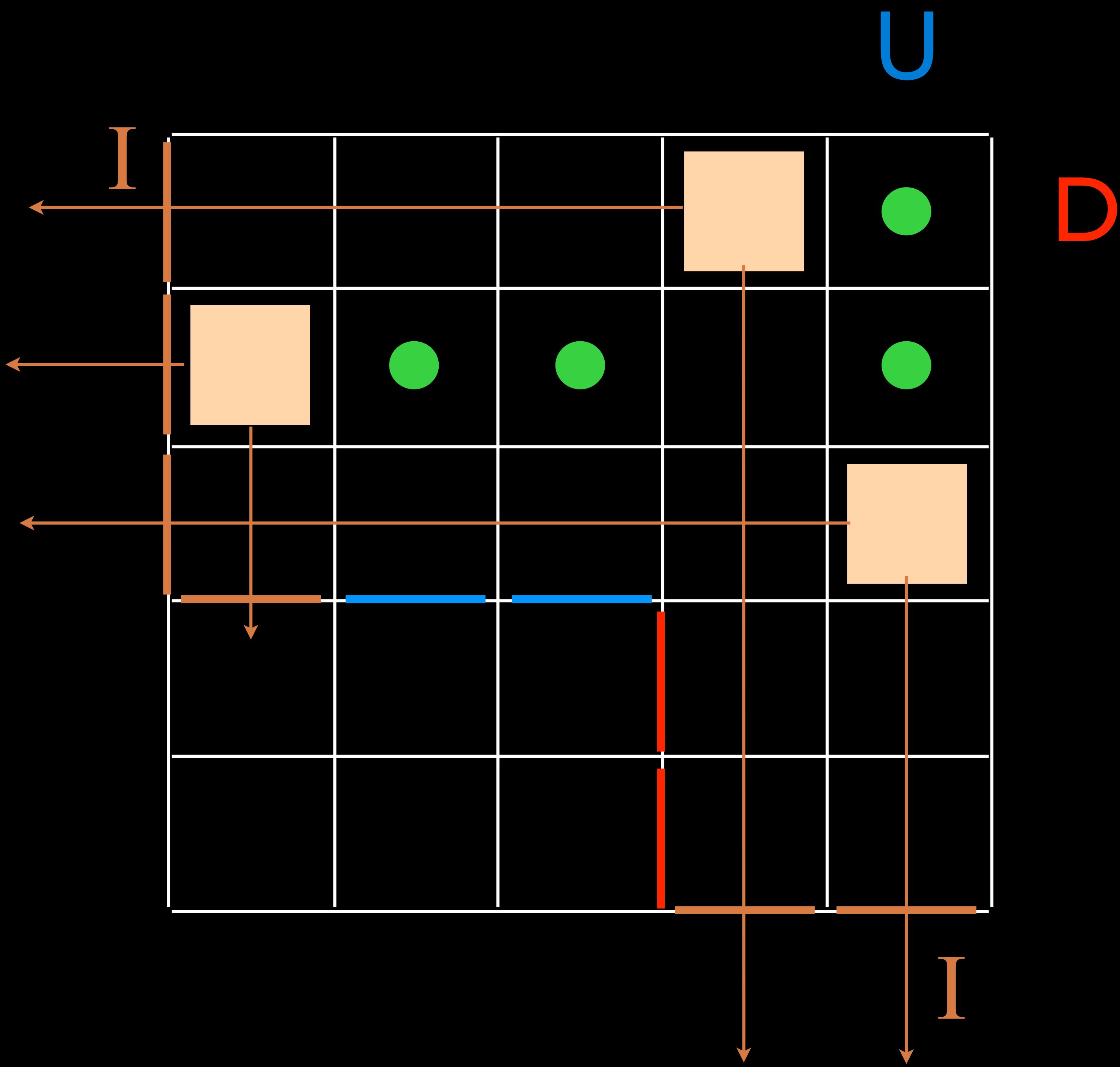
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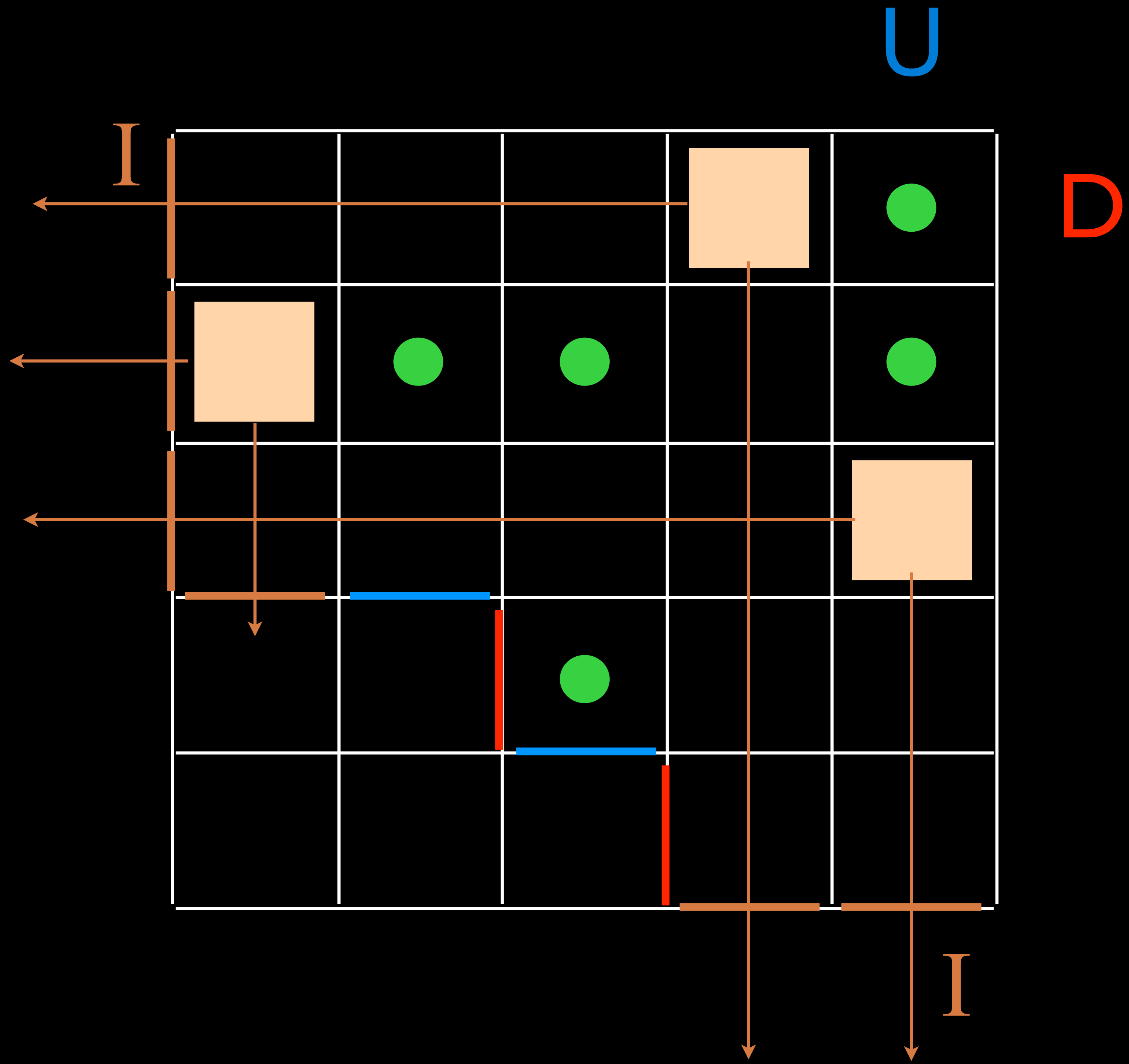

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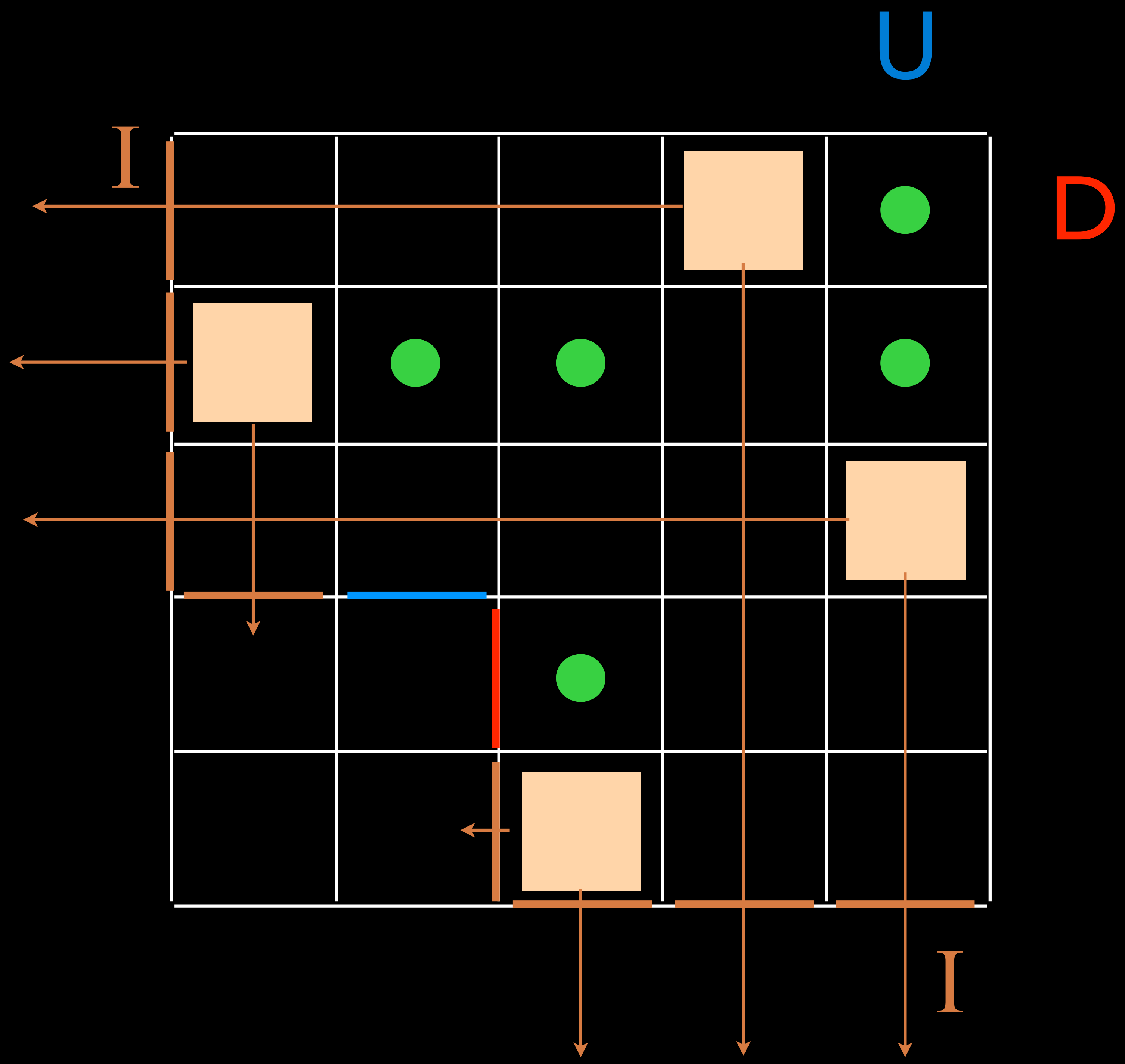
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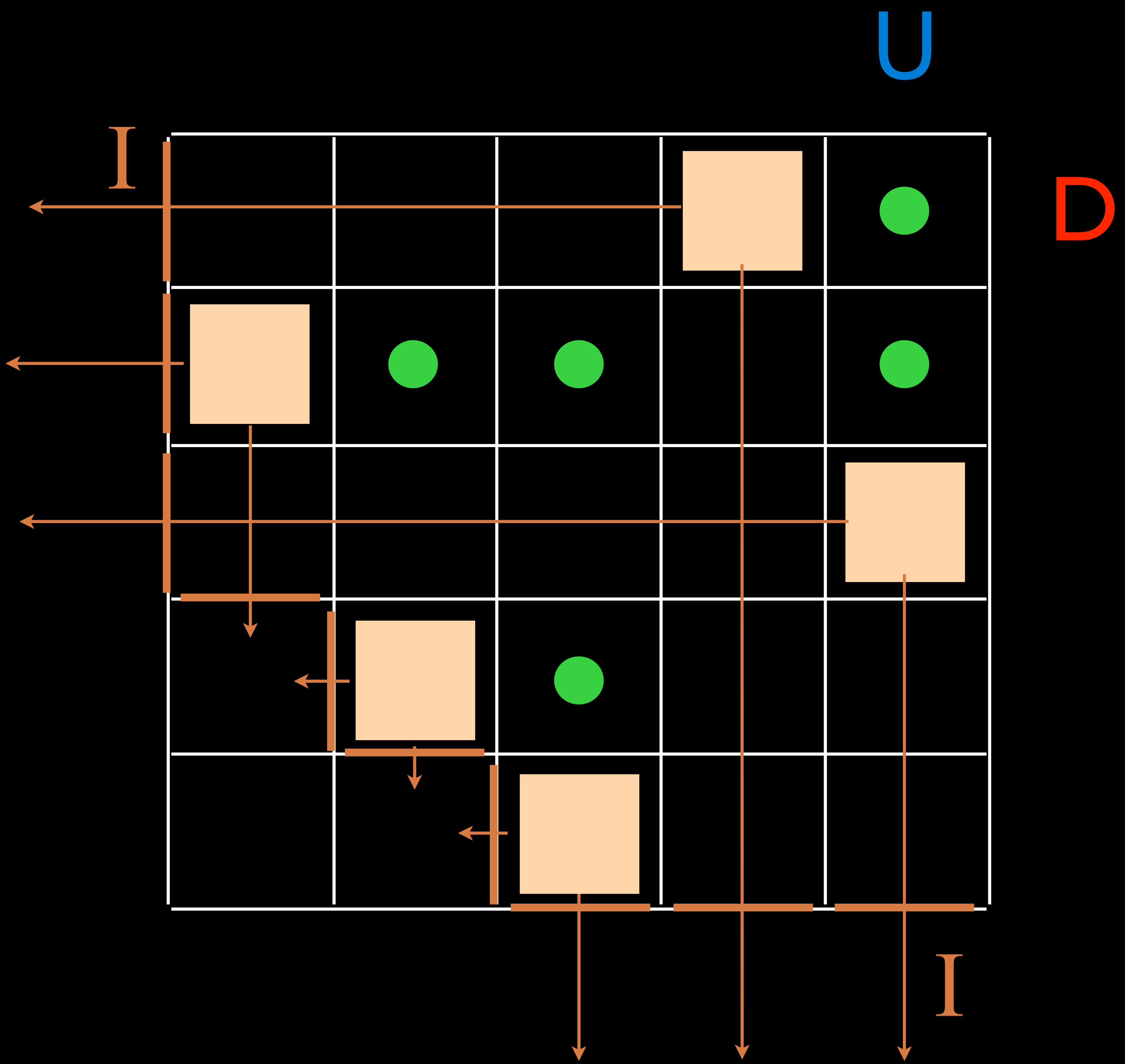
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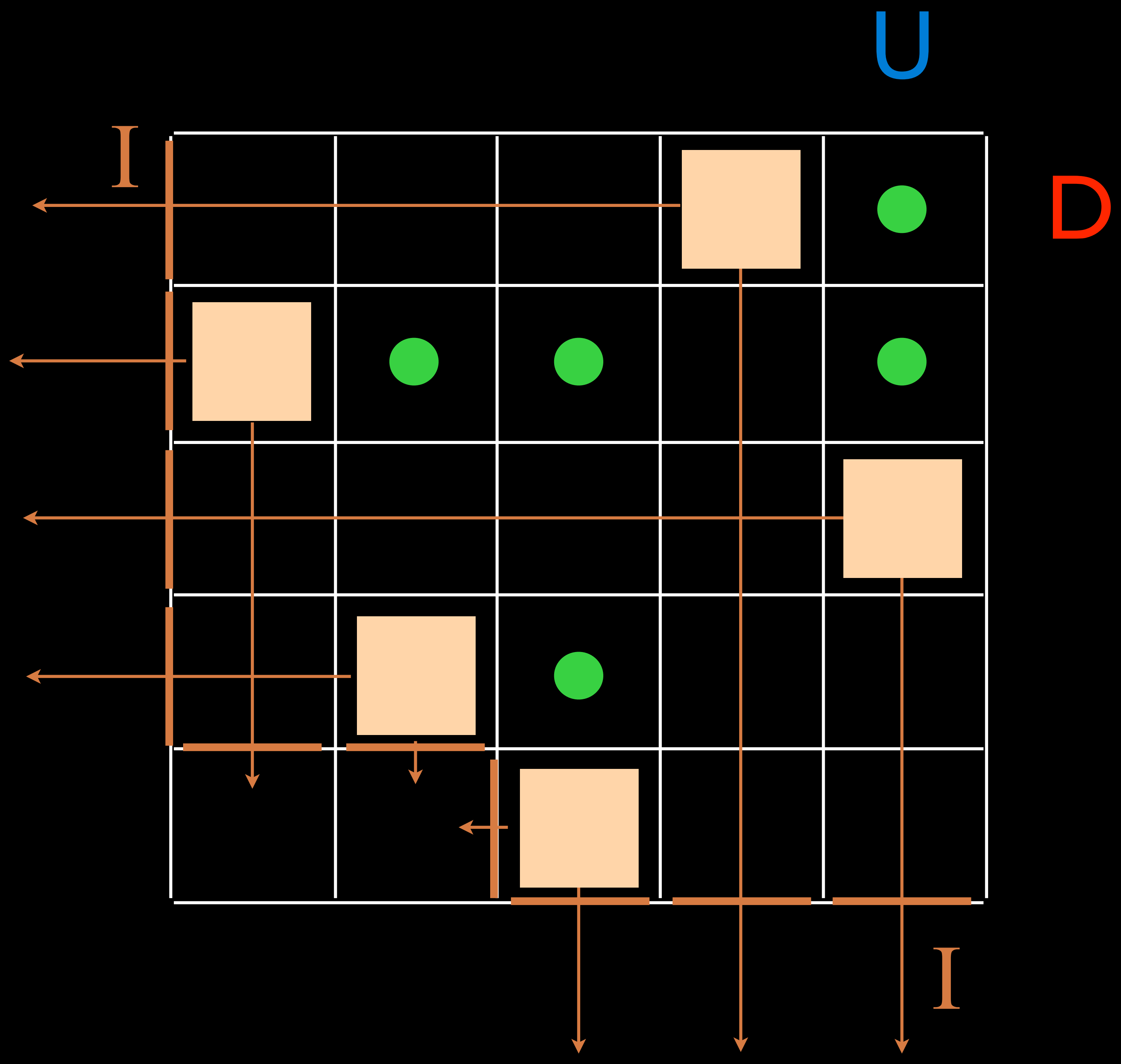
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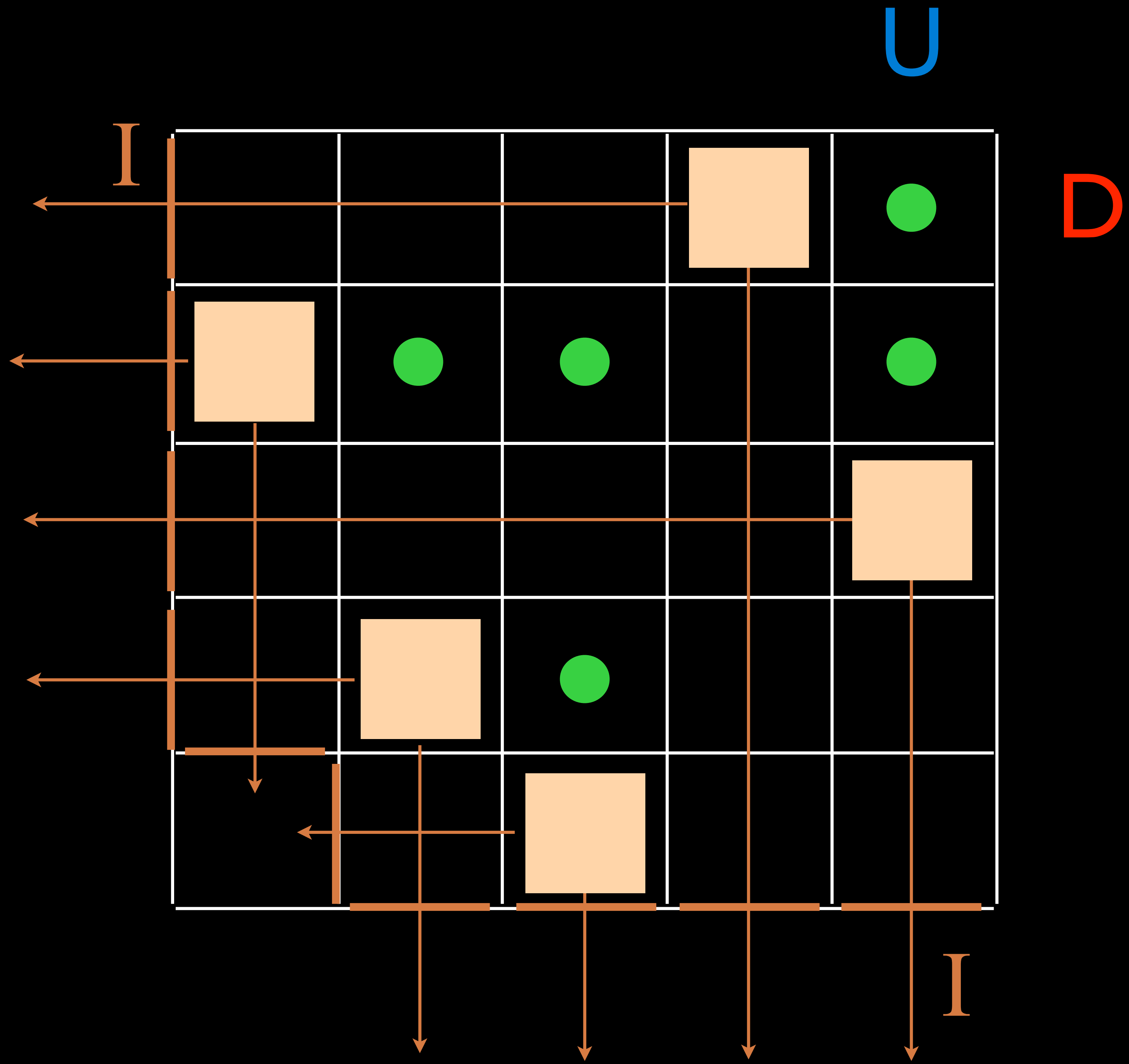
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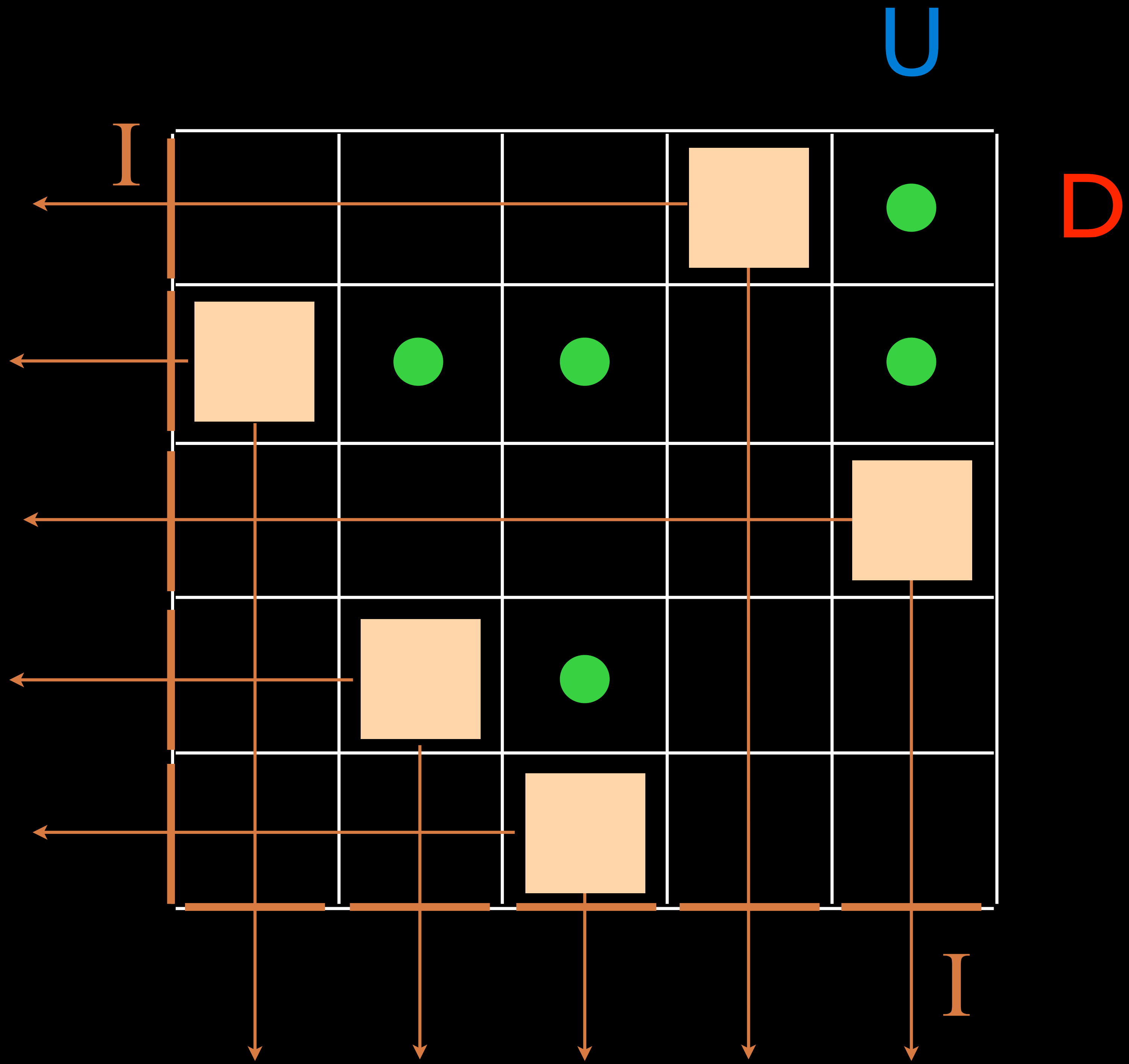
U
D
I
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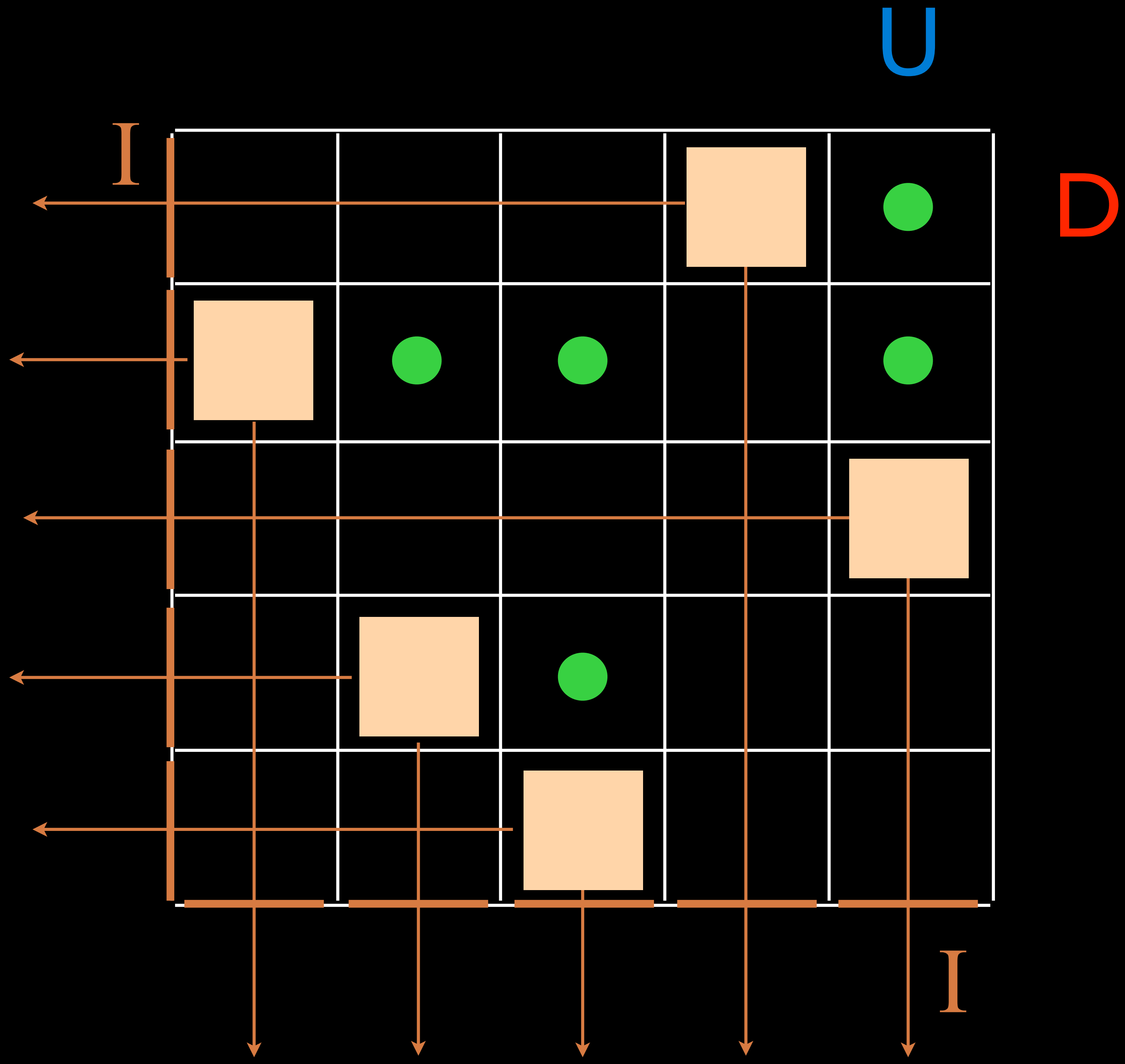
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

Quadratic algebra Q

5 rewriting rules

"complete" Q-tableau (5 labels)

↓ Q-tableau (2 labels)



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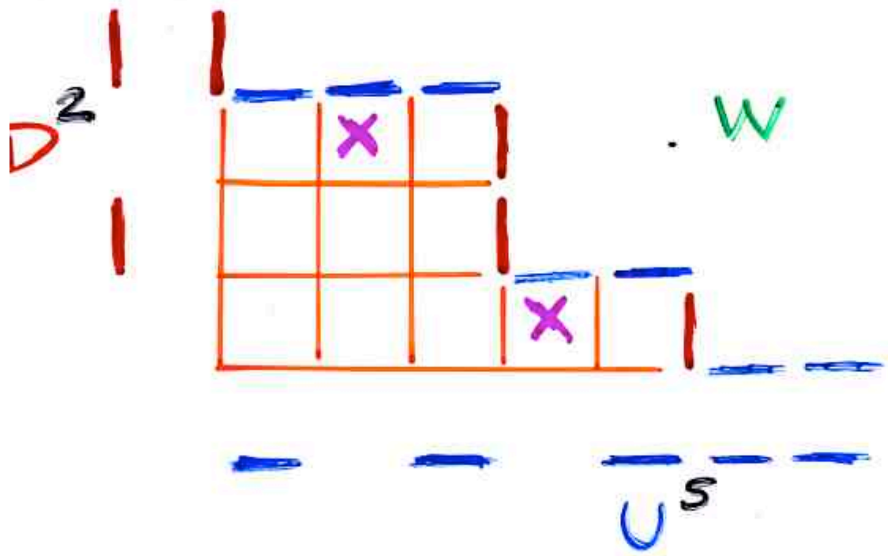
notation $w \rightarrow F_w$ diagram Ferrers



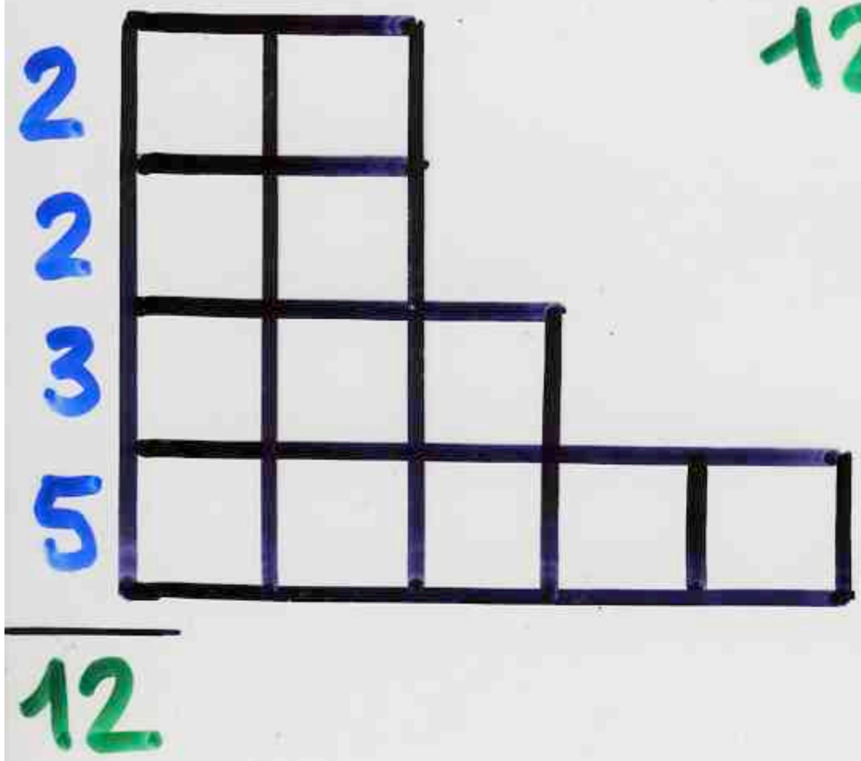
<u>Prop</u>. $c_{i,j}(w)$ = nb de "placement" de k tours sur F

avec $i = |w|_D - k$
$j = |w|_U - k$

An introduction to RSK

G. de B. Robinson, 1938

C. Schensted, 1961

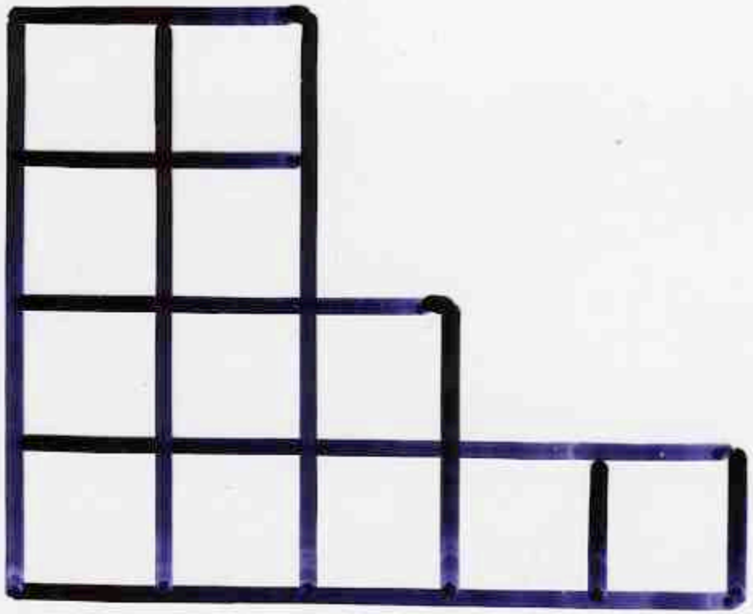


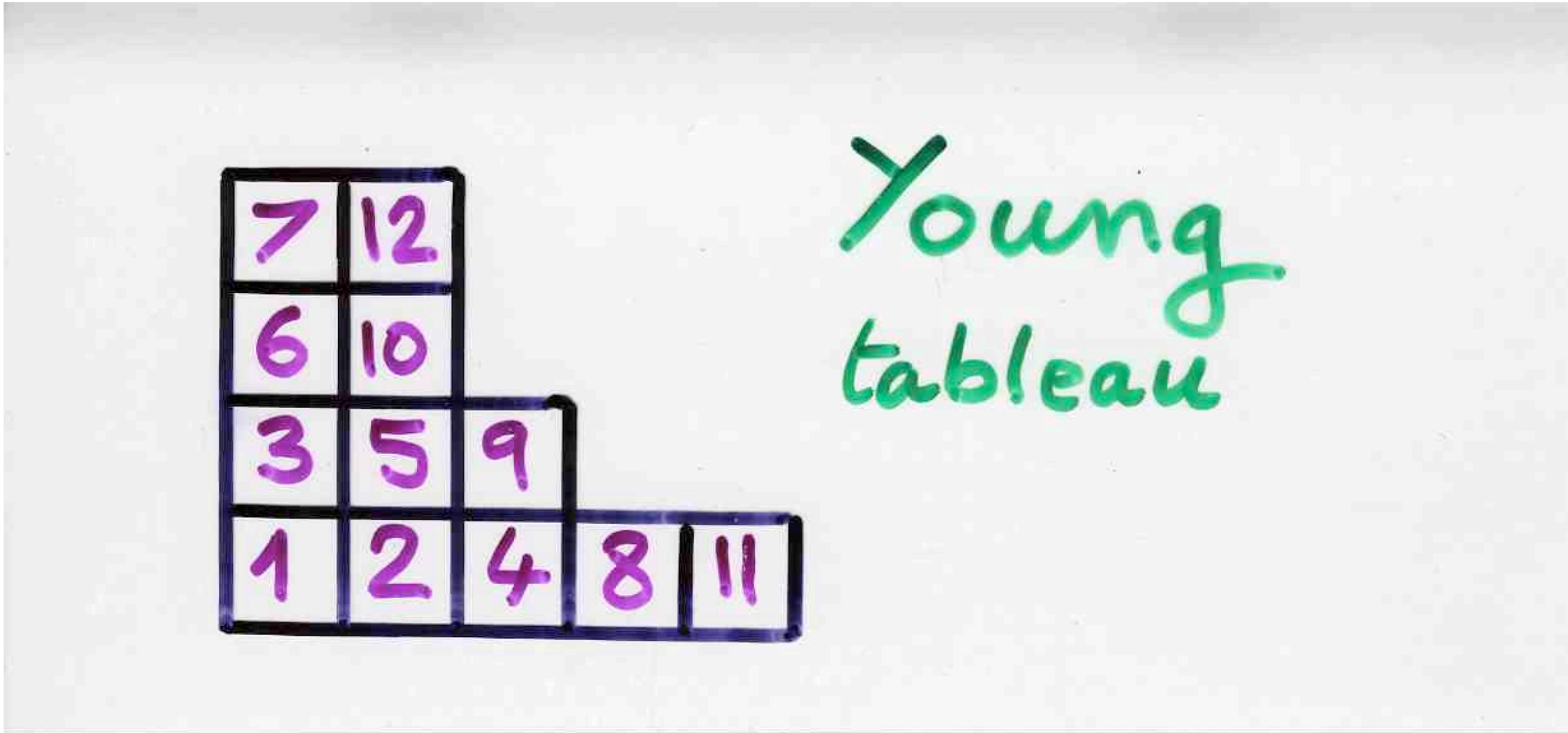
$12 = n = 5 + 3 + 2 + 2$

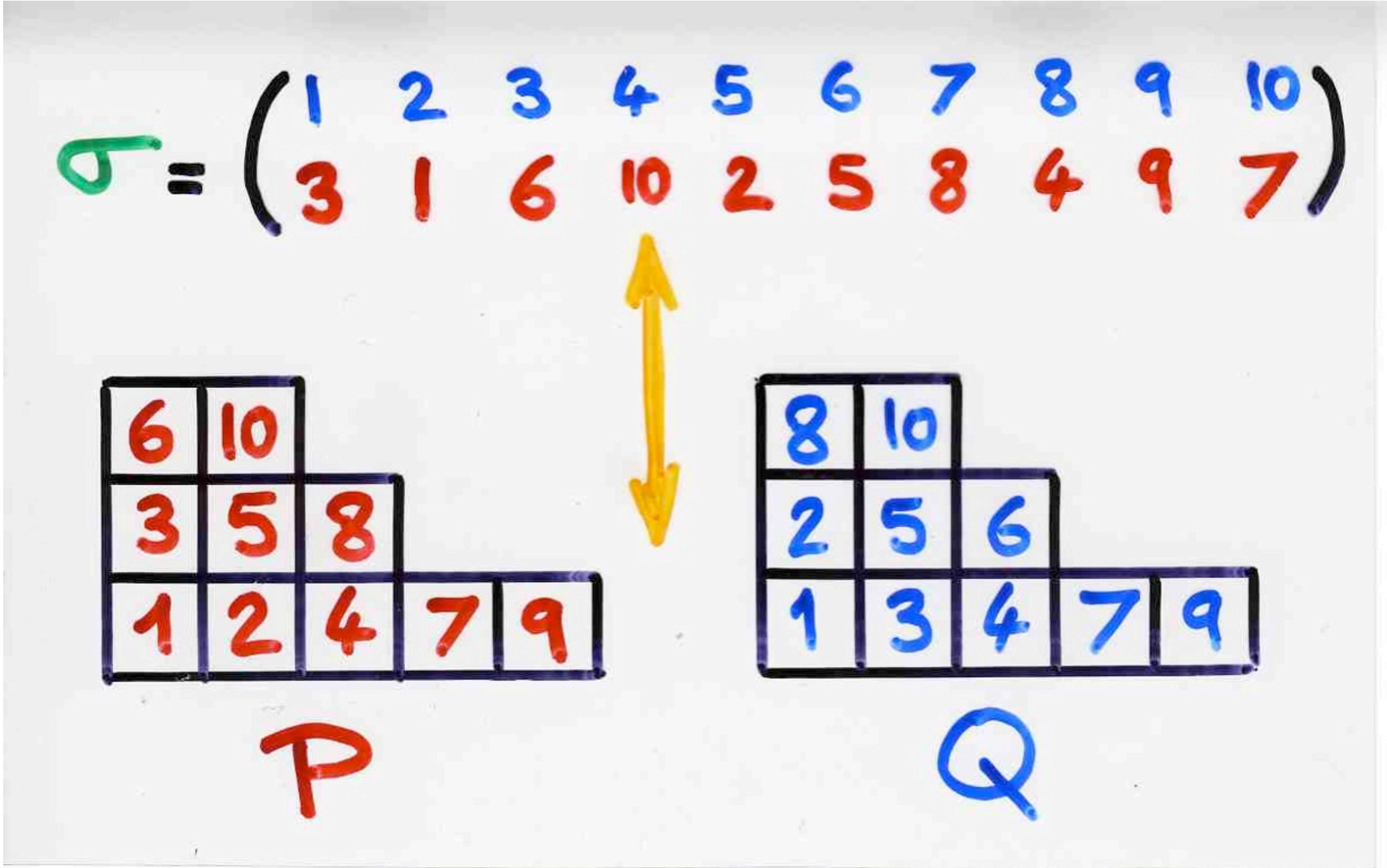
Ferrers

diagram

Partition of n




7 12
6 10
3 5 9
1 2 4 8 11
Young tableau



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

3					

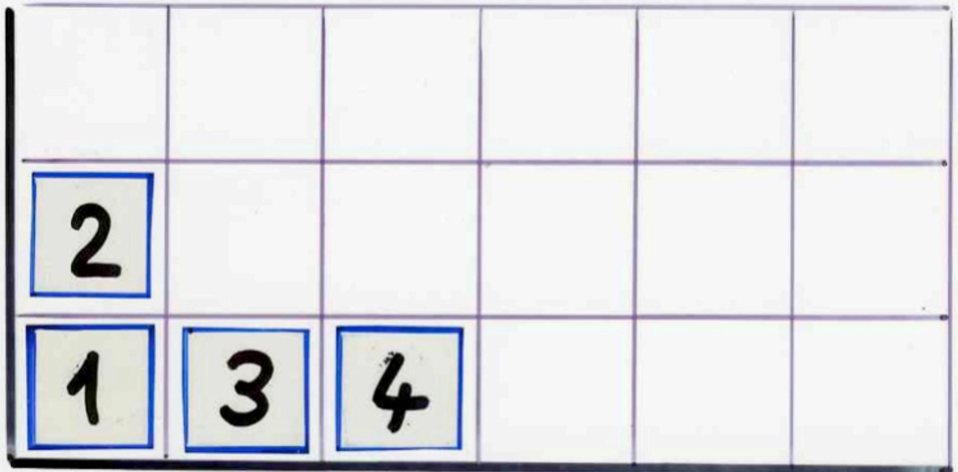
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

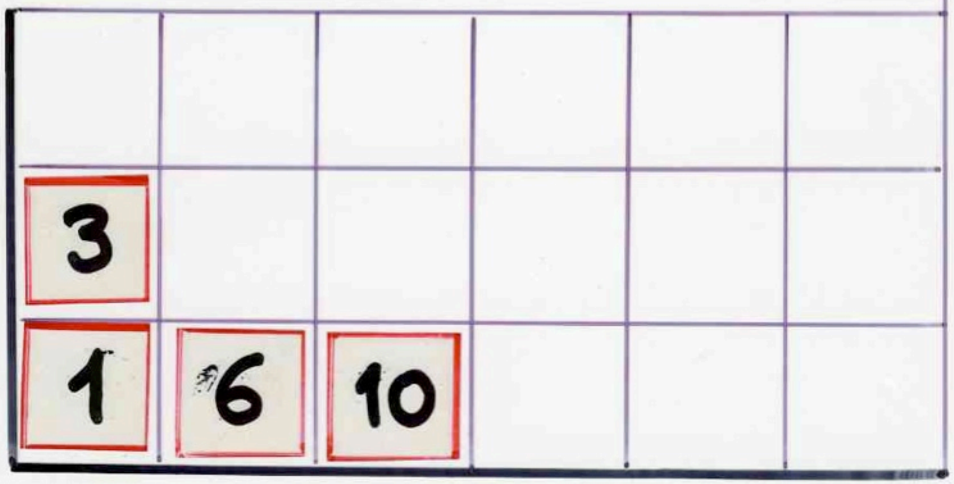
2					
1					

3					
1					

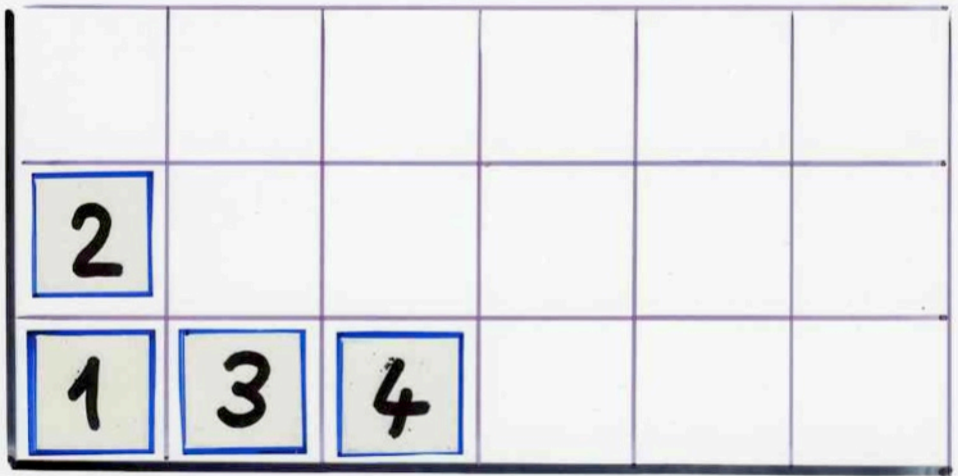
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

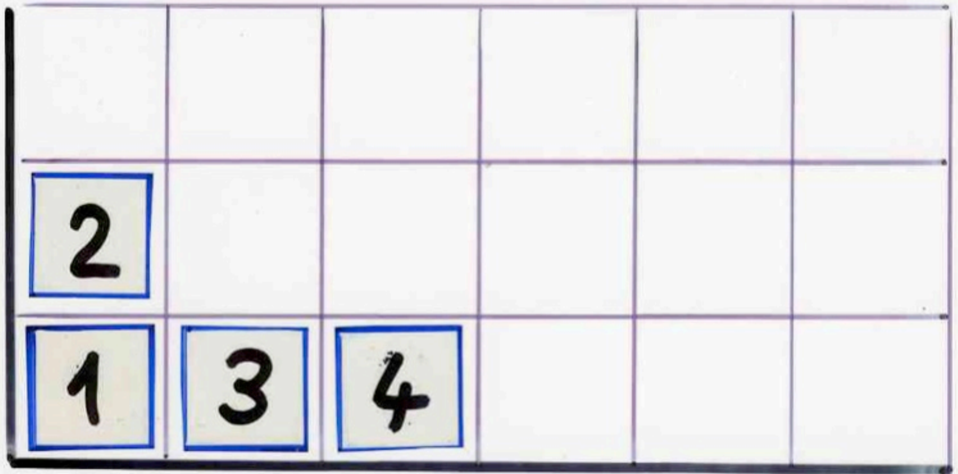


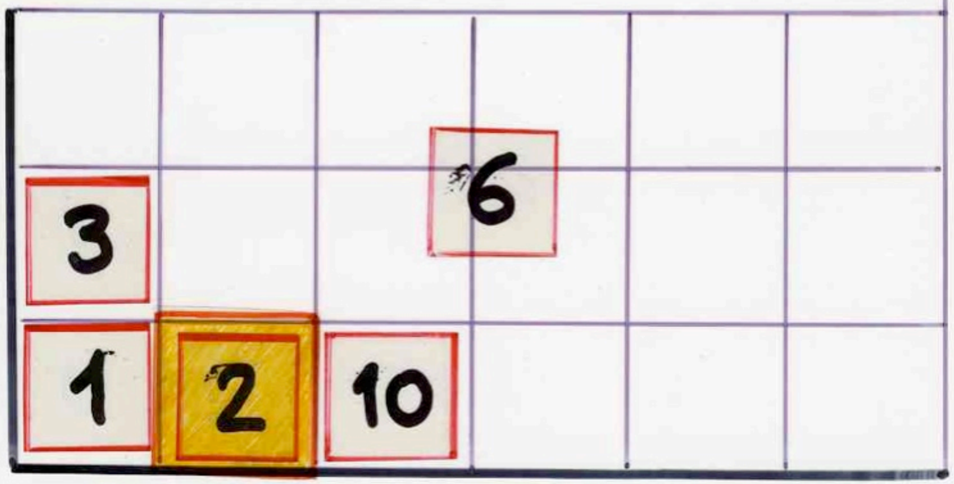


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

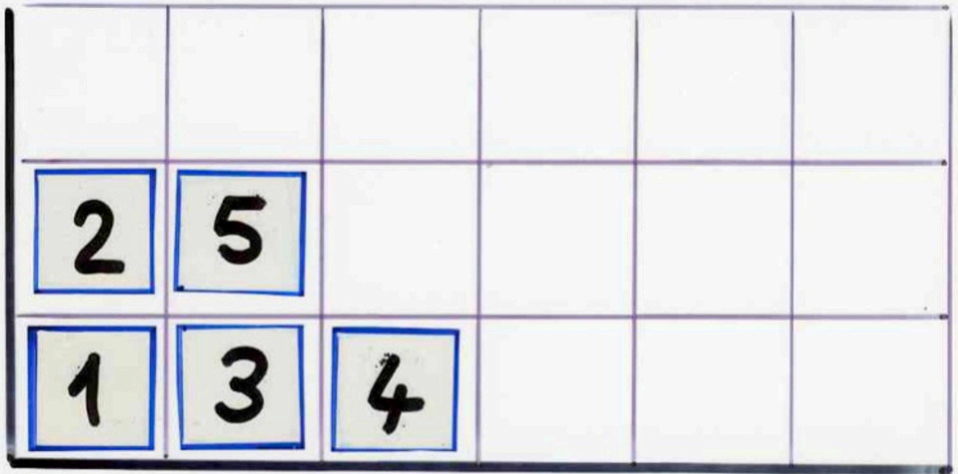


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

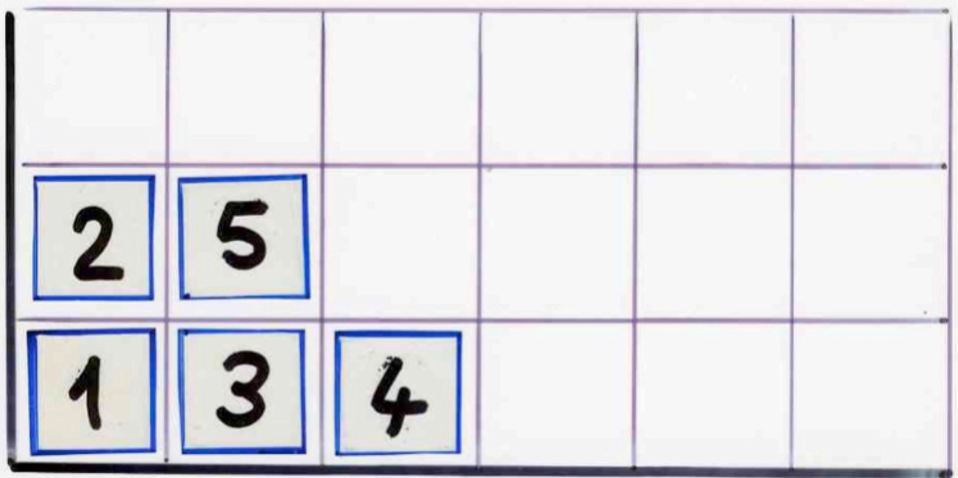


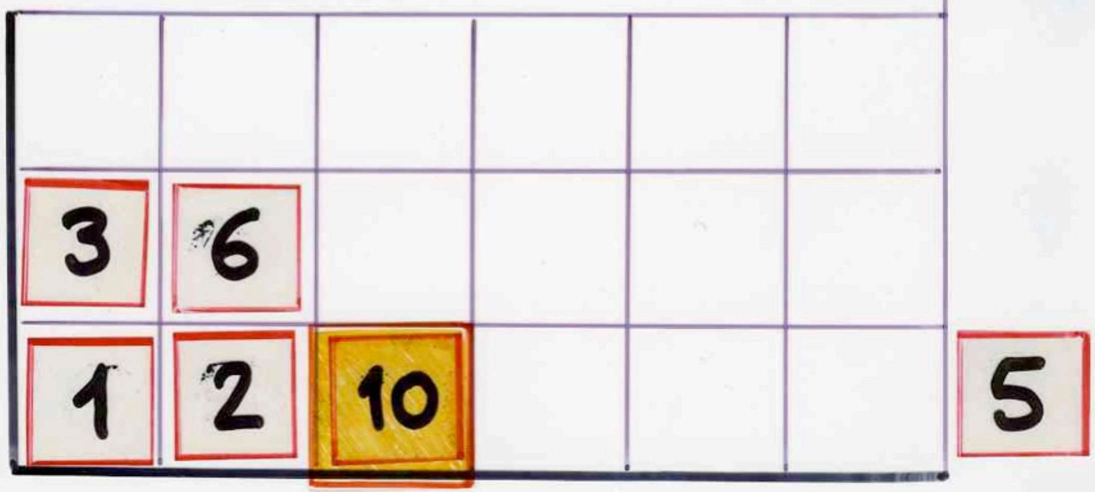


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

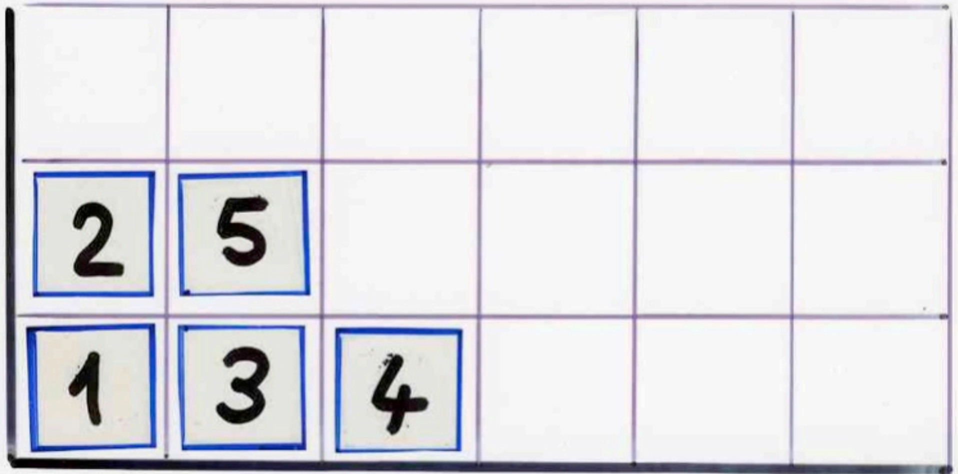


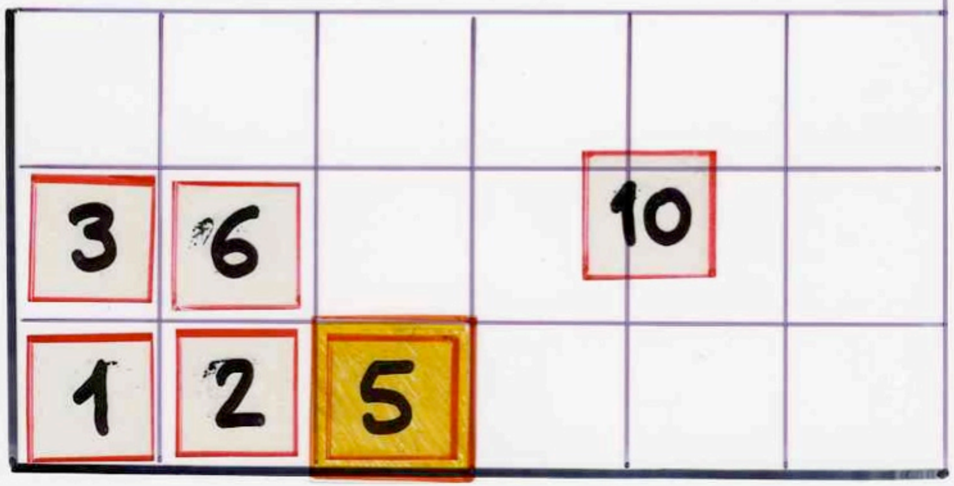
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

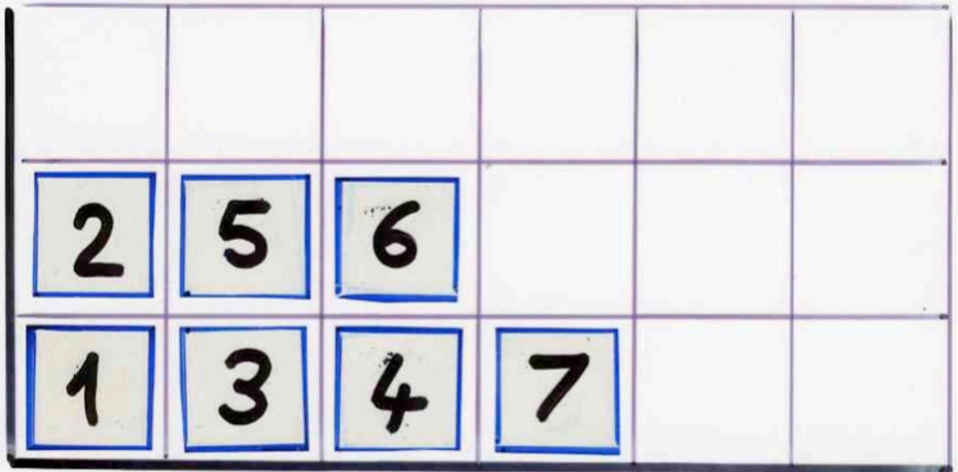
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

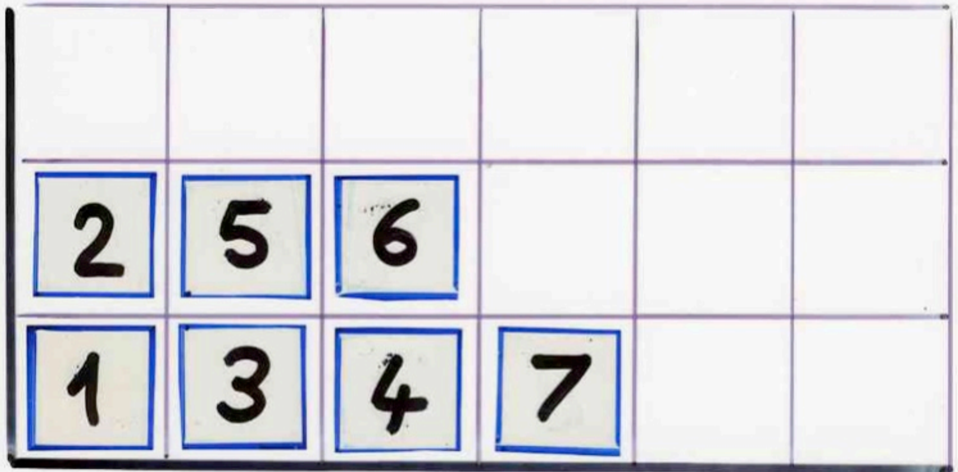
2	5	6			
1	3	4	7		

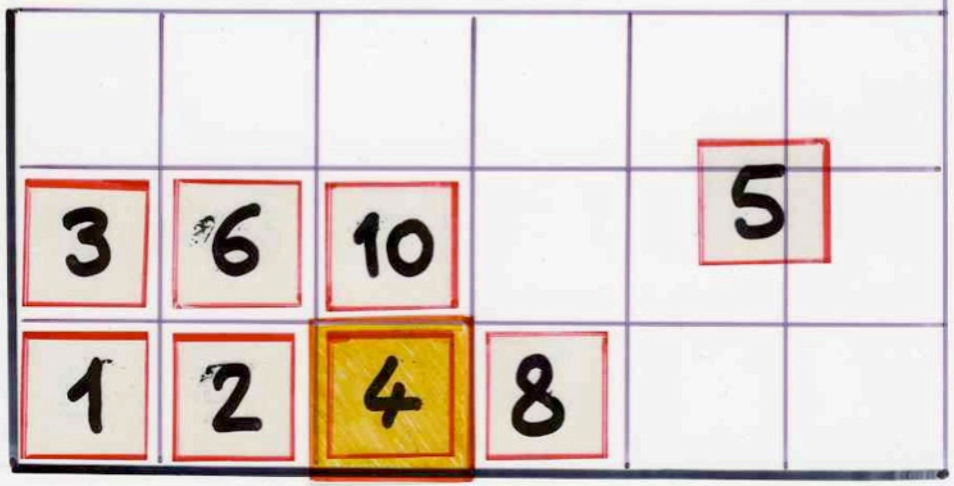
3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

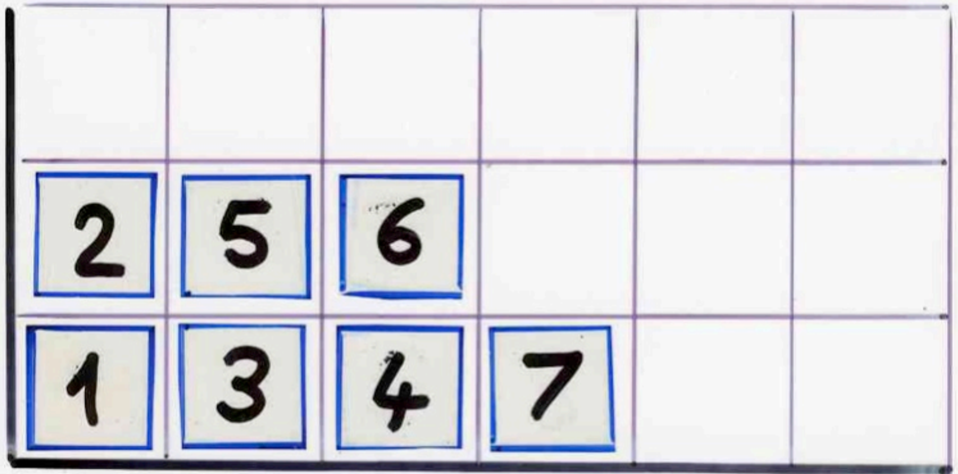


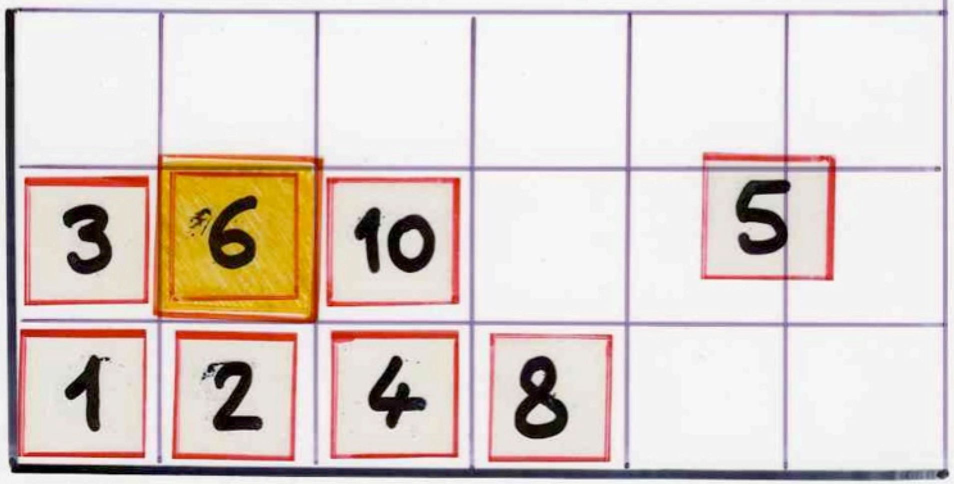
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



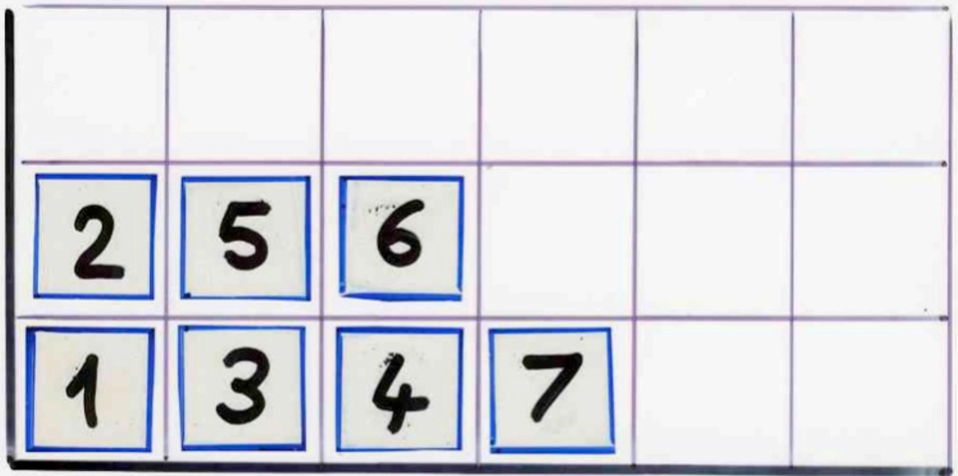


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

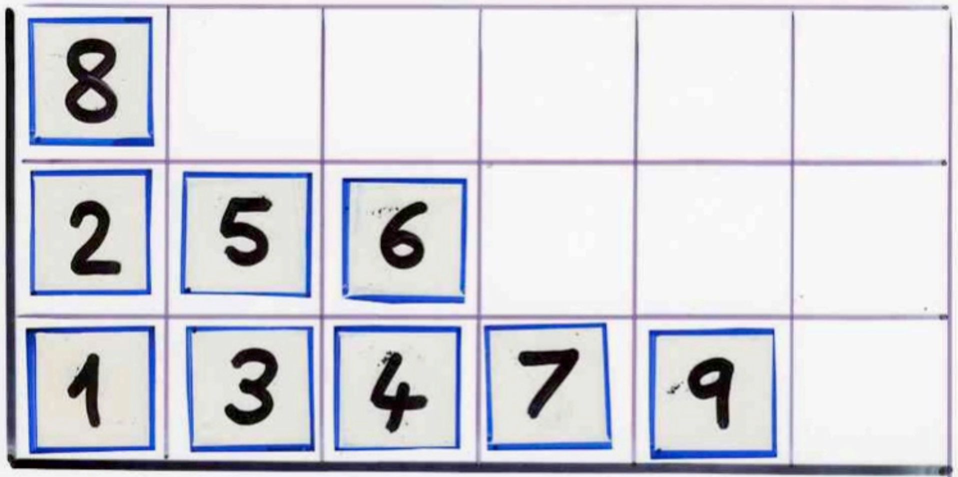
6					
3	5	10			
1	2	4	8		

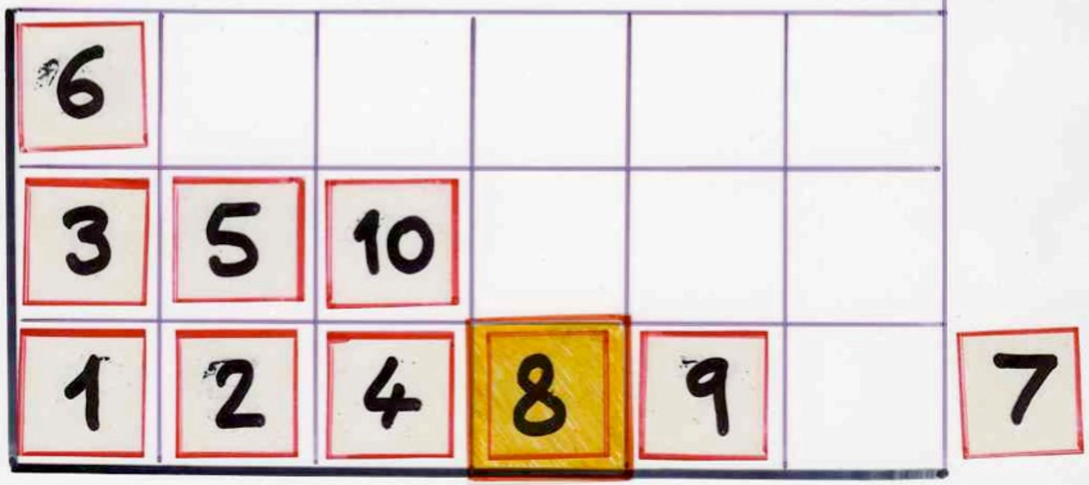
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

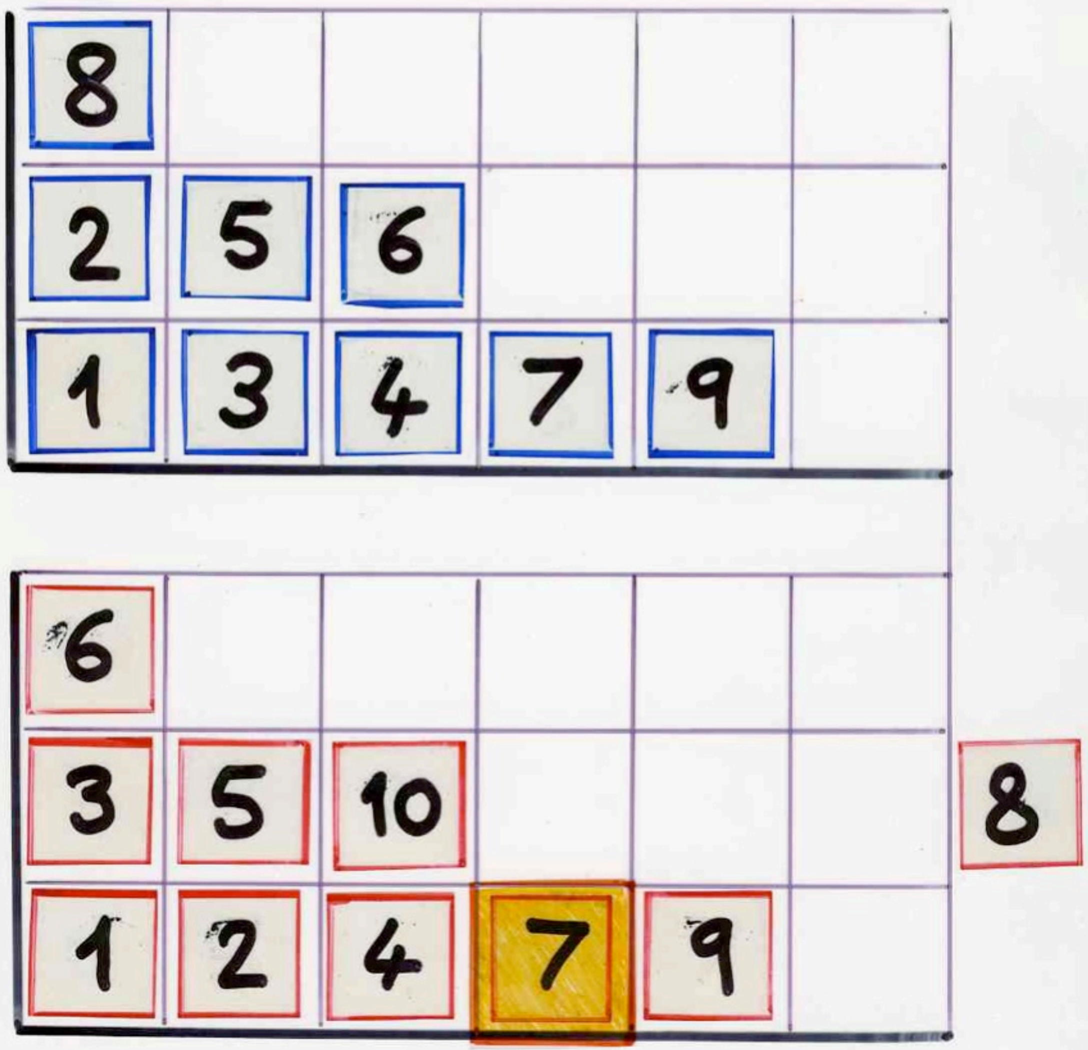
6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

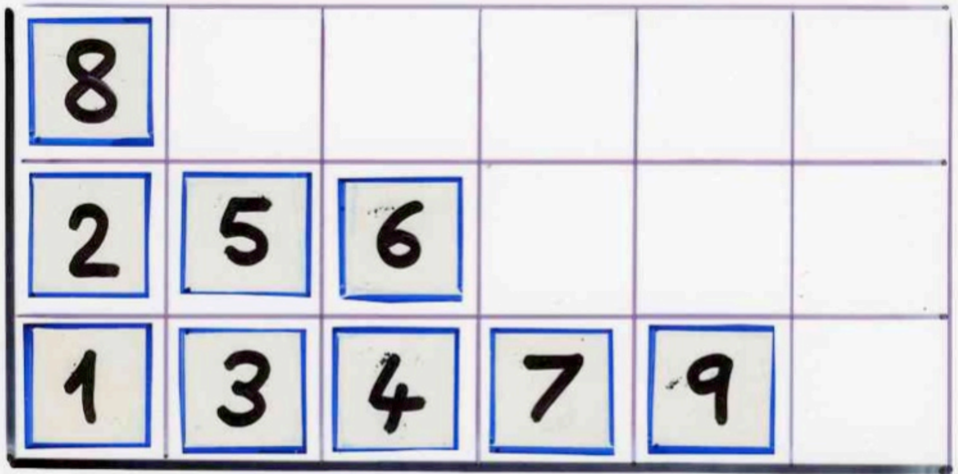




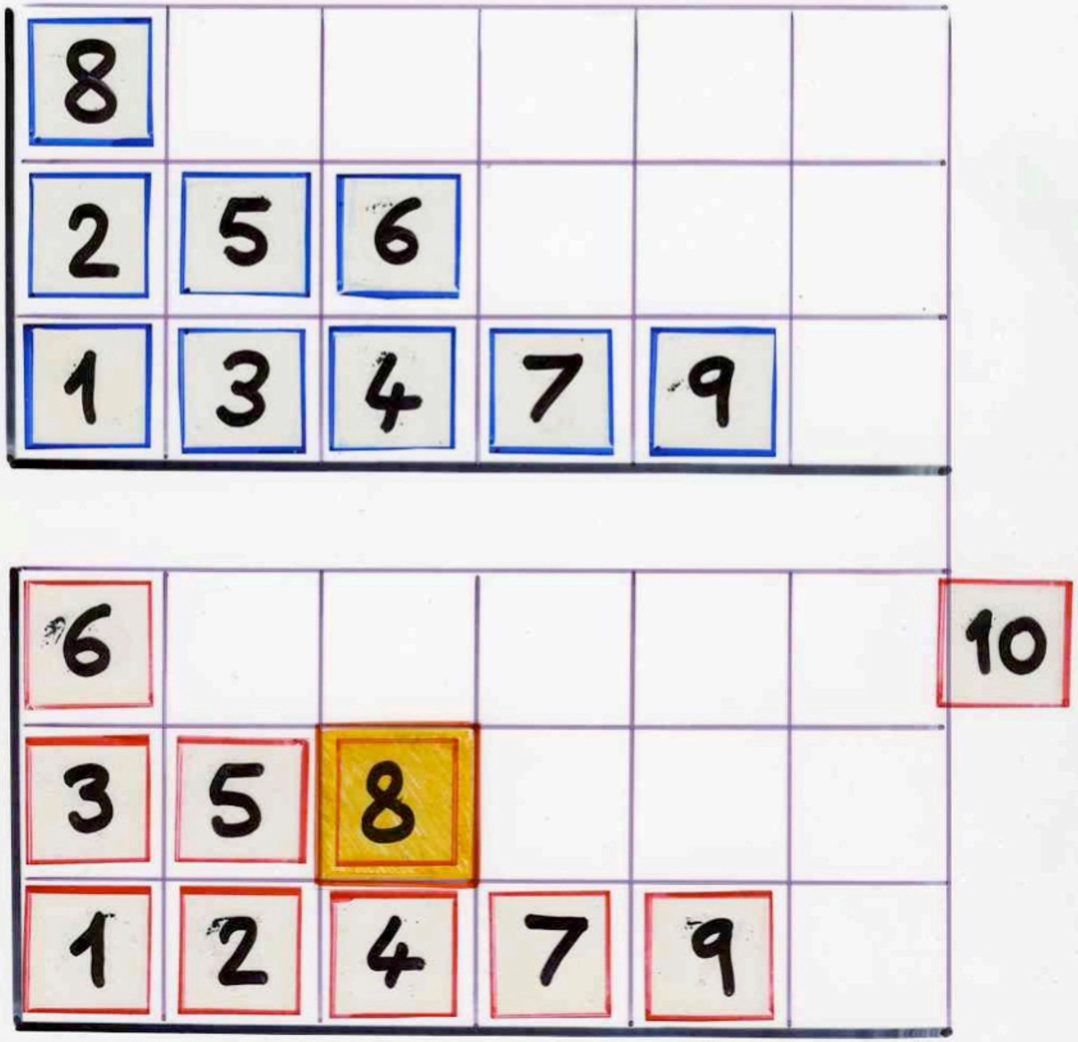
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

$$\sigma \longleftrightarrow (P, Q)$$

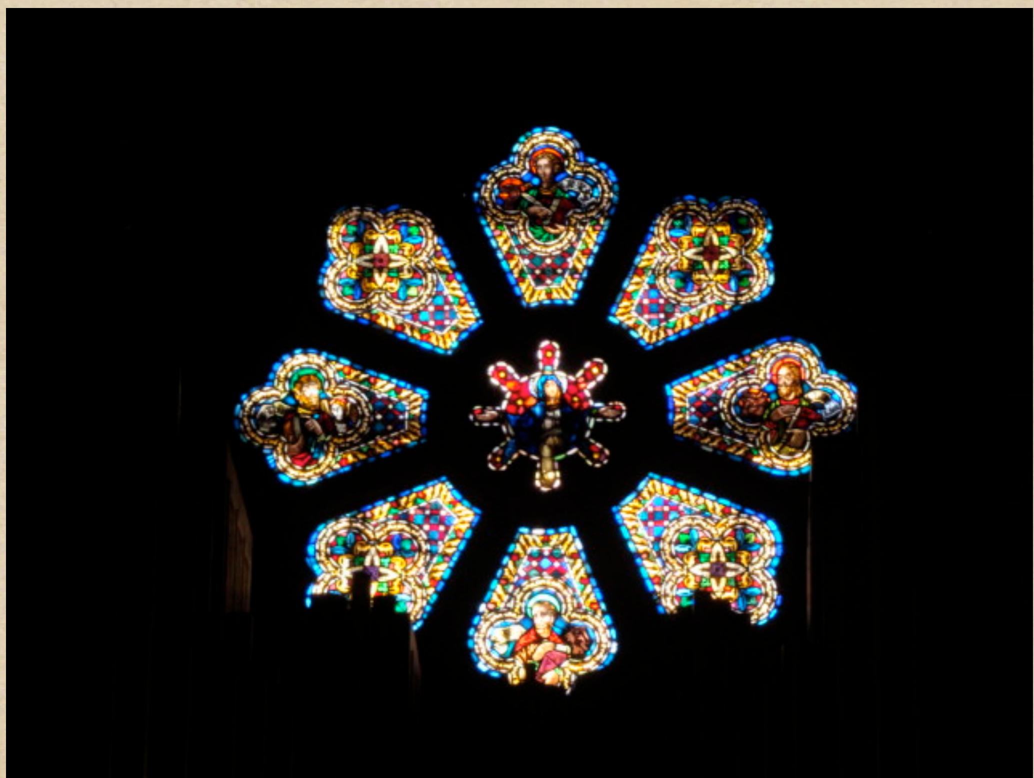
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

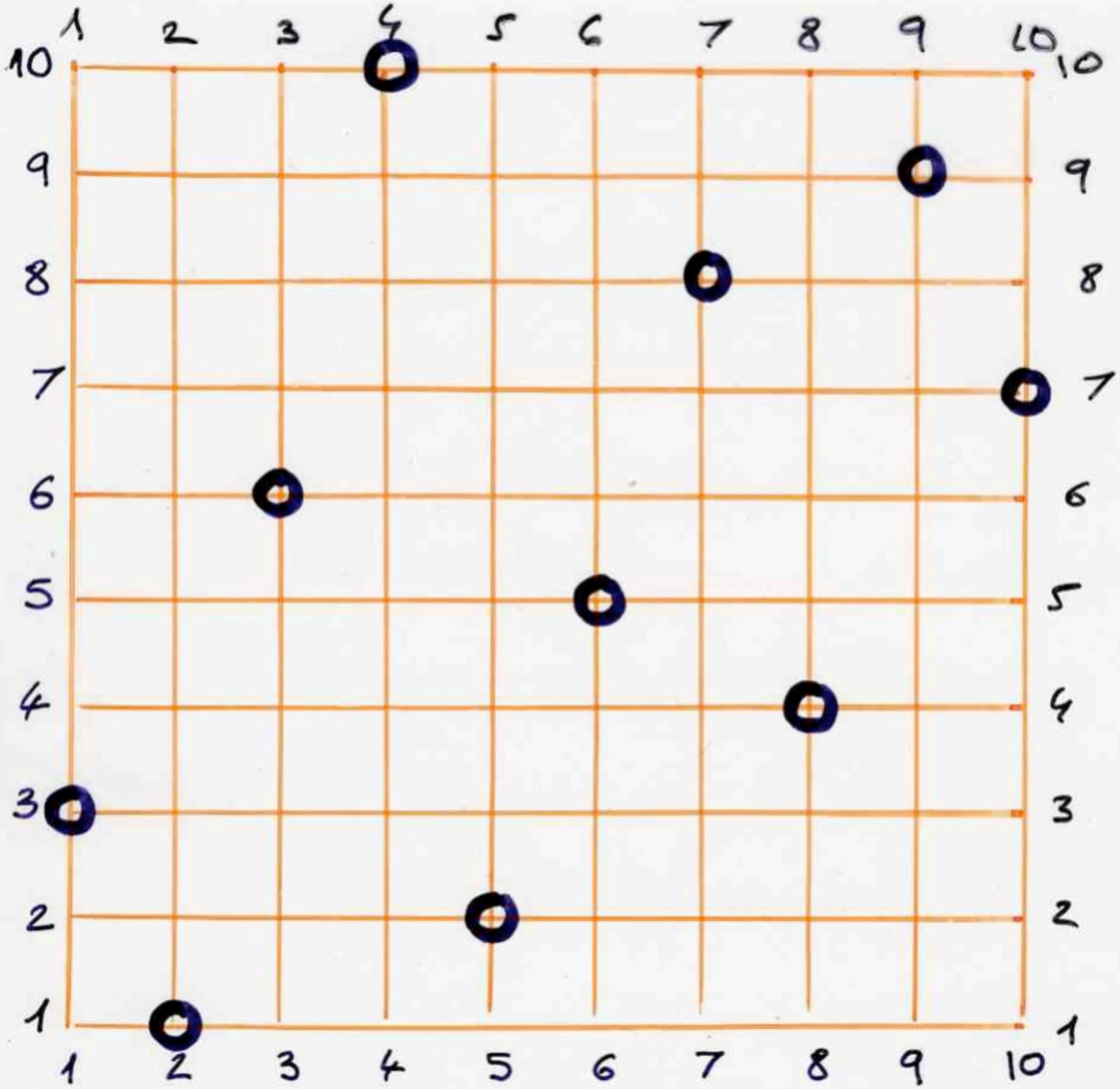
Donald Knuth (1972)

" The unusual nature of these coincidences might lead us to suspect that some sort of withcraft is operating behind the scenes "

A geometric version of RSK with "light" and "shadow lines"

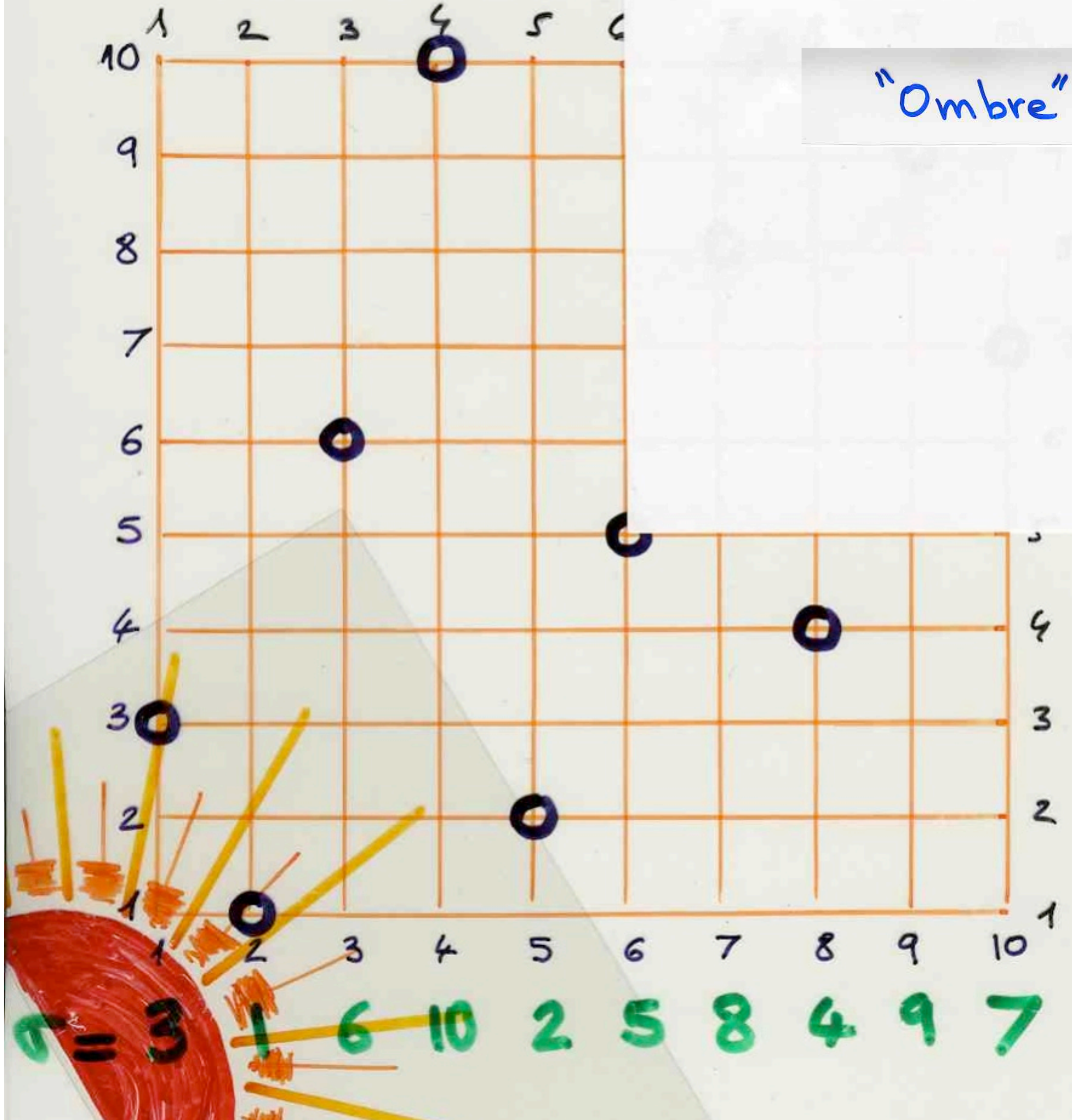
xgv, 1976





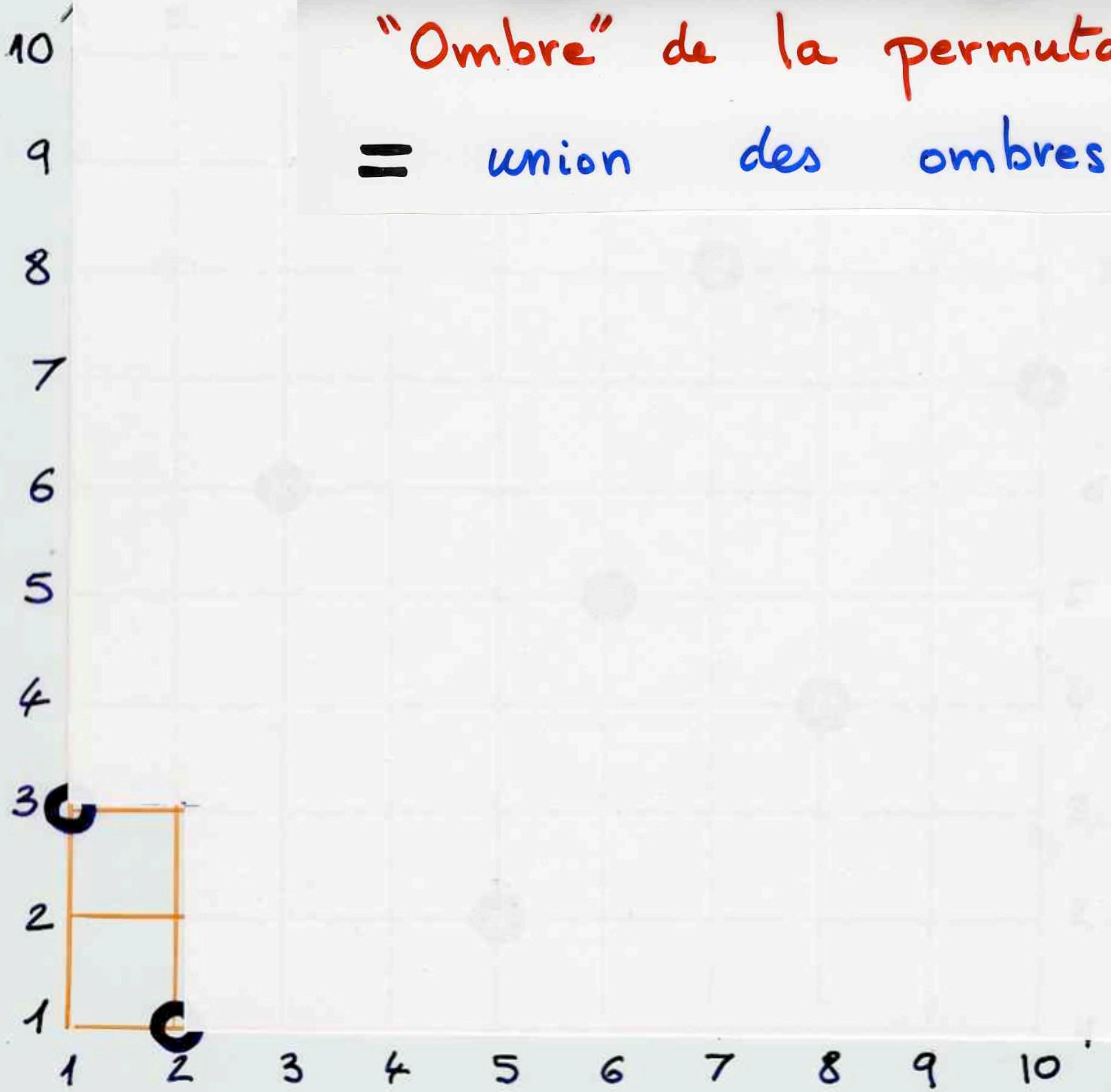
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$

"Ombre" d'un point

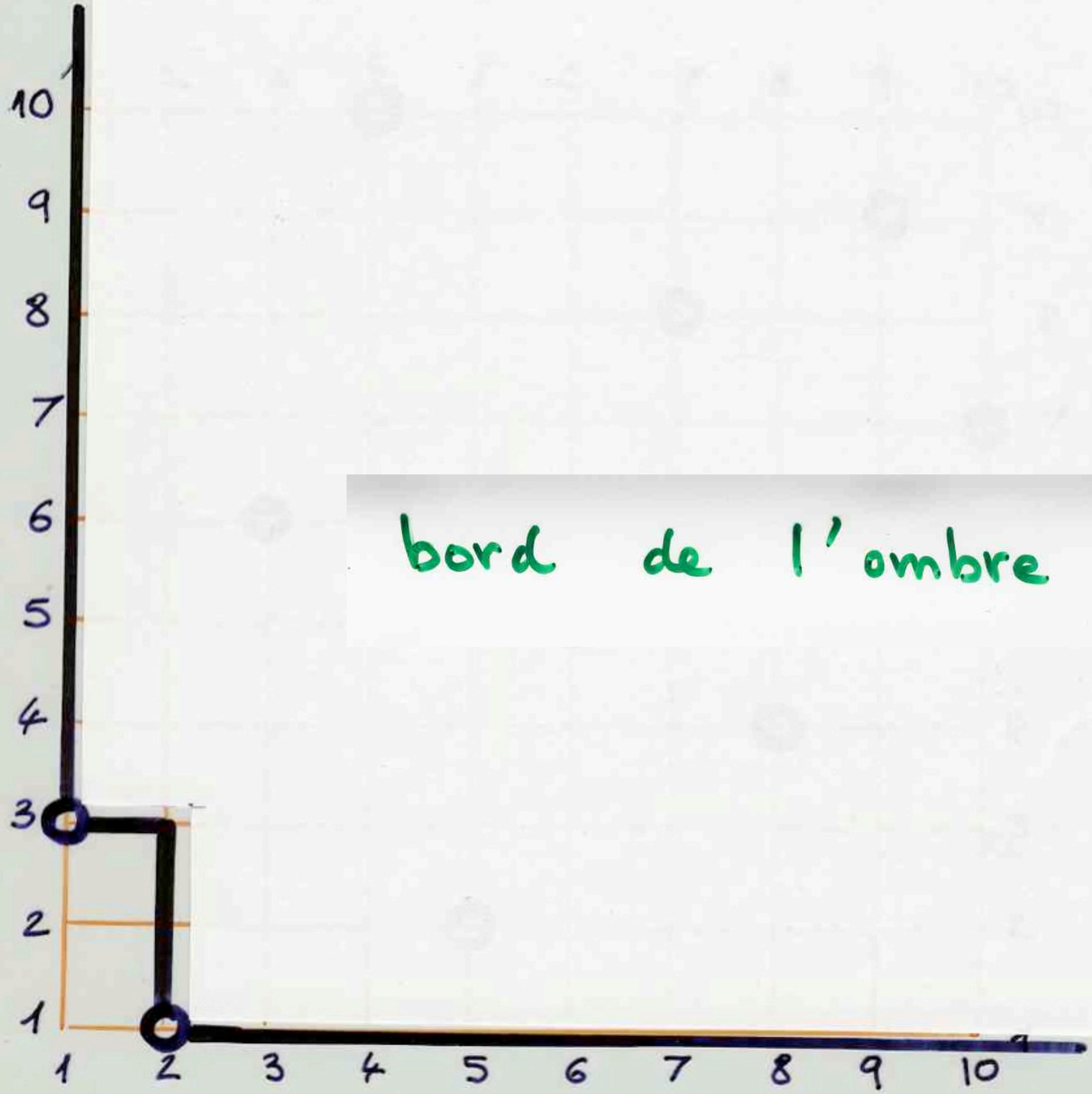


"Ombre" de la permutation

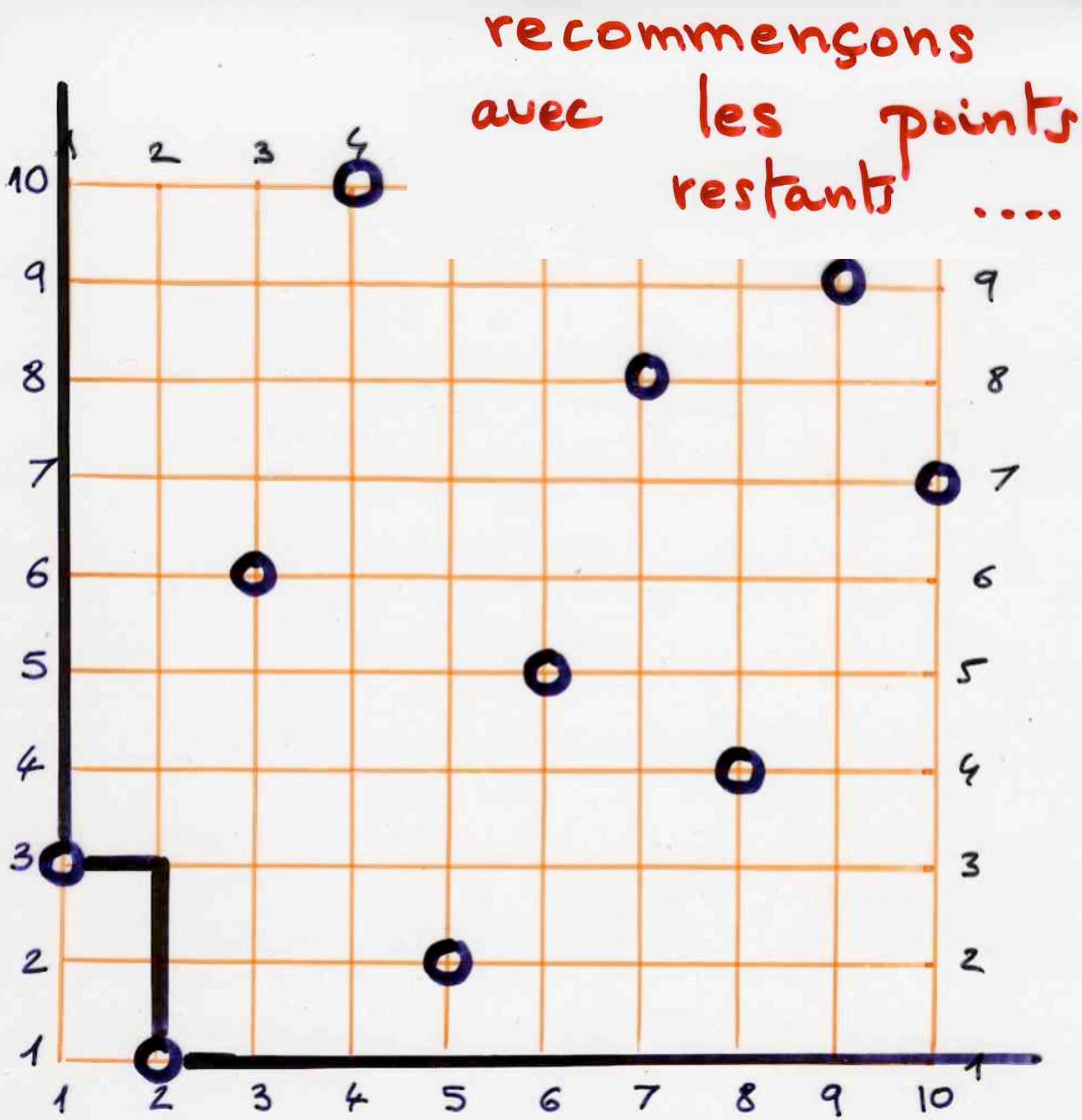
= union des ombres



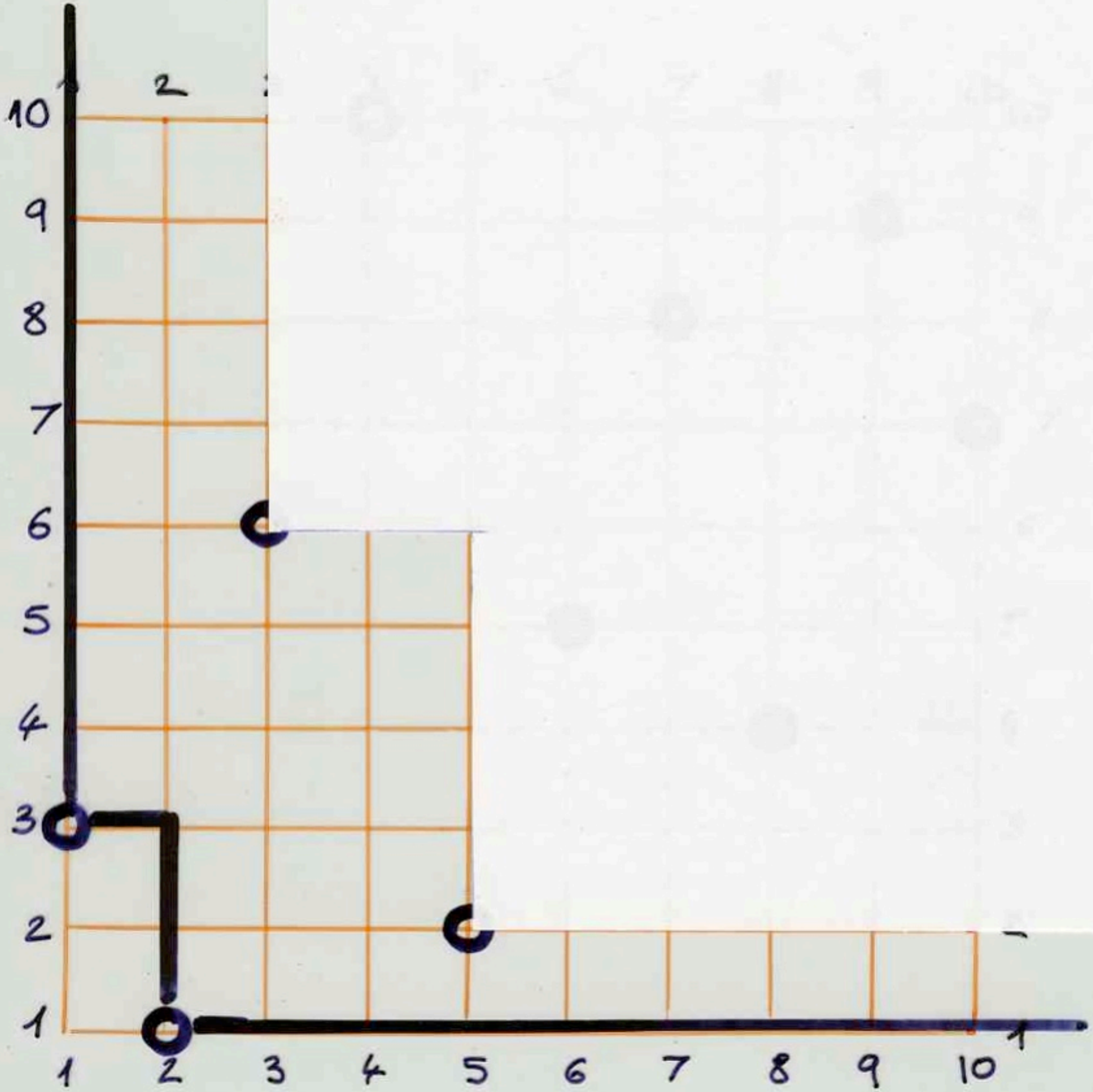
σ = 3 1 6 10 2 5 8 4 9 7



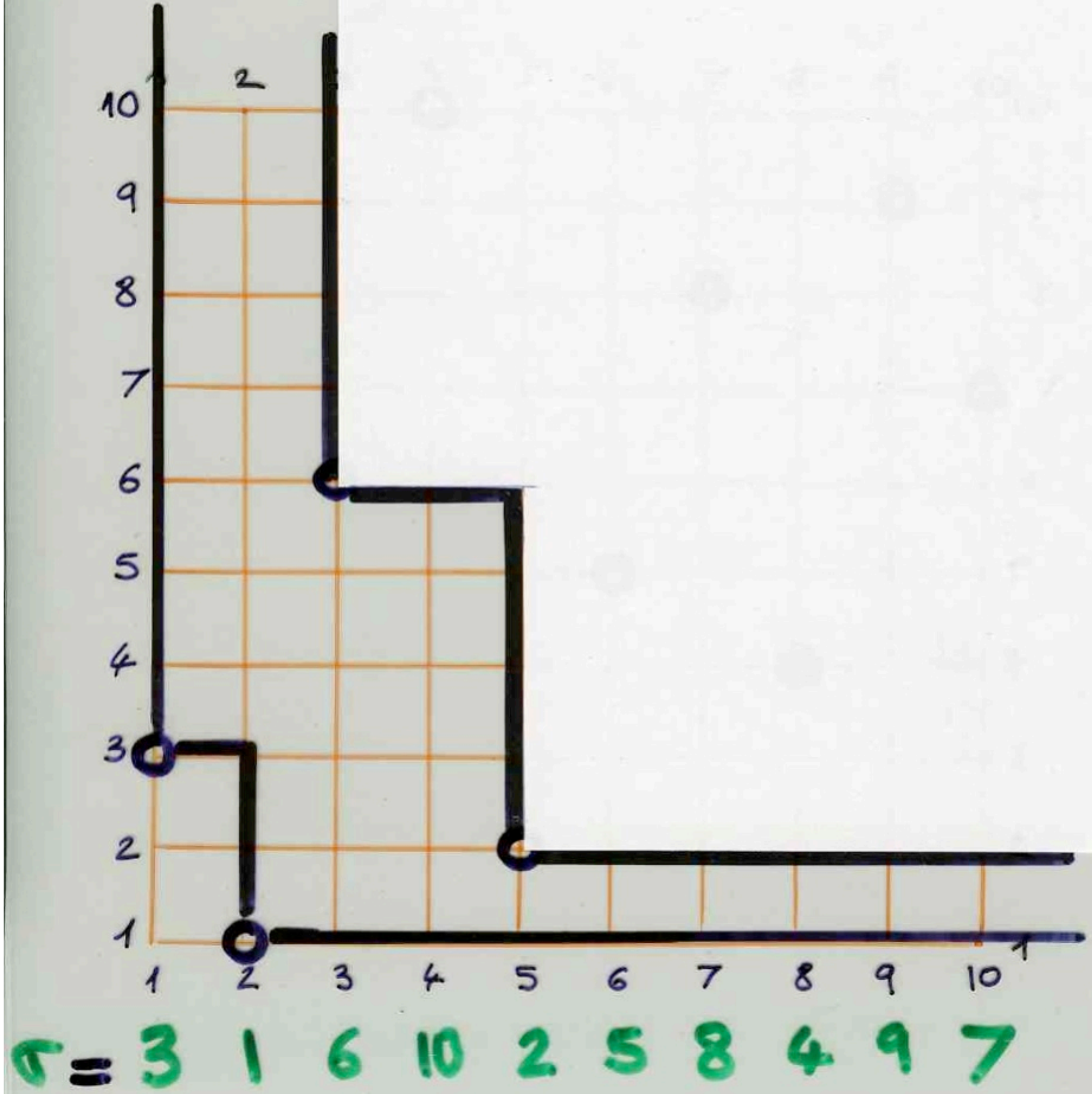
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



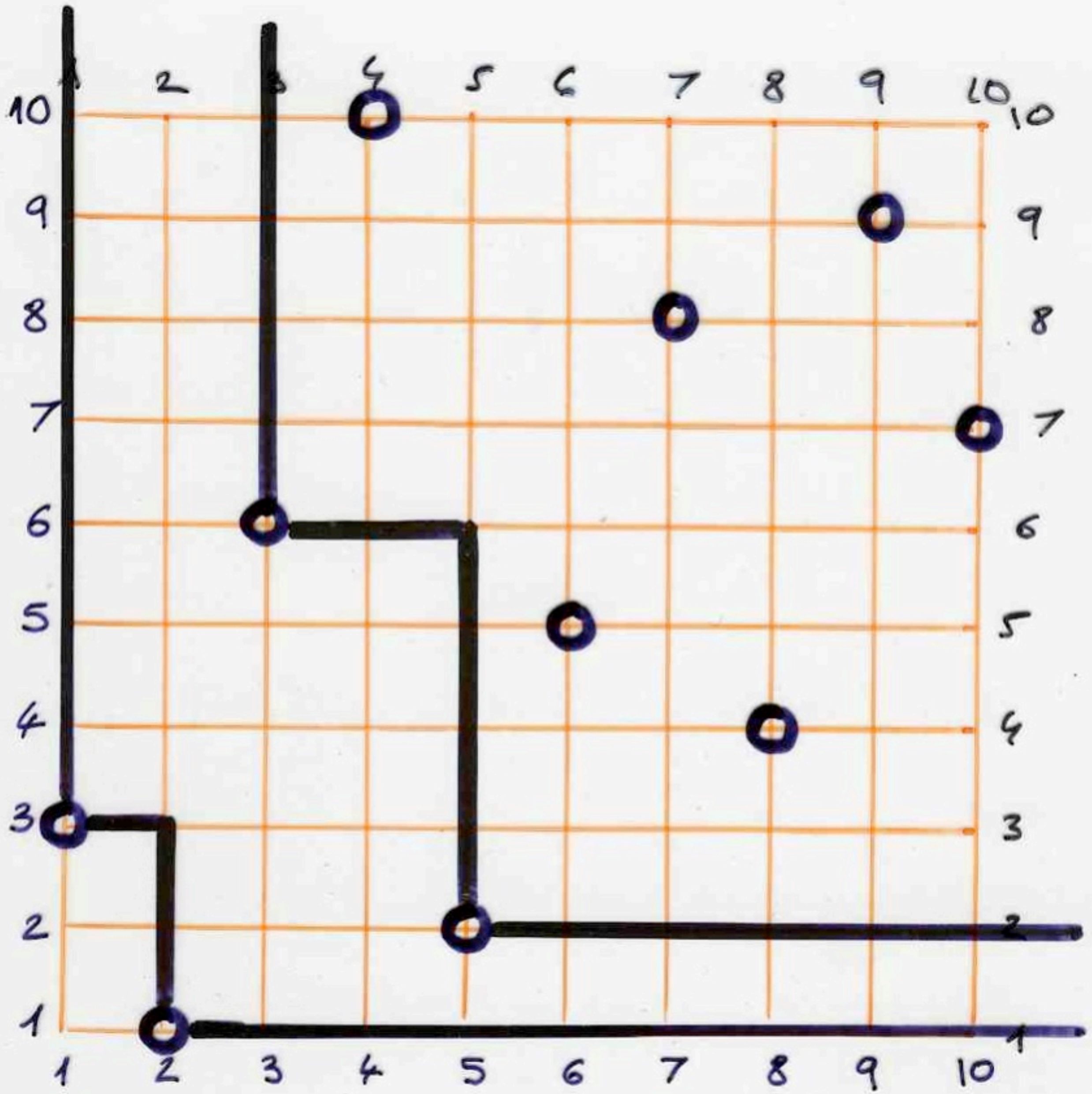
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$

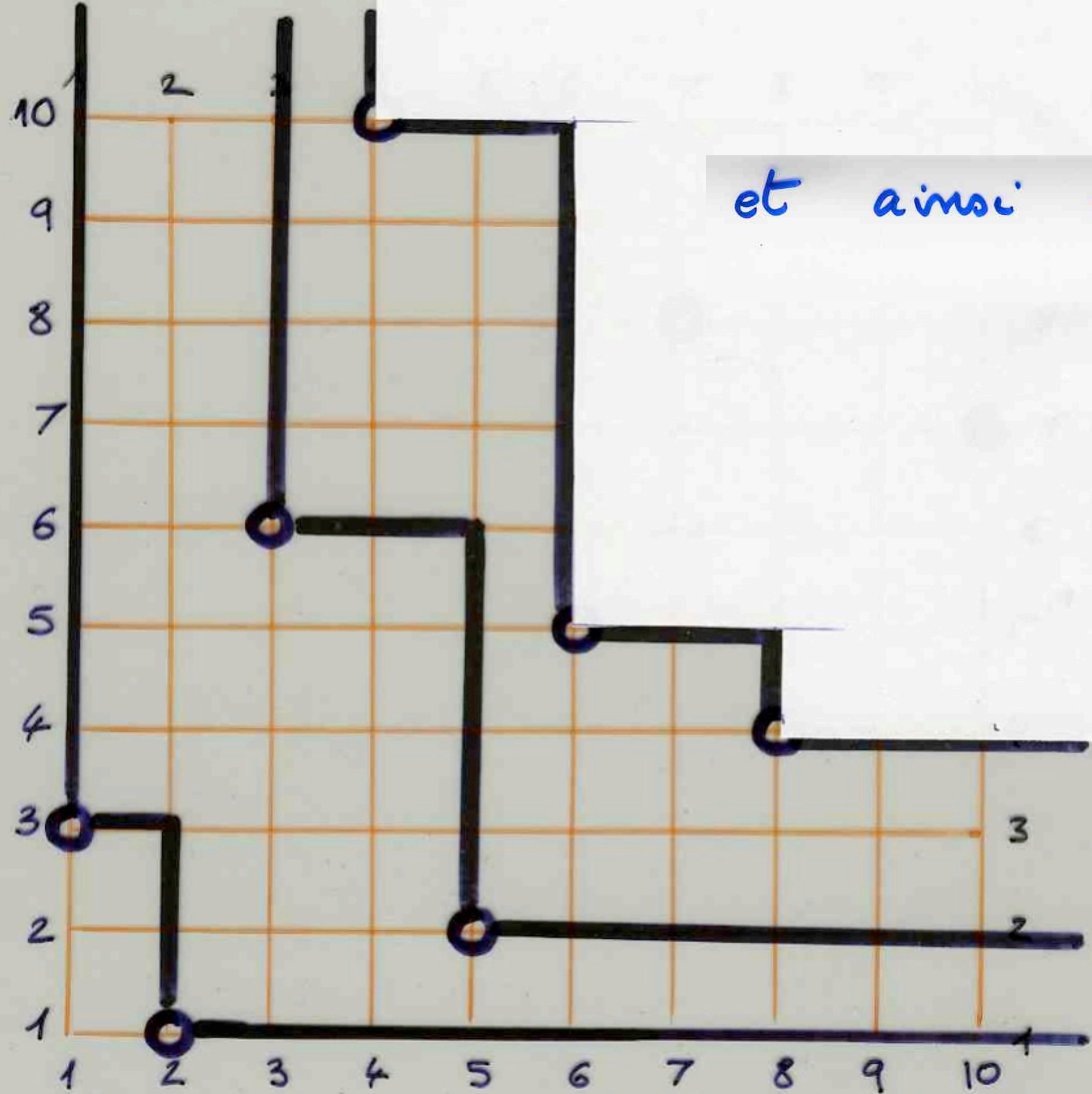


σ = 3 1 6 10 2 5 8 4 9 7

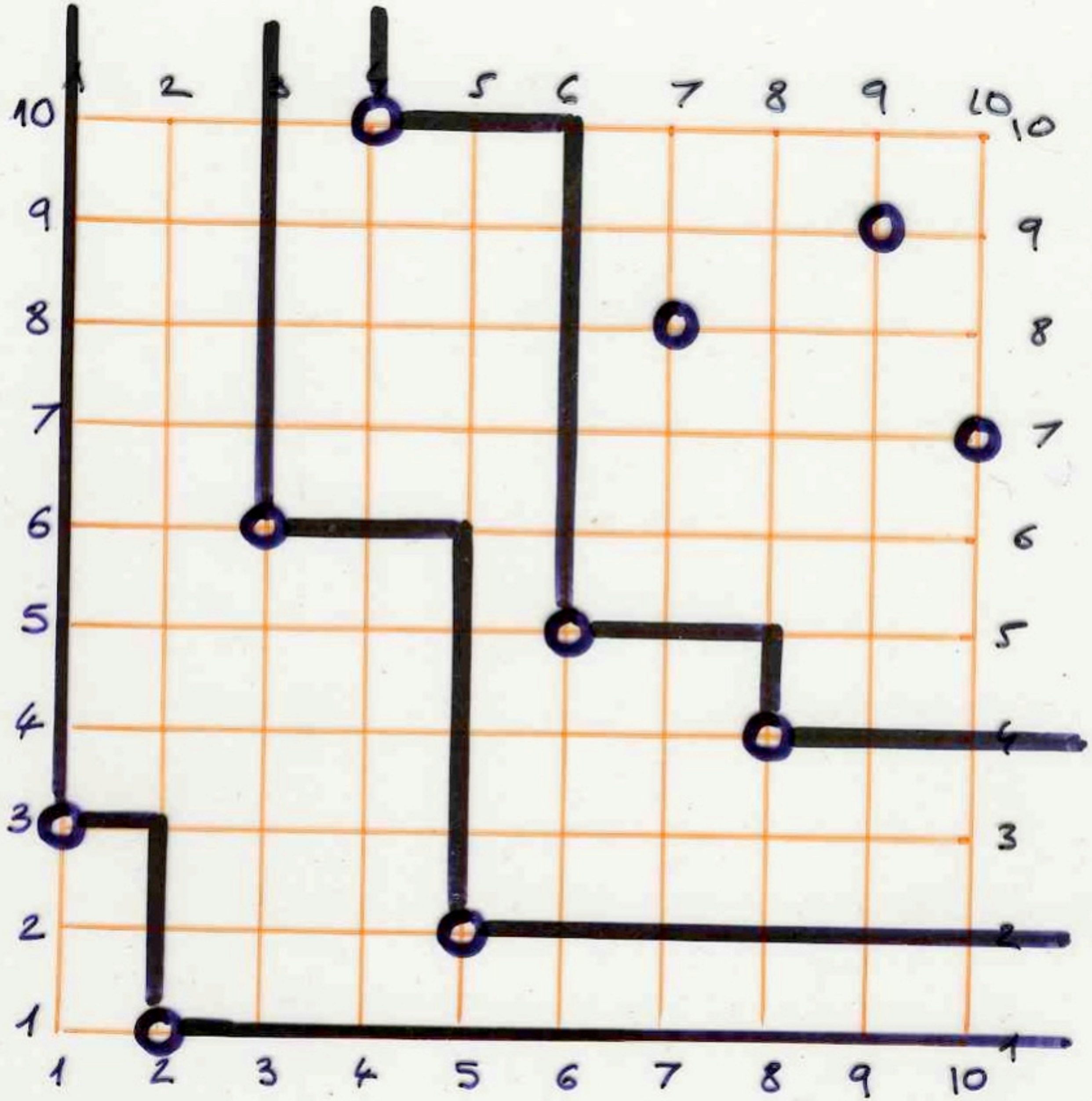


$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

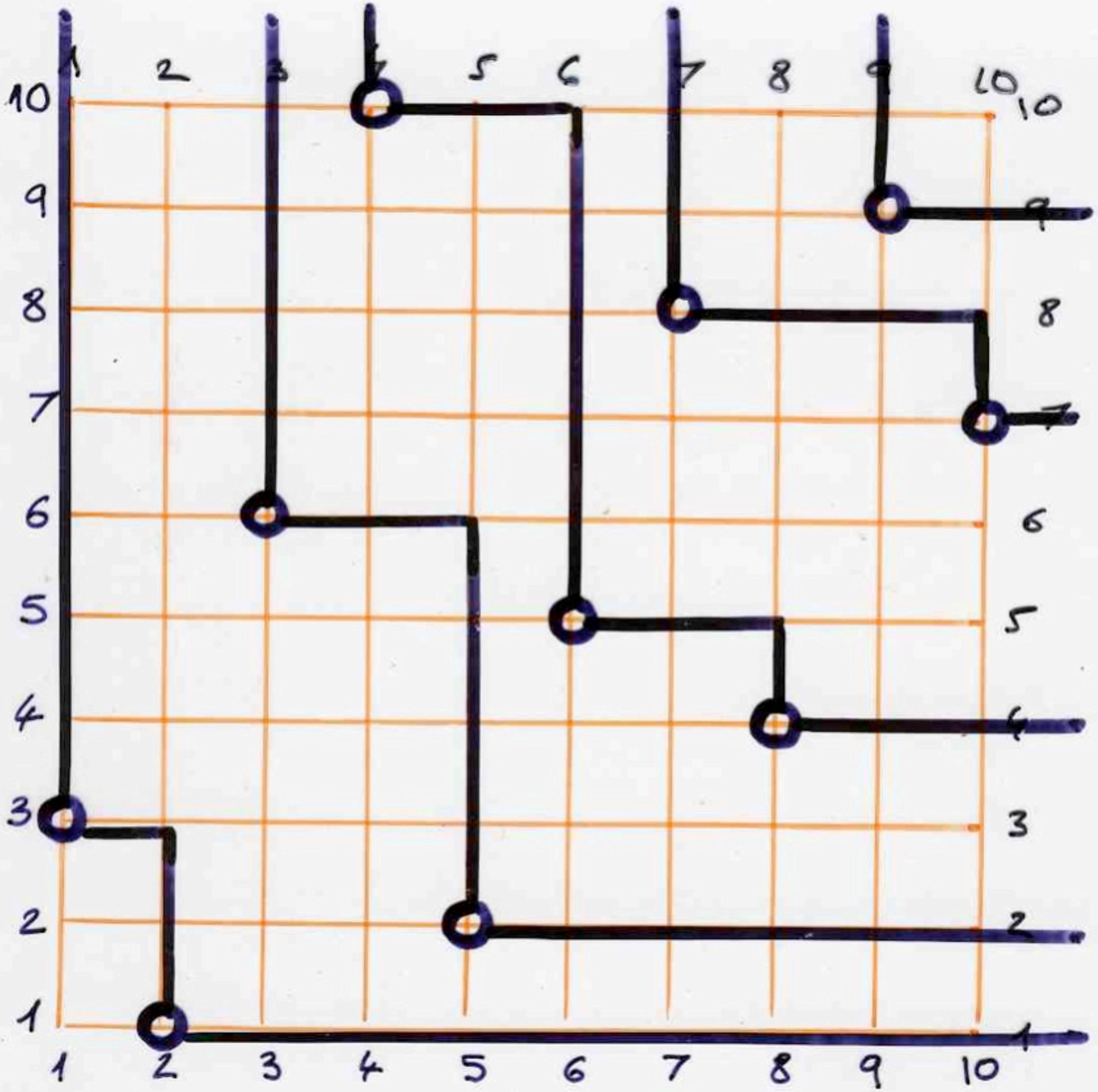
et ainsi de suite ...



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

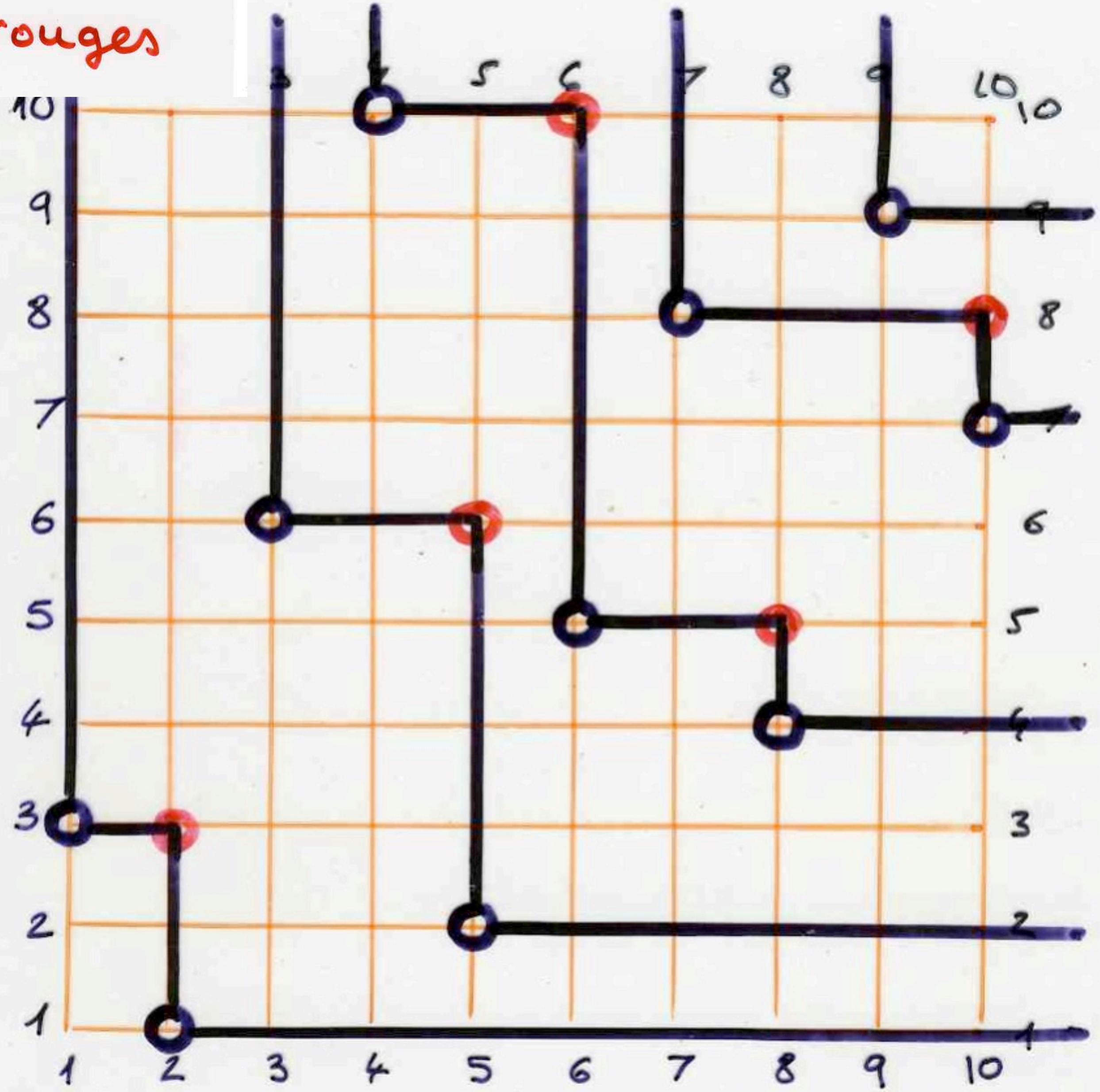


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

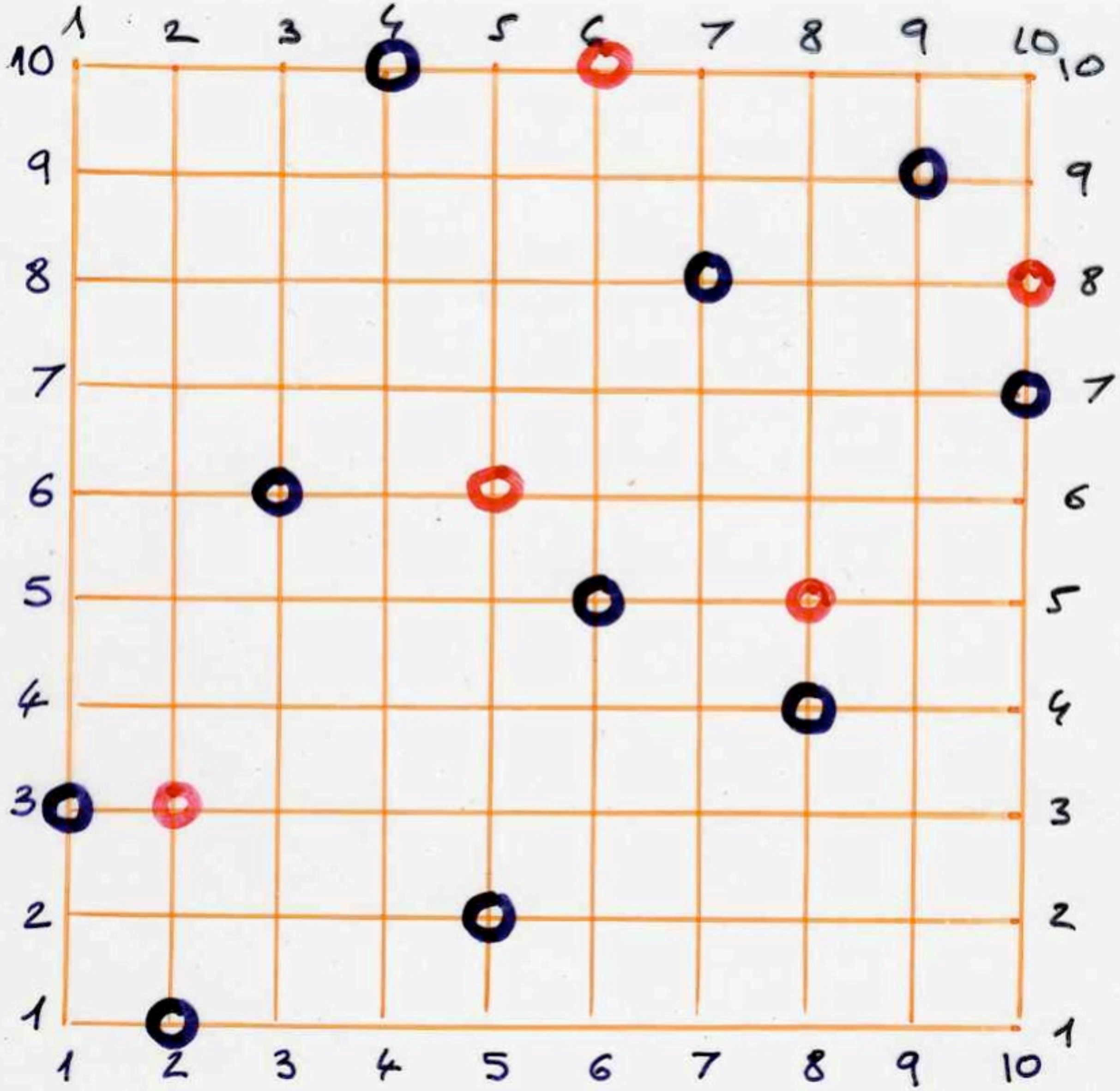


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

des nouveaux points
les rouges

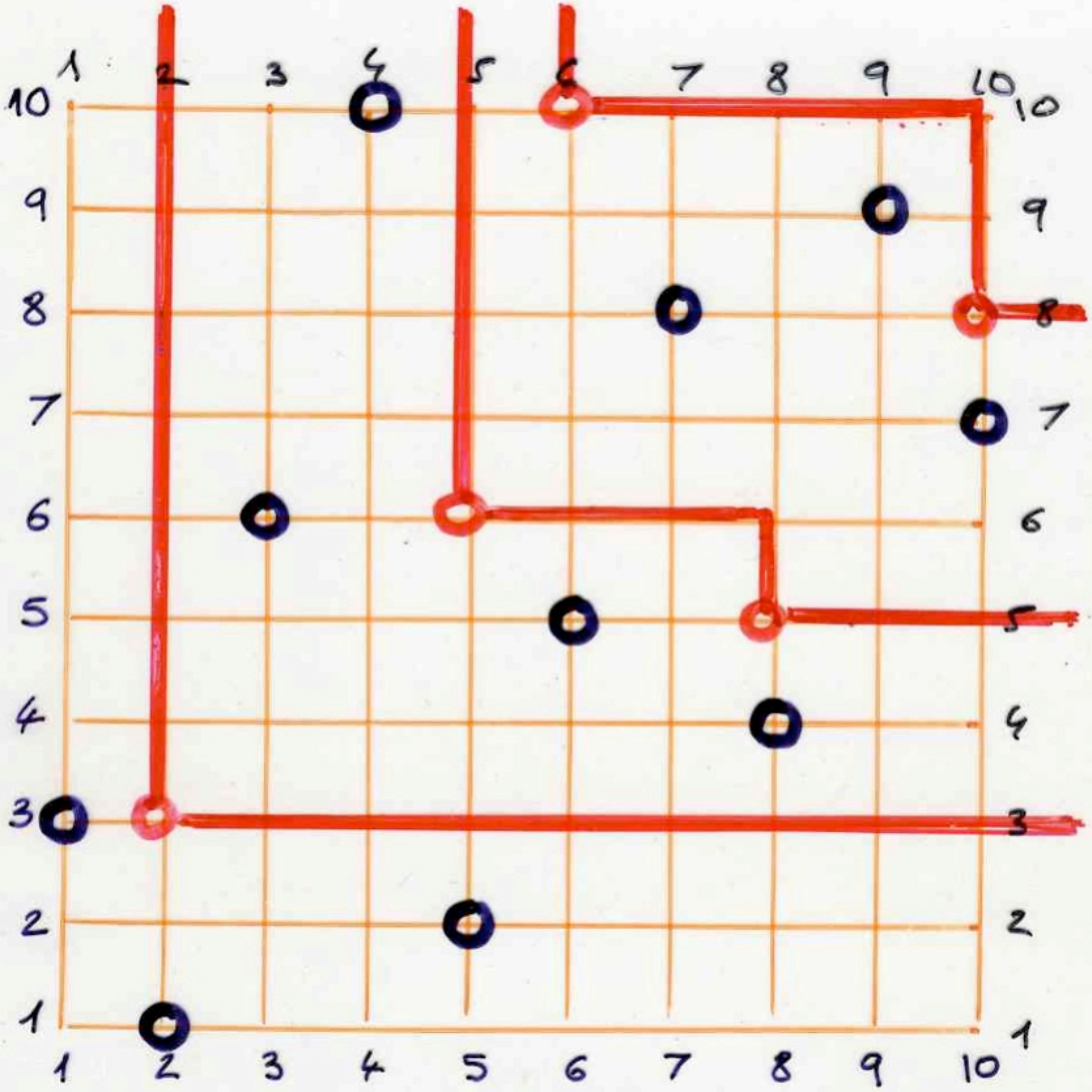


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

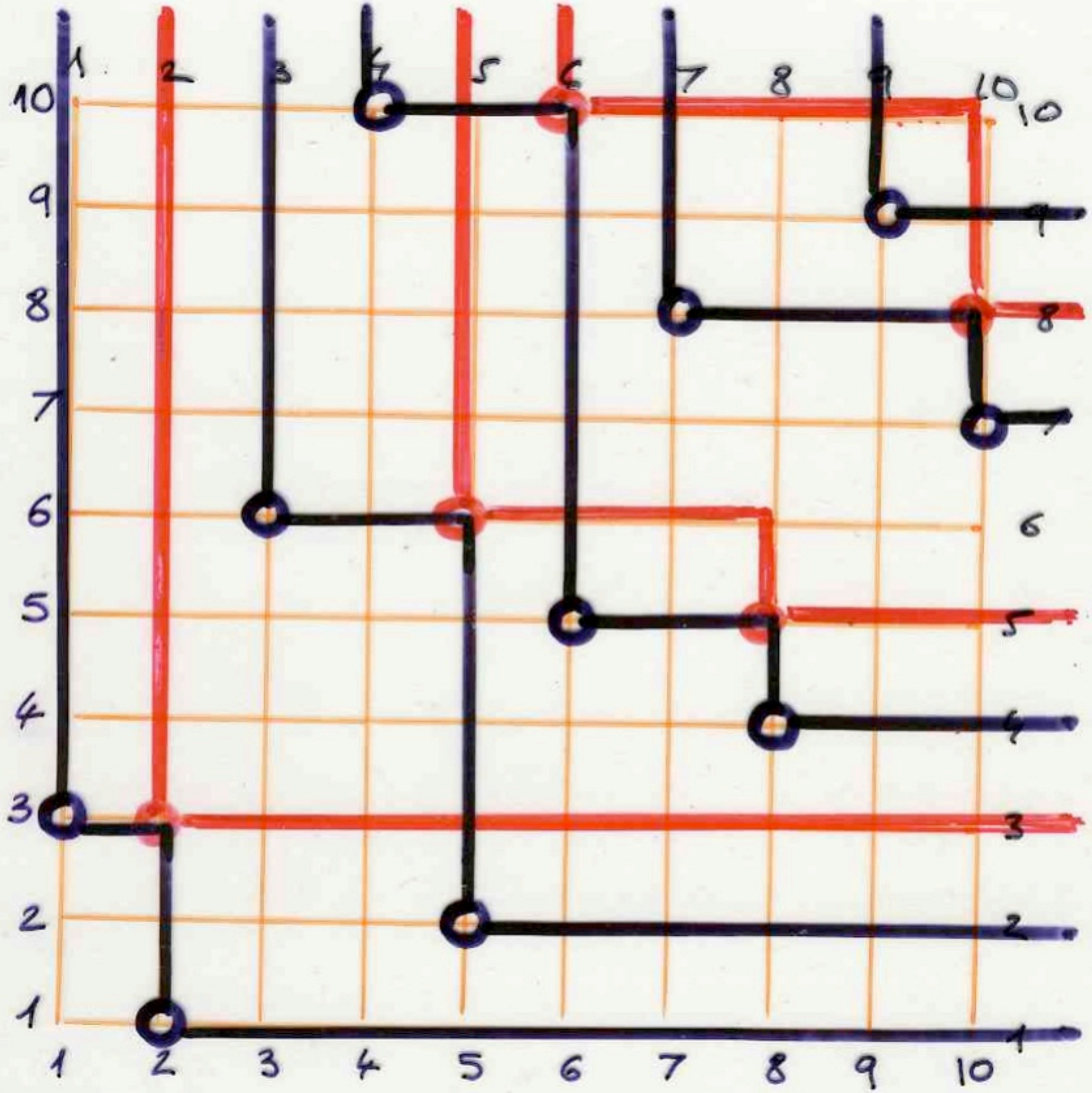


$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

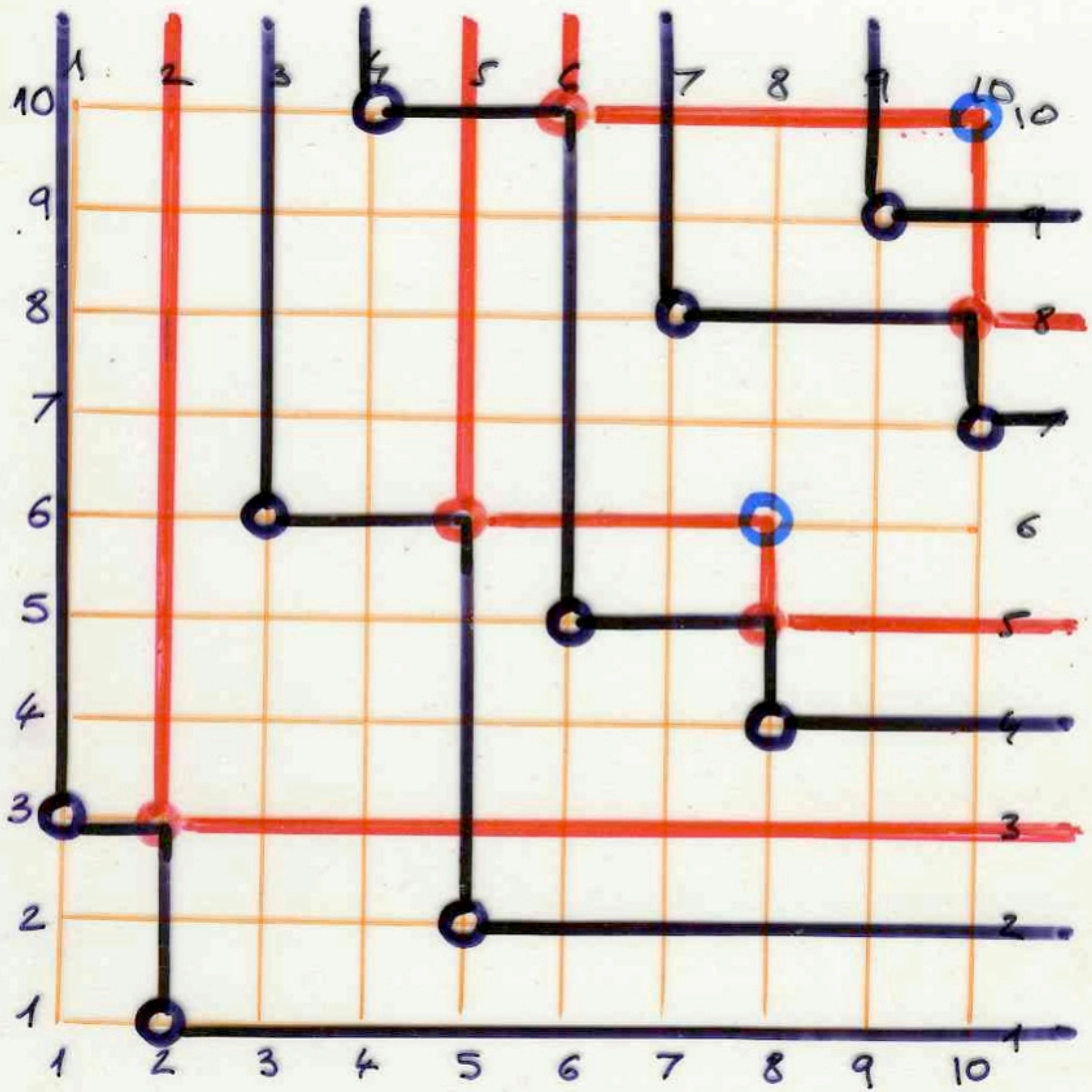
Répétons sur les points rouges la construction des bords d'ombres.



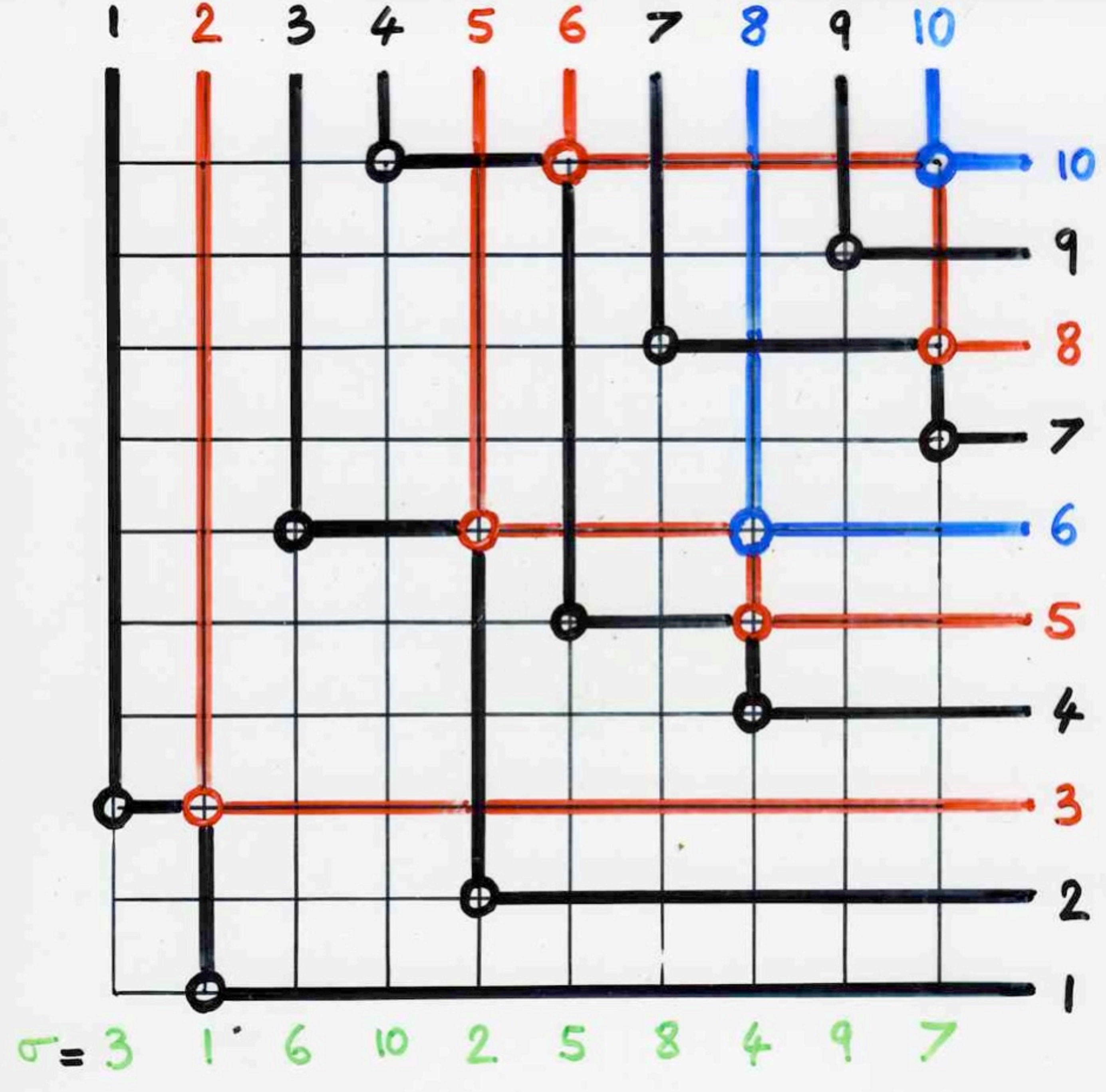
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



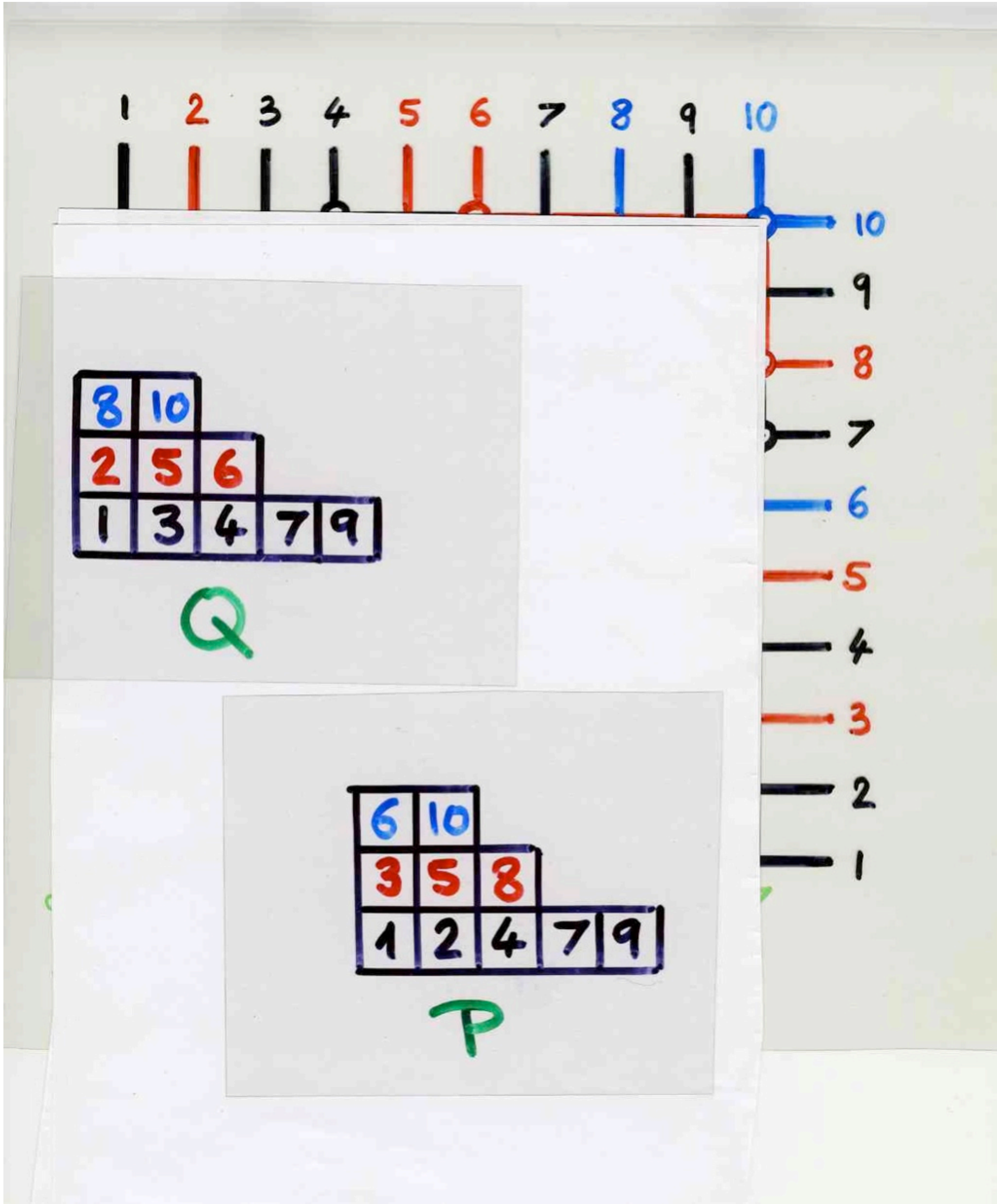
$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$



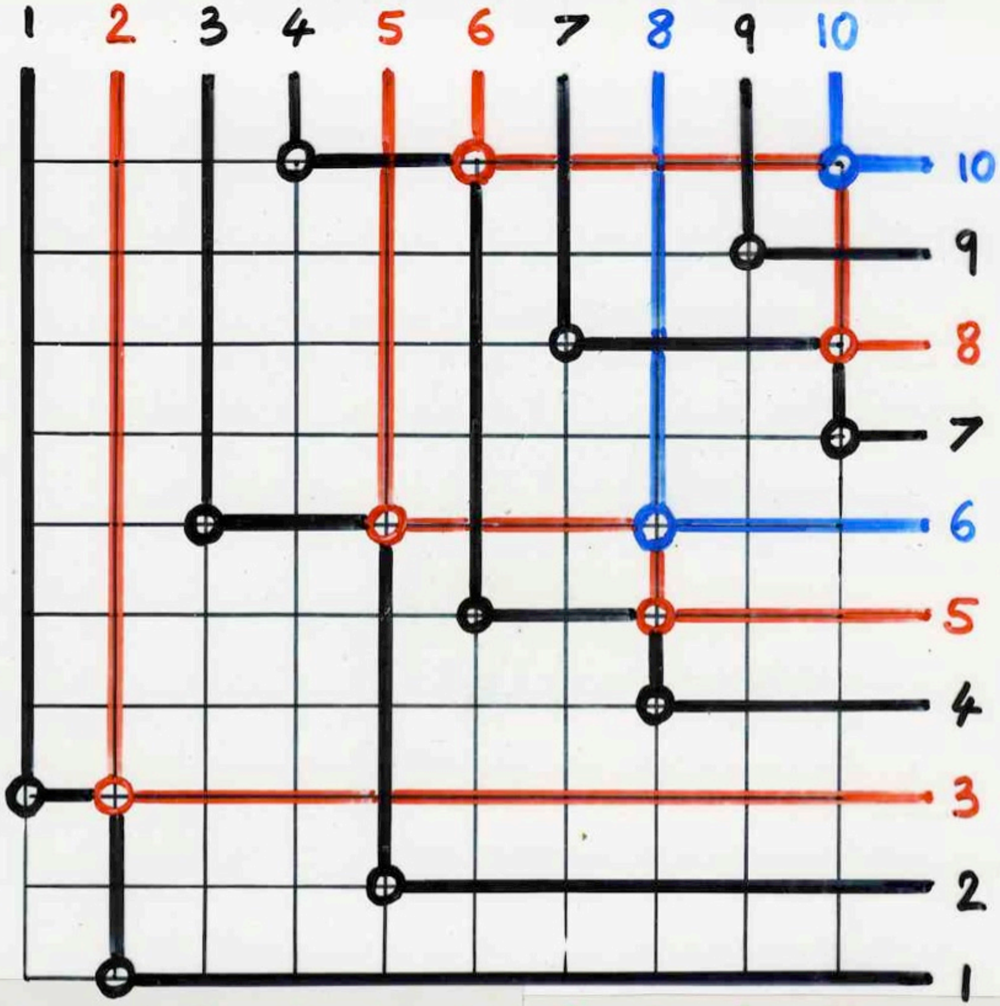
1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7

1 2 3 4 5 6 7 8 9 10

10 9 8 7 6 5 4 3 2 1



1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P



σ = 3 1 6 10 2 5 8 4 9 7

6	10			
3	5	8		
1	2	4	7	9

P

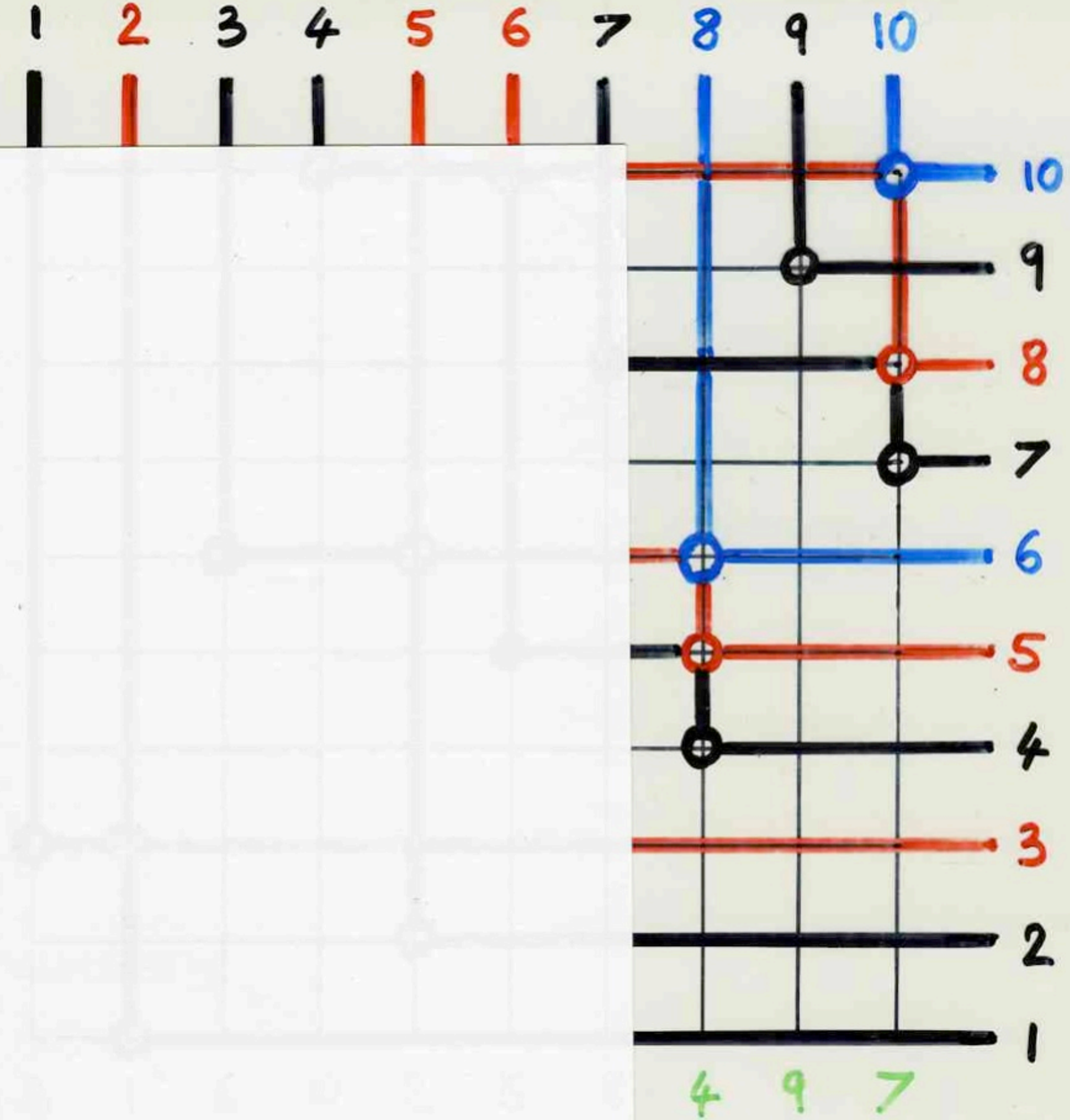
8	10			
2	5	6		
1	3	4	7	9

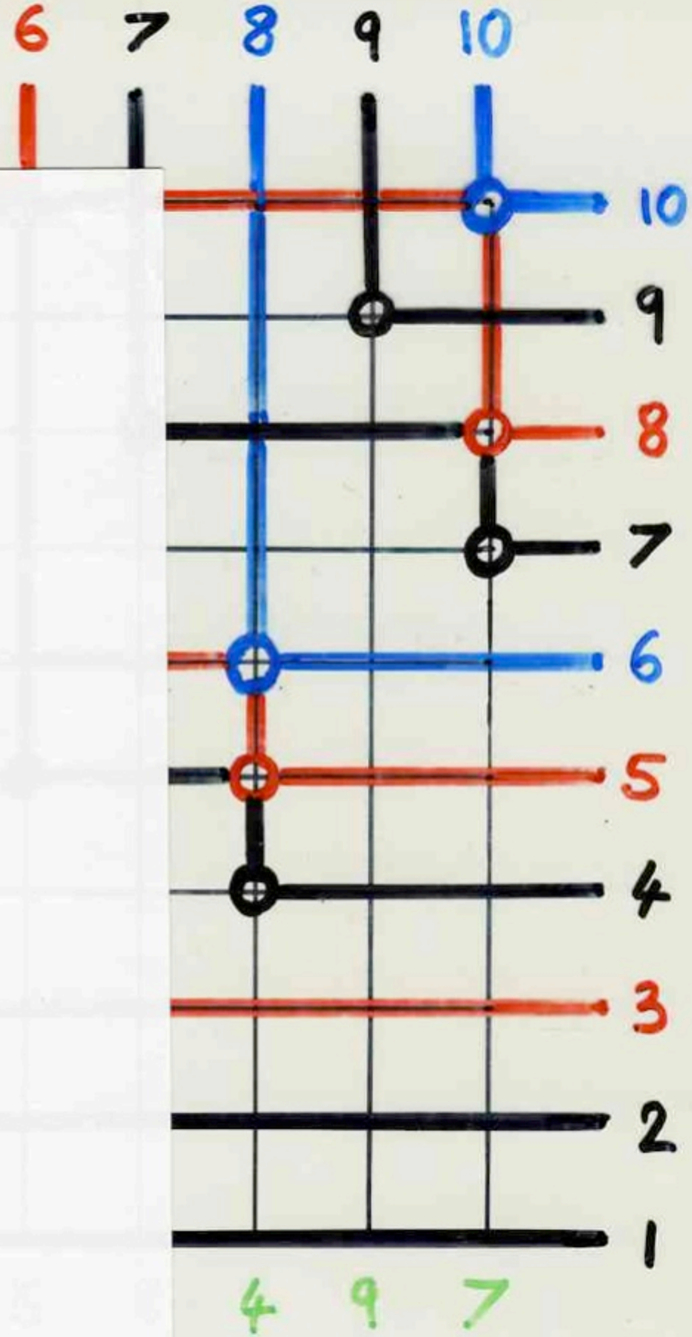
Q

$$\sigma \longleftrightarrow (P, Q)$$

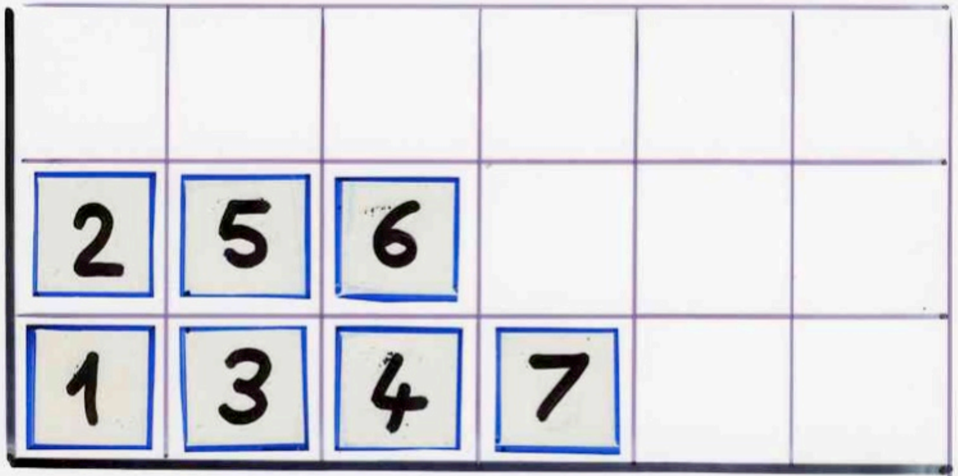
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

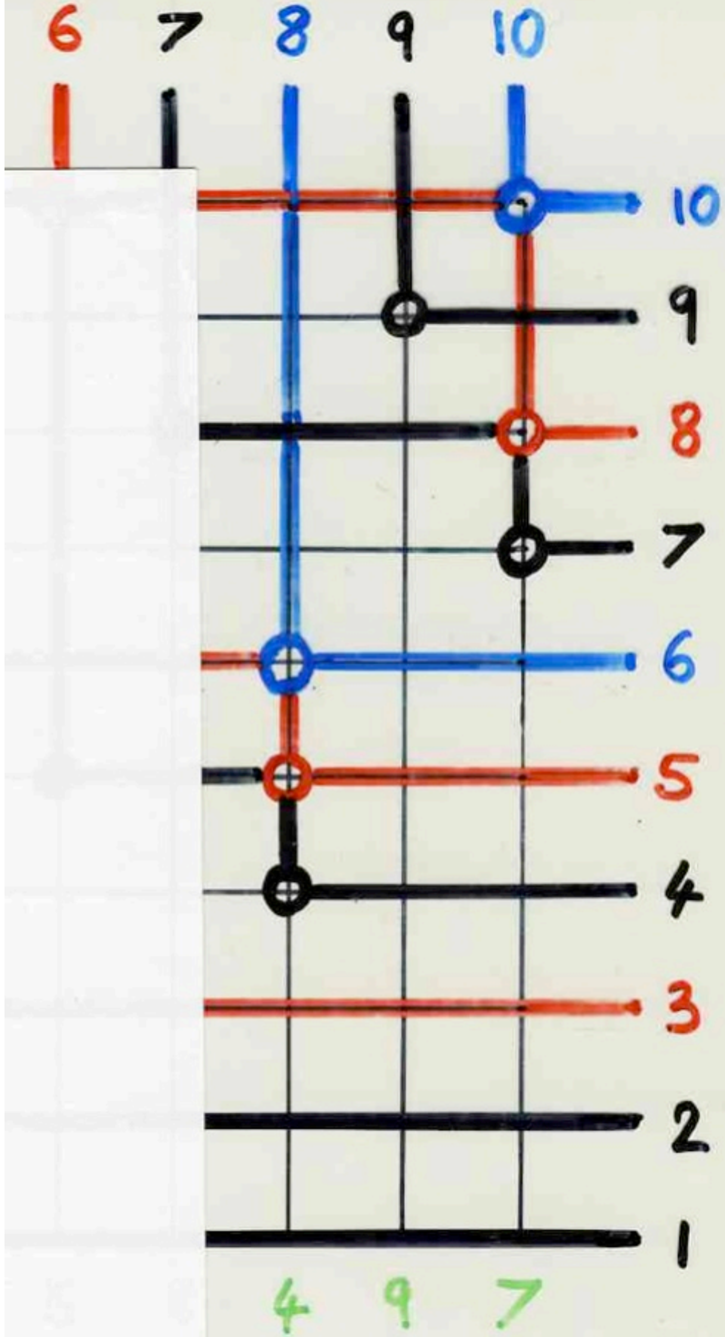
proof of the equivalence
insertions --- geometric constrution


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
4 9 7

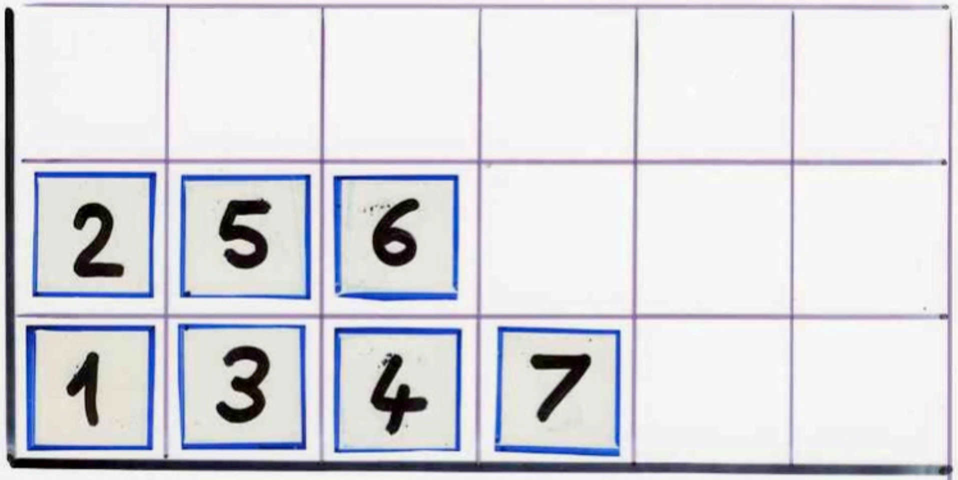


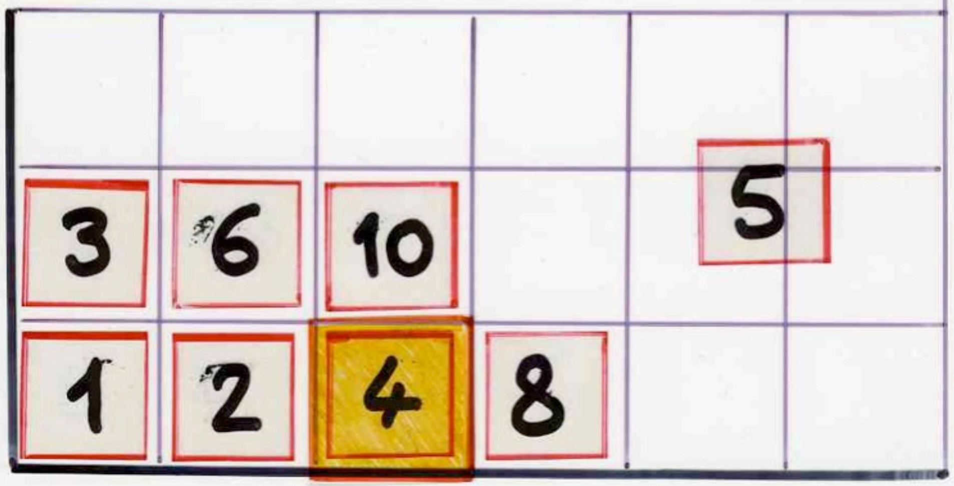
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

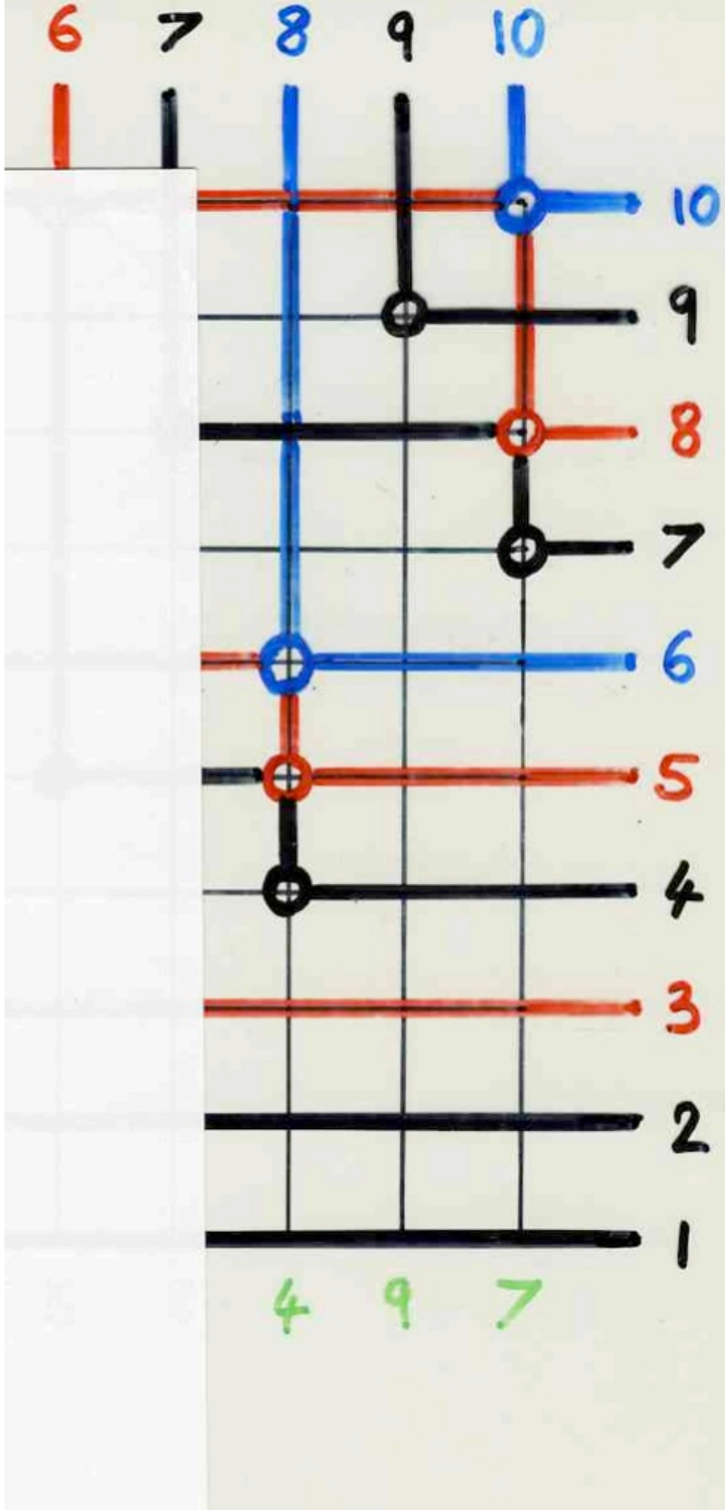




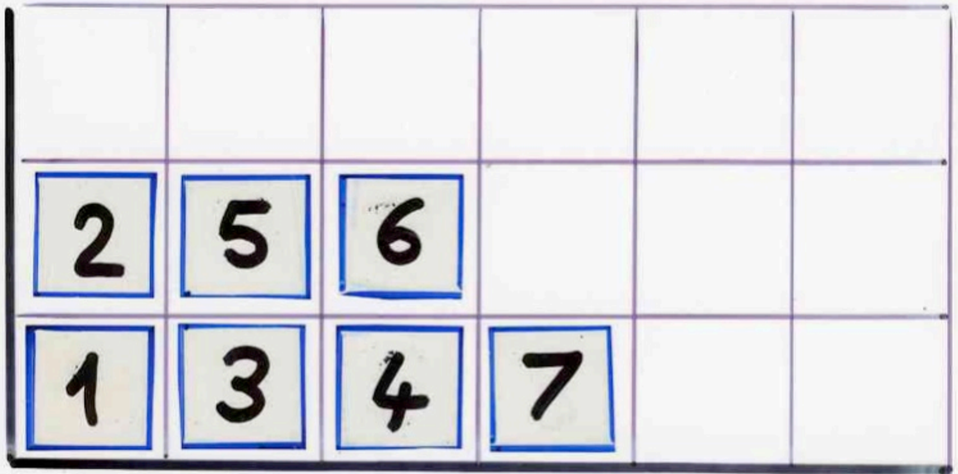
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

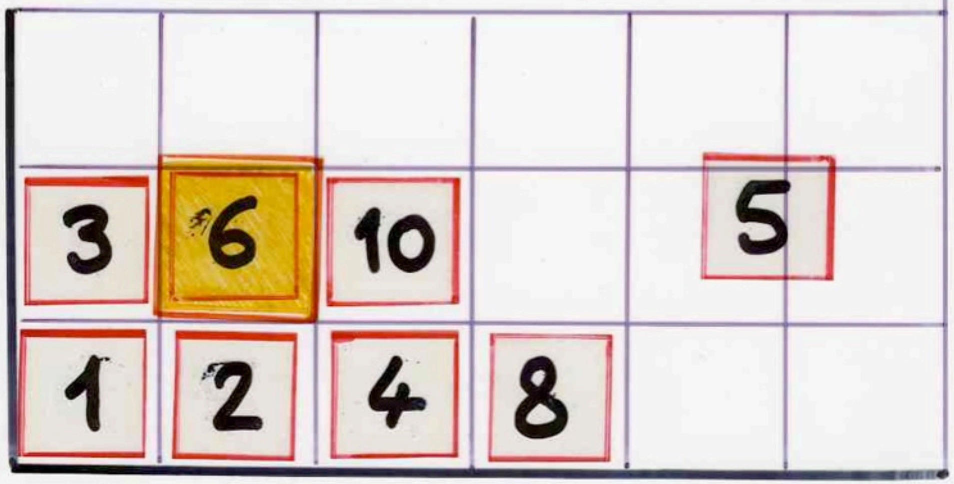


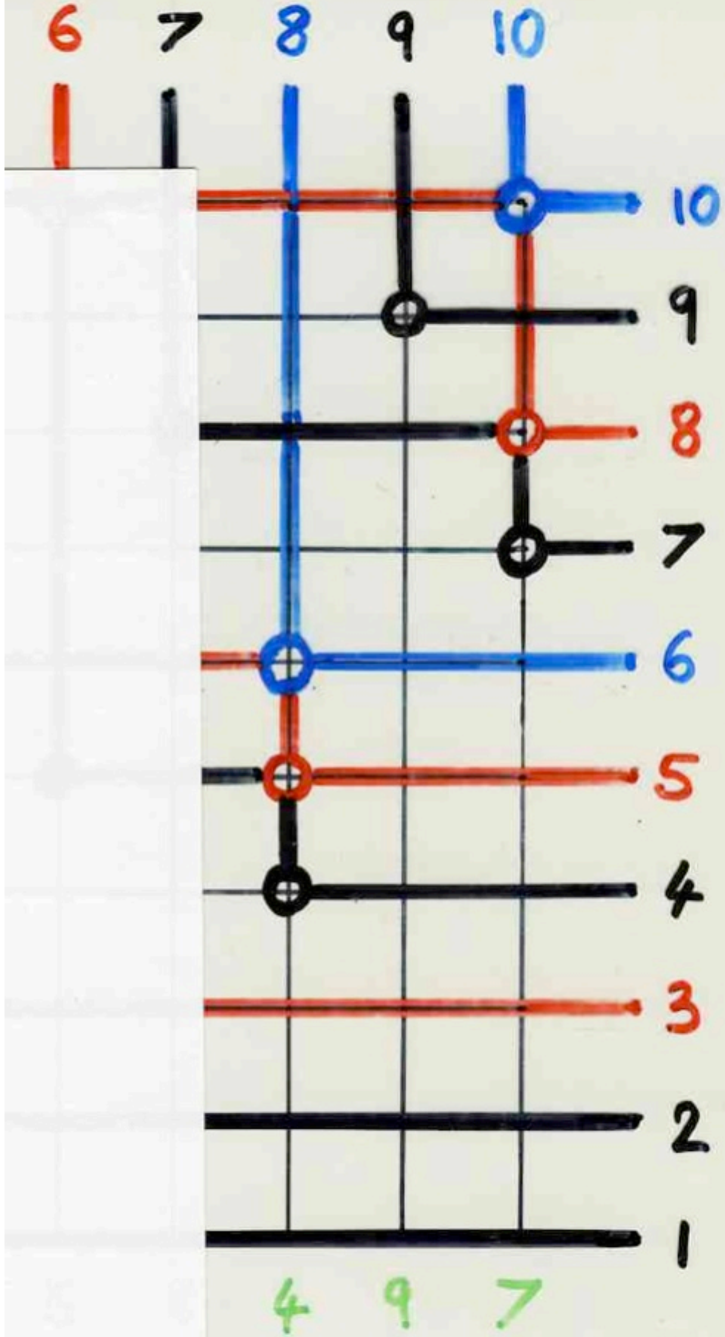




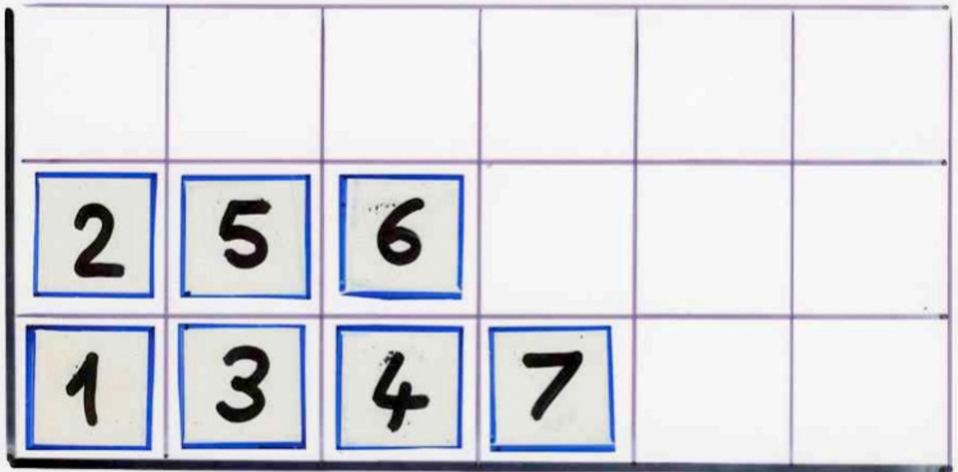
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

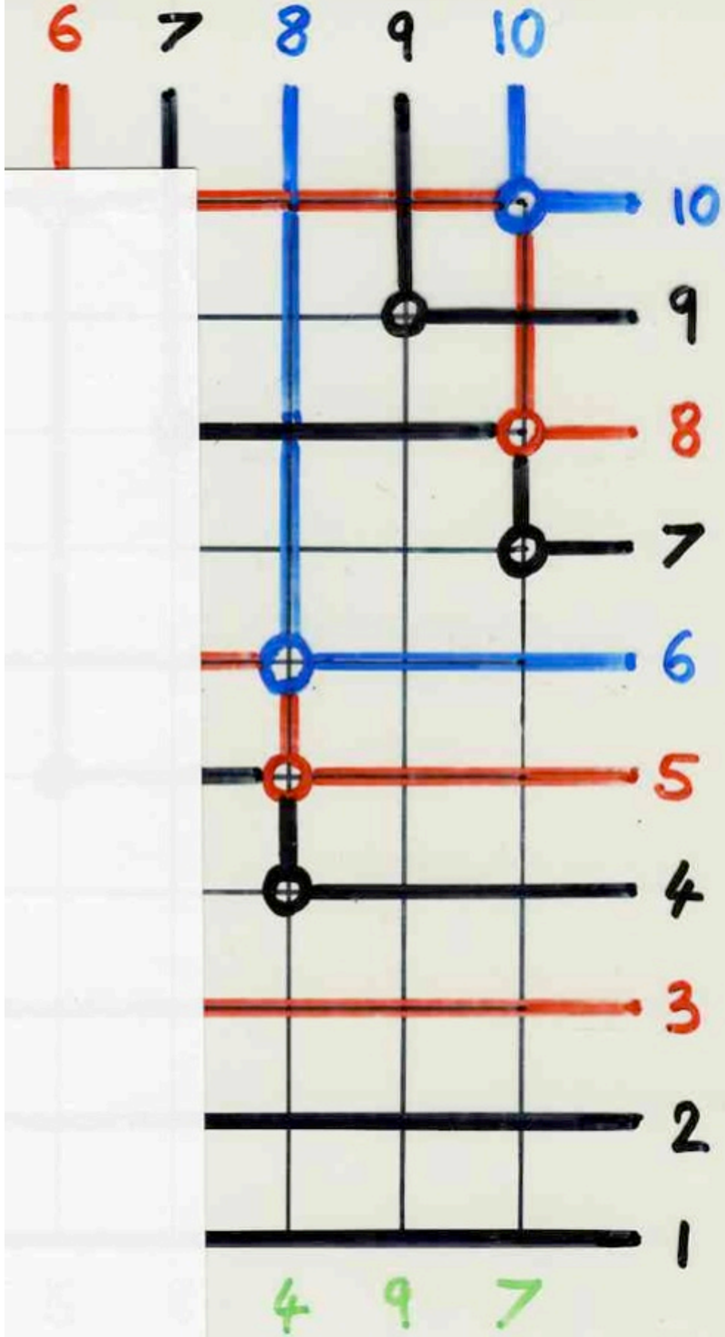






1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

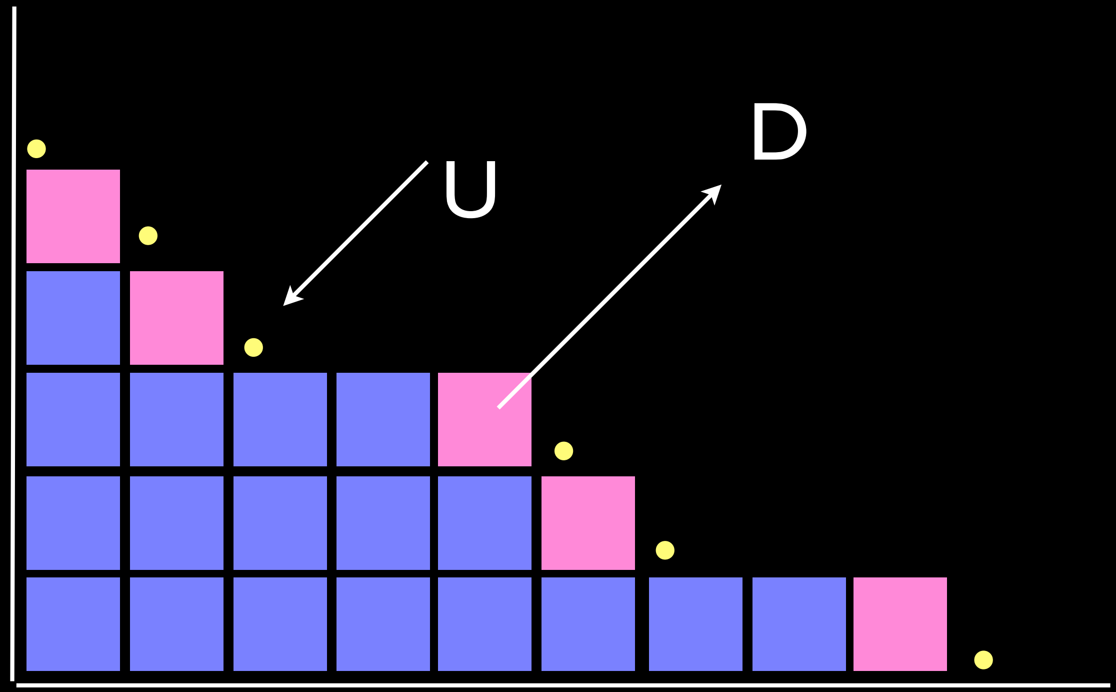
6					
3	5	10			
1	2	4	8		

representation of the operators U, D

Sergey Fomin

(with C. K.)

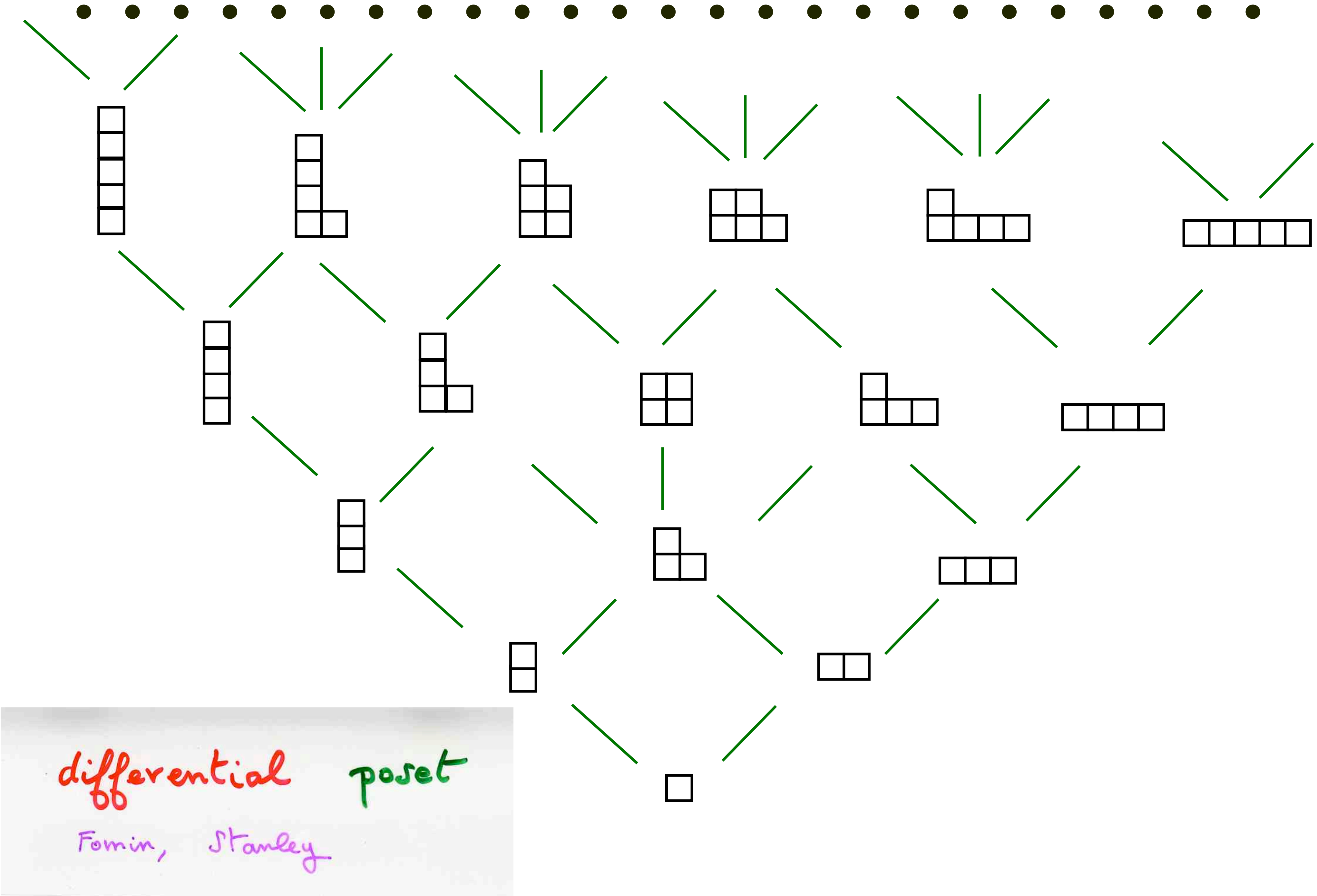
Operators U and D

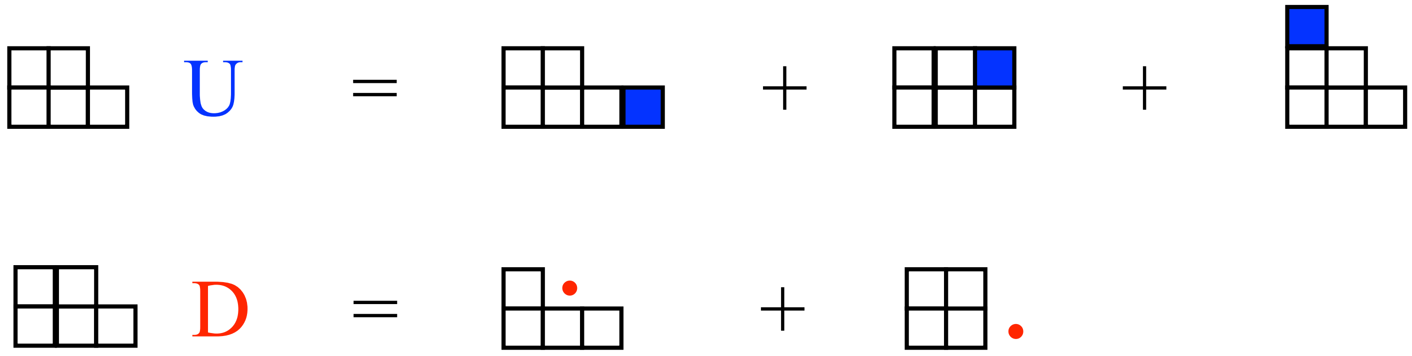


adding
or deleting
a cell in
a Ferrers
diagram

Young lattice

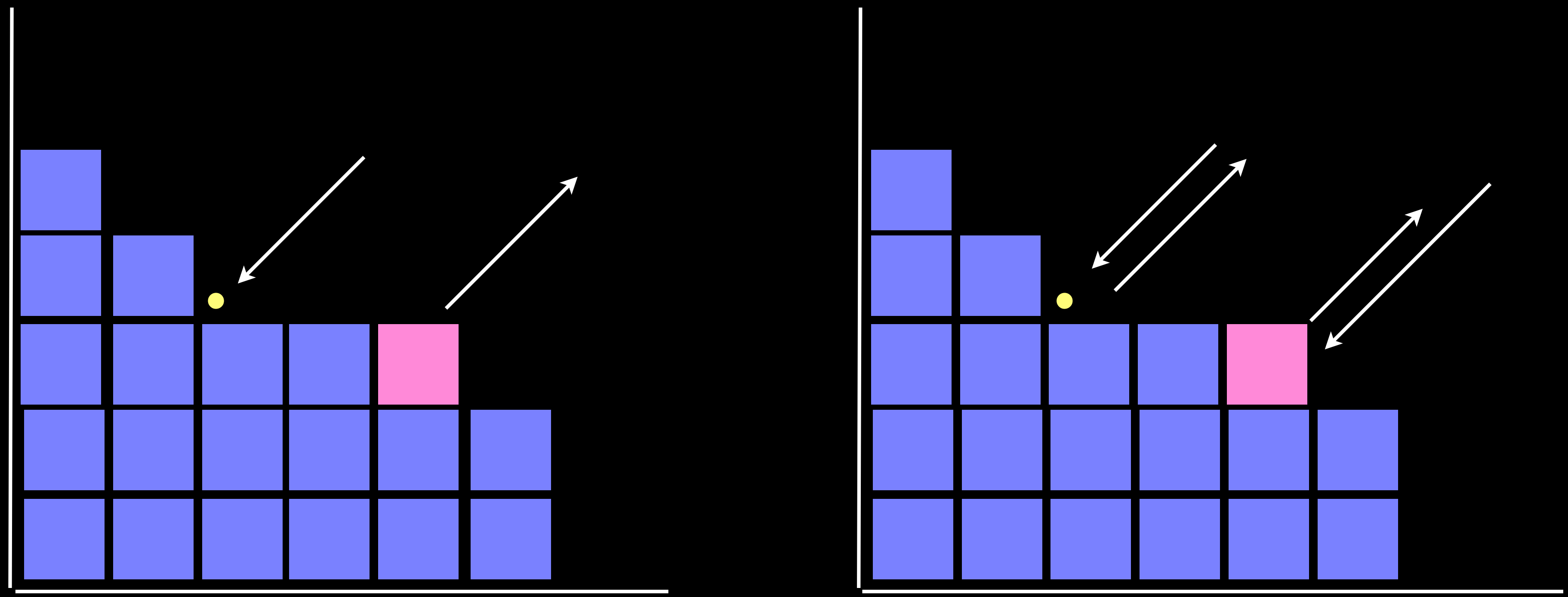
Young lattice

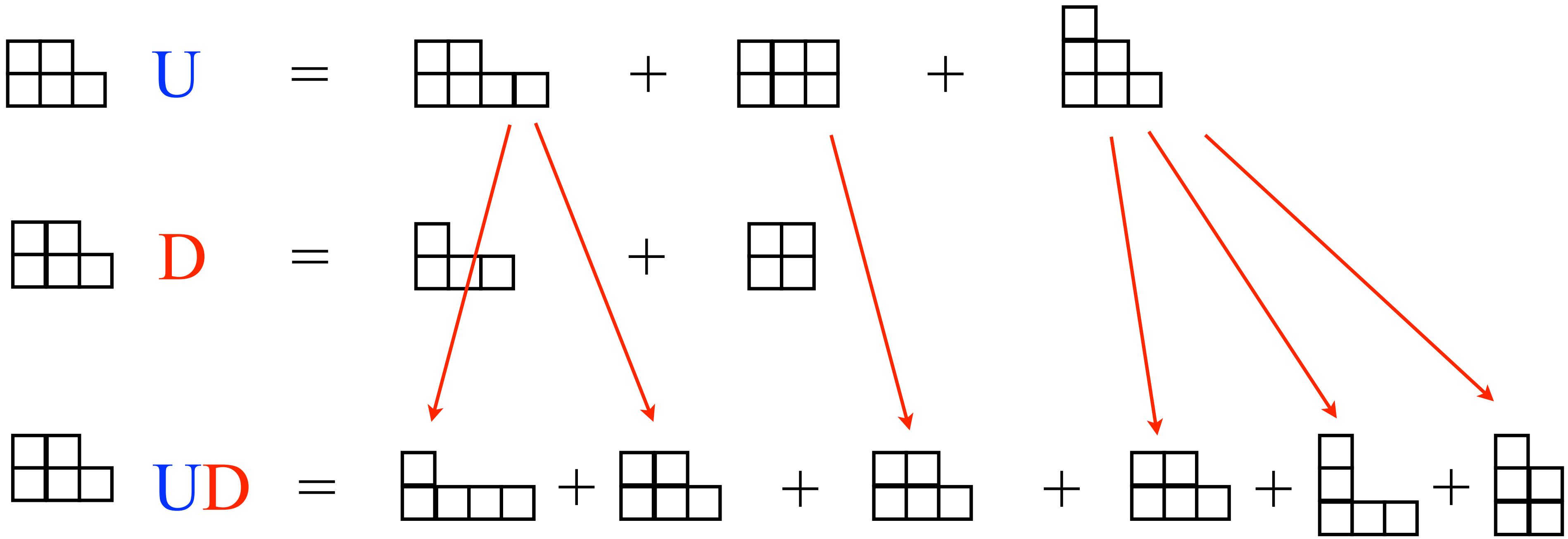


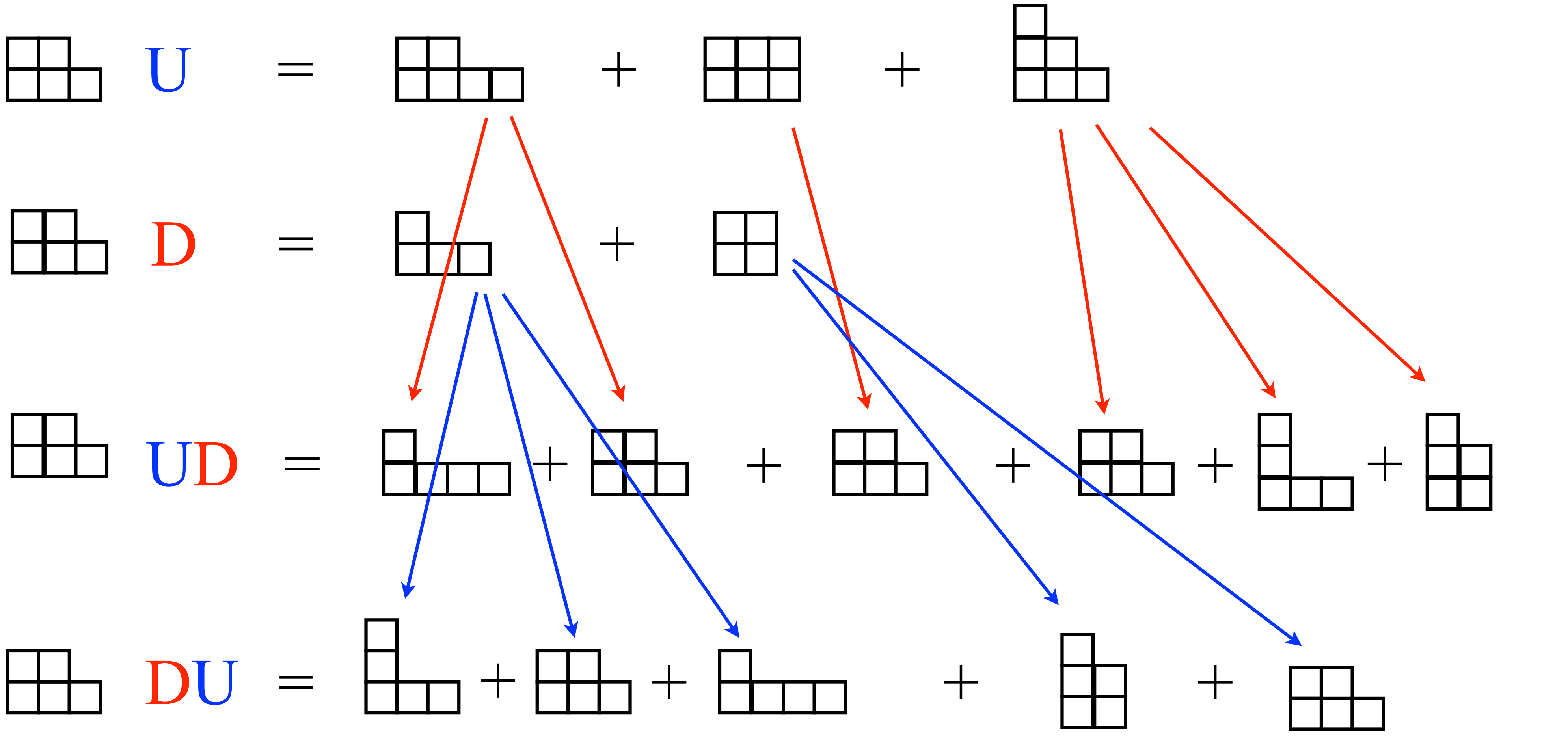

U
D

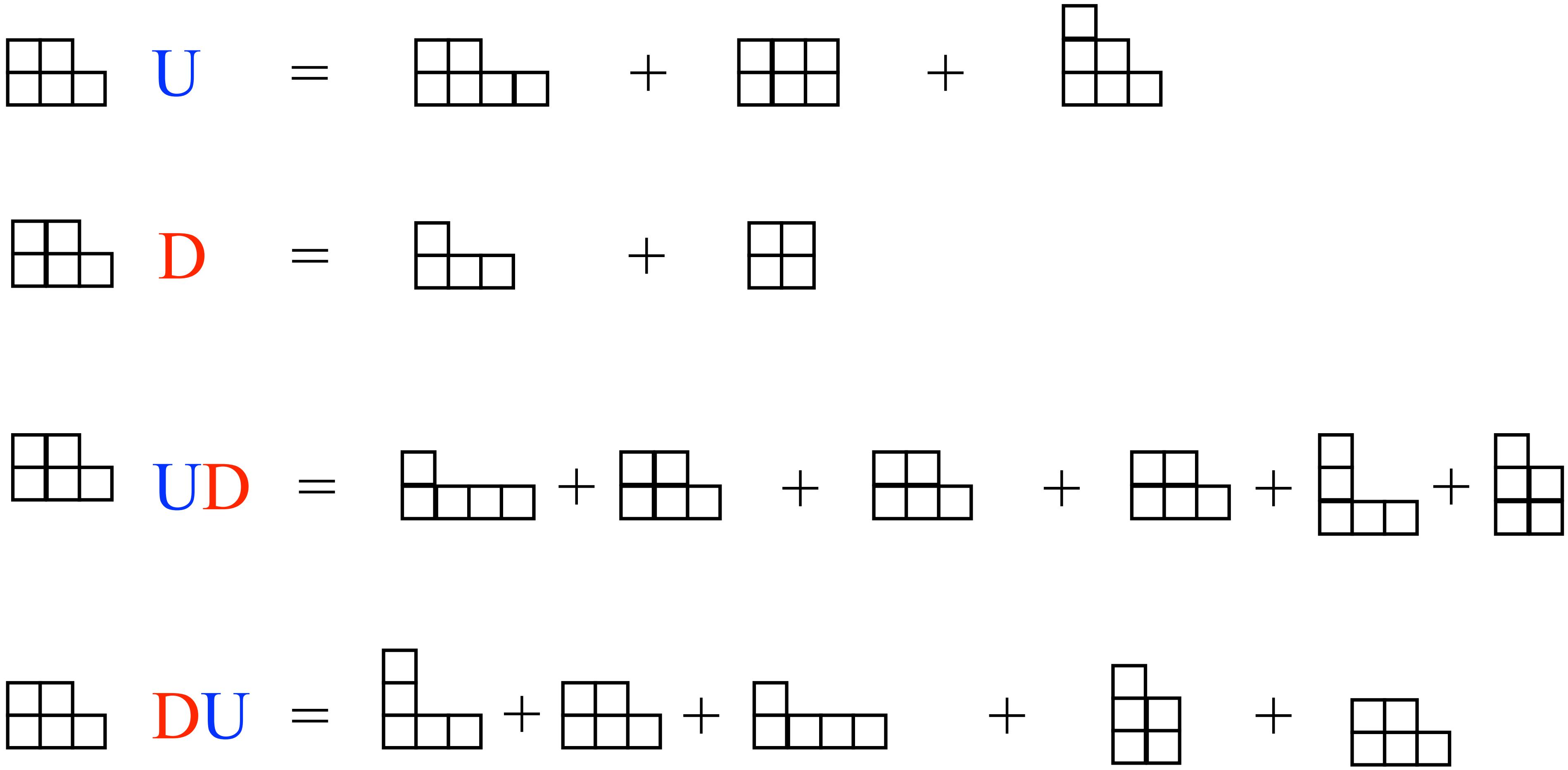
Heisenberg commutation relation

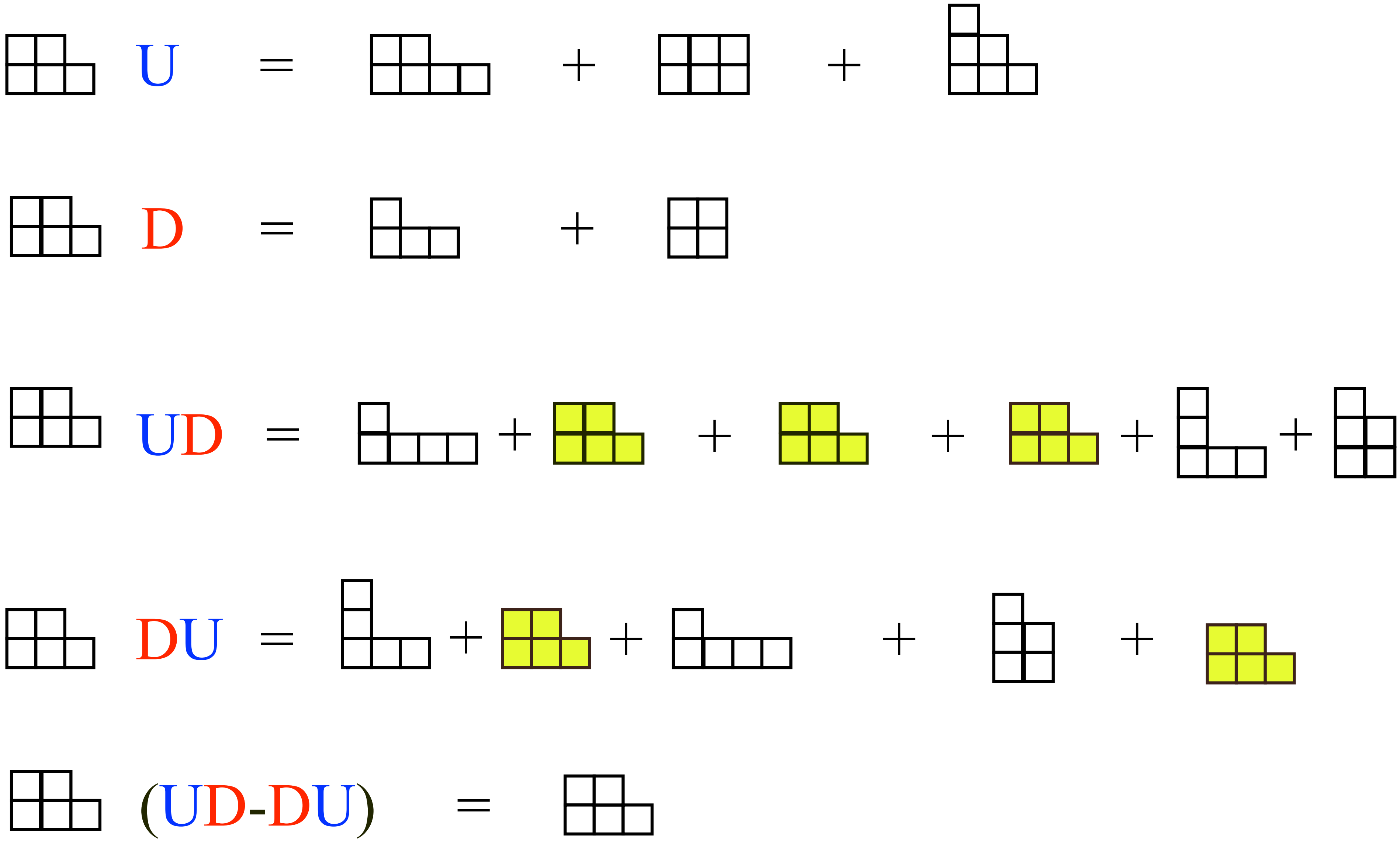
$$UD = DU + I$$




U
D
UD

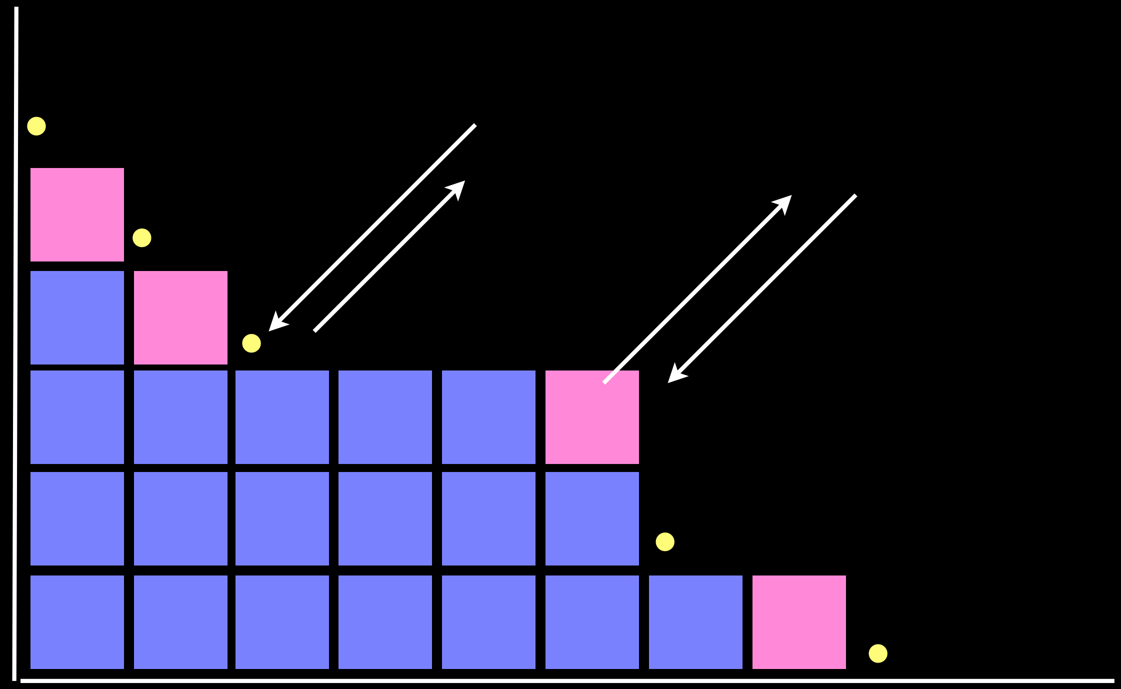

U =
D =
UD =
DU =


U
D
UD
DU



U =
D =
UD =
DU =
(UD-DU) =

UD = DU + I

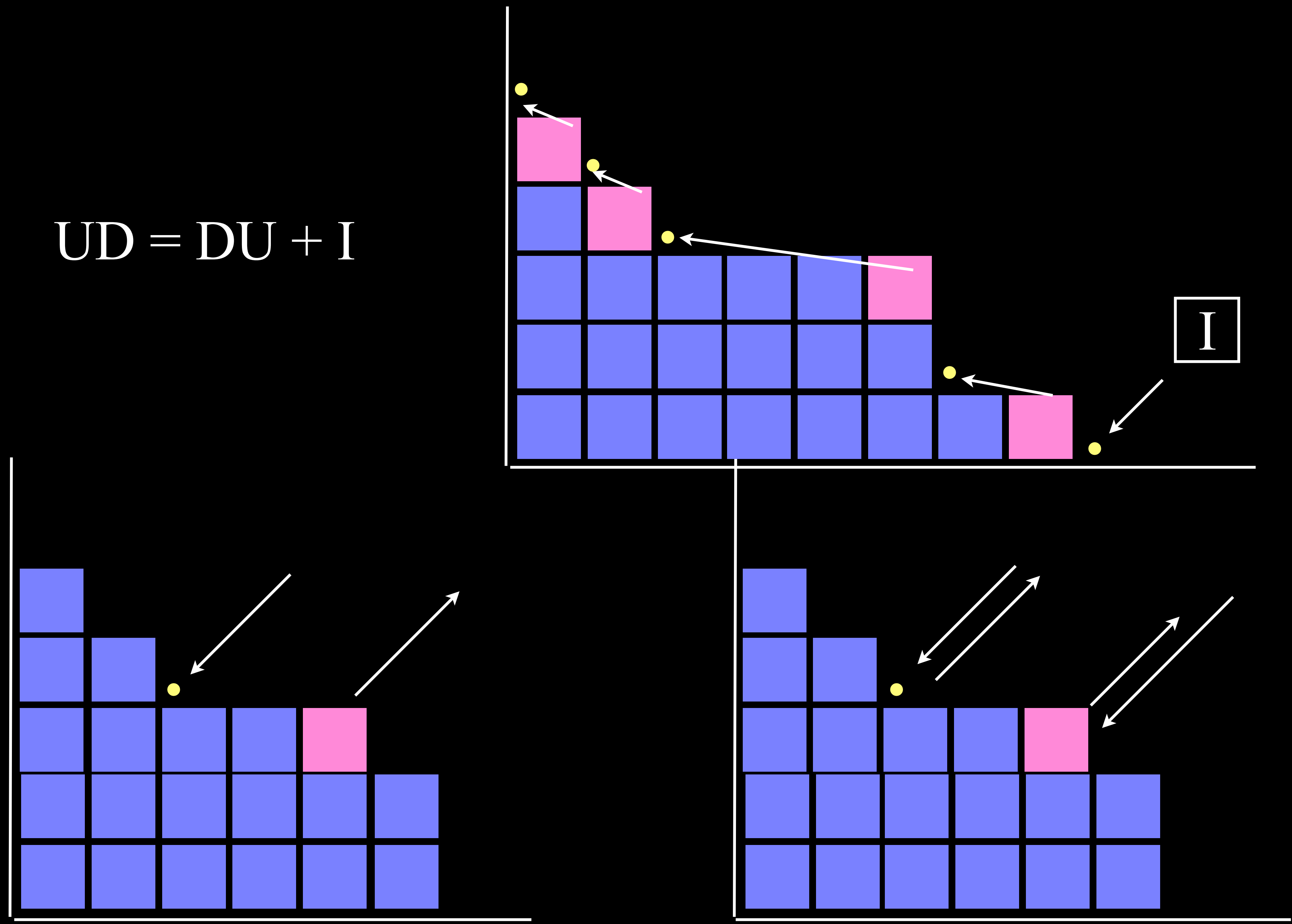


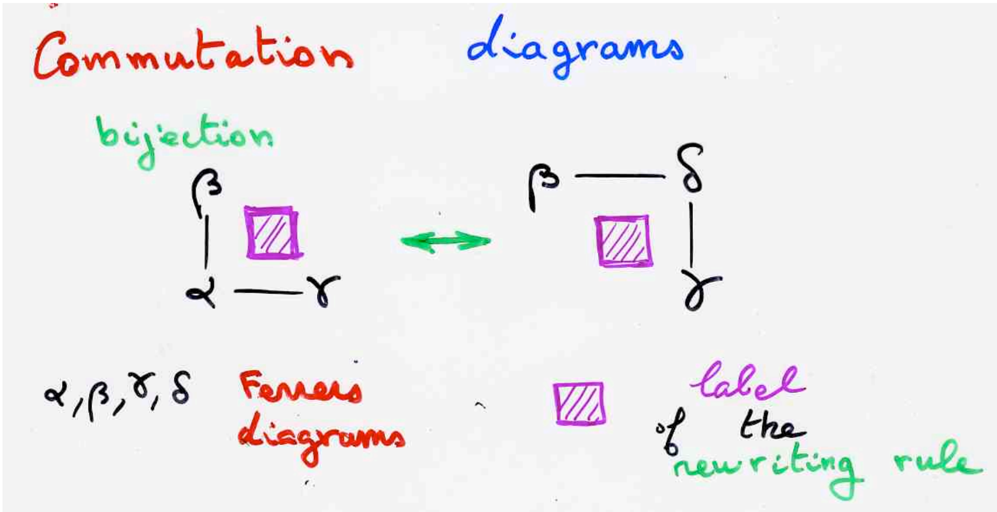
The cellular Ansatz
second part:

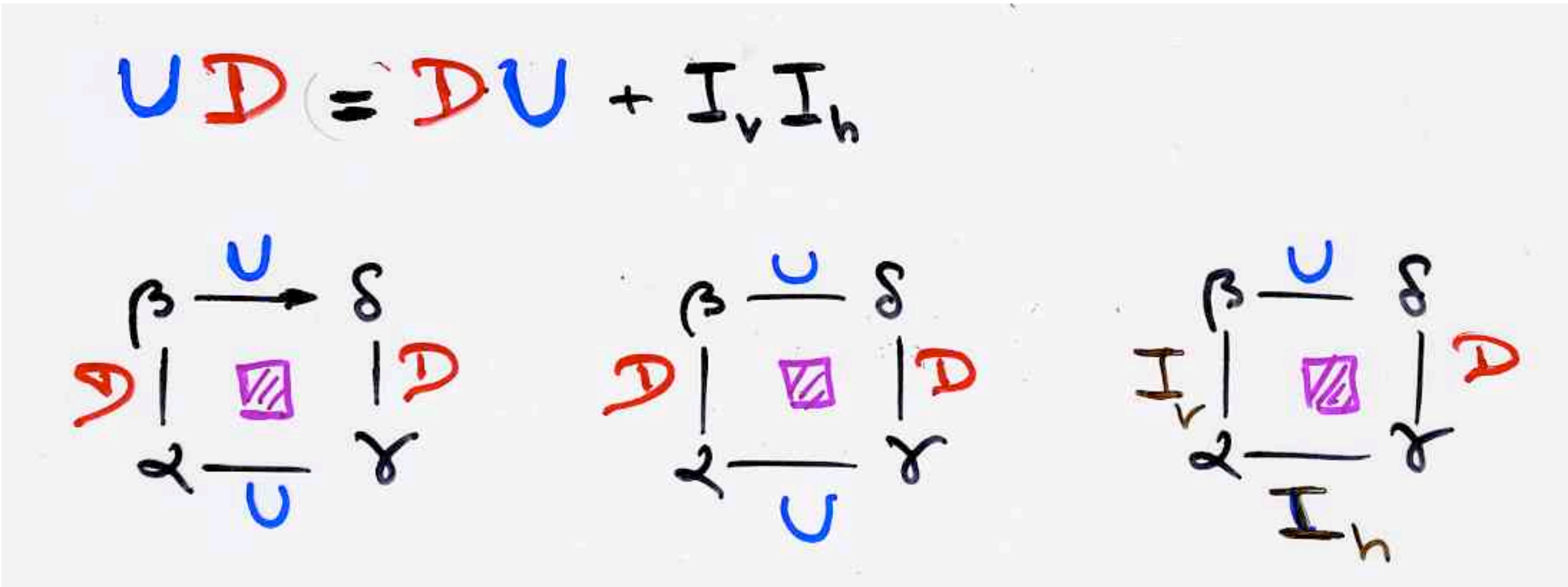
guided construction
of a bijection
(from the representation of U and D)

combinatorial “representation” of the commutation relation $UD = DU + I$

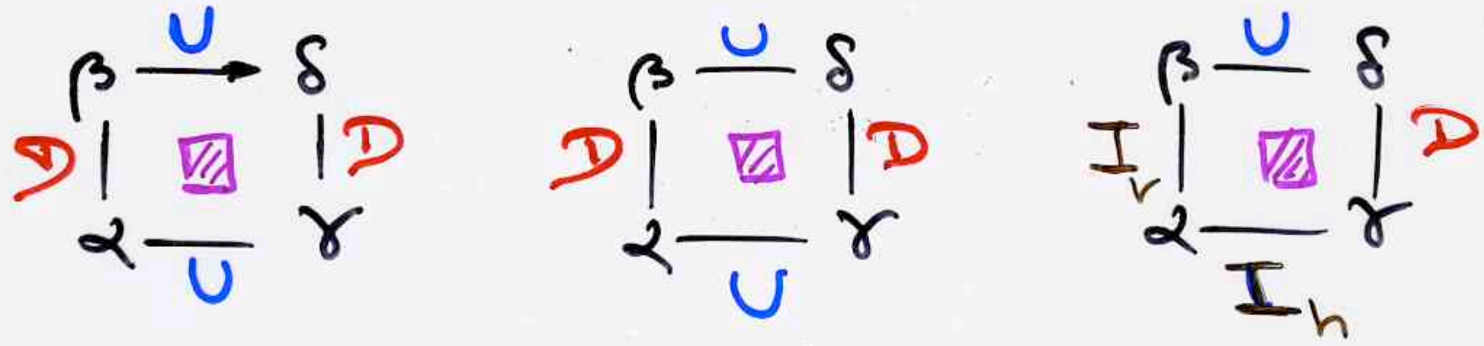
$$UD = DU + I$$

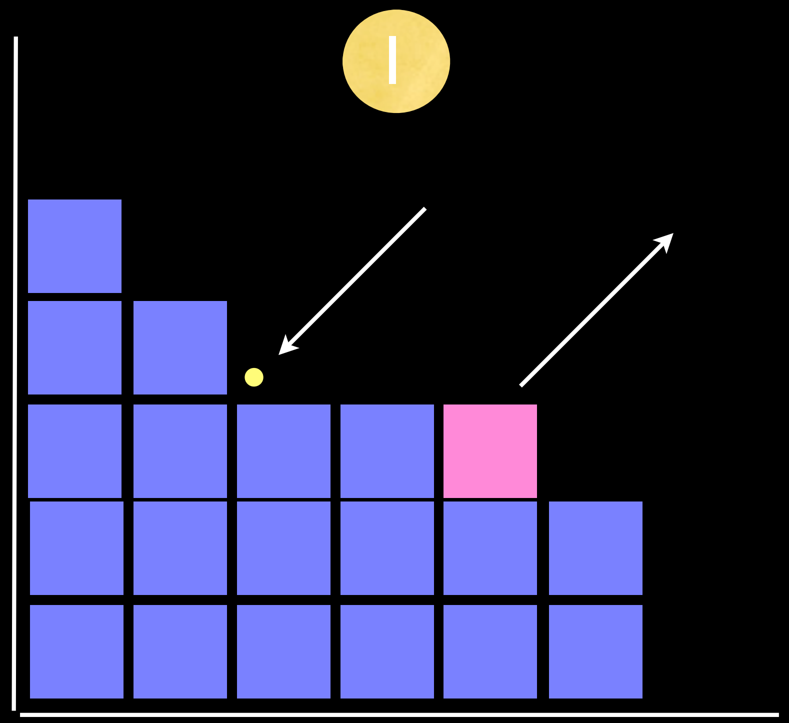



Commutation diagrams
bijection
α, β, γ, δ Ferrers diagrams
label of the rewriting rule


UD = DU + $I_v I_h$
U
D
I_v
I_h

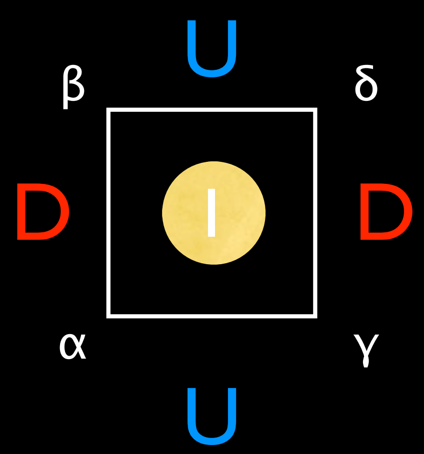
$$UD = DU + I_v I_h$$



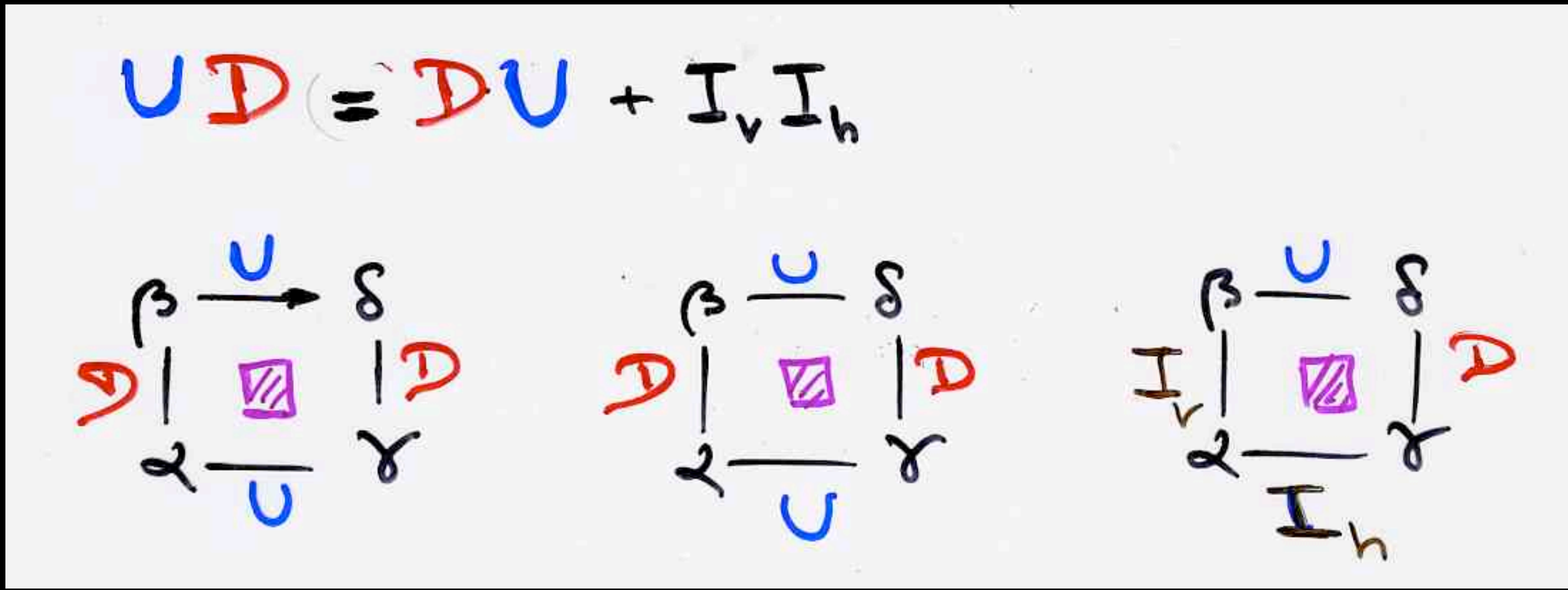


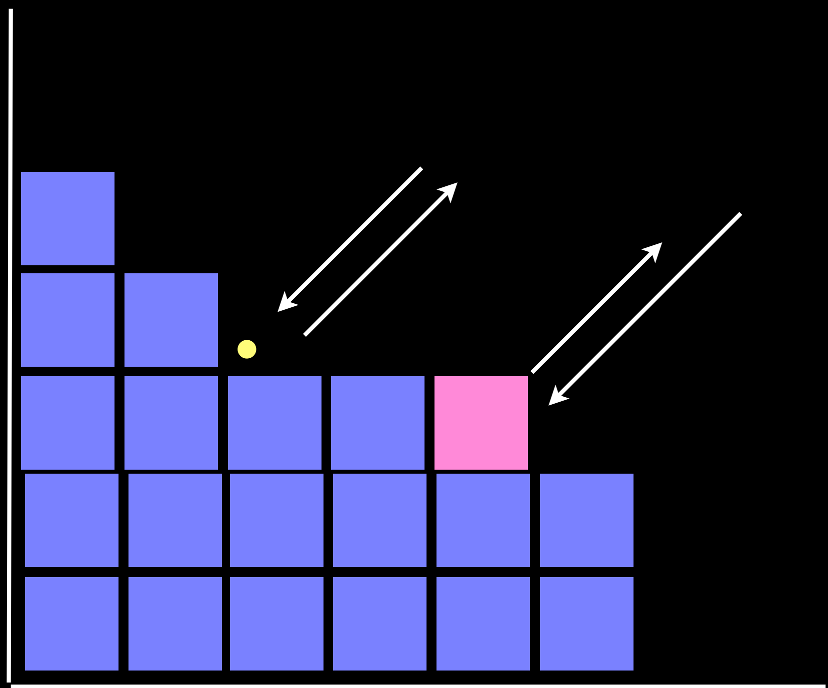
β

β ≠ γ



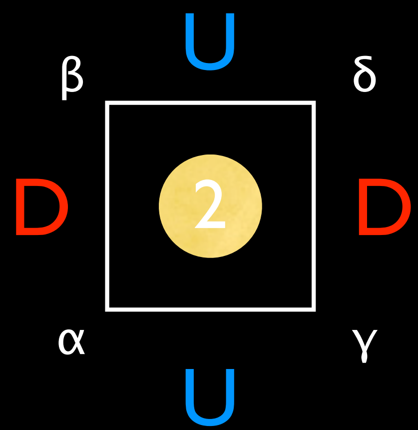
δ = β ∪ γ





2

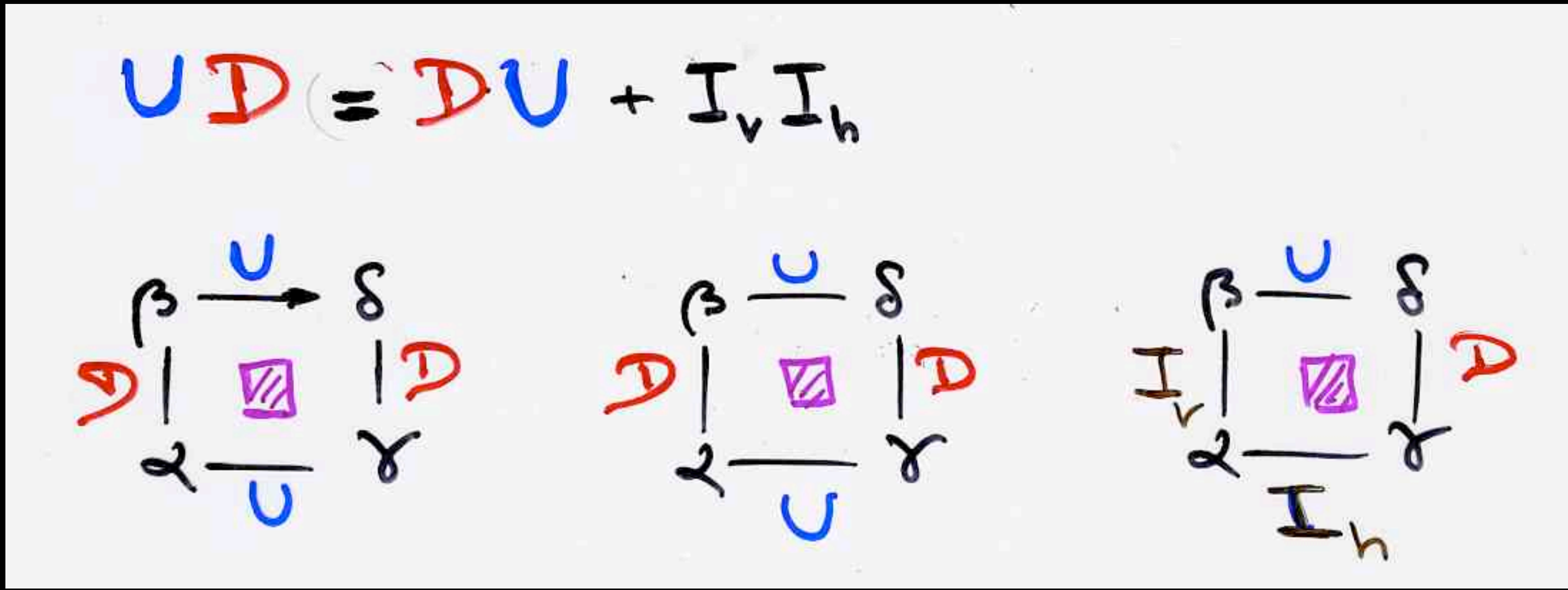
β = γ

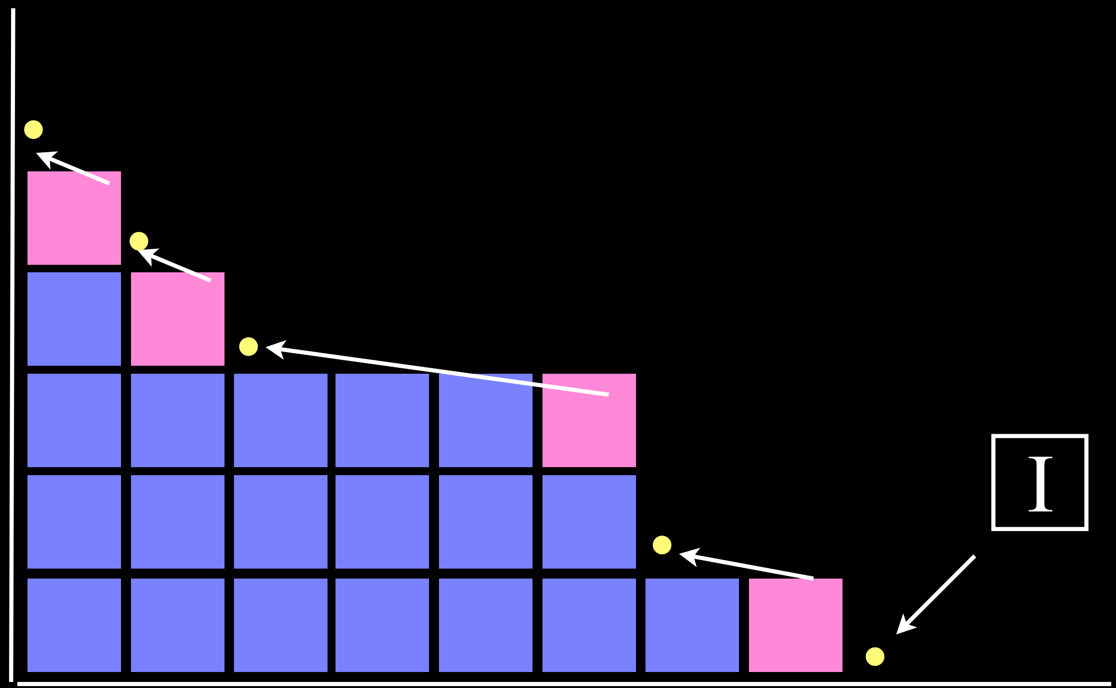


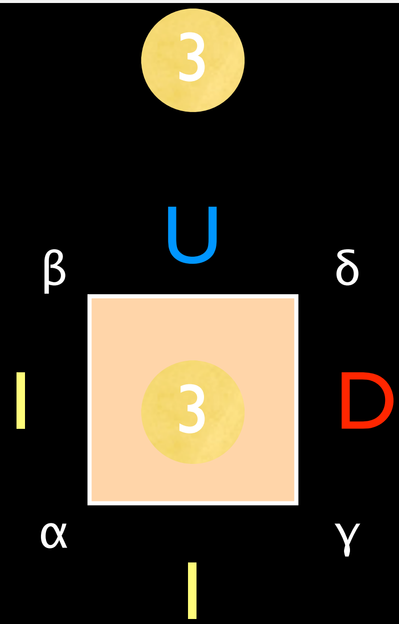
$\beta = \gamma = \alpha + (i)$

$\delta = \beta + (i+1)$

$\beta = \gamma$





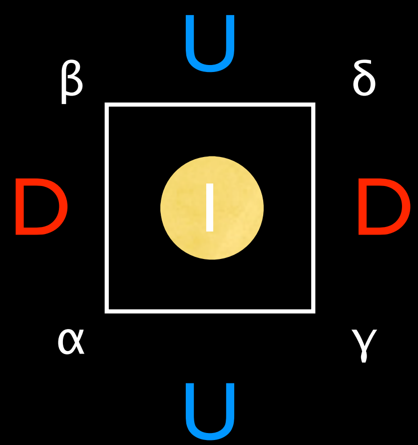


$\alpha = \beta = \gamma$

$\delta = \alpha + (I)$

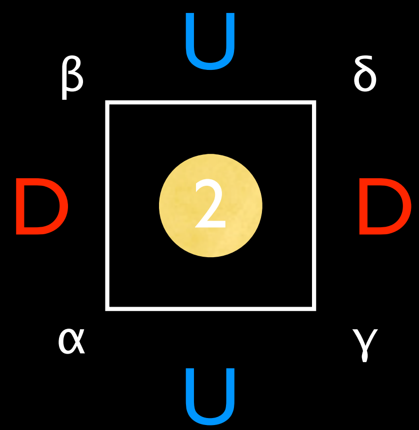
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$\beta \neq \gamma$



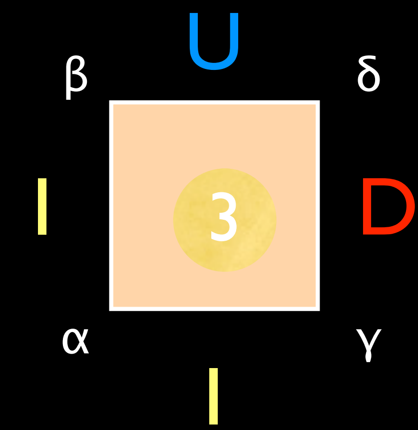
$\delta = \beta \cup \gamma$

$\beta = \gamma$



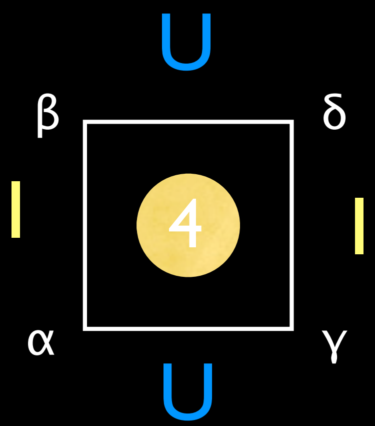
$\beta = \gamma = \alpha + (i)$

$\delta = \beta + (i+1)$



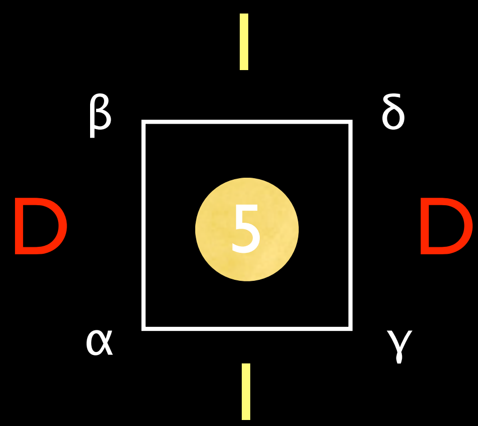
$\alpha = \beta = \gamma$

$\delta = \alpha + (1)$



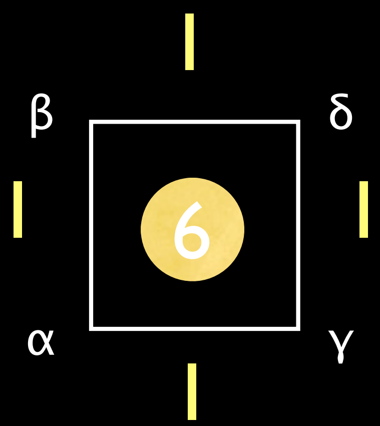
$\alpha = \beta$

$\delta = \gamma = \beta + (i)$

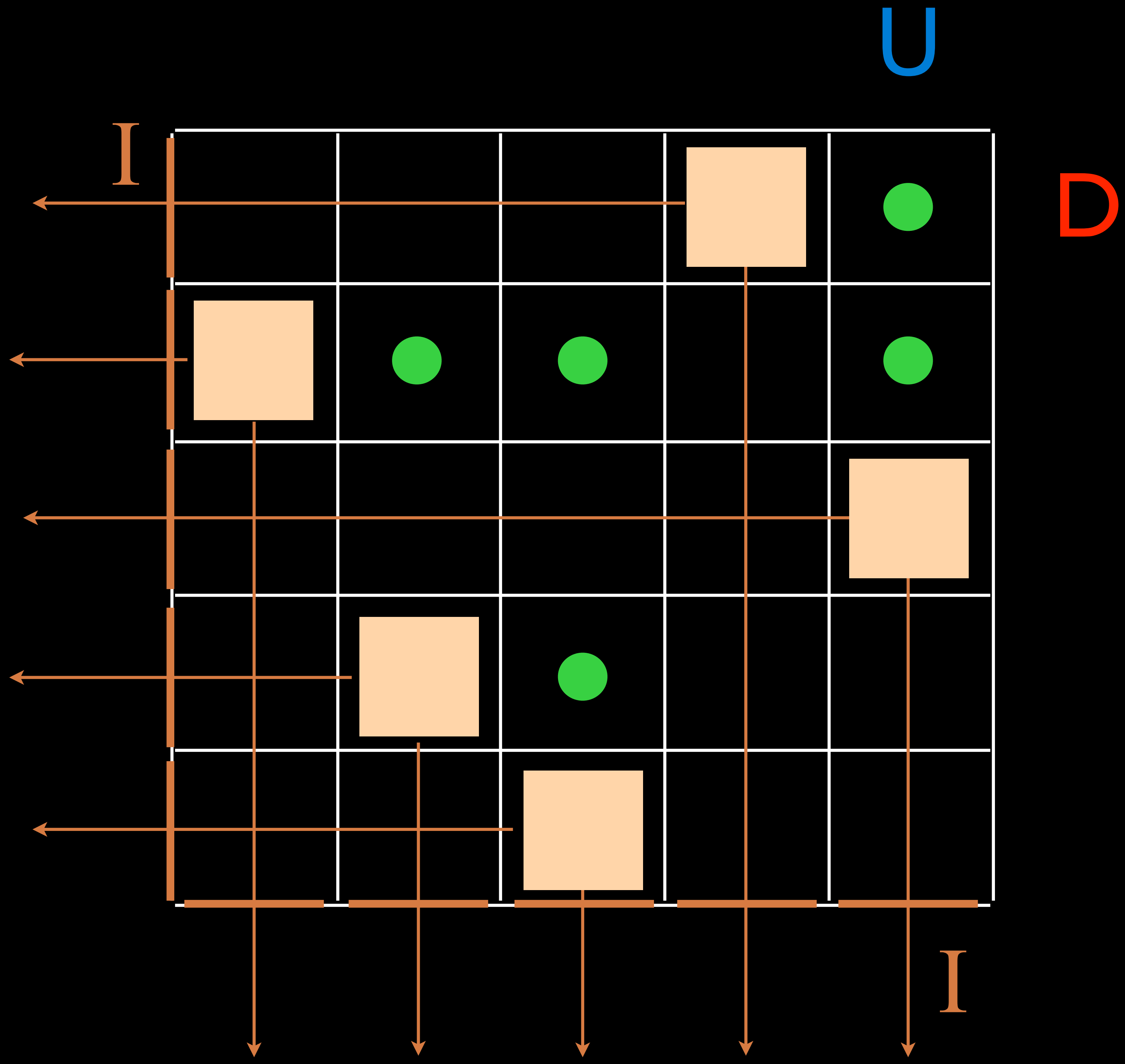


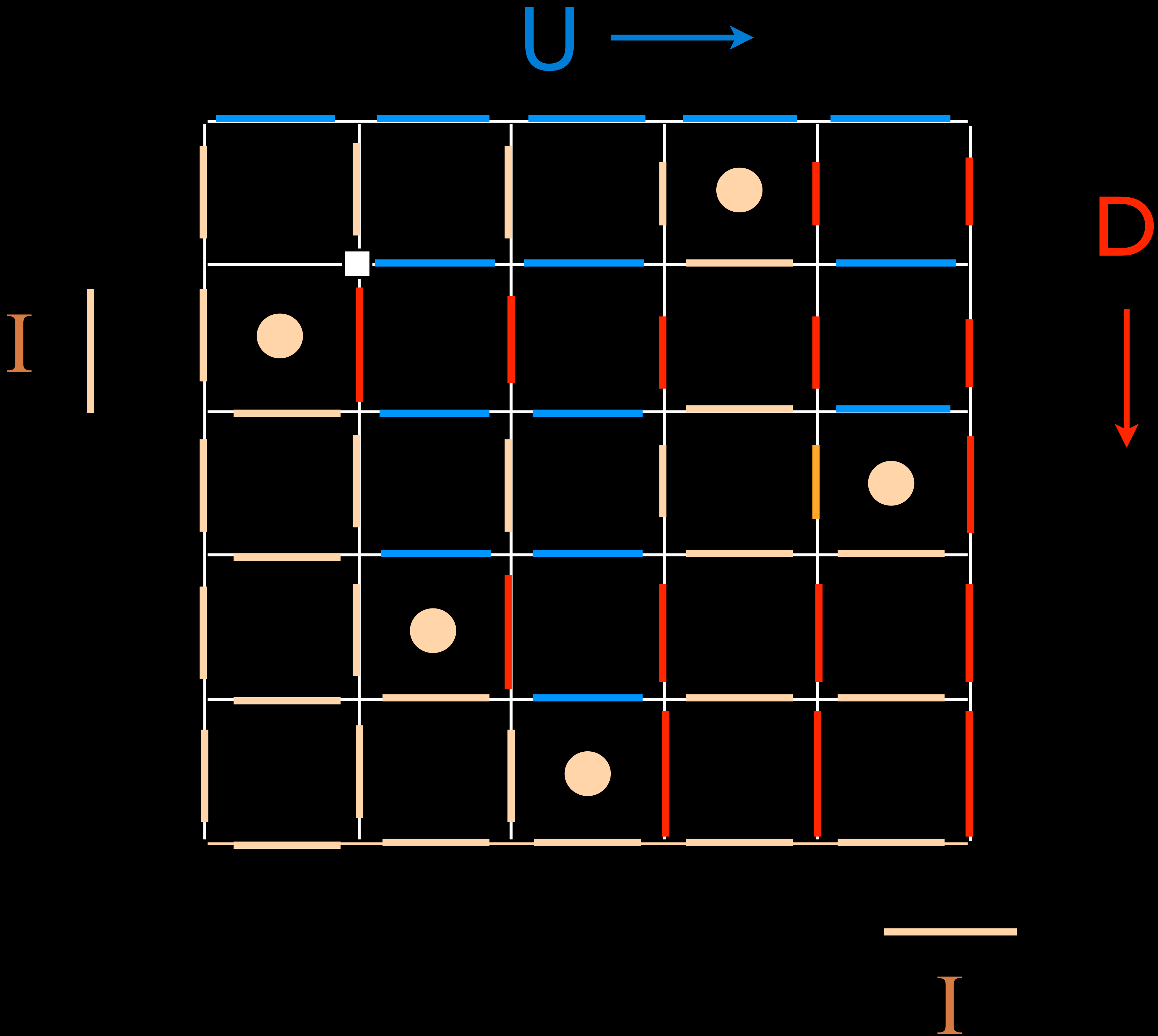
$\alpha = \gamma$

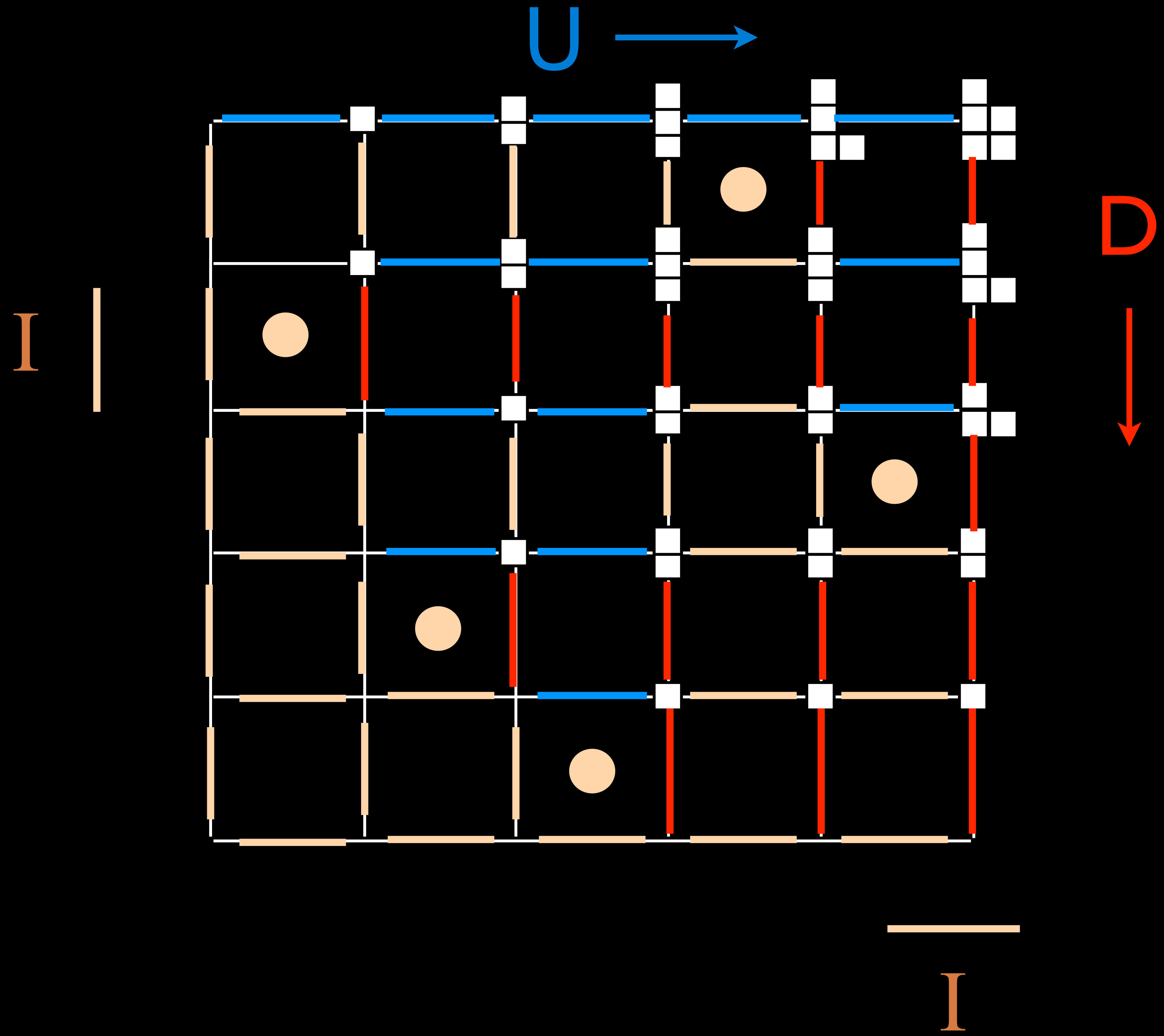
$\delta = \beta = \alpha + (i)$



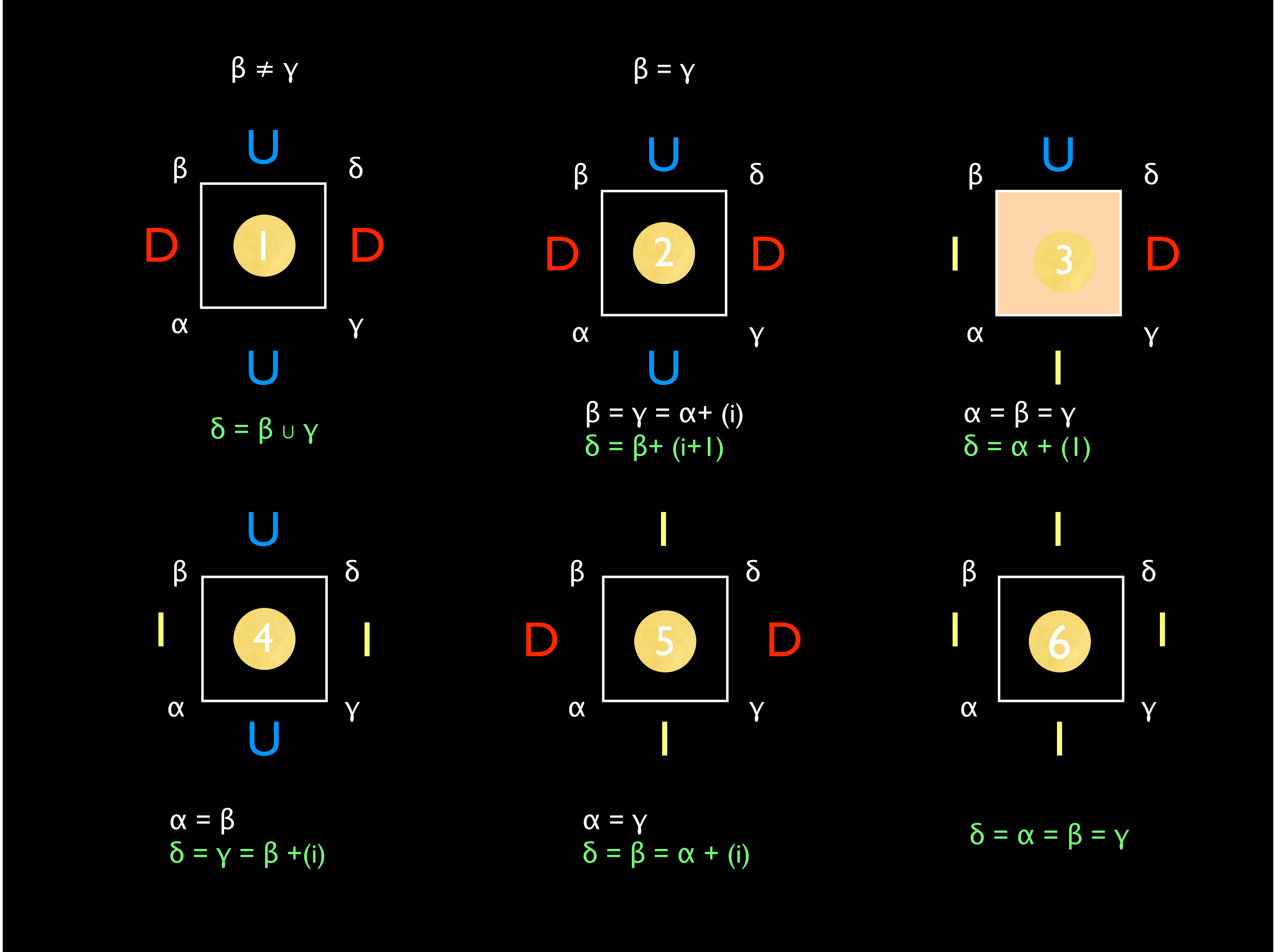
$\delta = \alpha = \beta = \gamma$


U
D
I
I

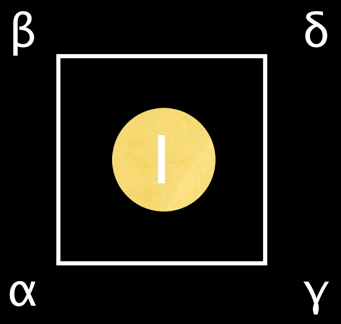

U
D
I
I



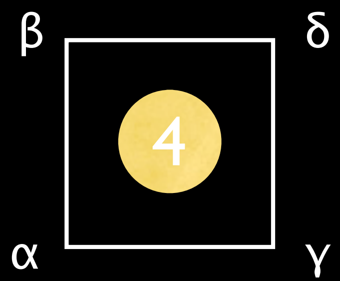
U
D
I
I

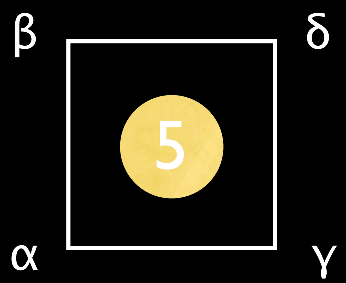

β ≠ γ
β = γ
U
β
δ
D
1
D
α
γ
U
δ = β ∪ γ
U
β
δ
D
2
D
α
γ
U
β = γ = α+ (i)
δ = β+ (i+1)
U
β
δ
1
3
D
α
γ
1
α = β = γ
δ = α + (1)
U
β
δ
1
4
1
α
γ
U
α = β
δ = γ = β +(i)
1
β
δ
D
5
D
α
γ
1
α = γ
δ = β = α + (i)
1
β
δ
1
6
1
α
γ
1
δ = α = β = γ

$\beta \neq \gamma$



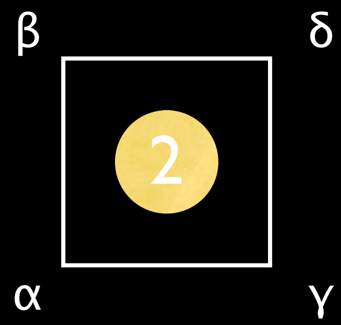
$\delta = \beta \cup \gamma$





$\beta = \gamma$

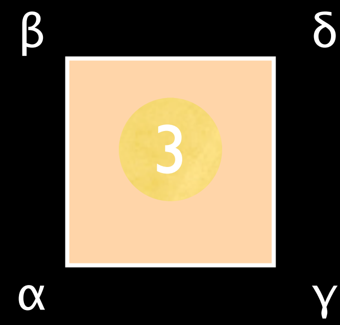
$\alpha \neq \beta$



$\beta = \gamma = \alpha + (i)$

$\delta = \beta + (i+1)$

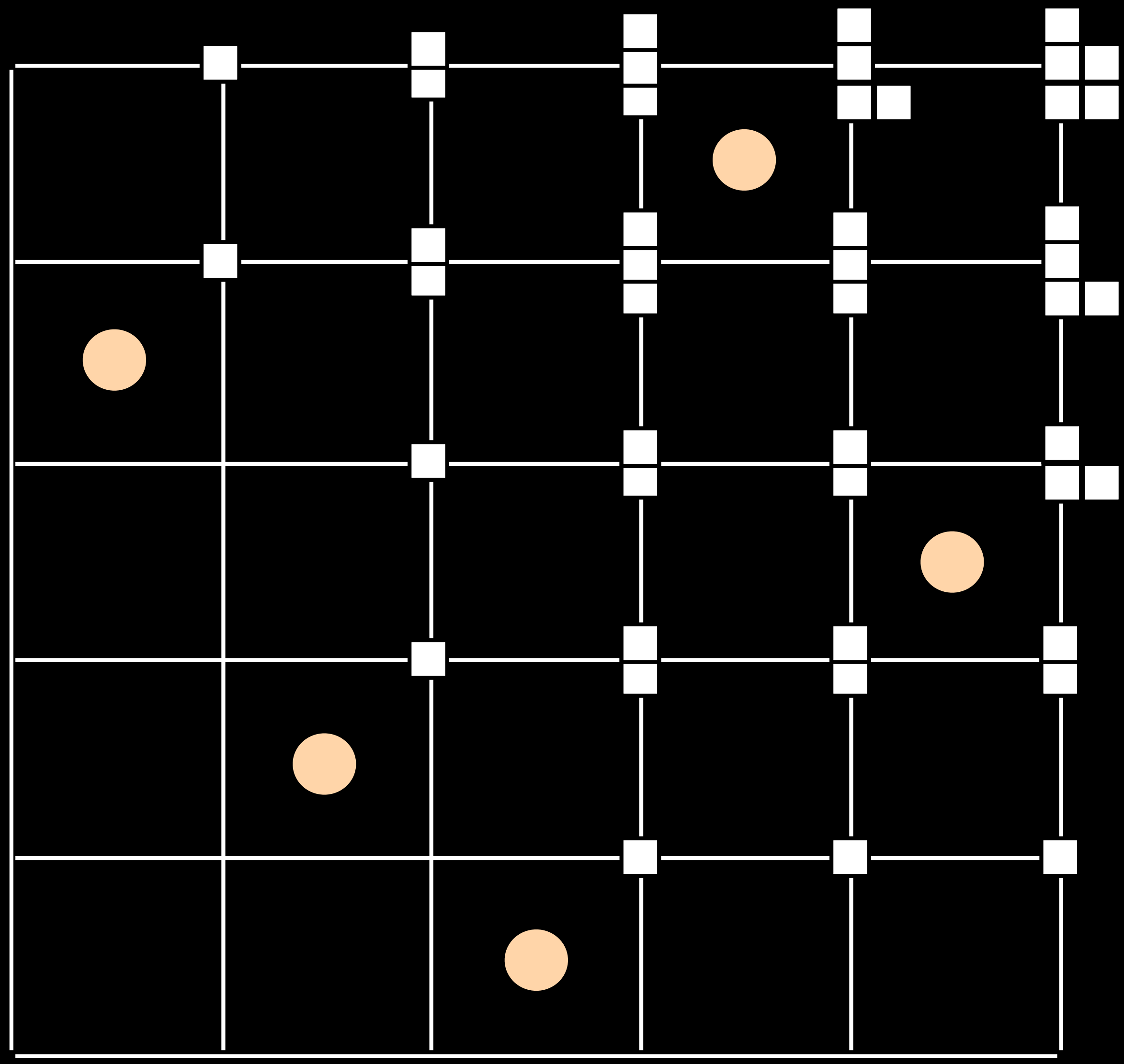
$\alpha = \beta = \gamma$



$\delta = \alpha + (1)$

$\alpha = \beta = \gamma$

$\delta = \alpha = \beta = \gamma$



RSK with Fomin's "local rules"

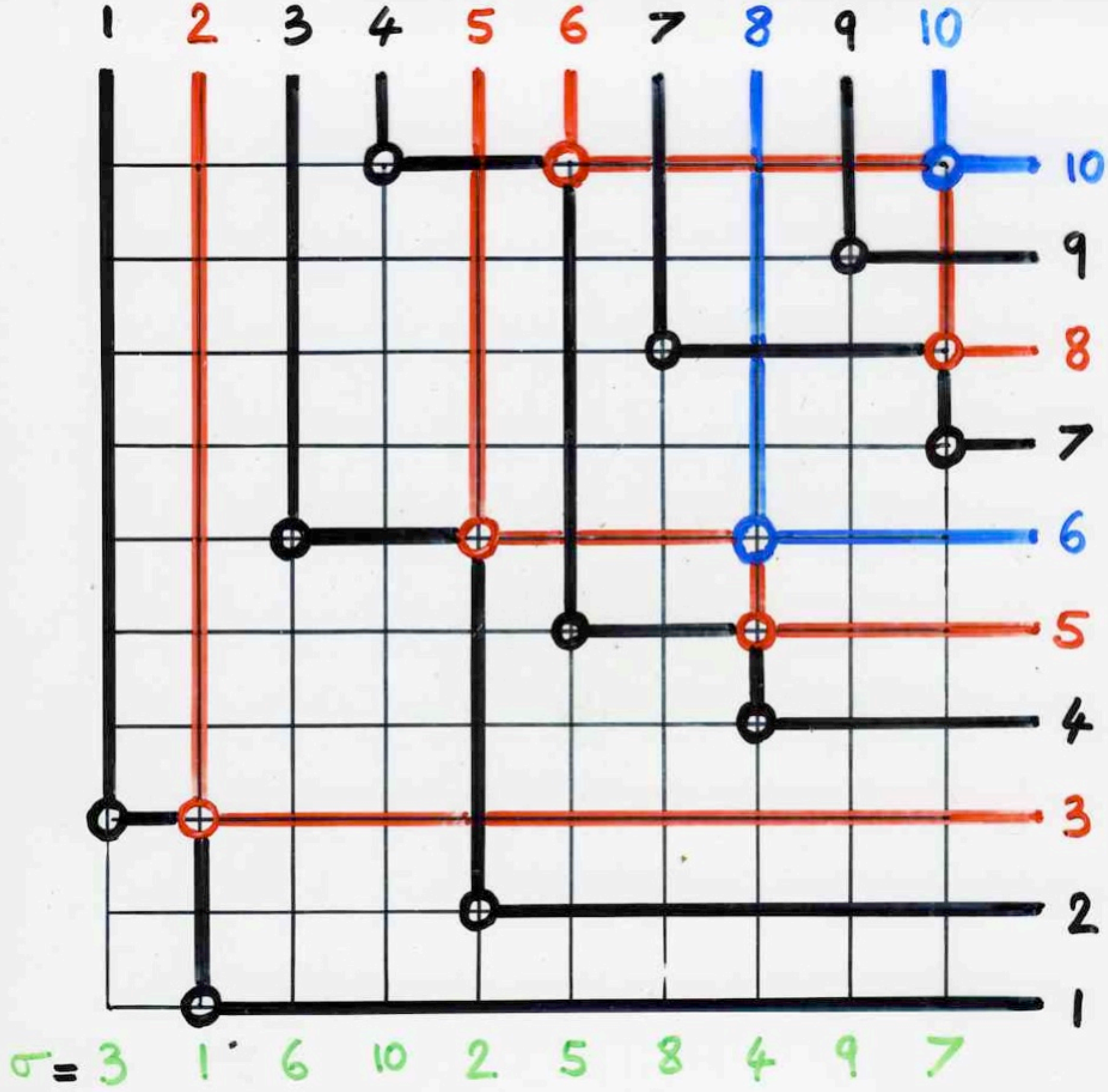
$UD = q\,DU + 1$

Sergey Fomin

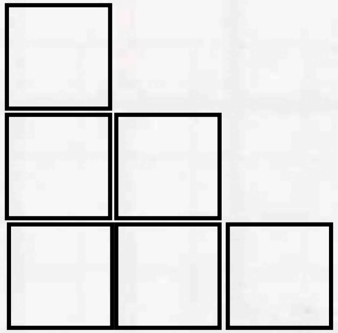
(with C. K.)

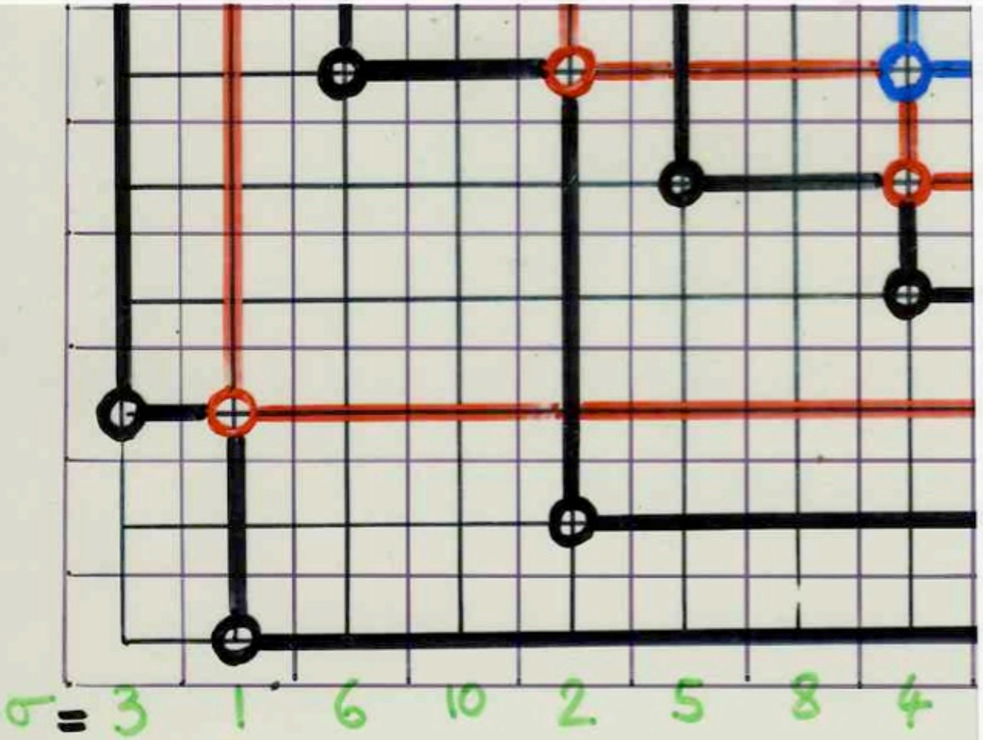
local RSK and geometric RSK

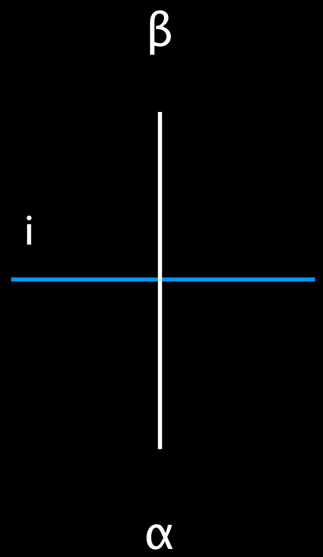
(the geometric construction with "light" and "shadow" for RSK leads to a simple proof of the fact that RSK and the "local rules" give the same bijection)

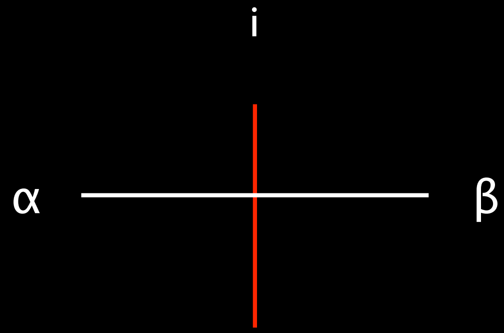


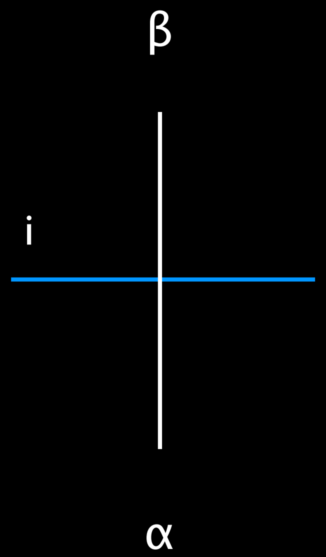
1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7



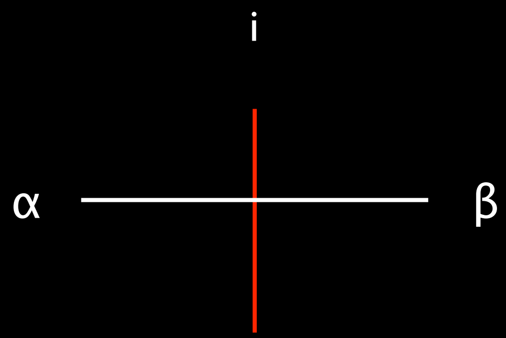

σ = 3 1 6 10 2 5 8 4

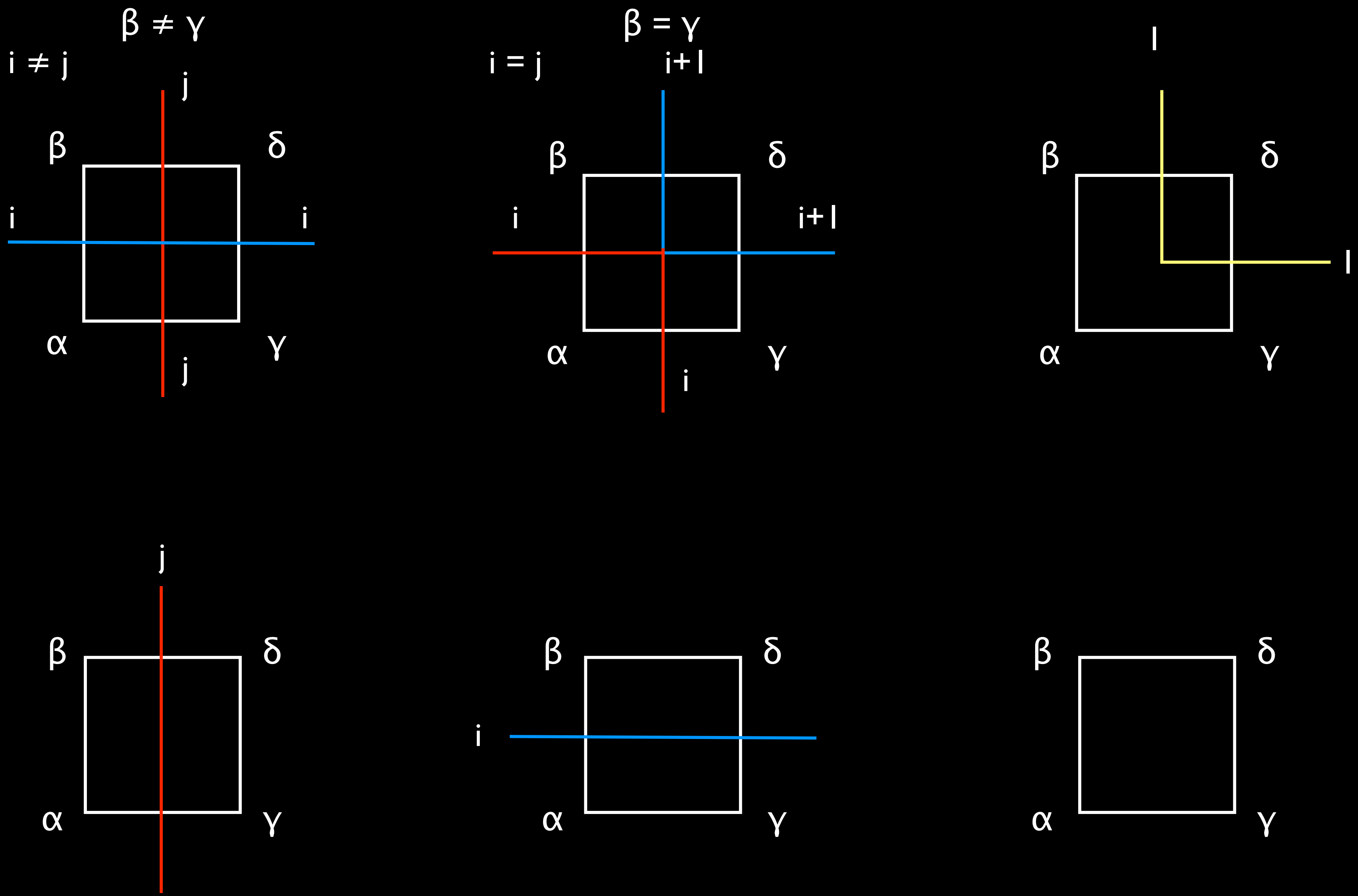

β
i
α

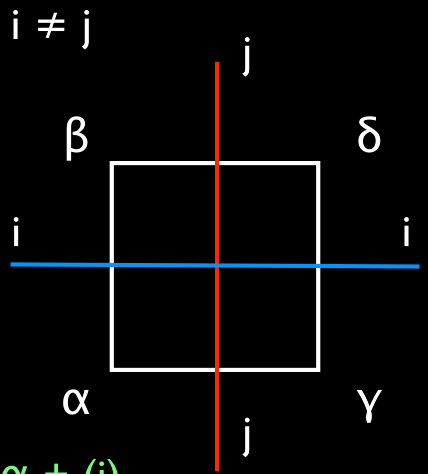

i
α
β



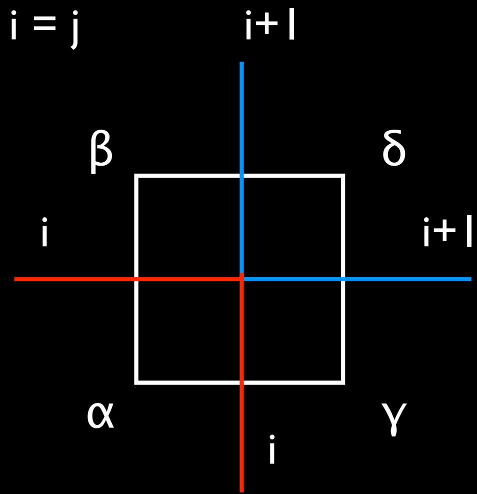
$\beta = \alpha + (i)$



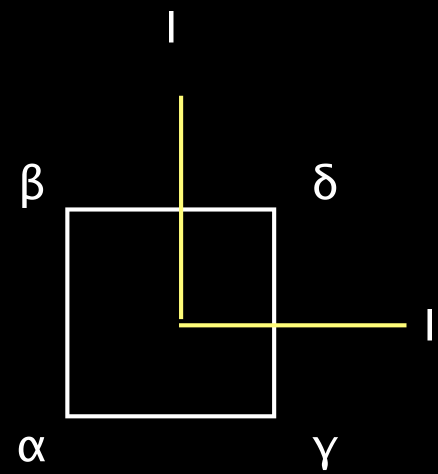

β ≠ γ
i ≠ j
j
β
δ
i
i
α
j
γ
β = γ
i = j
i+1
β
δ
i
i+1
α
i
γ
1
β
δ
1
α
γ
j
β
δ
α
γ
β
δ
i
α
γ
β
δ
α
γ



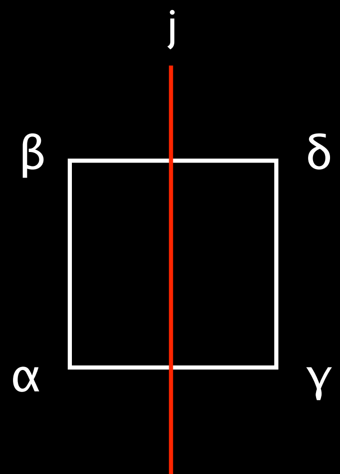
$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$



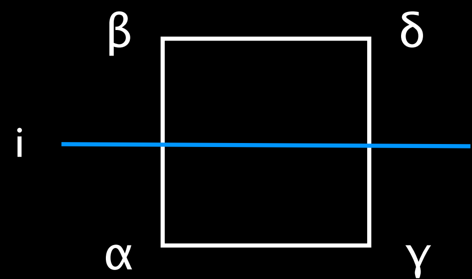
$\beta = \gamma = \alpha + (i)$
$\delta = \alpha + (i) + (i{+}1)$



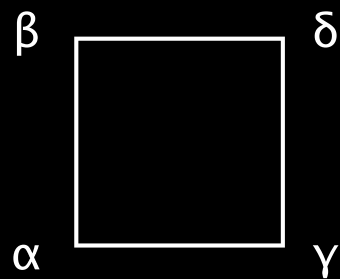
$\beta = \gamma = \alpha$
$\delta = \alpha + (l)$



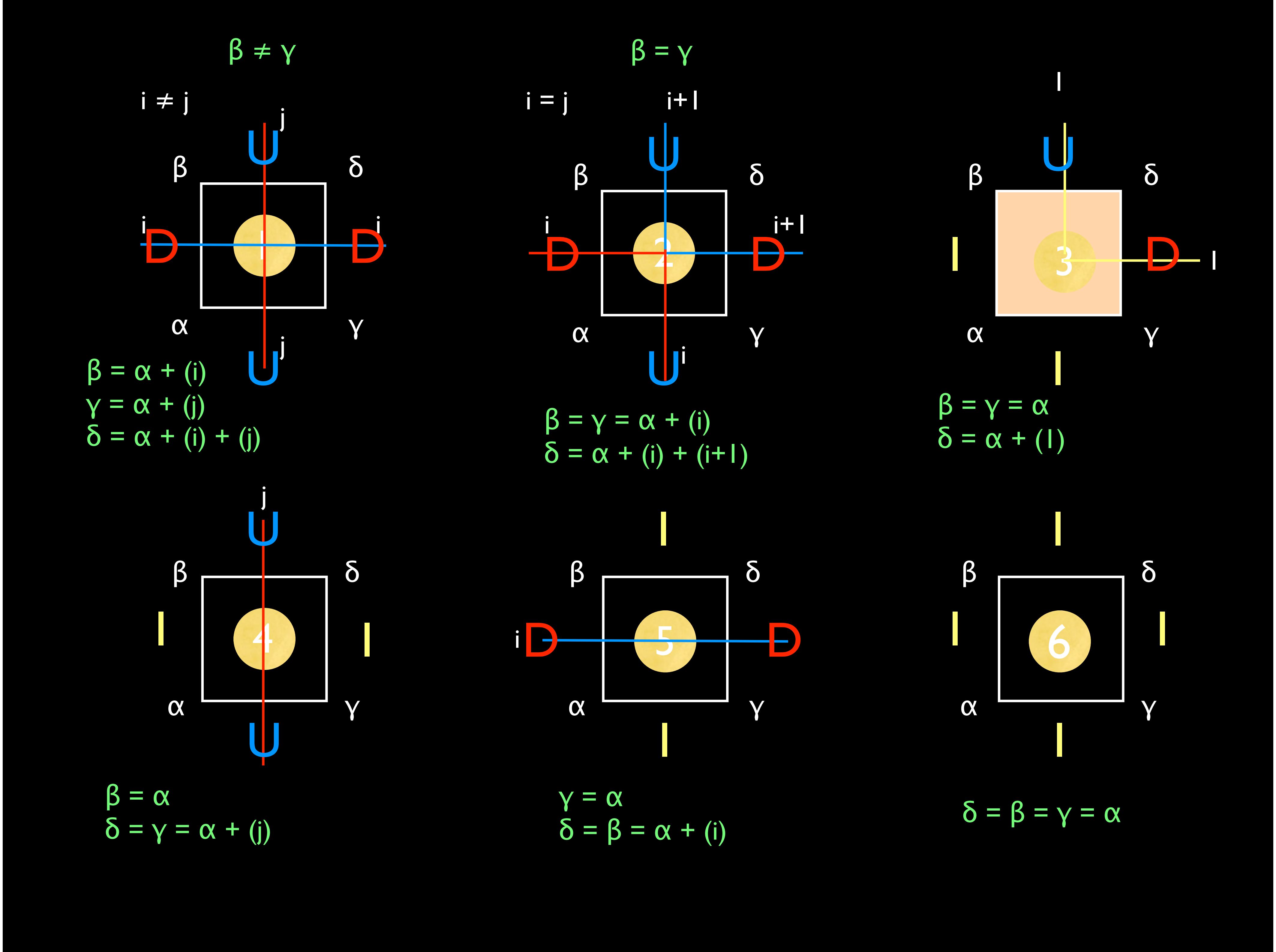
$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$



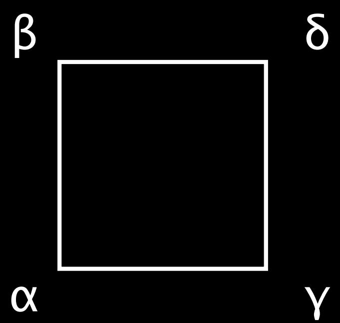
$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$



$\delta = \beta = \gamma = \alpha$


β ≠ γ
i ≠ j
j
β
δ
i
i
1
α
γ
j
β = α + (i)
γ = α + (j)
δ = α + (i) + (j)
β = γ
i = j
i+1
β
δ
i
i+1
2
α
γ
i
β = γ = α + (i)
δ = α + (i) + (i+1)
1
β
δ
1
3
1
α
γ
1
β = γ = α
δ = α + (1)
j
β
δ
1
4
1
α
γ
β = α
δ = γ = α + (j)
1
β
δ
i
5
α
γ
1
γ = α
δ = β = α + (i)
1
β
δ
1
6
1
α
γ
1
δ = β = γ = α

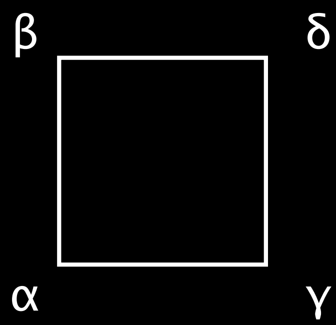
$\beta \neq \gamma$



$\delta = \beta \cup \gamma$

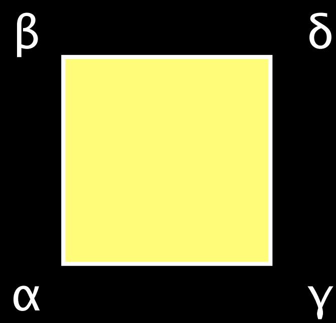
$\beta = \gamma$

$\beta = \gamma$
$\alpha \neq \beta$



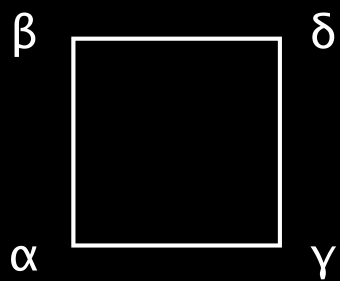
$\beta = \gamma = \alpha + (i)$
$\delta = \beta + (i+1)$

$\alpha = \beta = \gamma$

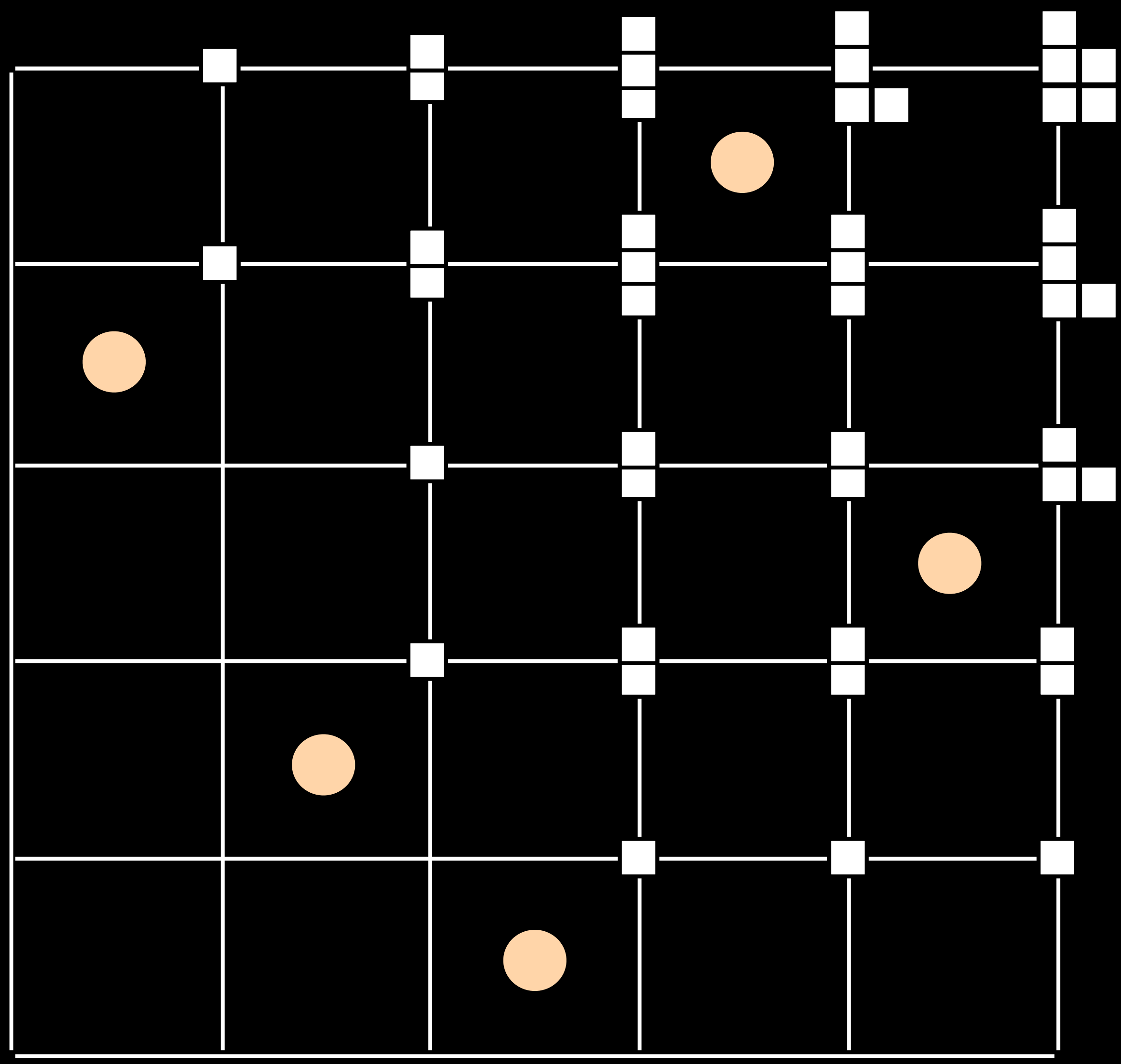


$\delta = \alpha + (1)$

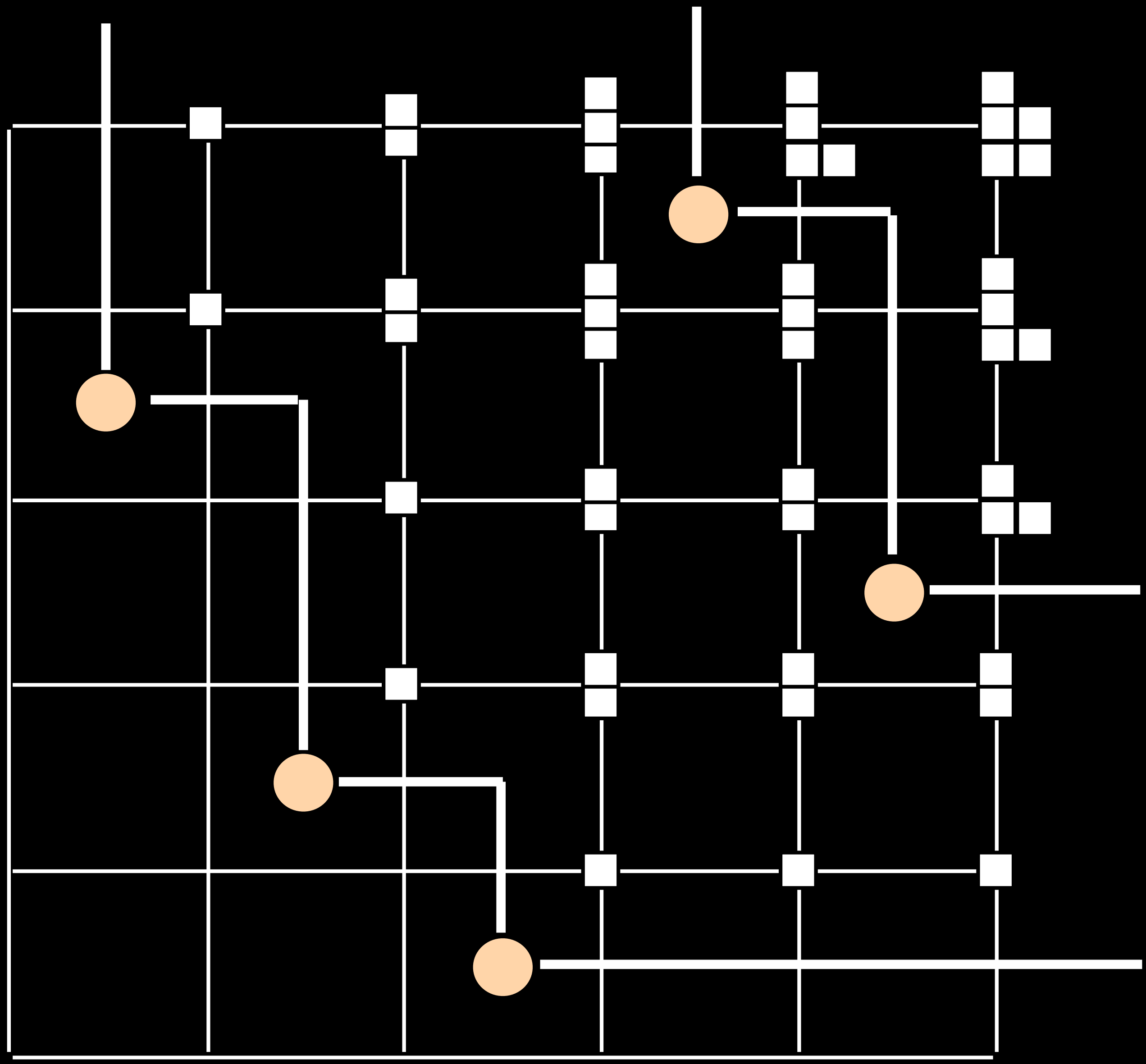
$\alpha = \beta = \gamma$

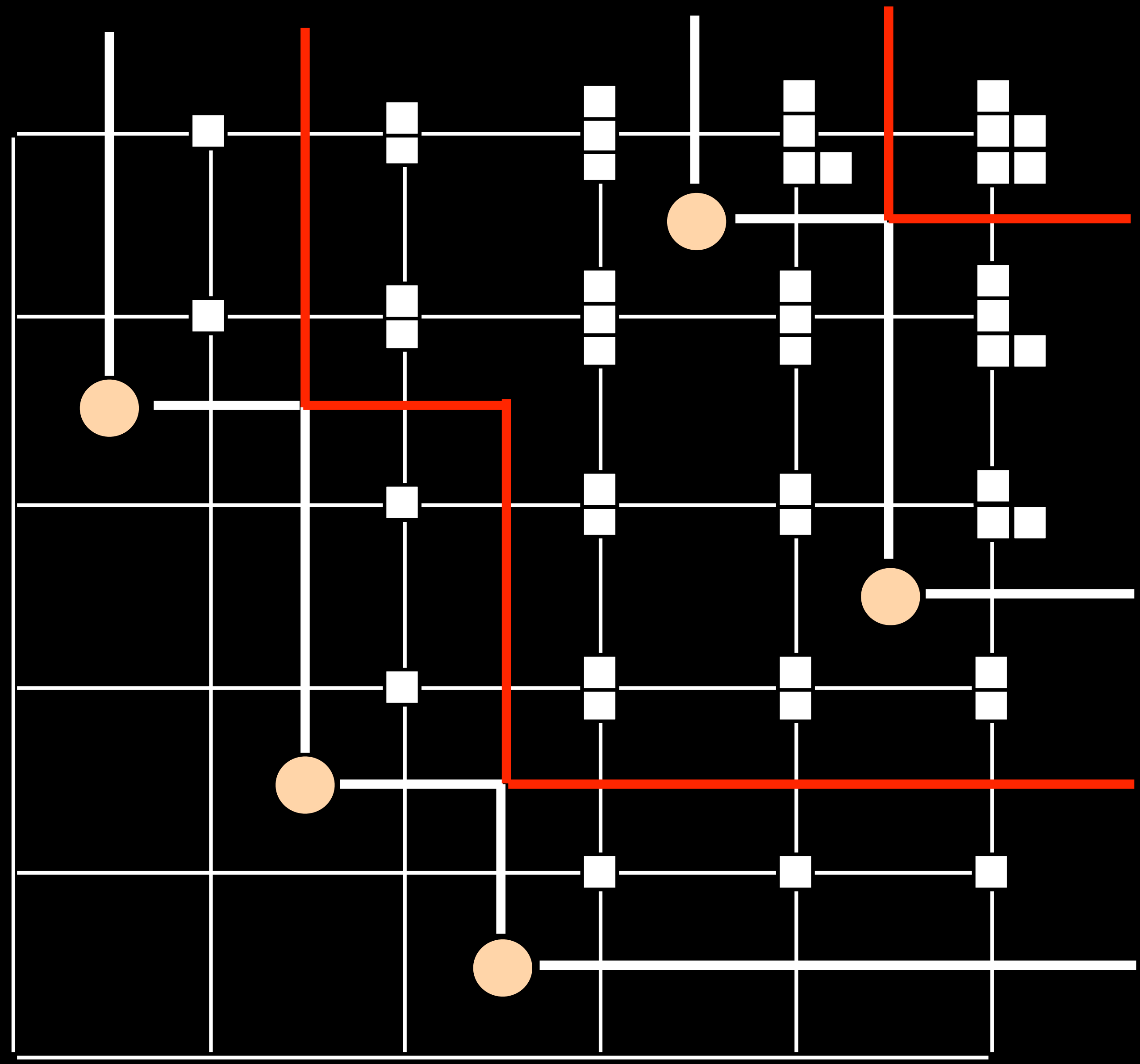


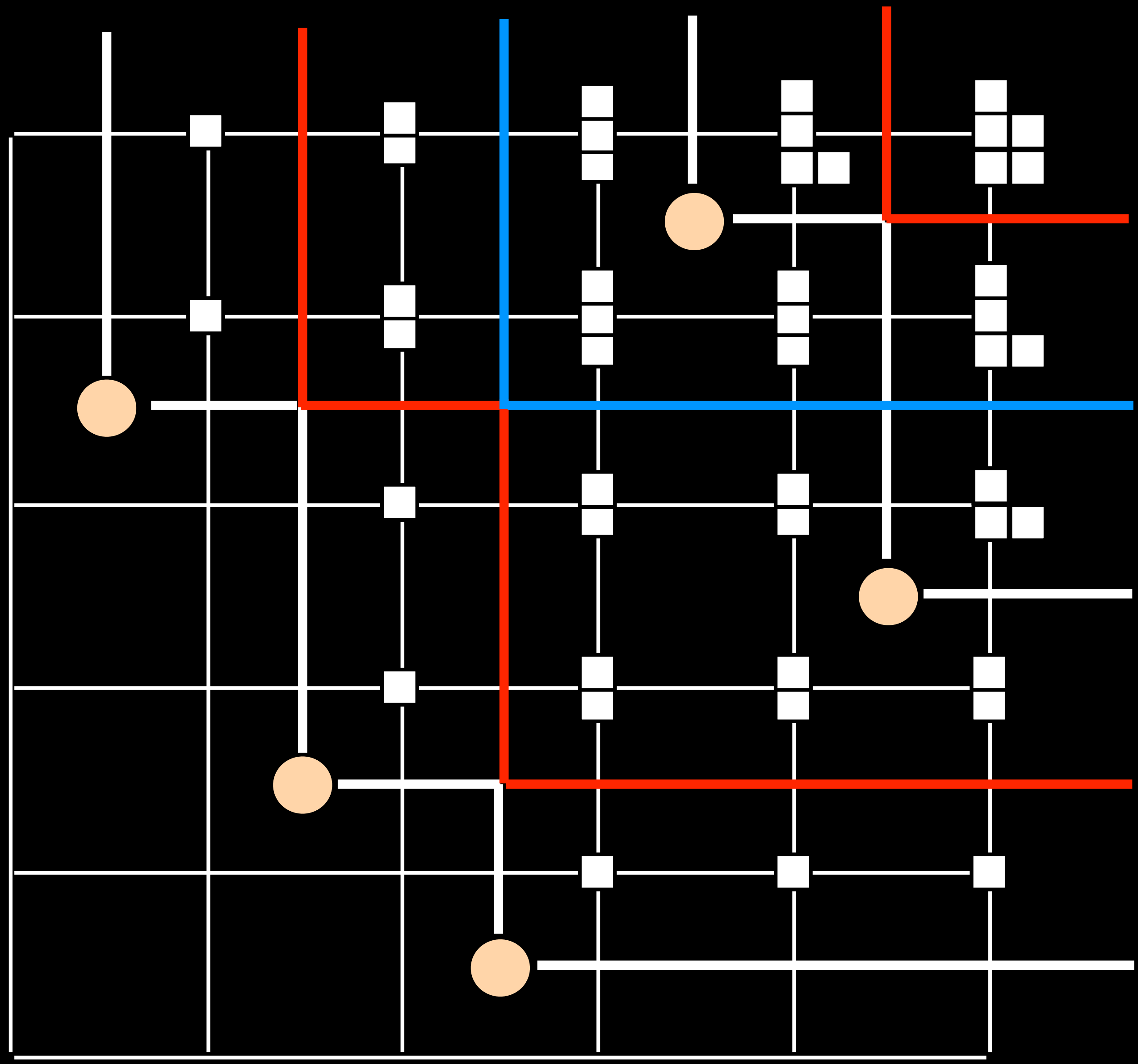
$\delta = \alpha = \beta = \gamma$

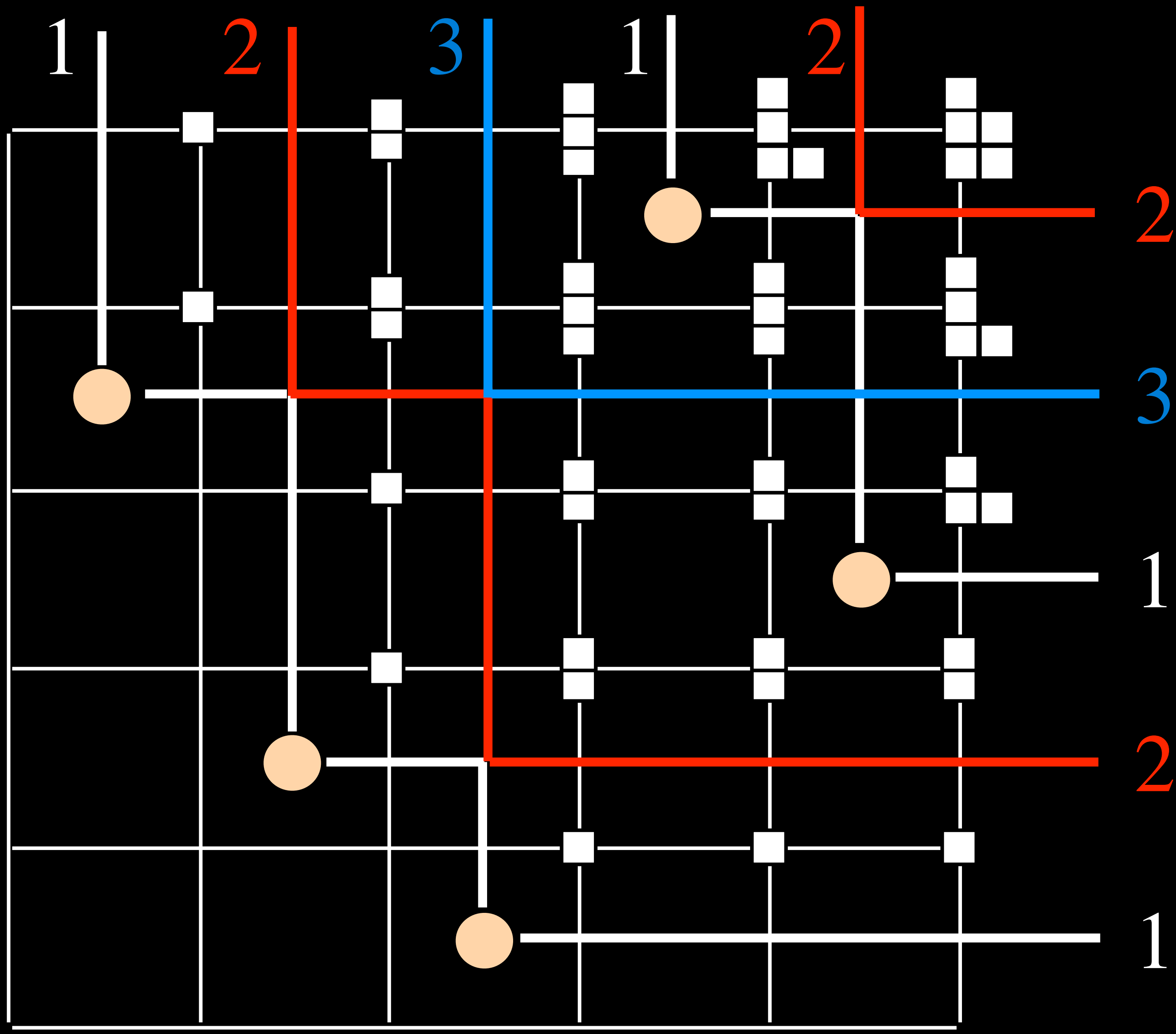


4 2 1 5 3

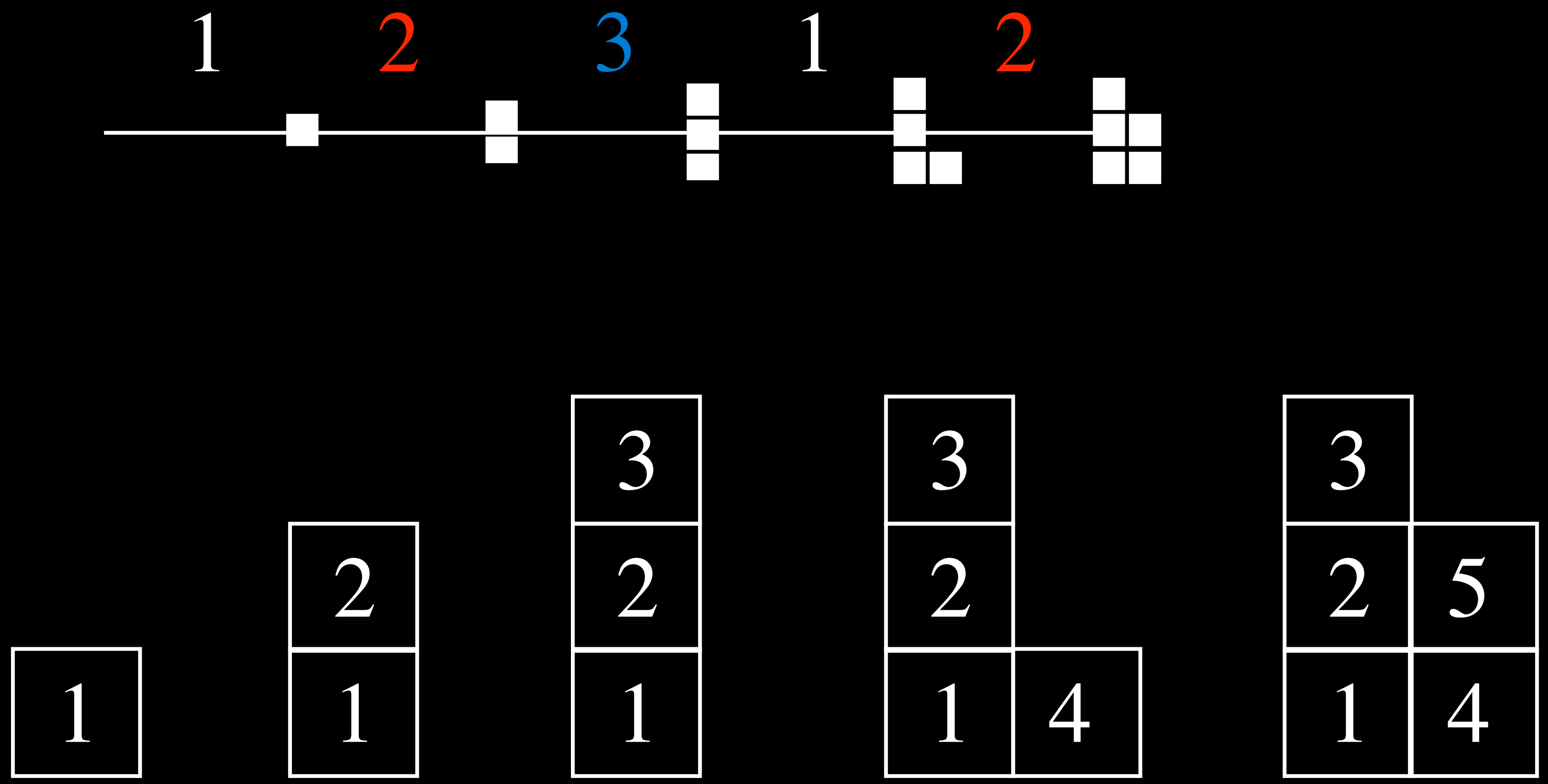






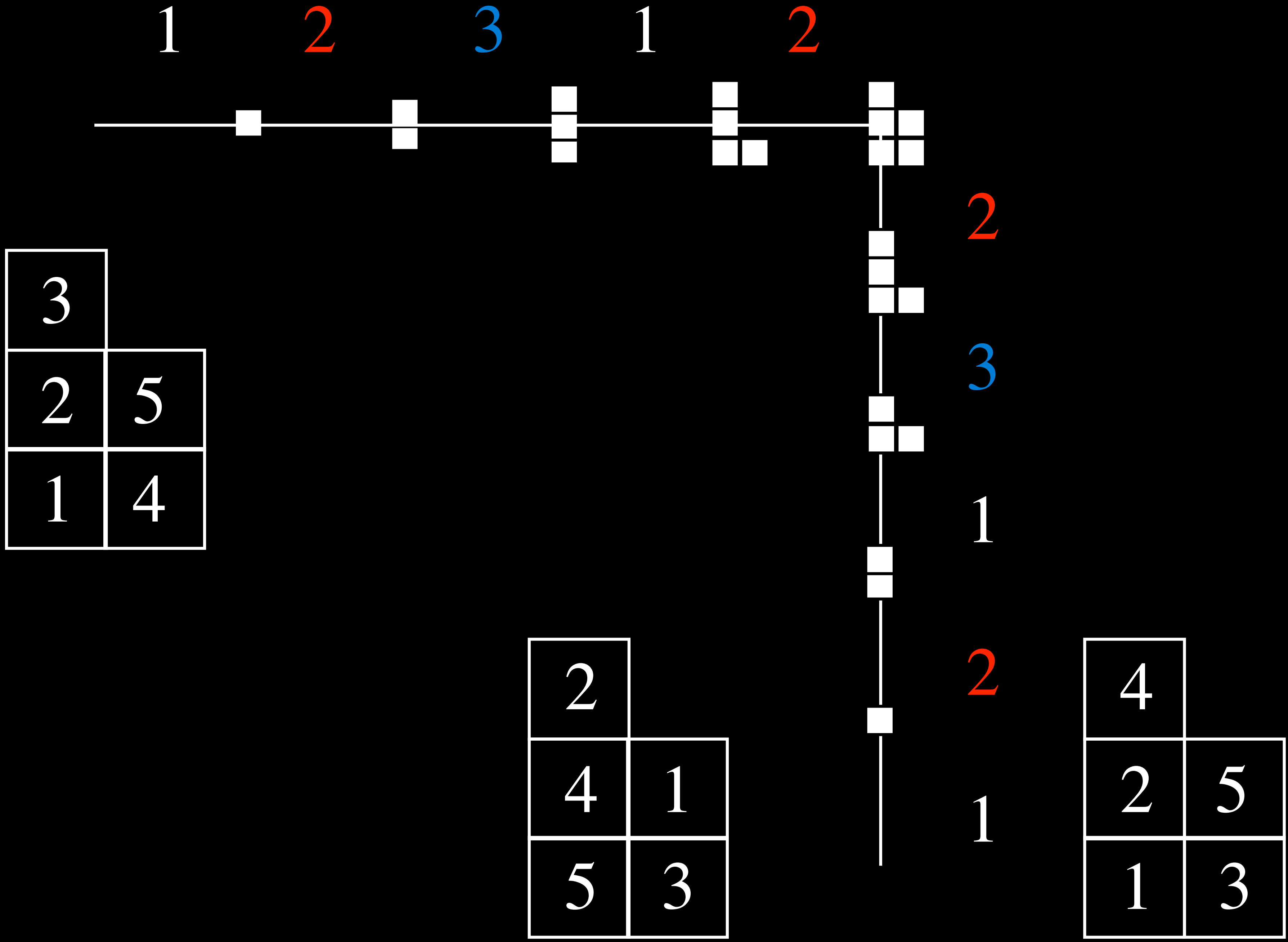


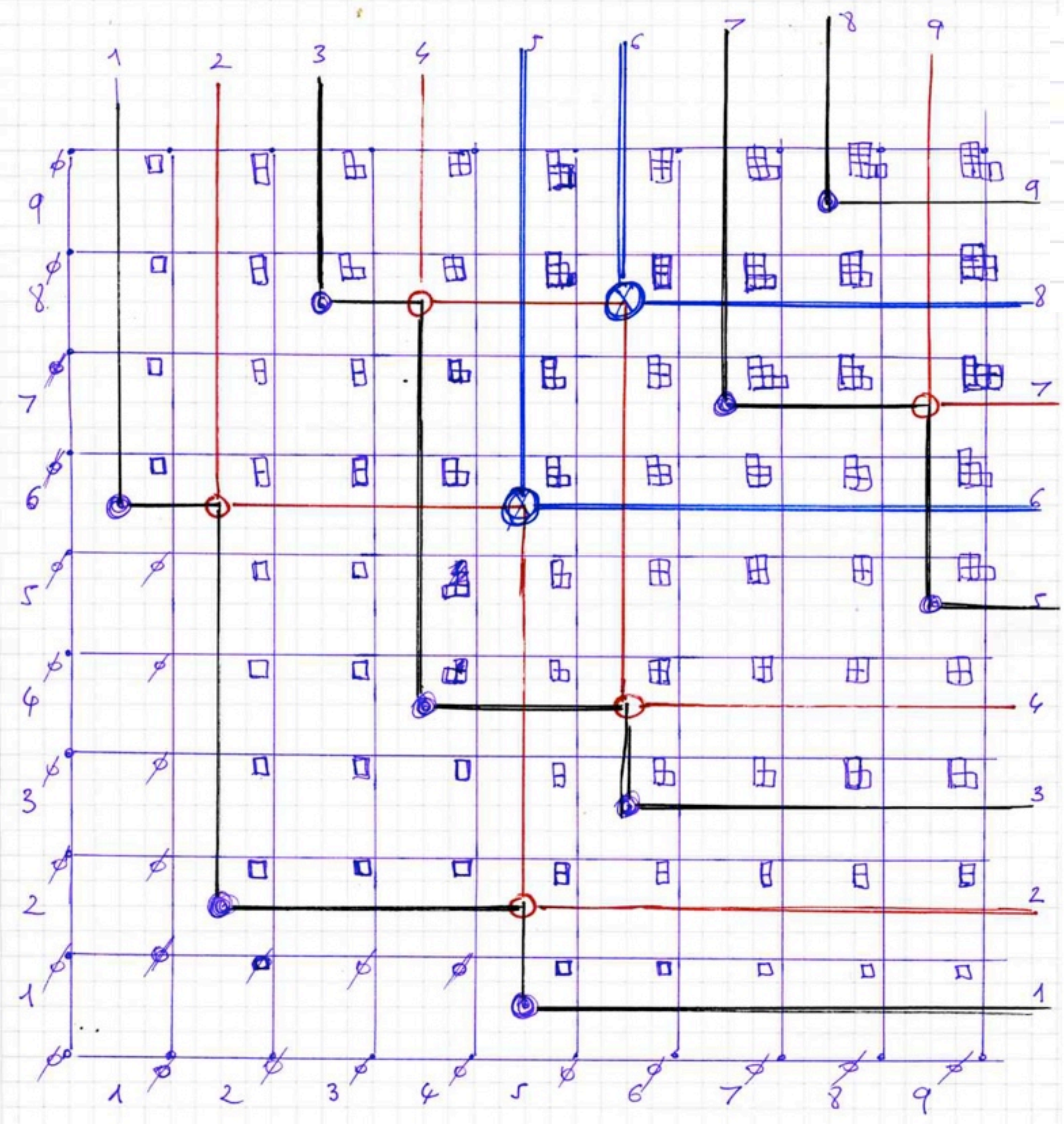
1
2
3
1
2
2
3
1
2
1



w = 1 2 3 1 2

Yamanuchi word


1 2 3 1 2
2
3
1
2
1
3
2 5
1 4
2
4 1
5 3
4
2 5
1 3



5	6		
2	4	9	
1	3	7	8

Q

another example with

6 2 8 4 1 3 7 9 5

6	8		
2	4	7	
1	3	5	9

P

[Sergey Fomin]

- **Schur operators and Knuth correspondences**, *[Journal of Combinatorial Theory, Ser.A]* **72** (1995), 277-292.
- **Duality of graded graphs**, *[Journal of Algebraic Combinatorics]* **3** (1994), 357-404.
- **Schensted algorithms for dual graded graphs**, *[Journal of Algebraic Combinatorics]* **4** (1995), 5-45.
- **Dual graphs and Schensted correspondences**, *Series formelles et combinatoire algebrique*, P.Leroux and C.Reutenauer, Ed., Montreal, LACIM, UQAM, 1992, 221-236.

- **[Finite posets and Ferrers shapes]** (with T.Britz, 41 pages)

[Advances in Mathematics] **158** (2000), 86-127.
A survey on the Greene-Kleitman correspondence; many proofs are new.

- **[Knuth equivalence, jeu de taquin, and the Littlewood-Richardson rule]** (30 pages)

Appendix 1 to Chapter 7 in: [R.P.Stanley], *[Enumerative Combinatorics, vol.2]*,
Cambridge University Press, 1999.

Richard P. Stanley

- **Differential posets, *J. Amer. Math. Soc.* 1 (1988), 919-961.**
- **Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.),**

The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145-165.

Xavier Gérard Viennot

- Une forme géométrique de la correspondance de Robinson-Schensted, in “Combinatoire et Représentation du groupe symétrique” (D. Foata ed.) Lecture Notes in Mathematics n° 579, pp 29-68, 1976

Marc van Leeuwen

- [The Robinson-Schensted and Schützenberger algorithms, an elementary approach]

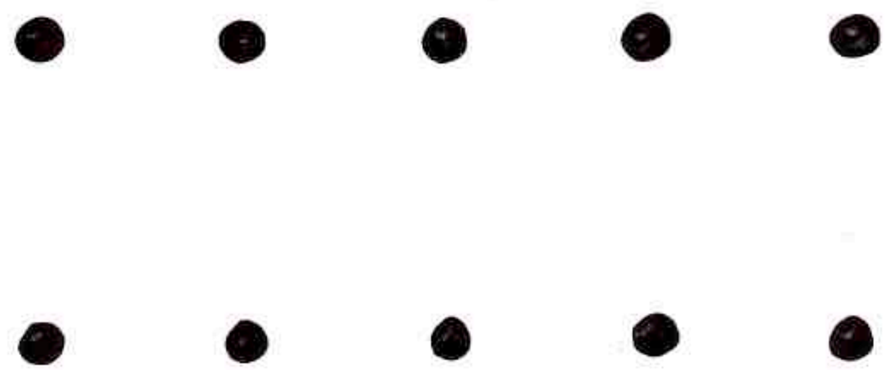
(a 272 Kb dvi file) [Electronic Journal of Combinatorics], [Foata Festschrift], [Vol 3(no.2), R15] (1996)

Guoniu Han

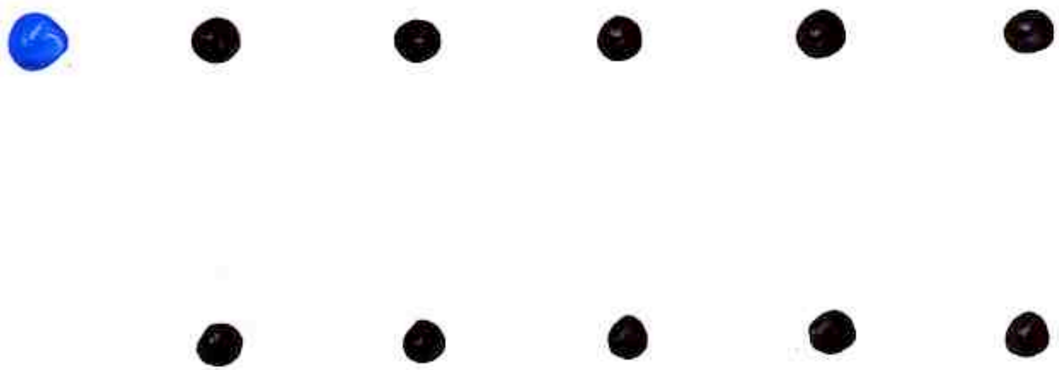
http://math.u-strasbg.fr/~guoniu/software/rsk/index.html
Autour de la correspondance de Robinson-Schensted
Exposé au SLC 52 et LascouxFest, 29/03/2004

other representations

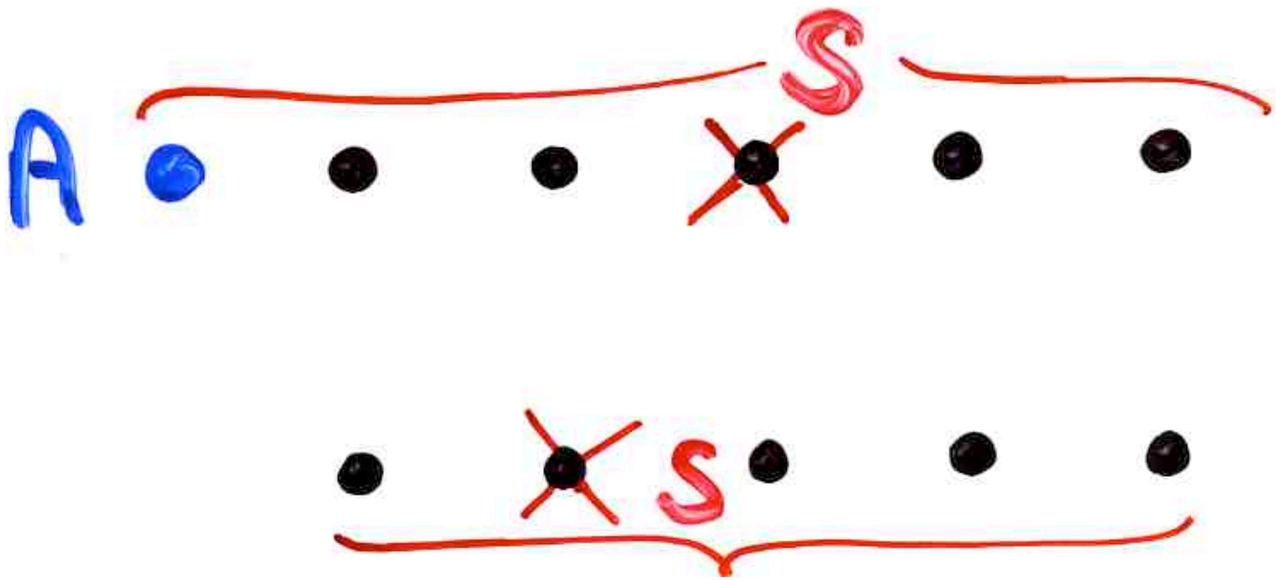
for $UD = DU + 1$

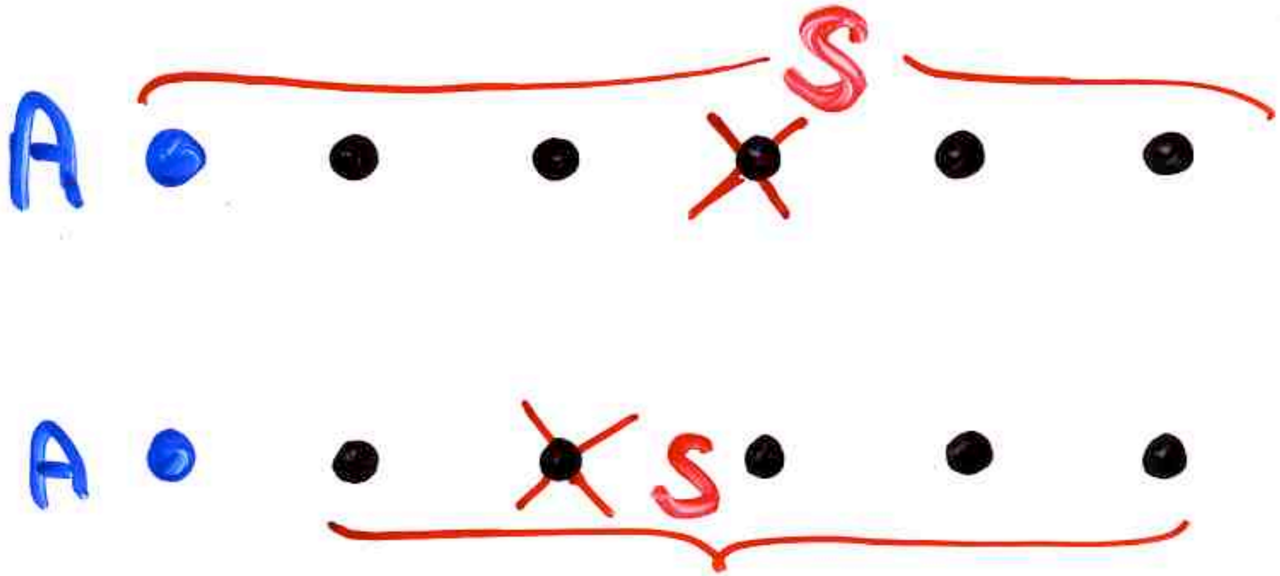


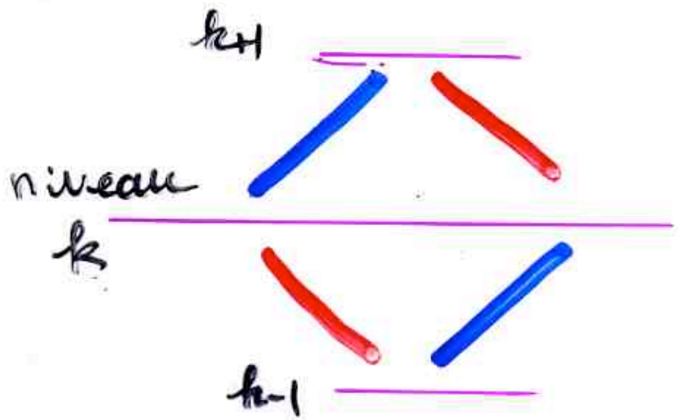
A




A
S


A
S
S


A
S
A
S



possibilités

AS $1 \times (k+1) = k+1$

SA $k \times 1 = k$

$$UD = DU + I$$

Prop. w mot de Dyck

$$c_{0,0}(w) = v_H(w)$$ valuation Hermite

$\lambda_k = k$

= nb d'histoires d'Hermite associées à w

polynôme d'Hermite $H_n(x)$

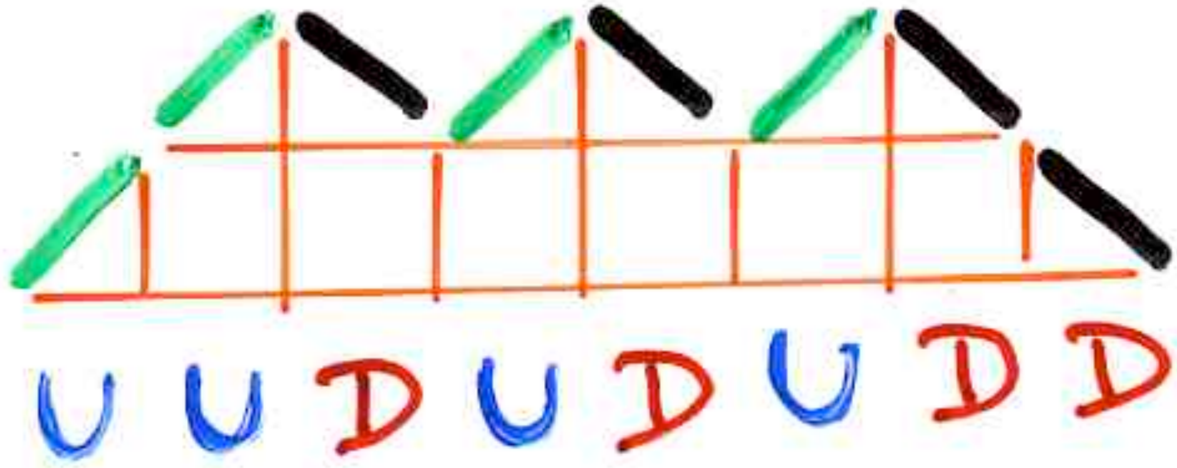
$\lambda_k = k$; $b_k = 0$

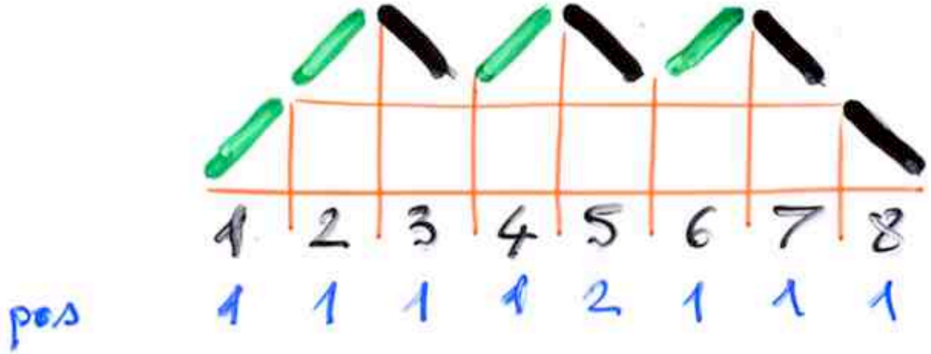
$(k \geqslant 1)$ $(k \geqslant 0)$

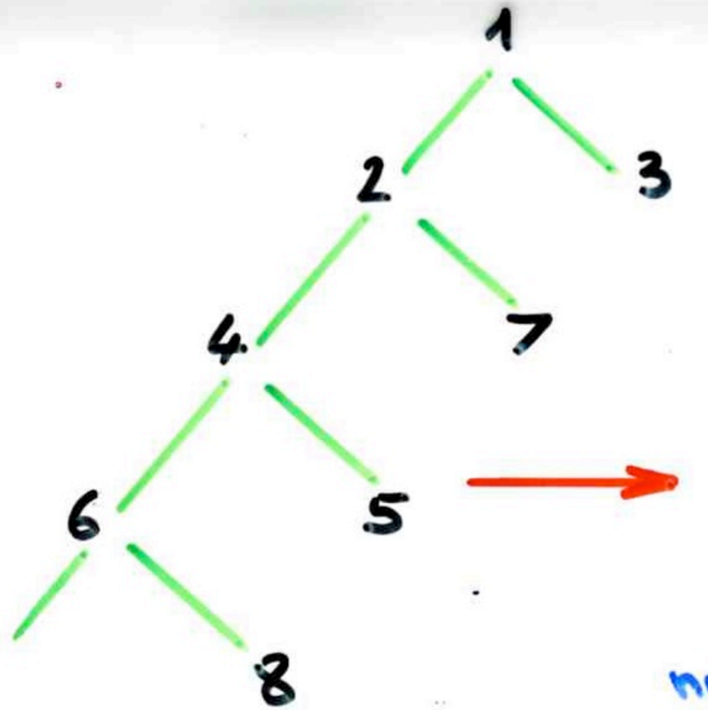
$a_k = 1$ $\begin{cases} b'_k = 0 \\ b''_k = 0 \end{cases}$ $c_k = k$

Histoires d'Hermite

ex :







Involution

$$\sigma = (1\,3)(2\,7)(4\,5)(6\,8)$$

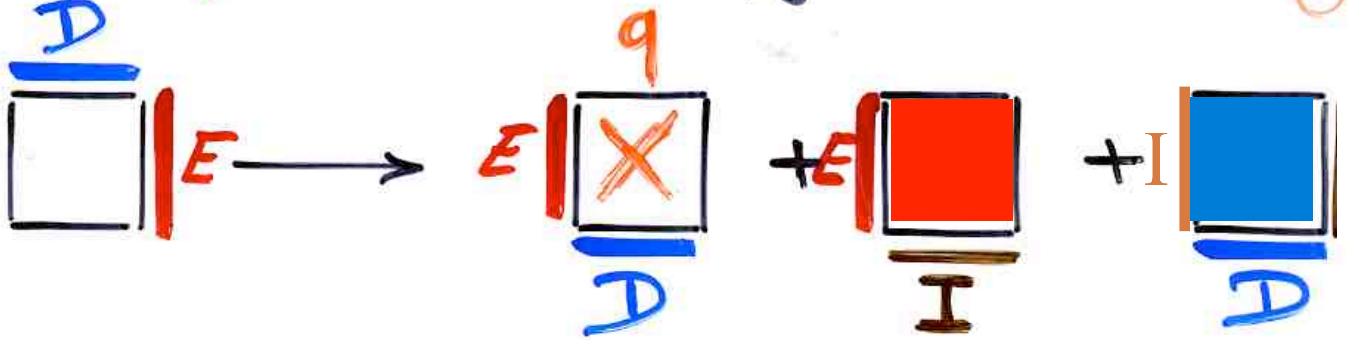
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 5 & 4 & 8 & 2 & 6 \end{pmatrix}$$

no fixed points

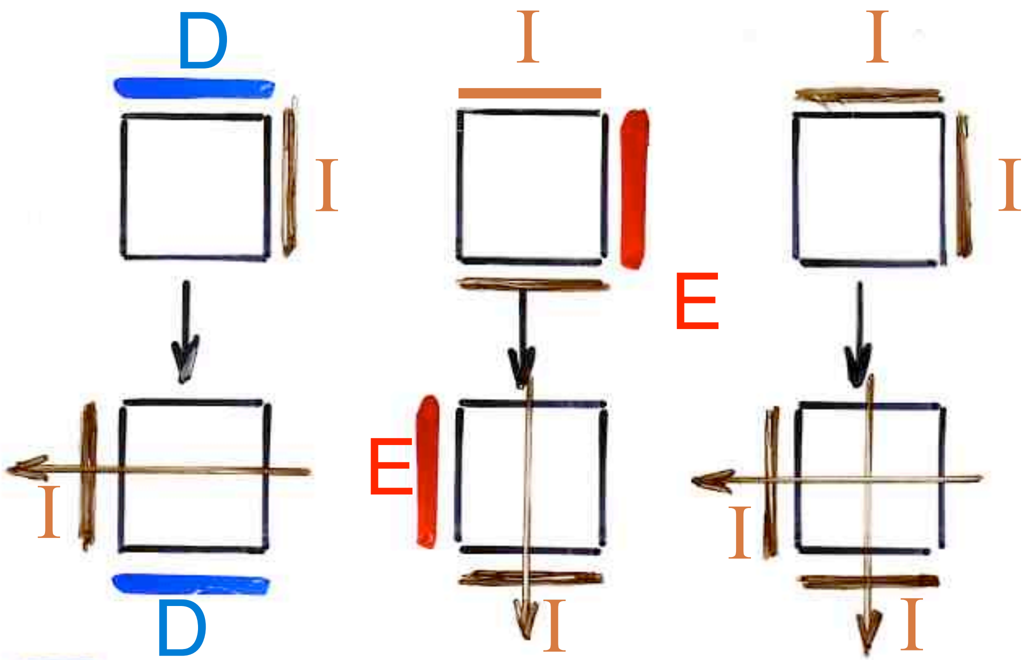
The PASEP algebra

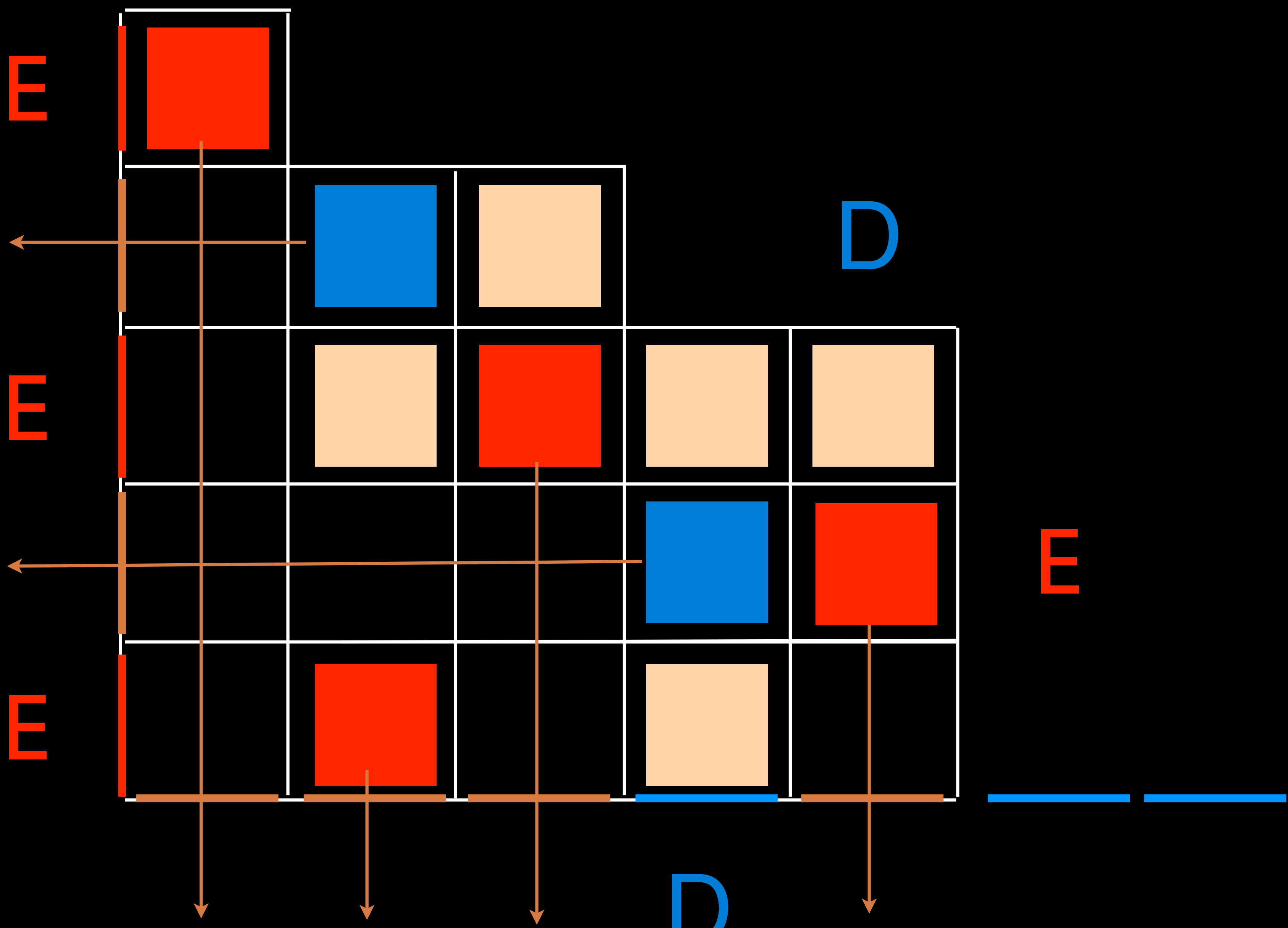
$$DE = qED + E + D$$

Proof: "planarization" of the rewriting rules

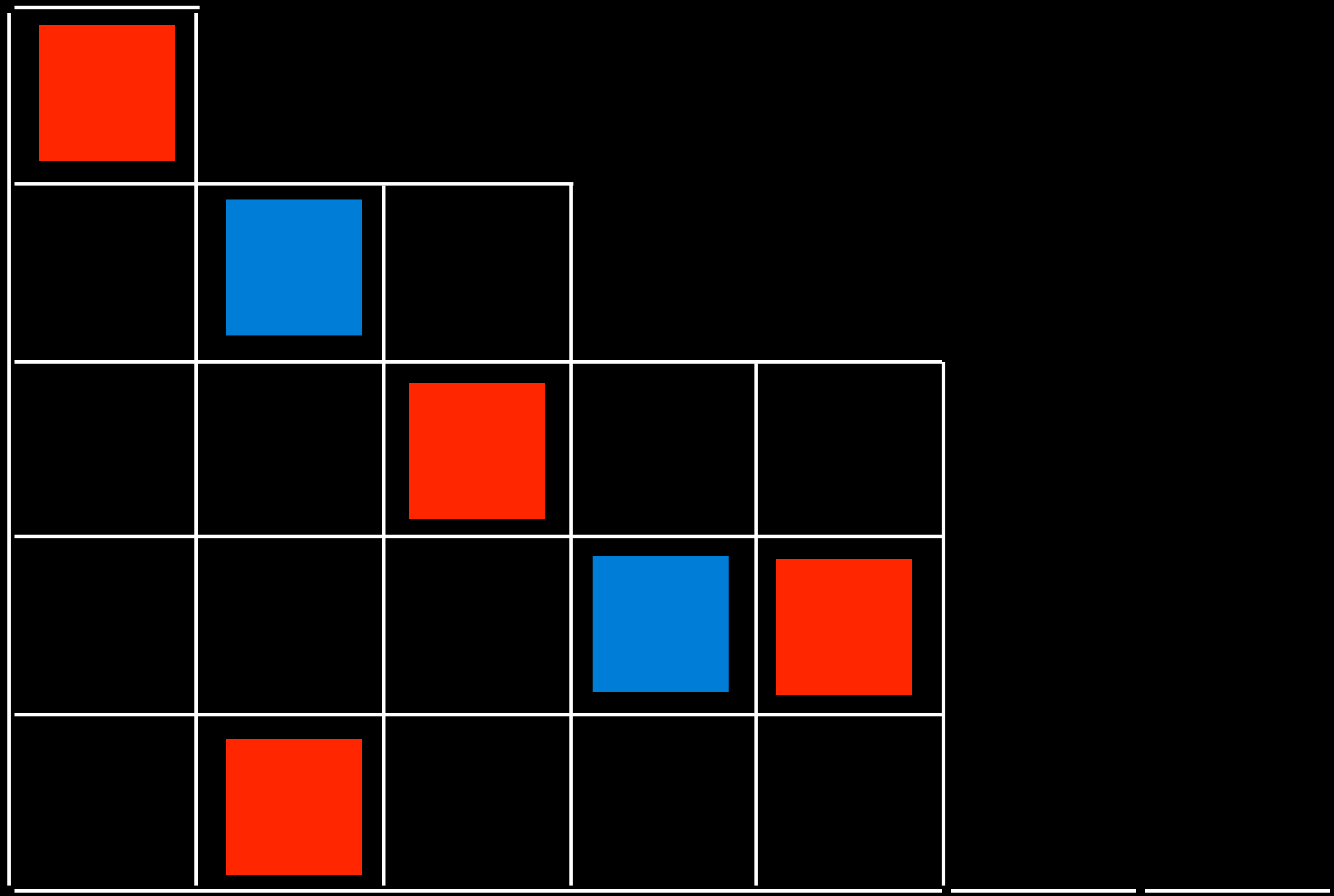


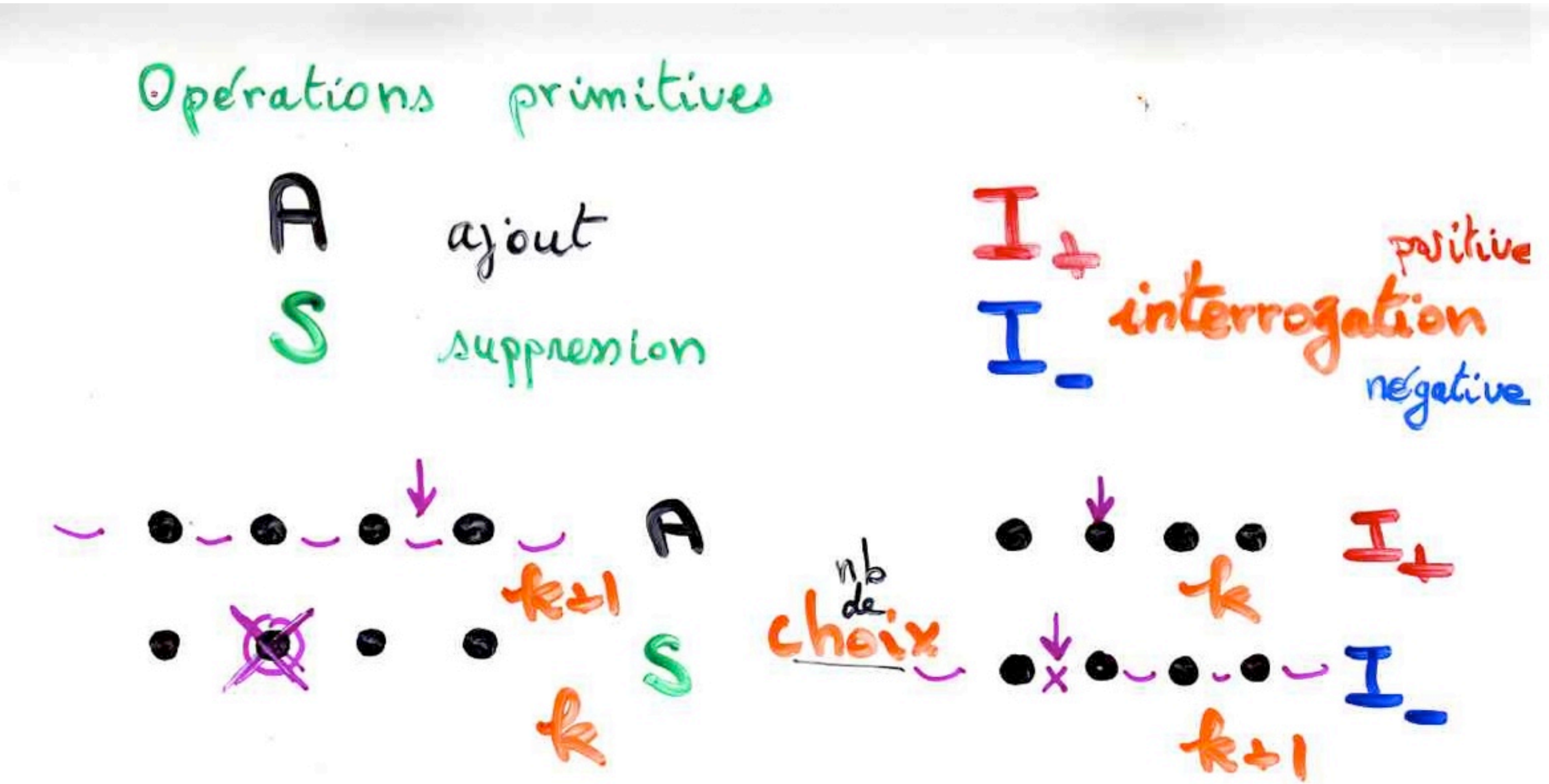
I identity



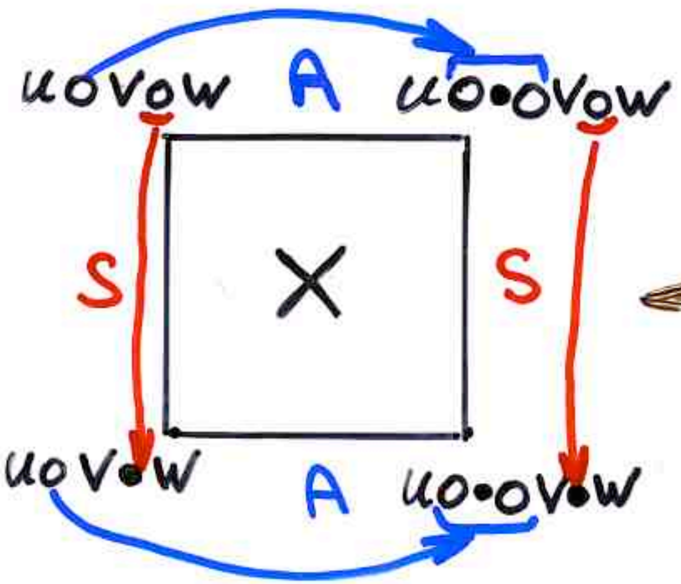

E
D
E
E
E
D

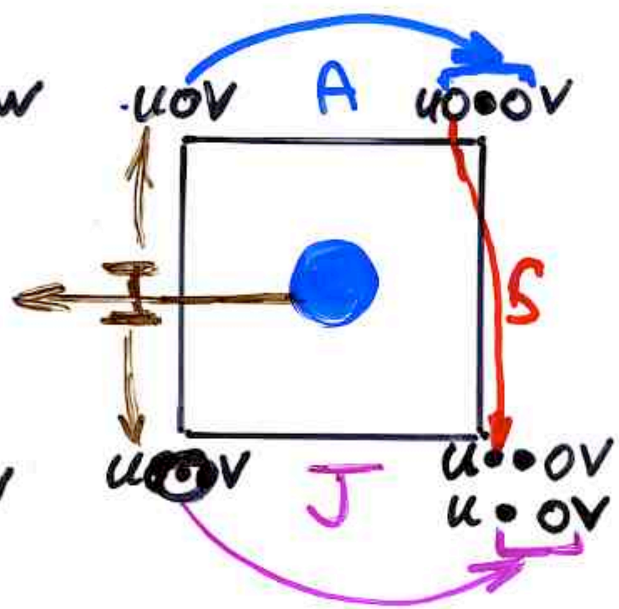
alternative tableau





number of choices for each
primitive operations


uovow
A
uo•ovow
S
X
S
uov•w
A
uo•ov•w


uov
A
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S
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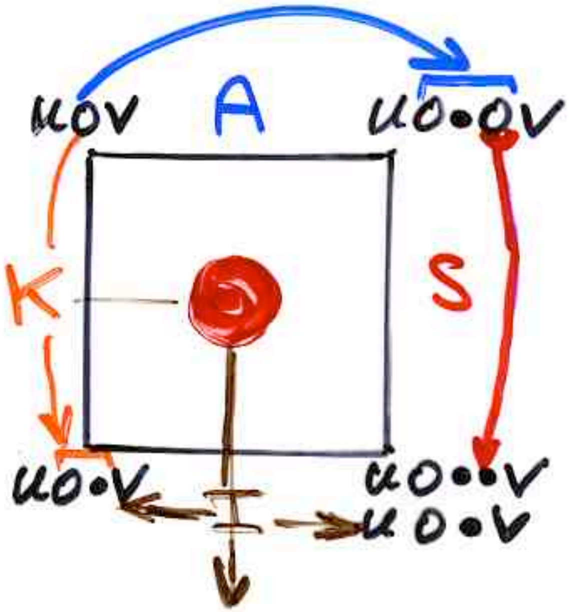
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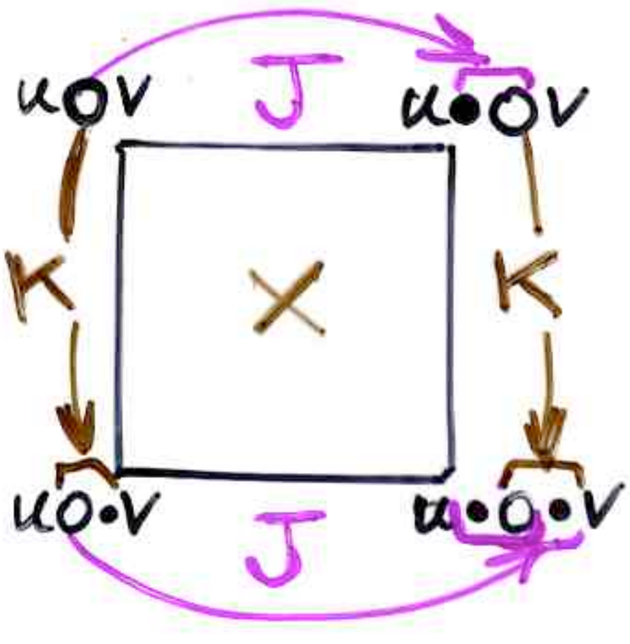
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K
K
uo•v
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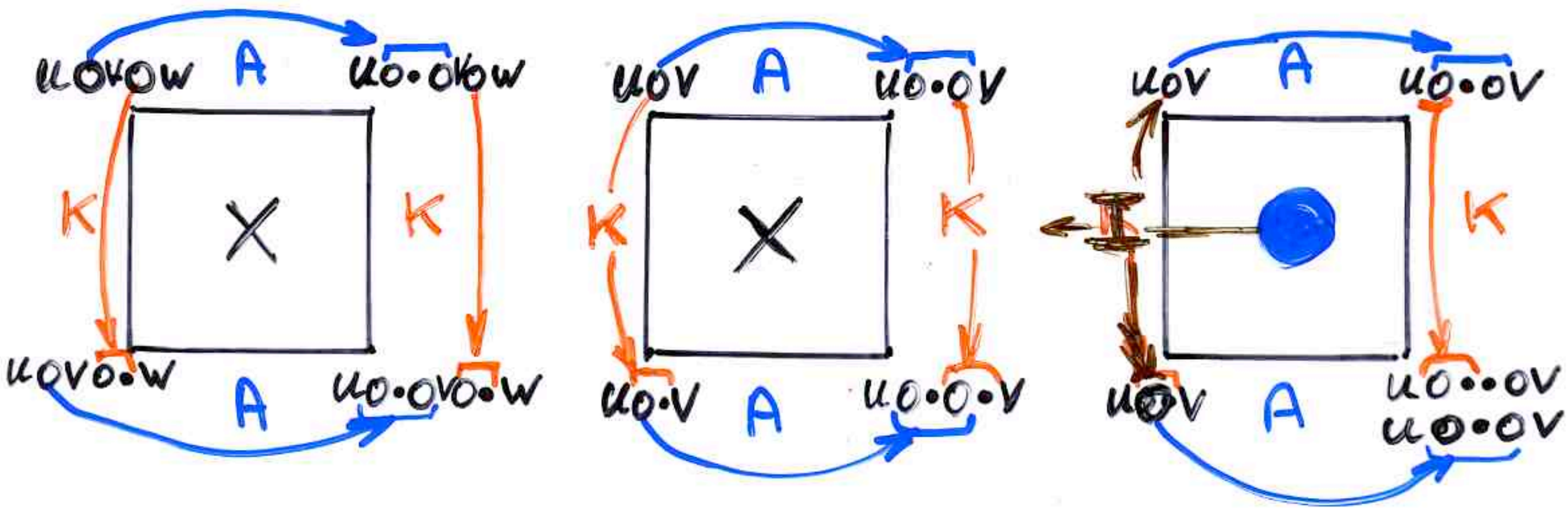
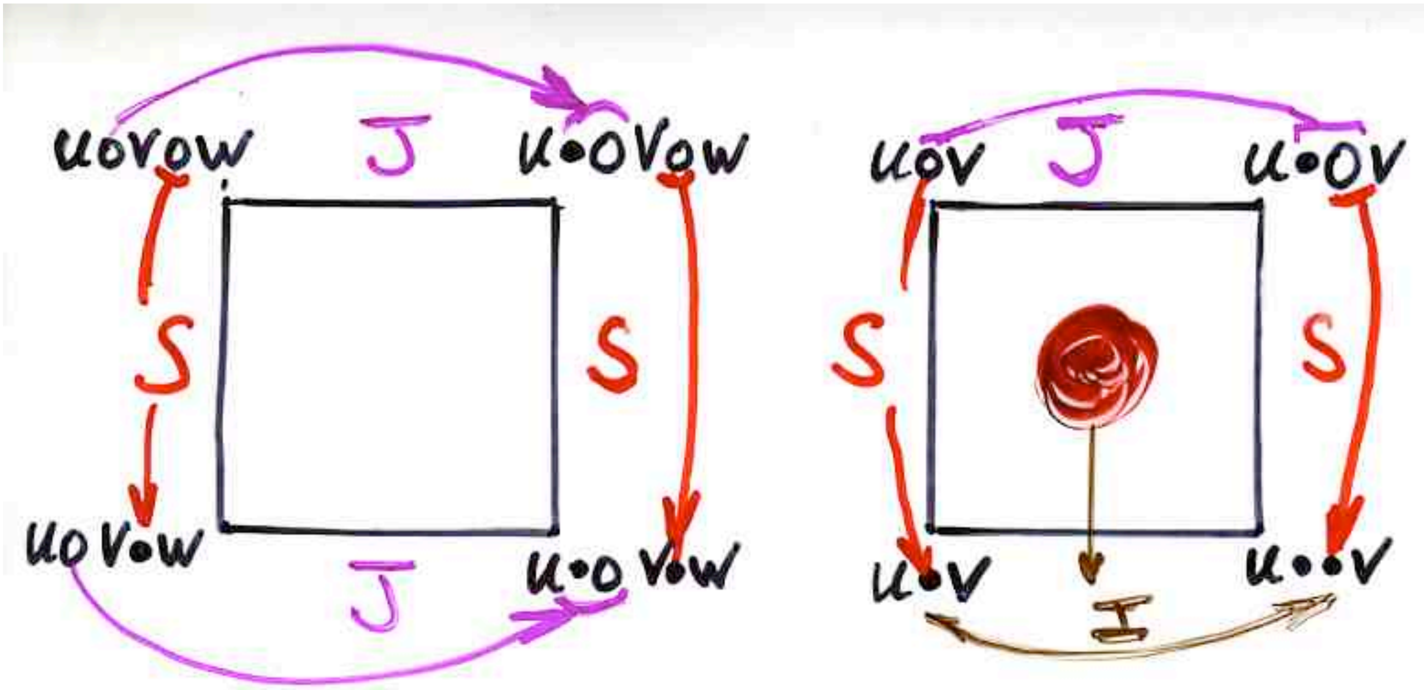
u•o•v


A
K
I


J
S
I

Lemma.

$$AS = SA + J + K$$

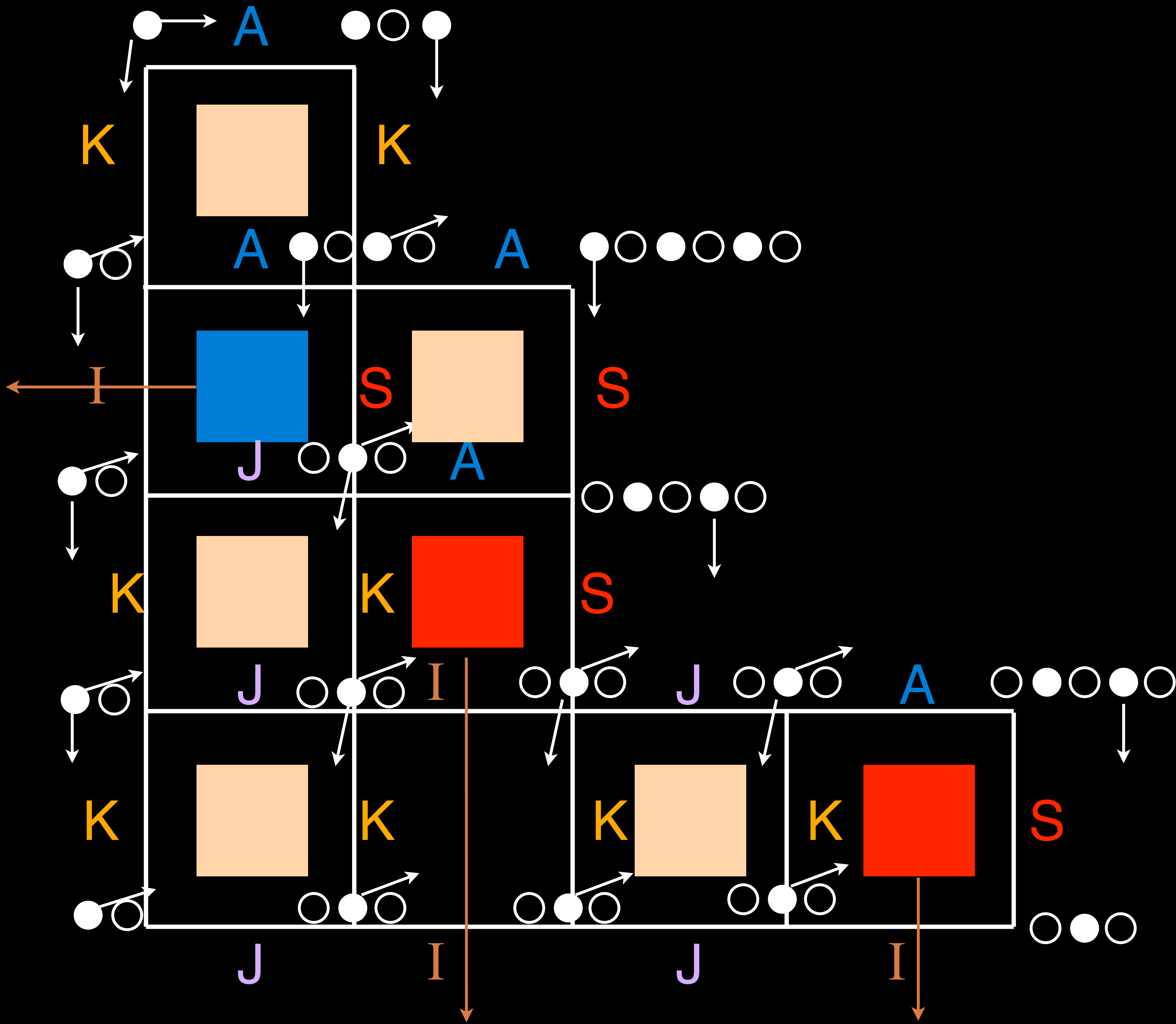
$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$


A
K
K
A
A
I
S
S
J
A
K
K
S
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I
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A
K
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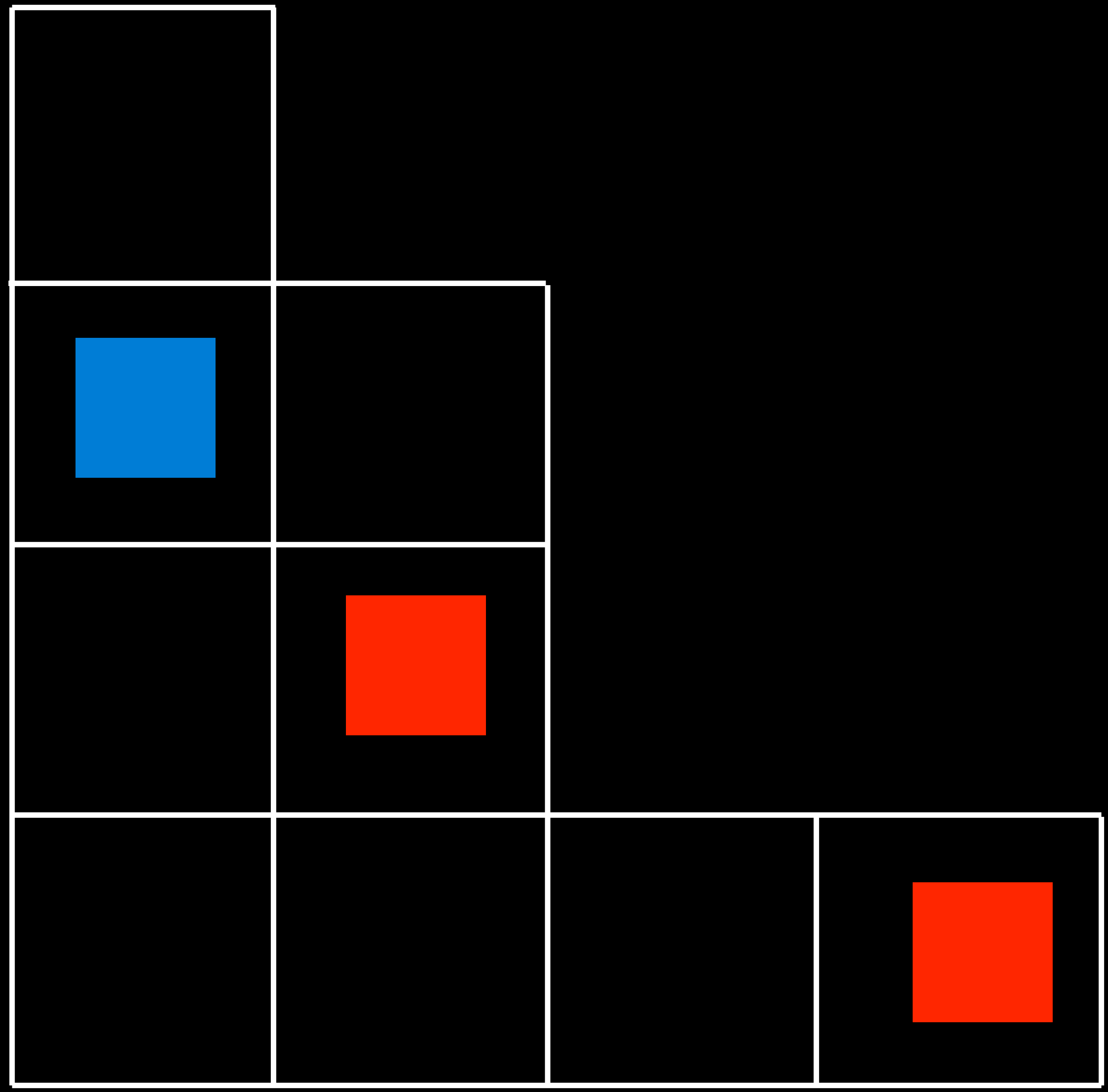
S
J
I
J
I



416978352

alternating sign matrices (ASM)
and a quadratic algebra

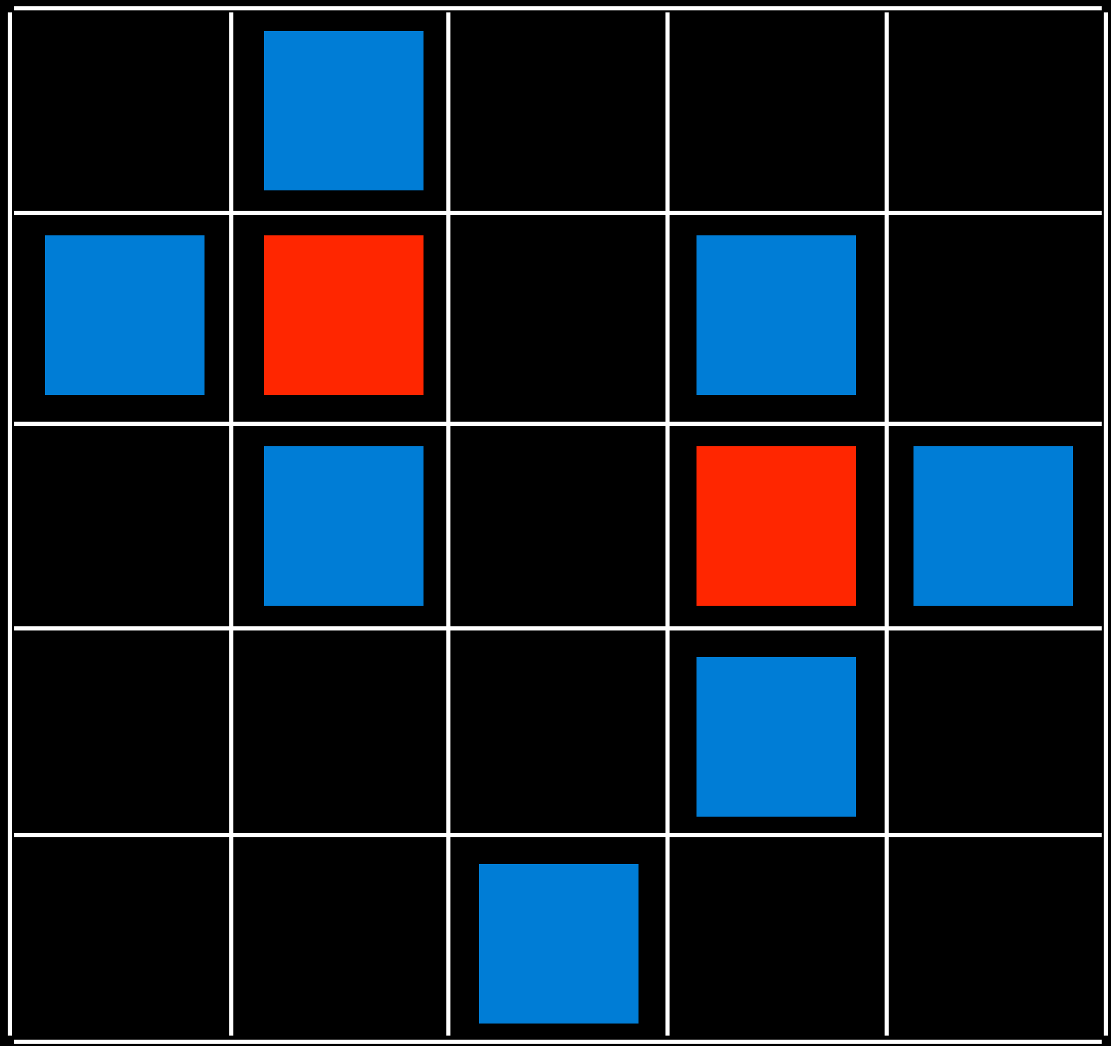
Def- **ASM** alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: $0, 1, -1$

(ii) sum of entries in each $\begin{cases} \text{row} \\ \text{column} \end{cases} = 1$

(iii) non-zero entries alternate in each $\begin{cases} \text{row} \\ \text{column} \end{cases}$



$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

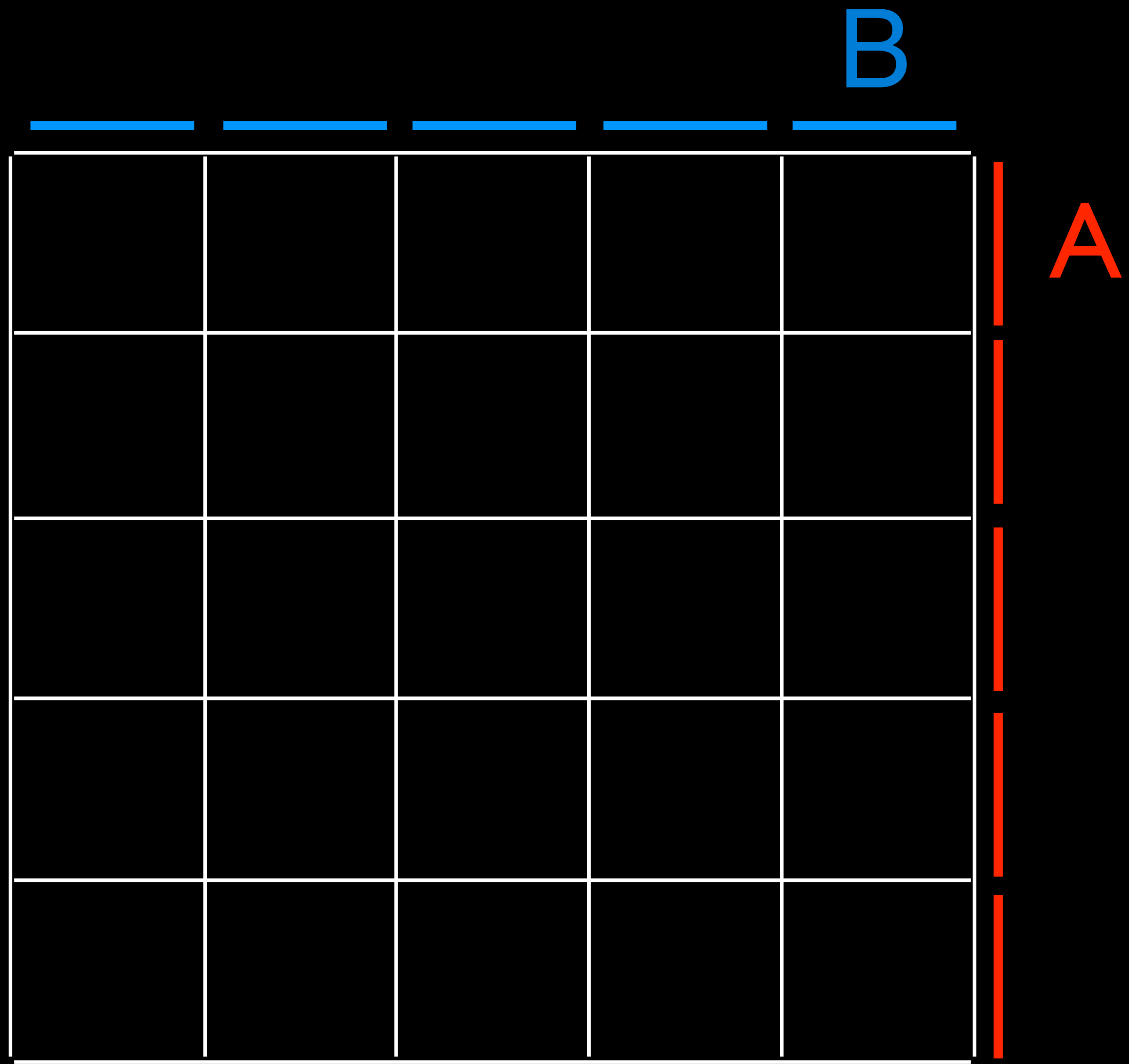
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

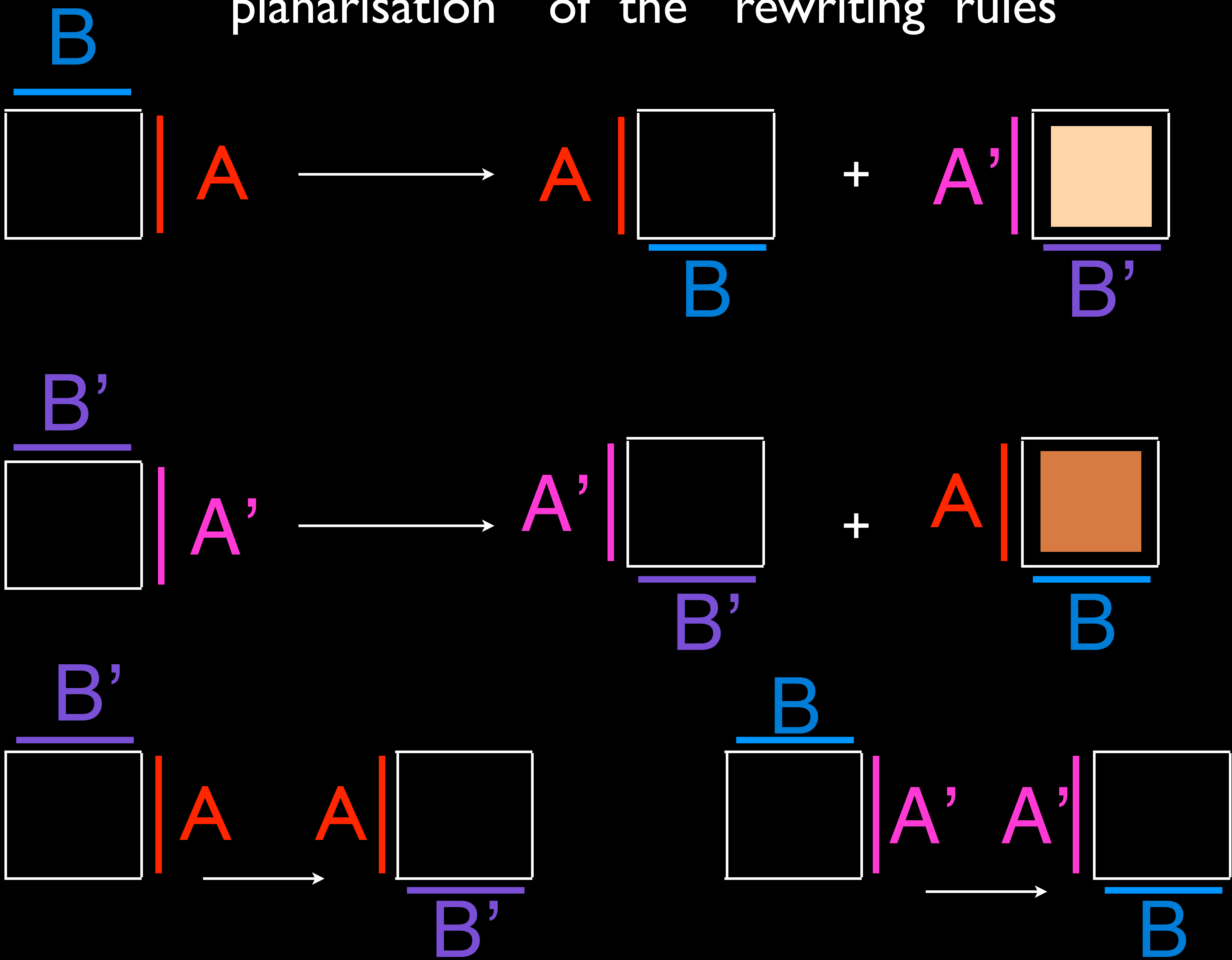
$$\sum c(u,v;w)\ \underbrace{u(A,A')}_{\text{word in } A,\,A'}\ \underbrace{v(B,B')}_{\text{word in } B,\,B'}$$

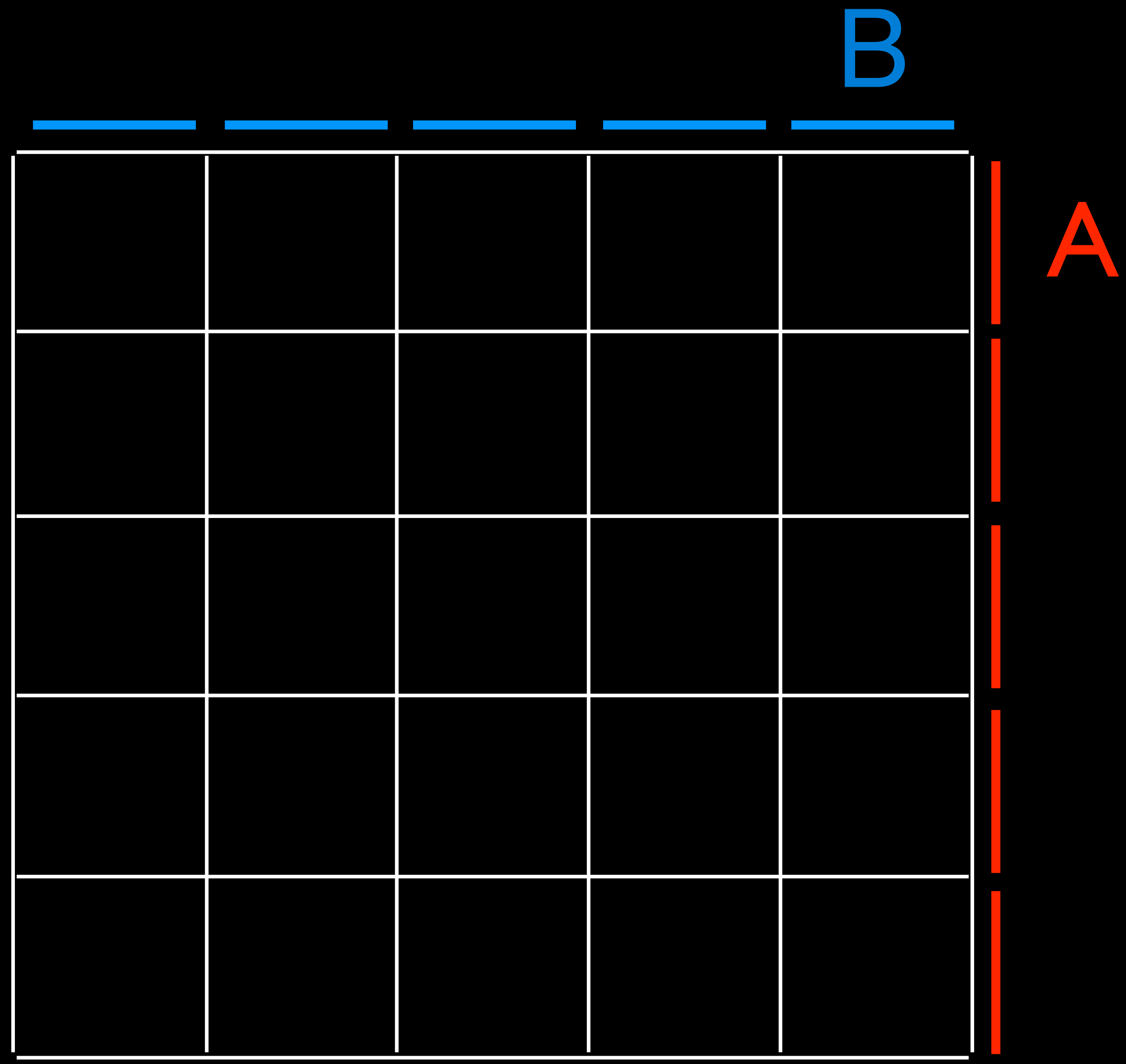
Prop. For $w = B^n A^n$
$u = A'^n$, $v = B'^n$

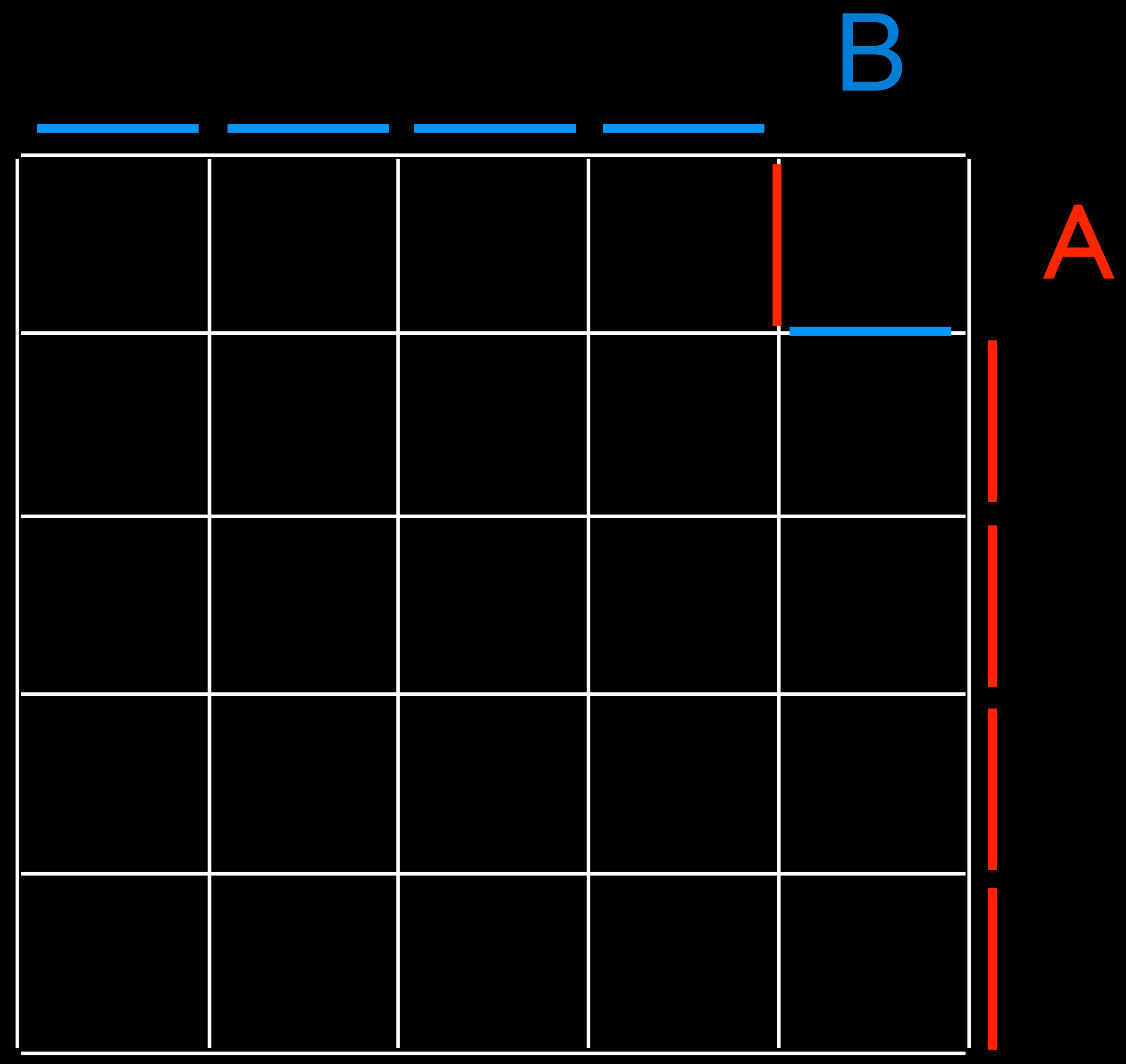
$c(u,v;w)$ = the number of $n \times n$ ASM (alternating sign matrices) “planar” proof:

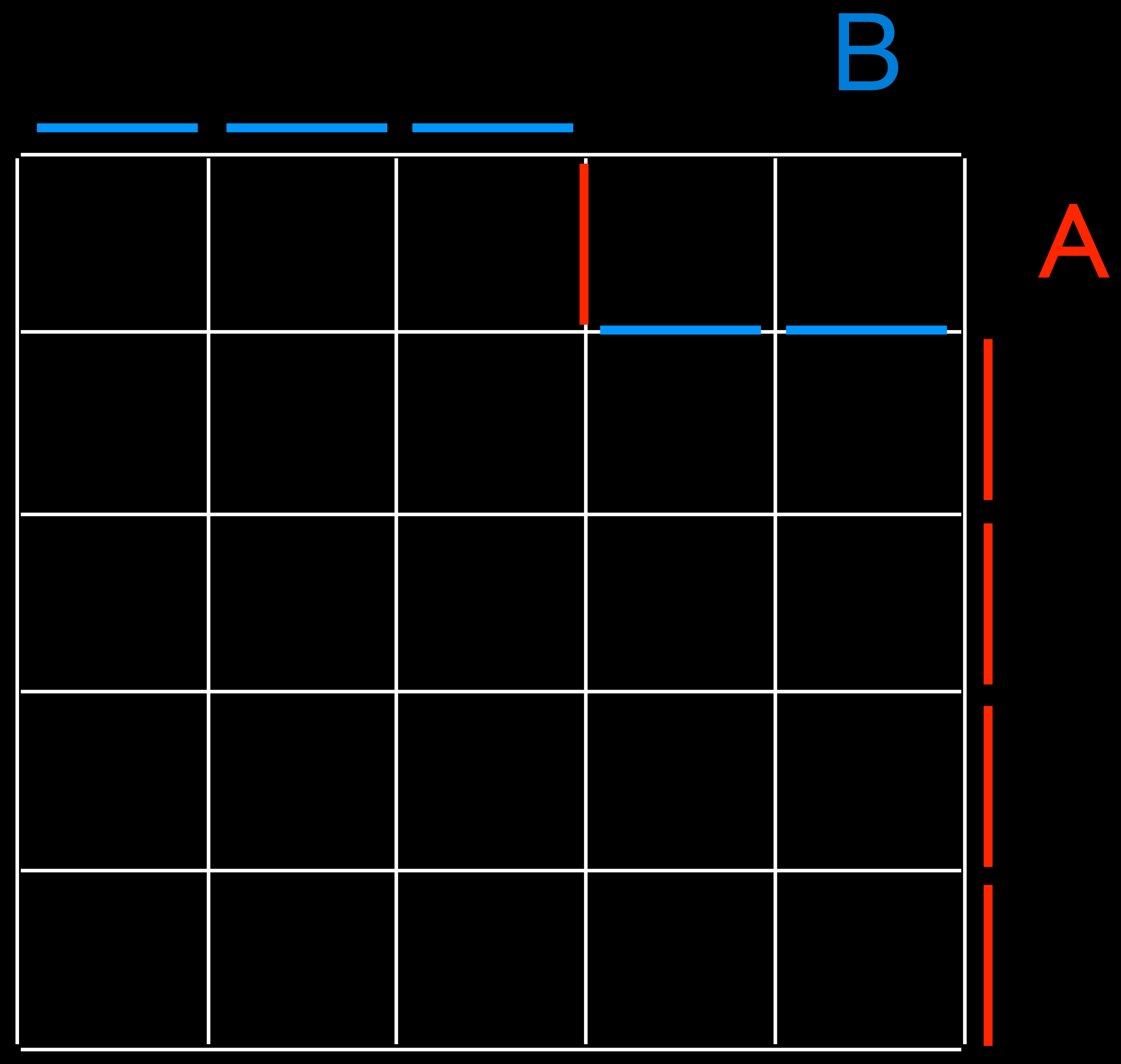


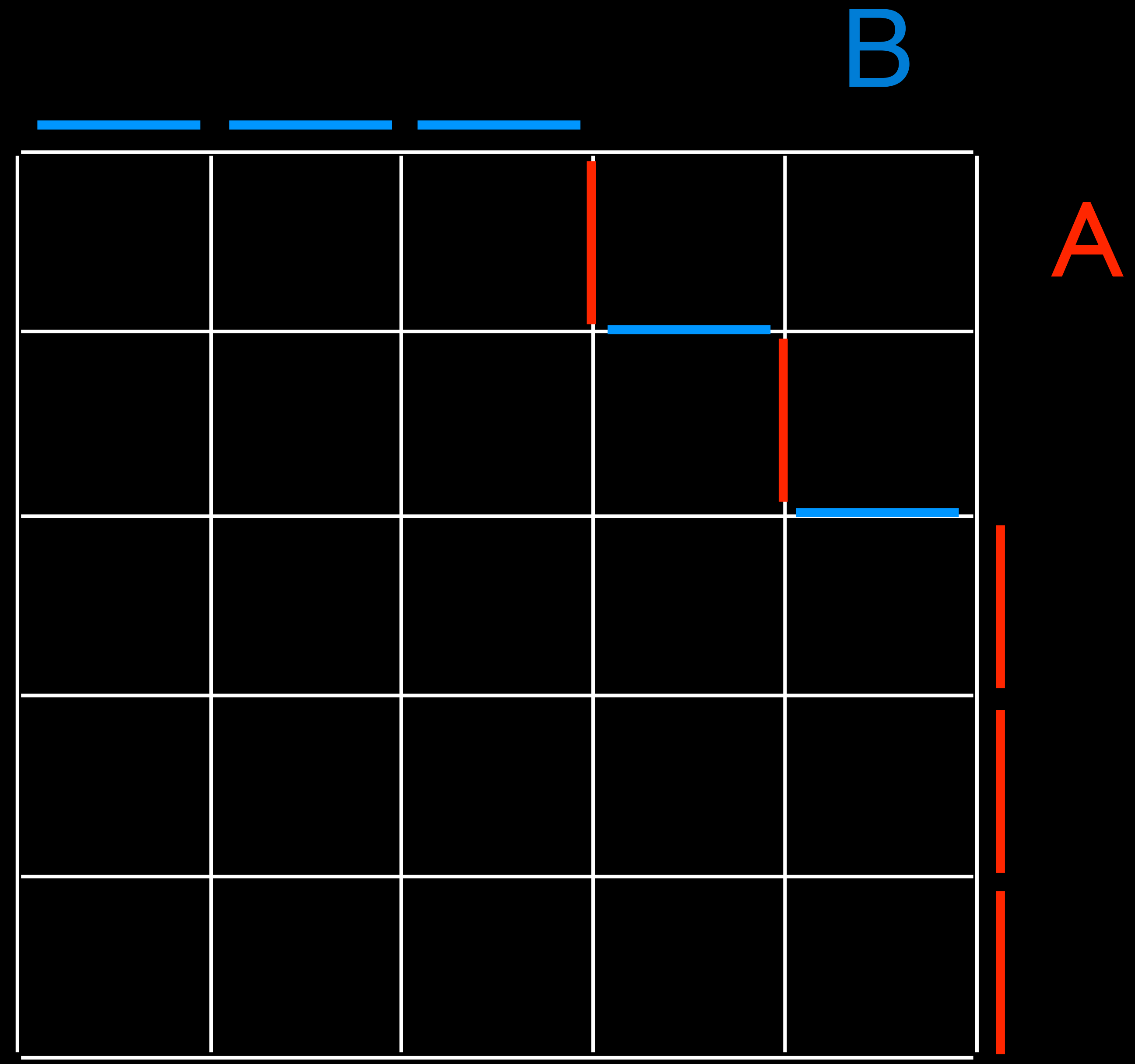
“planarisation” of the “rewriting rules”

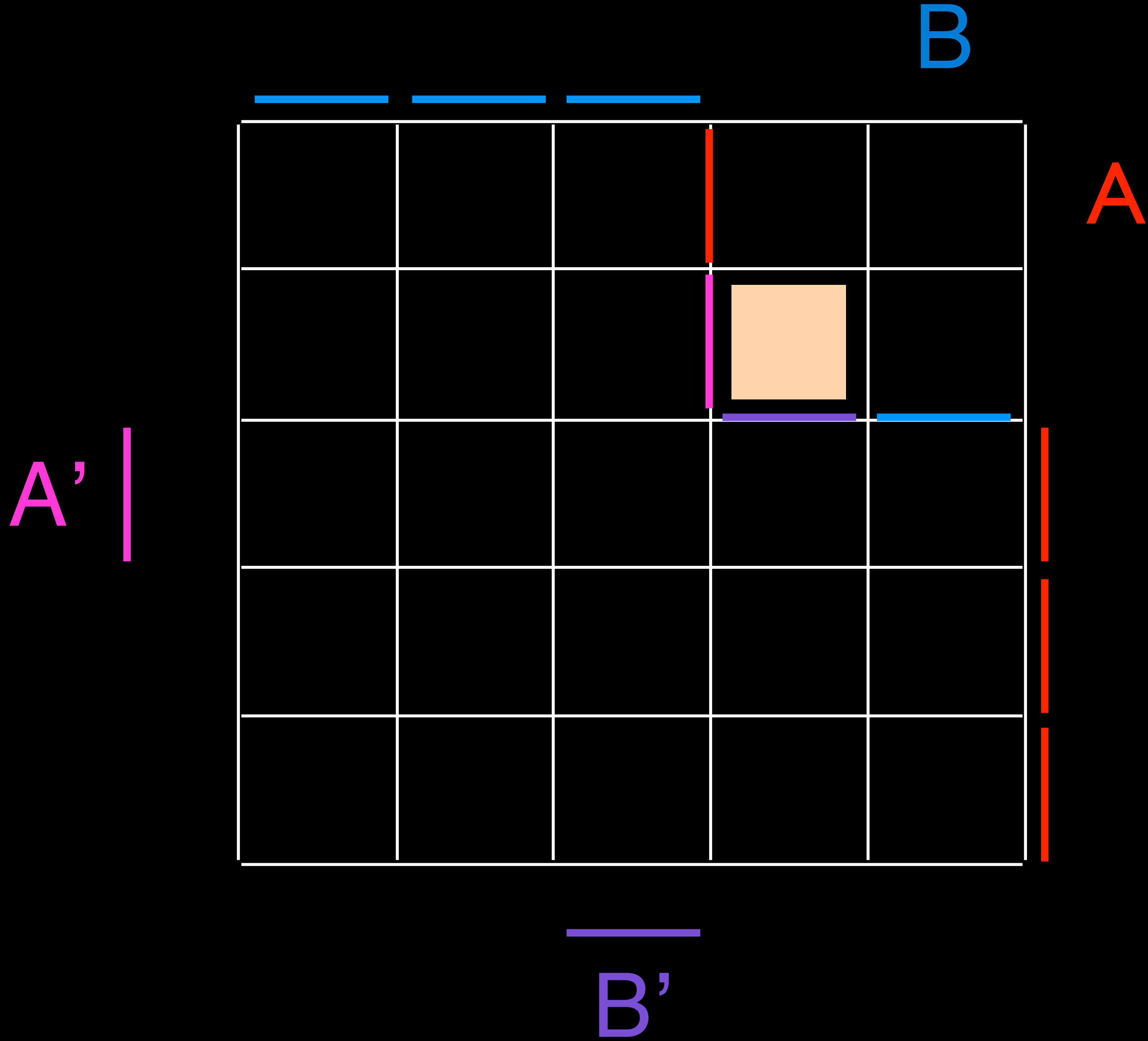


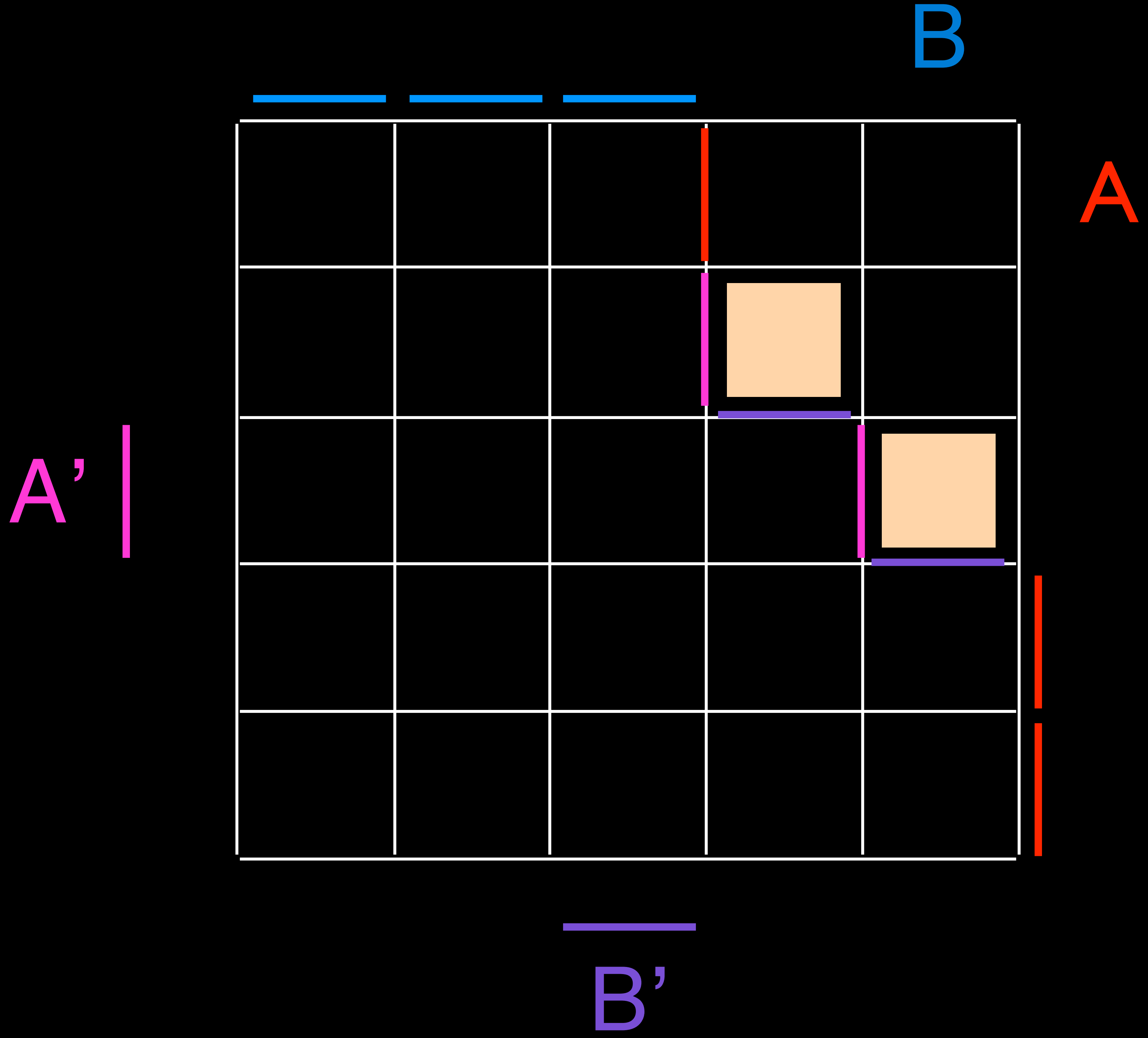

B
A

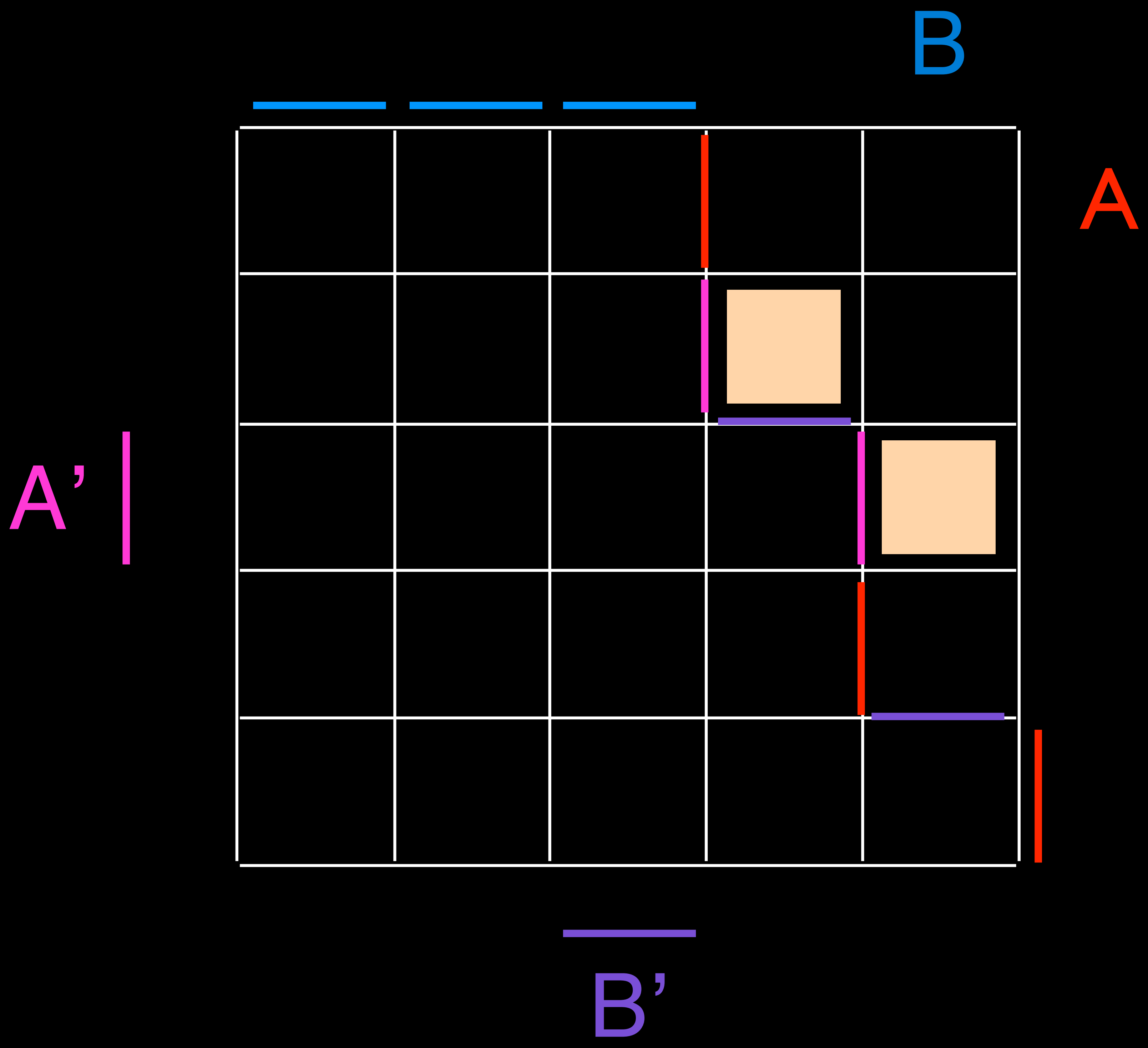

B
A

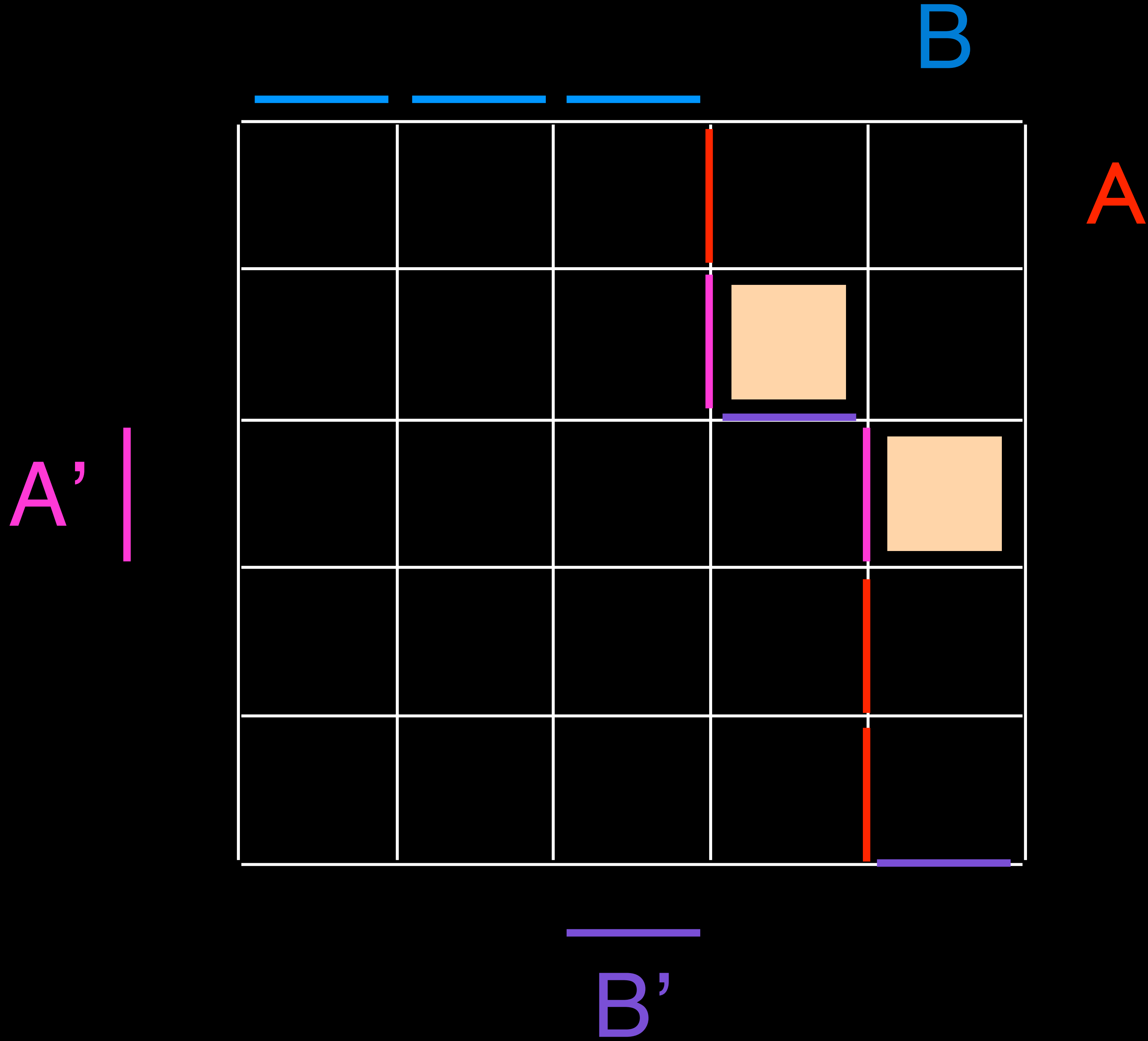

B
A

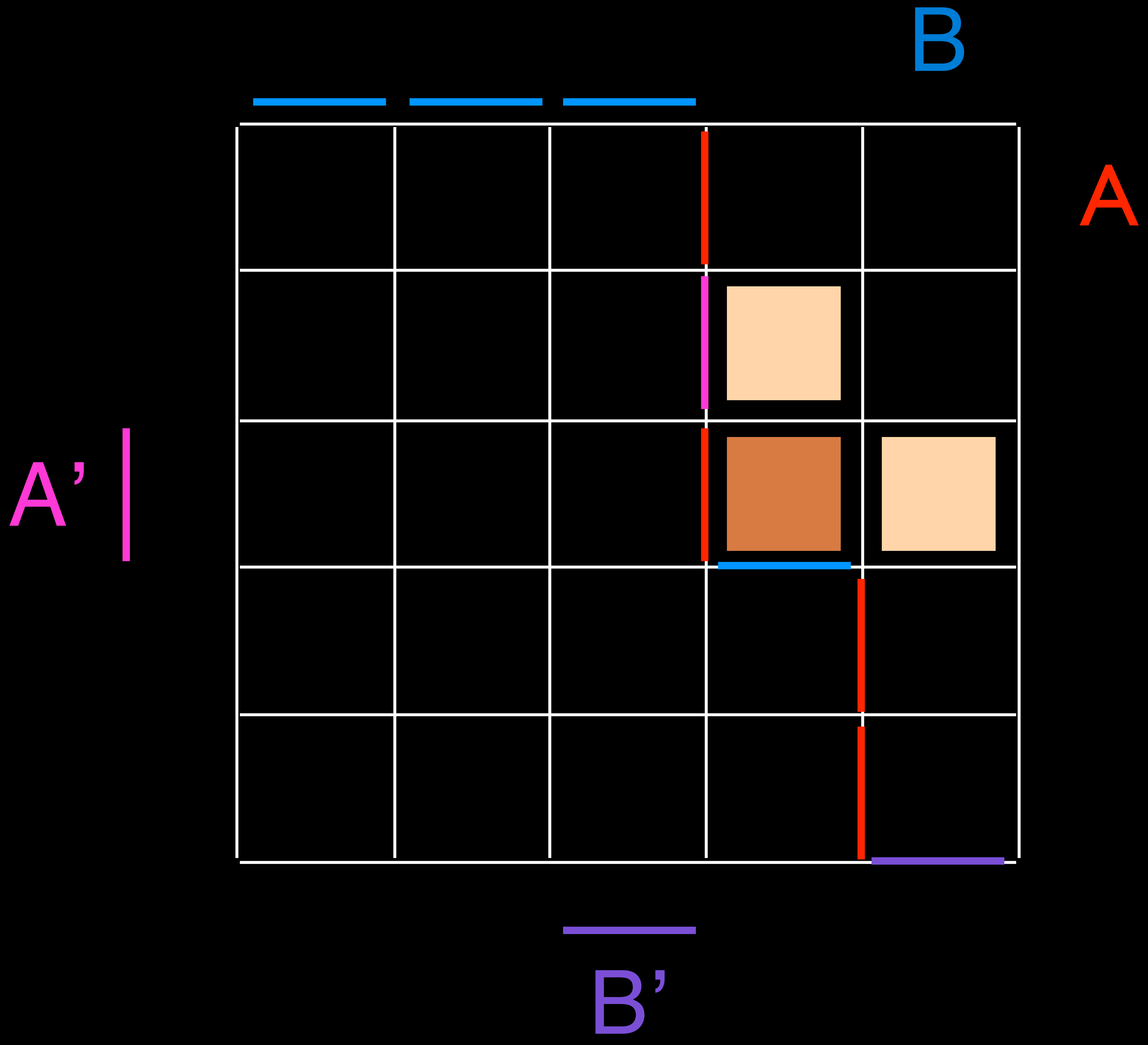

B
A

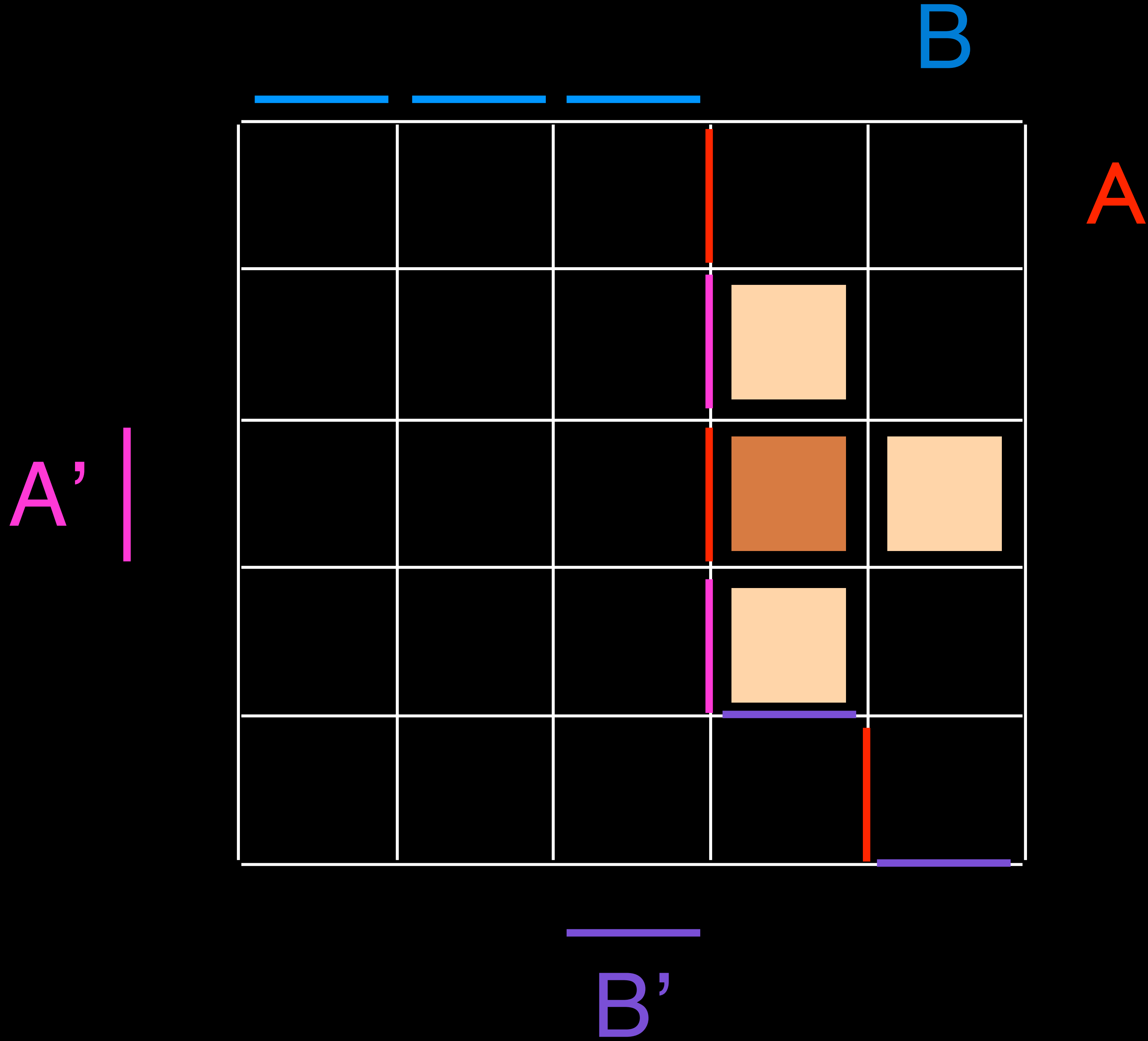

B
A
A'
B'

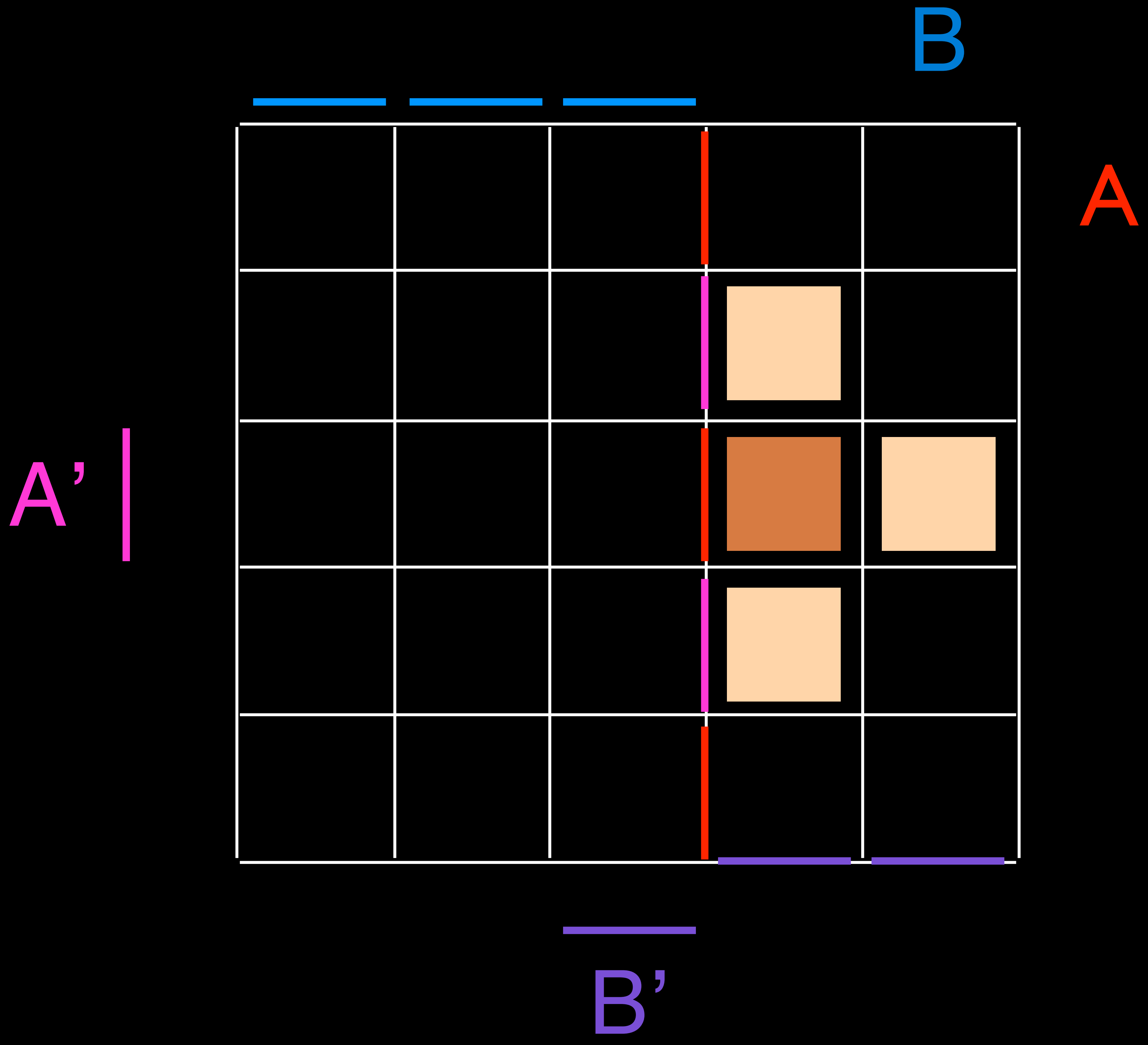

B
A
A’
B’


B
A
A’
B’


B
A
A'
B'


B
A
A'
B'


B
A
A'
B'


B
A
A'
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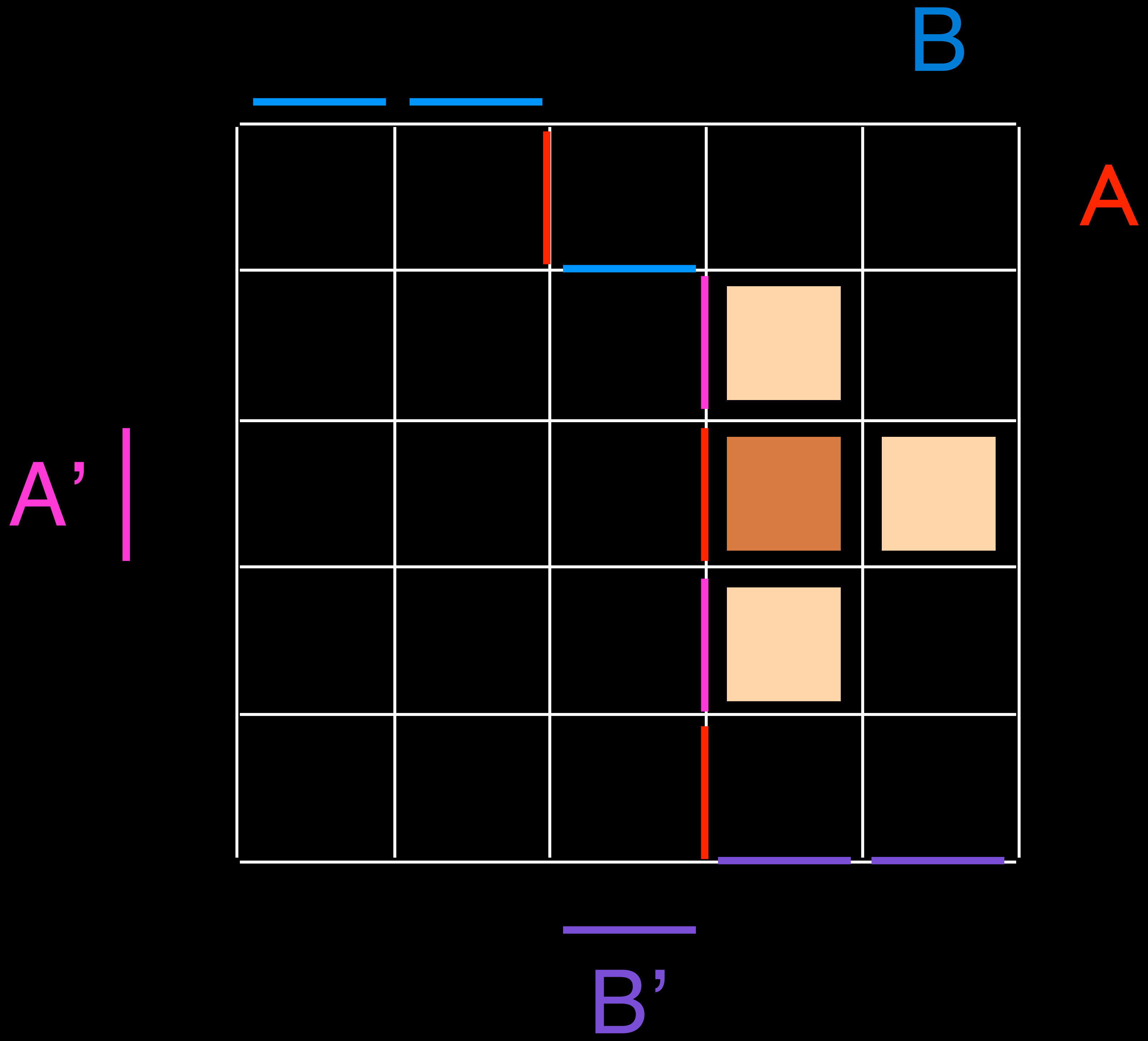
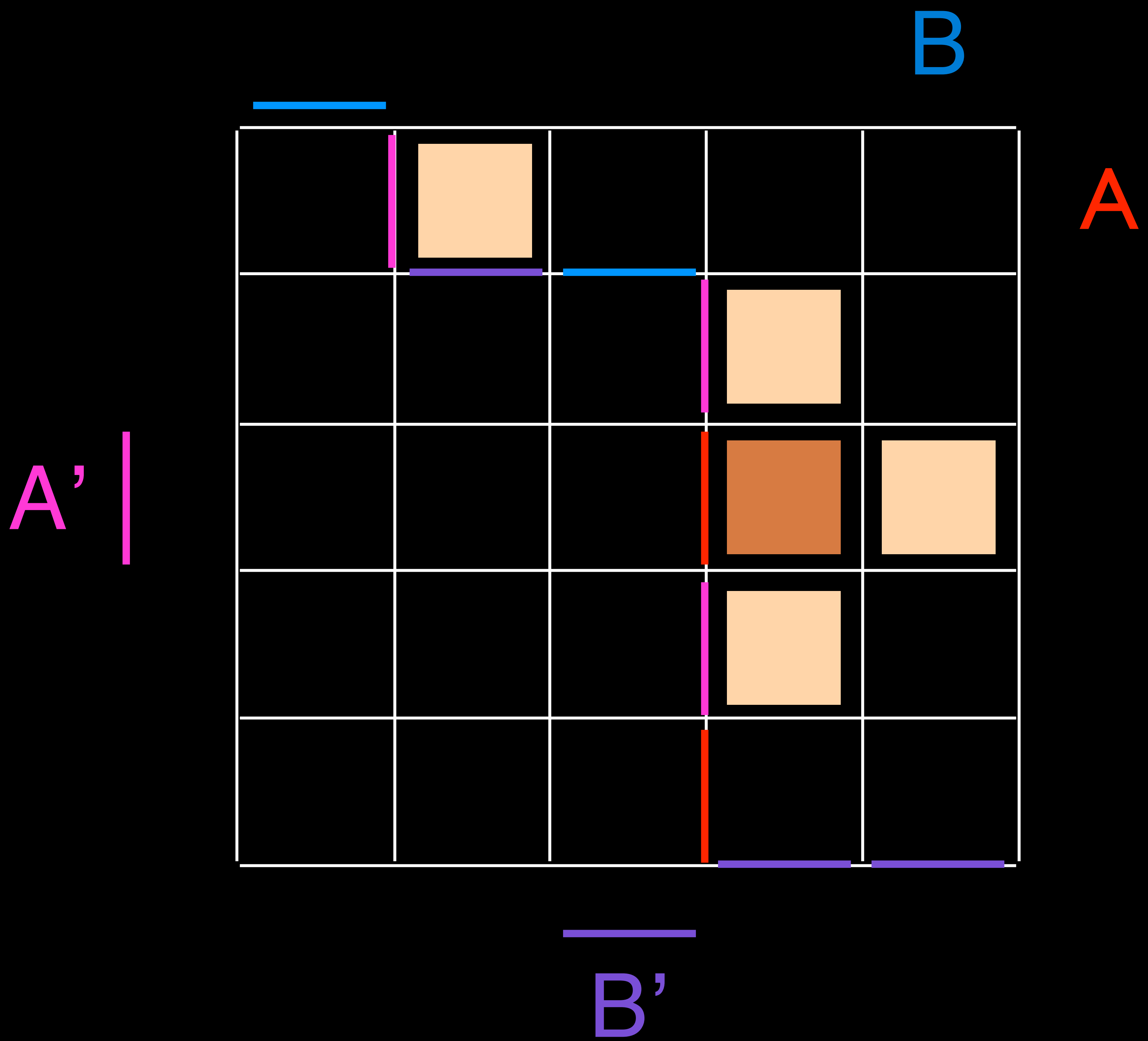
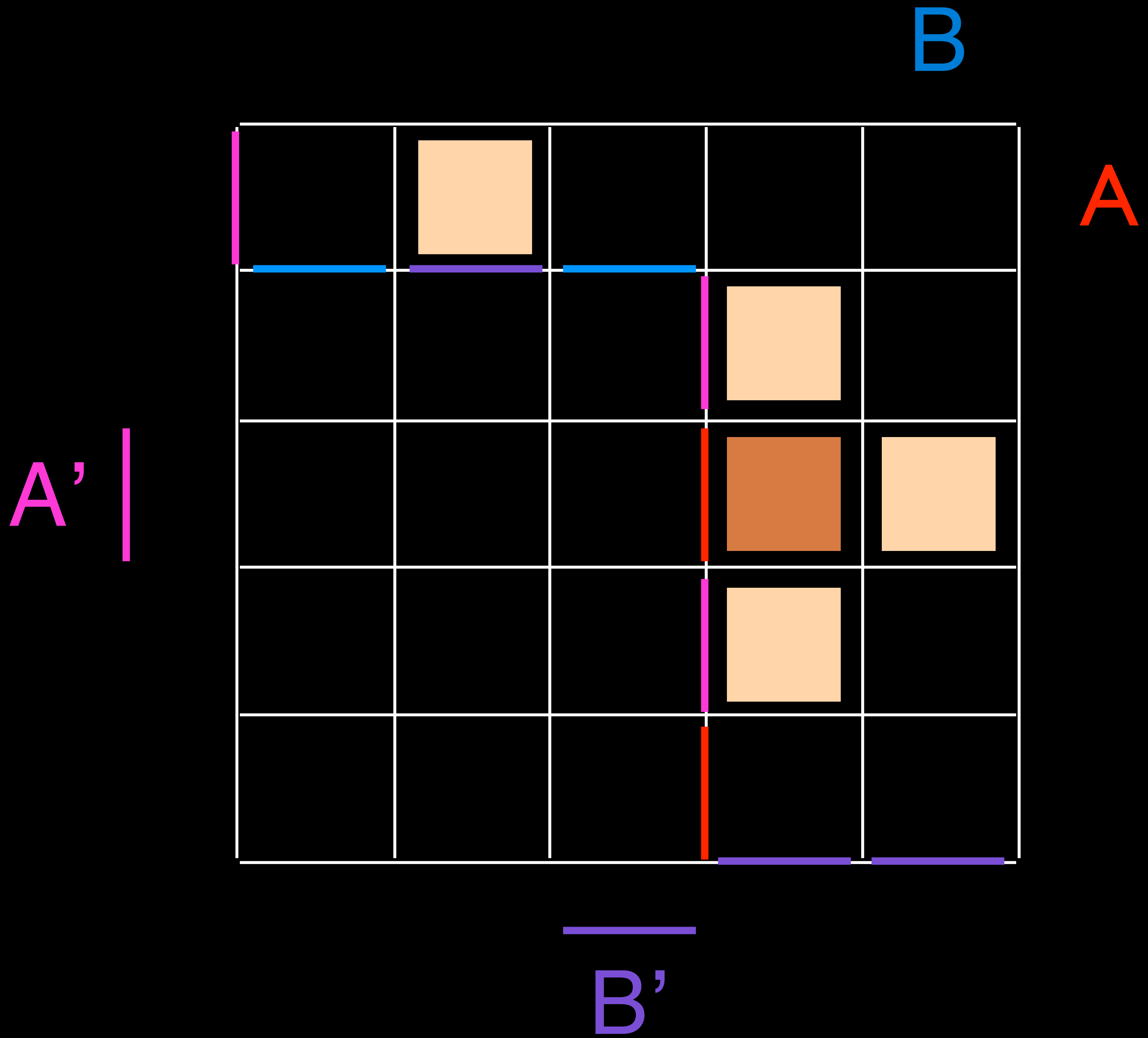

B
A
A'
B'


B
A
A'
B'


B
A
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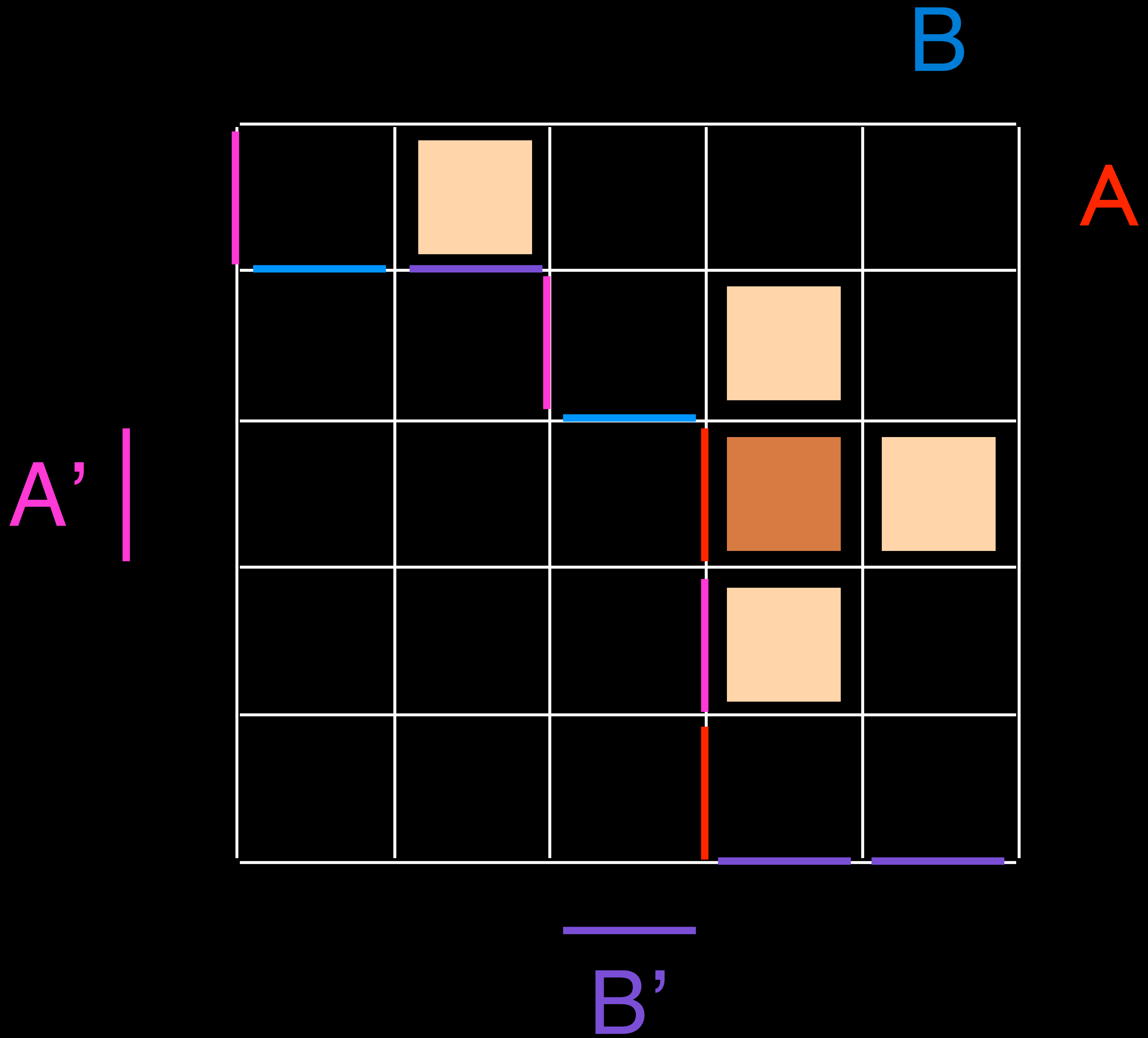
B
A
A'
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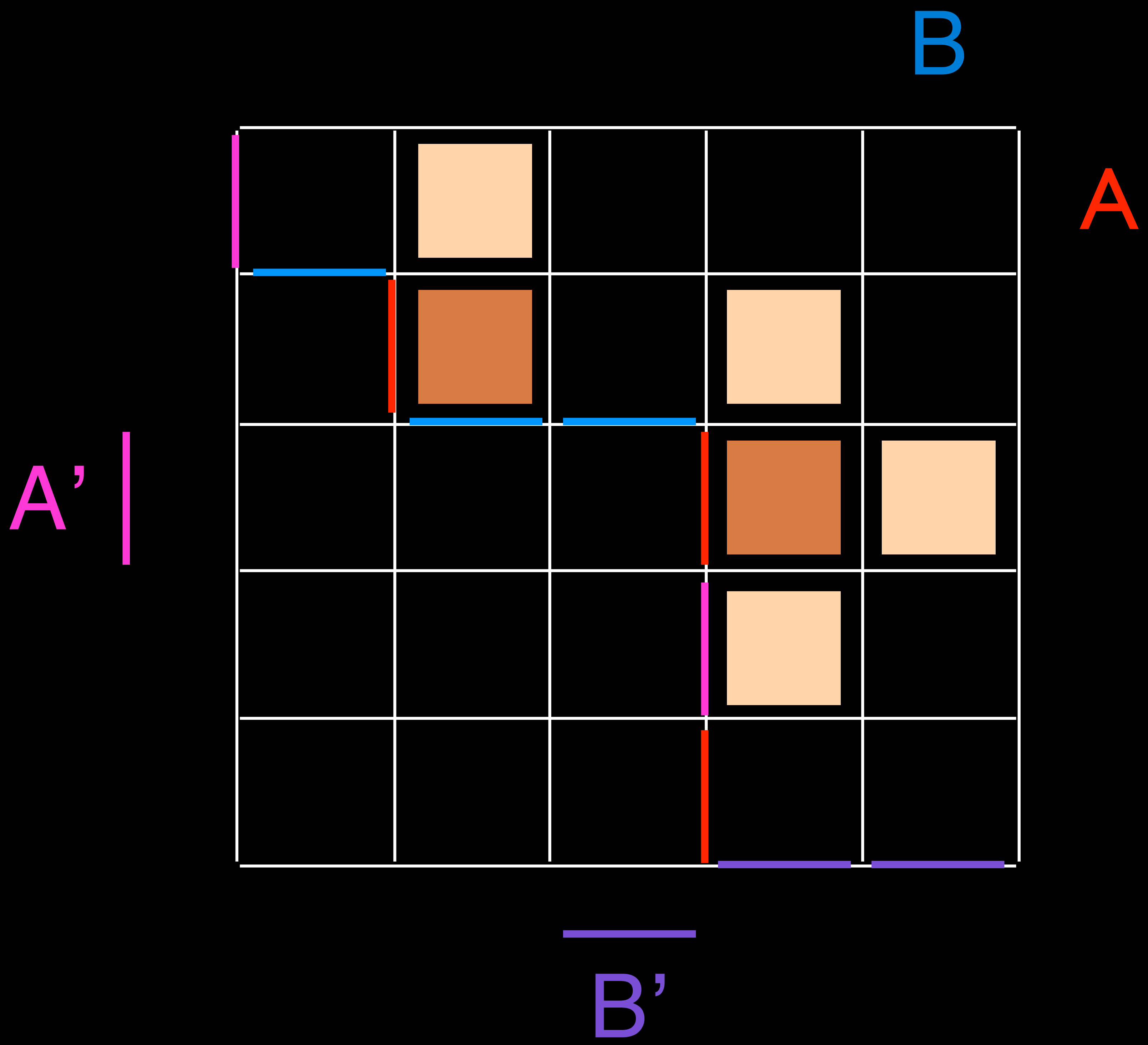
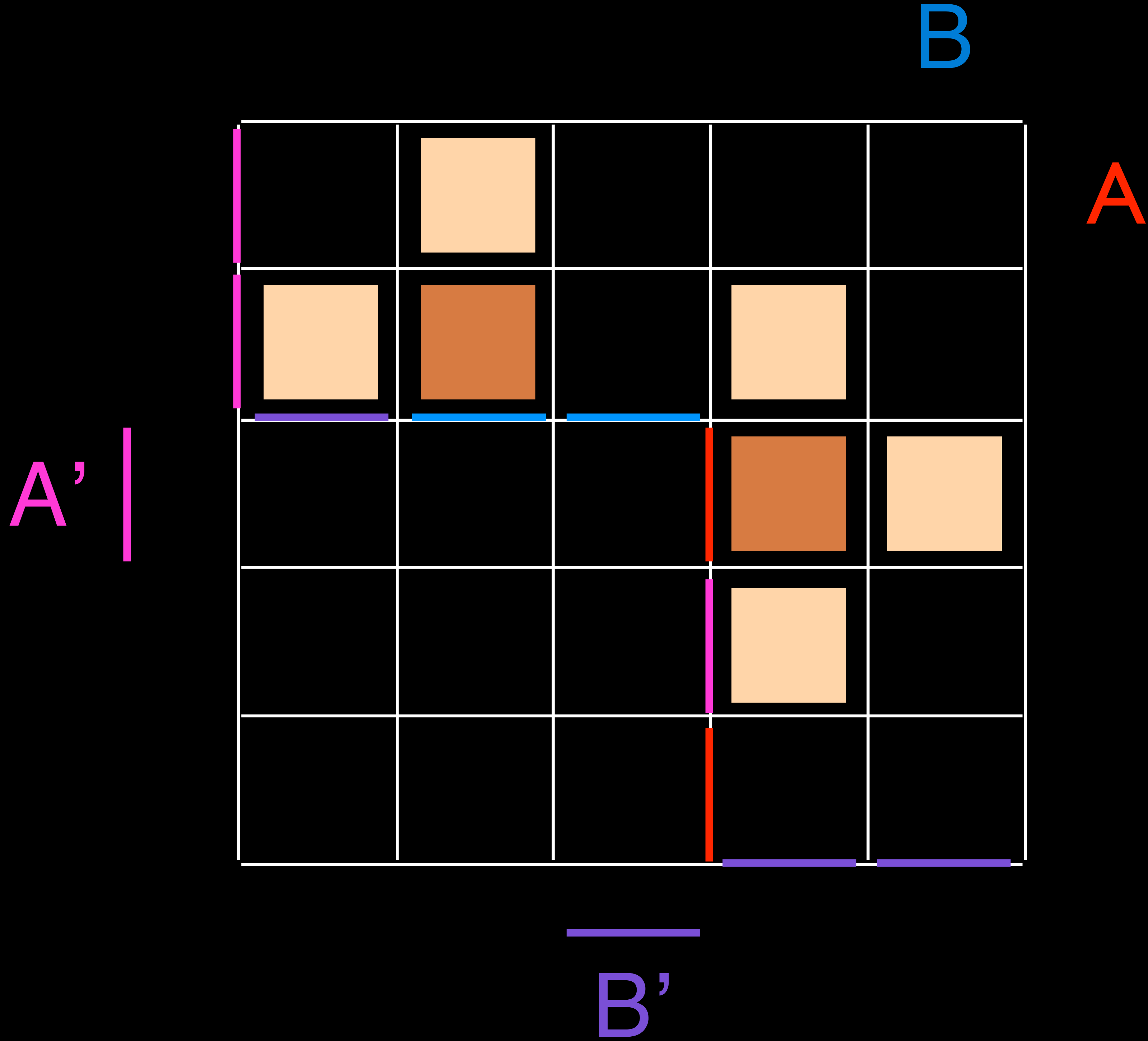


B
A
A'
B'


B
A
A'
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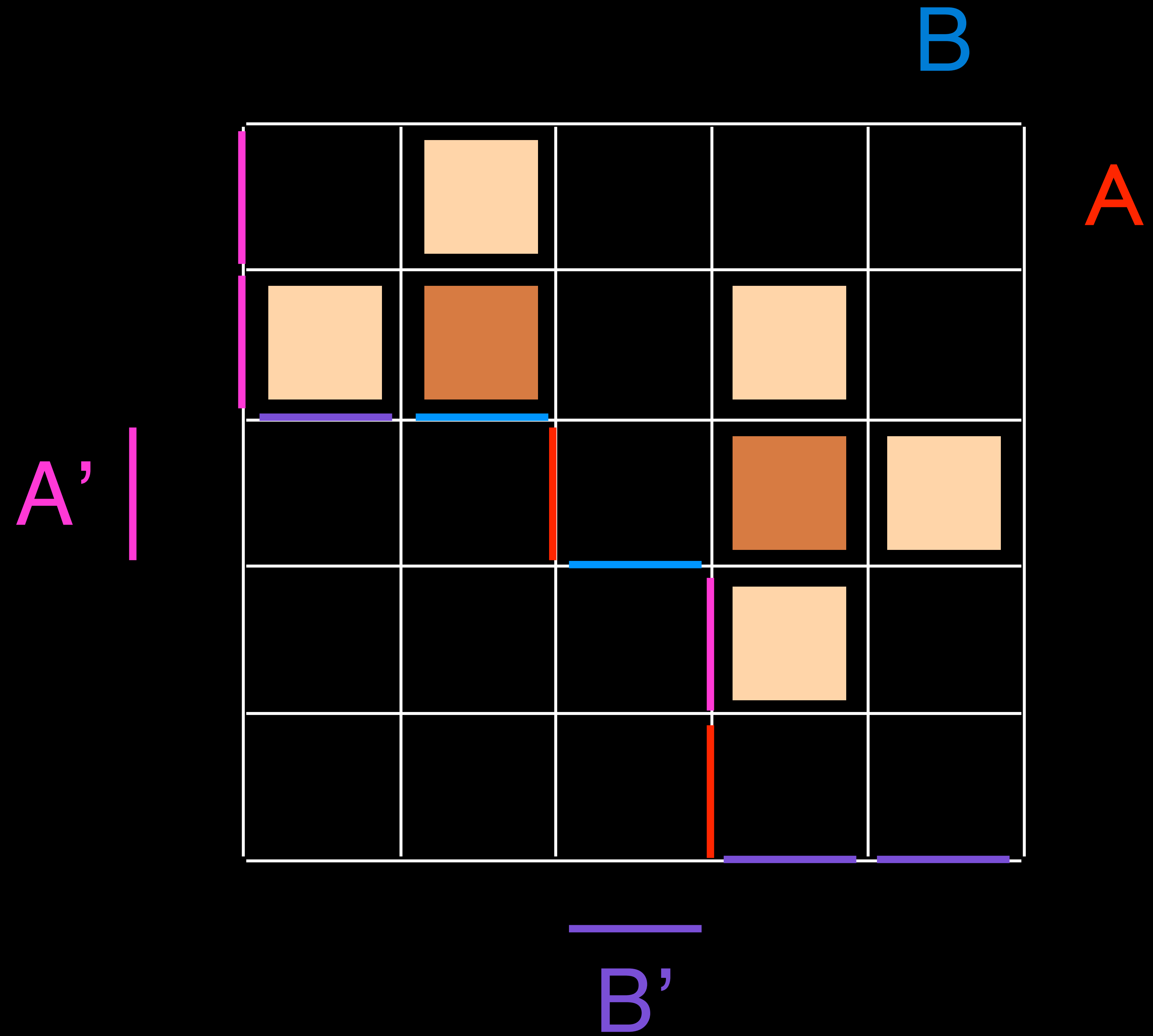
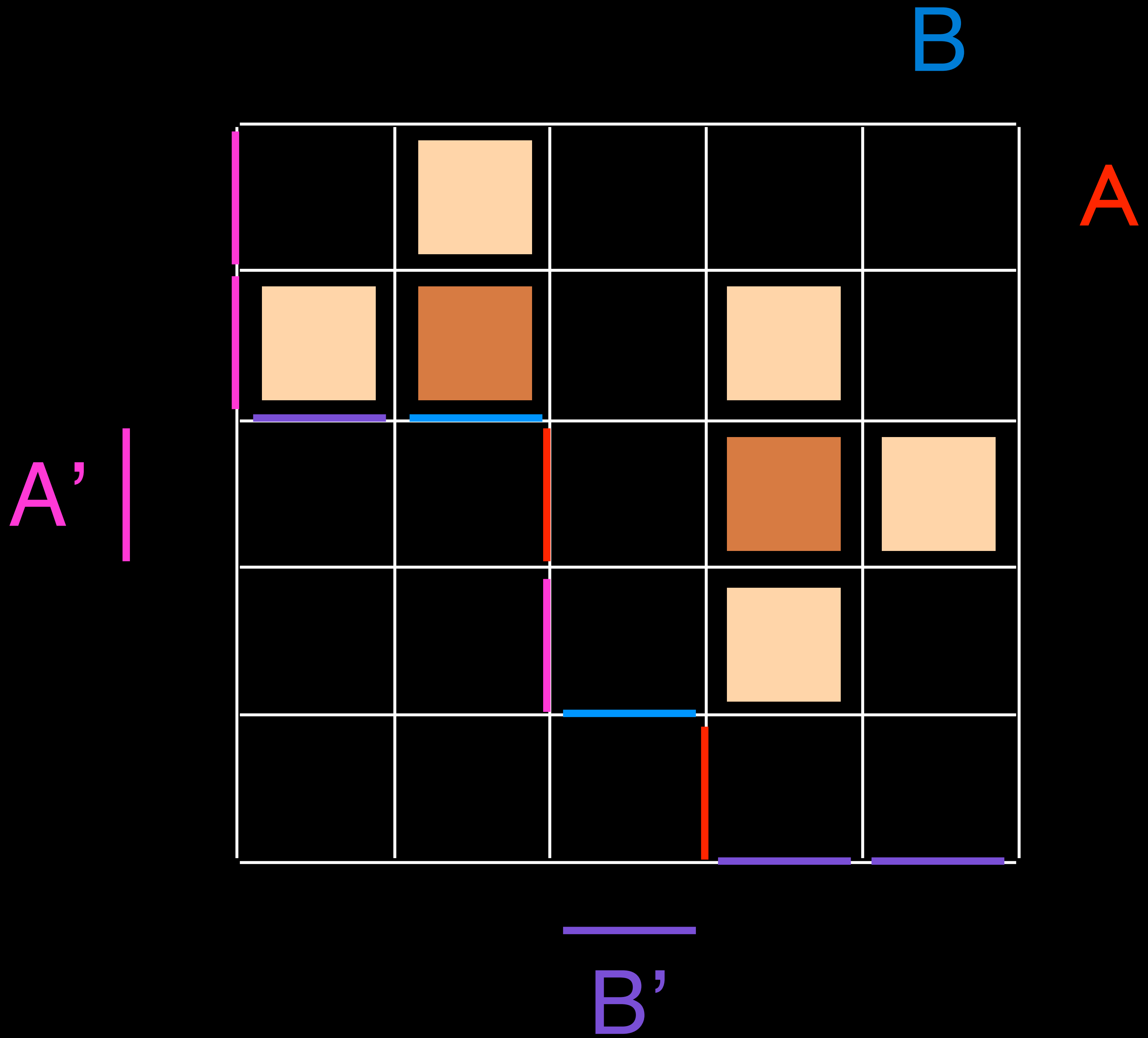
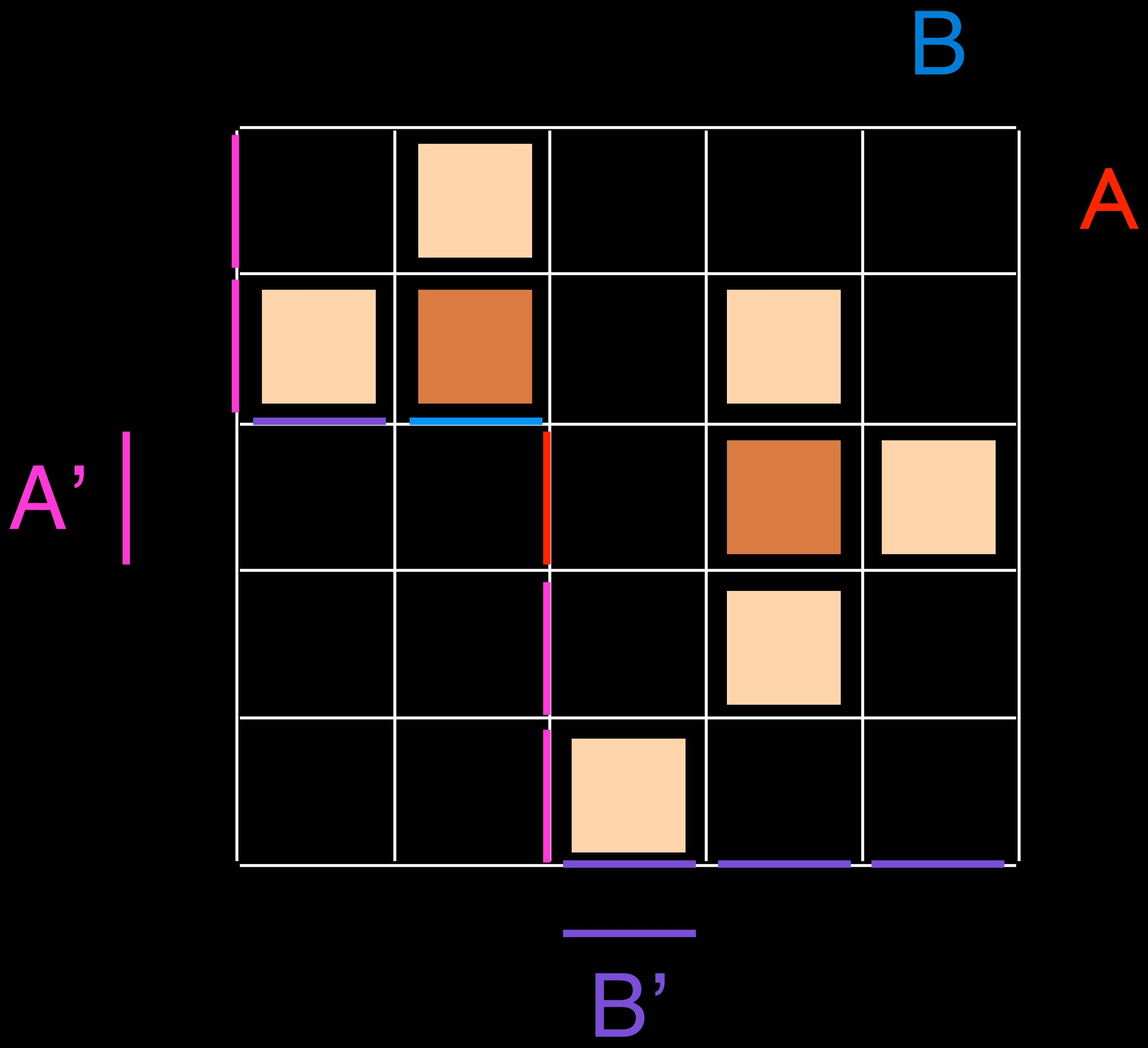
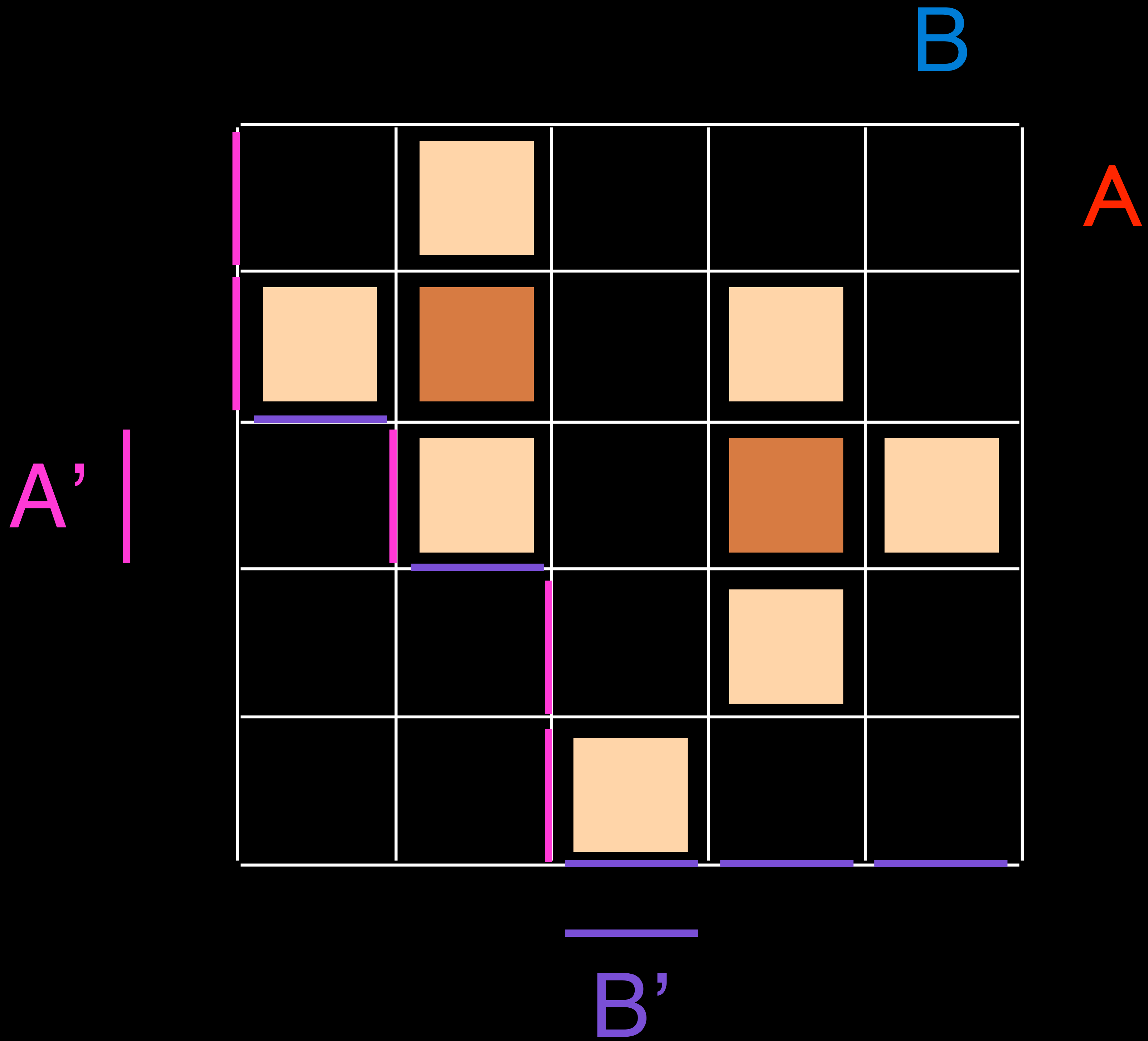

B
A
A’
B’


B
A
A’
B’


B
A
A'
B'


B
A
A’
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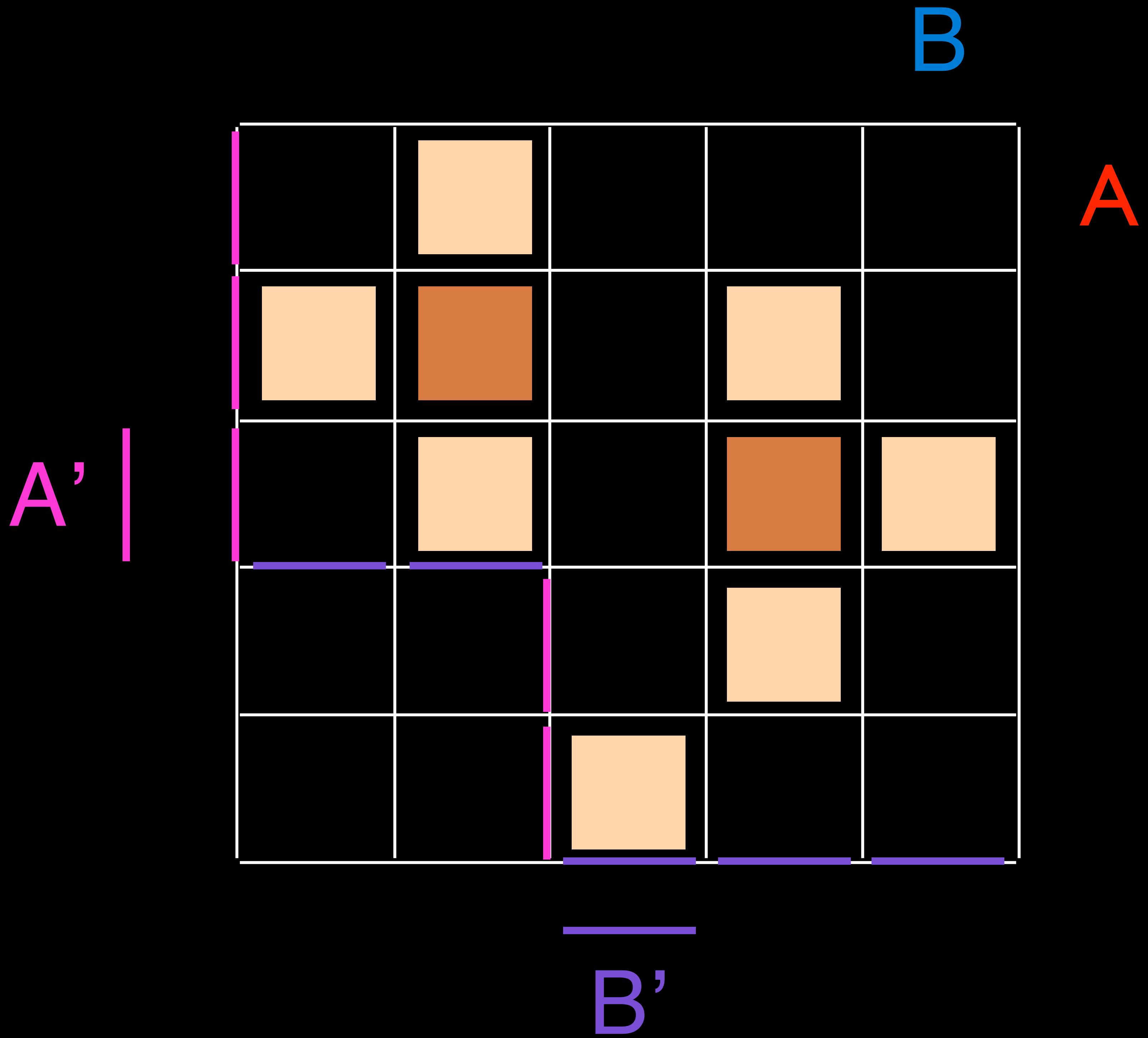
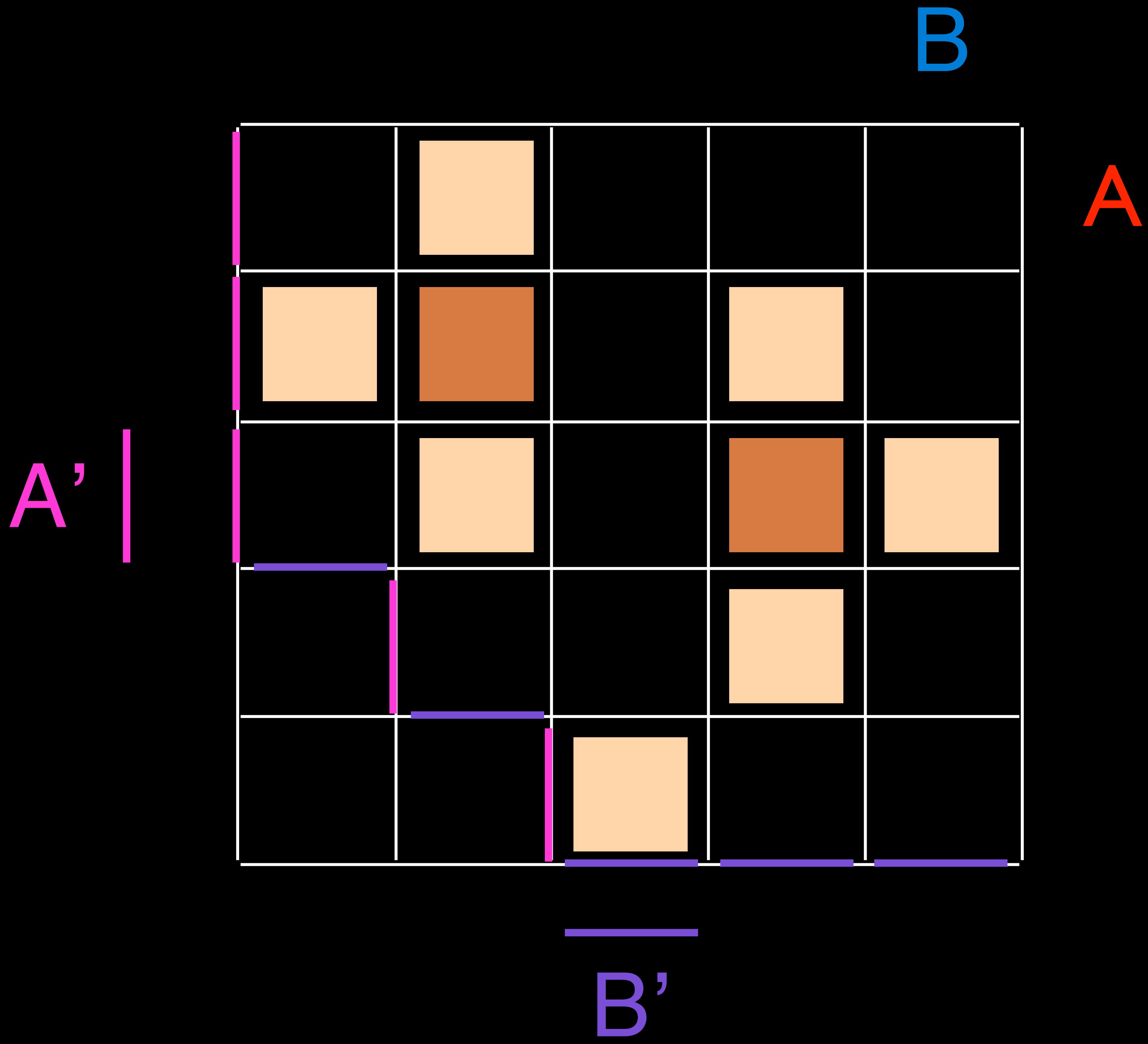
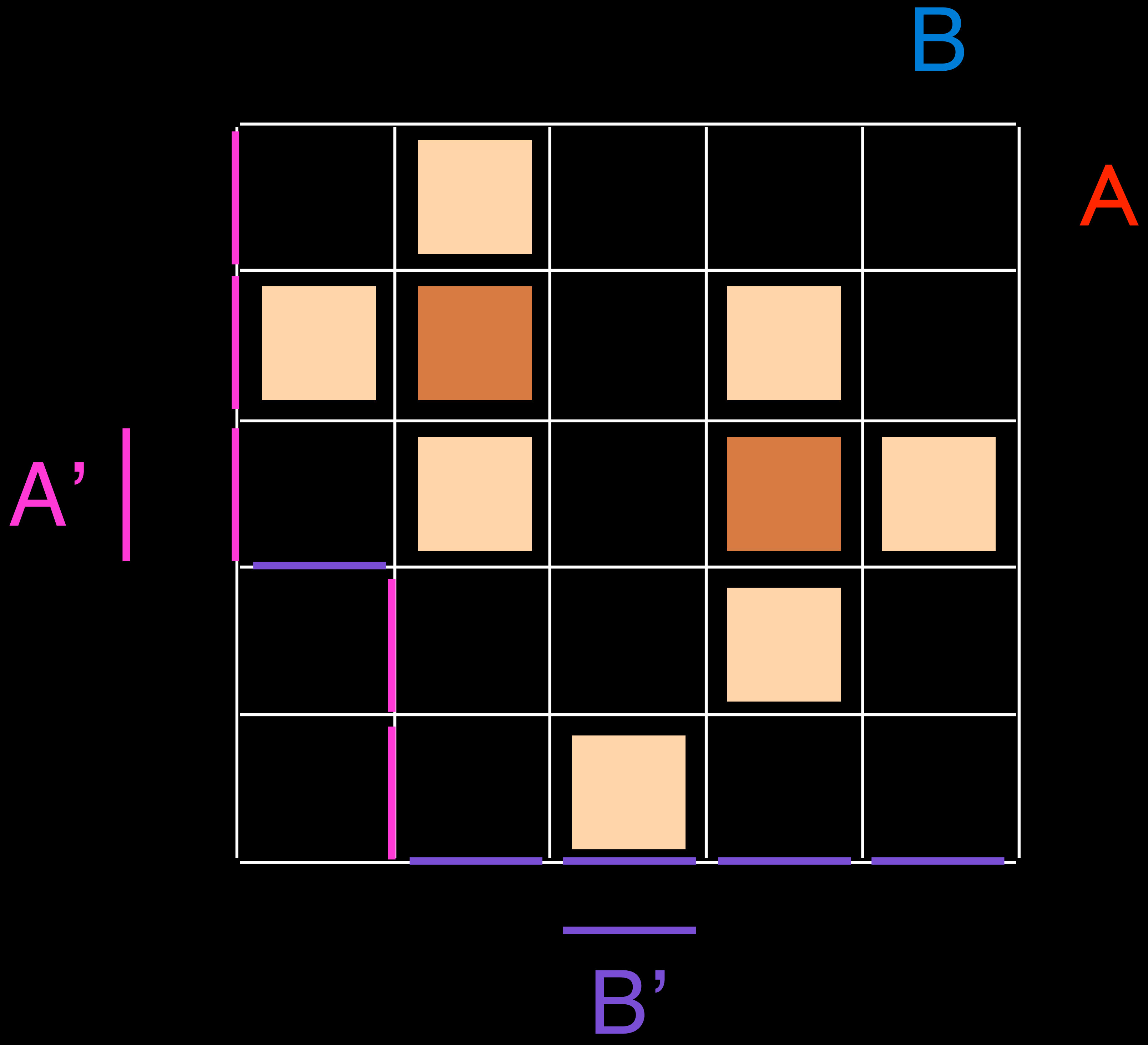

B
A
A'
B'



B
A
A'
B'


B
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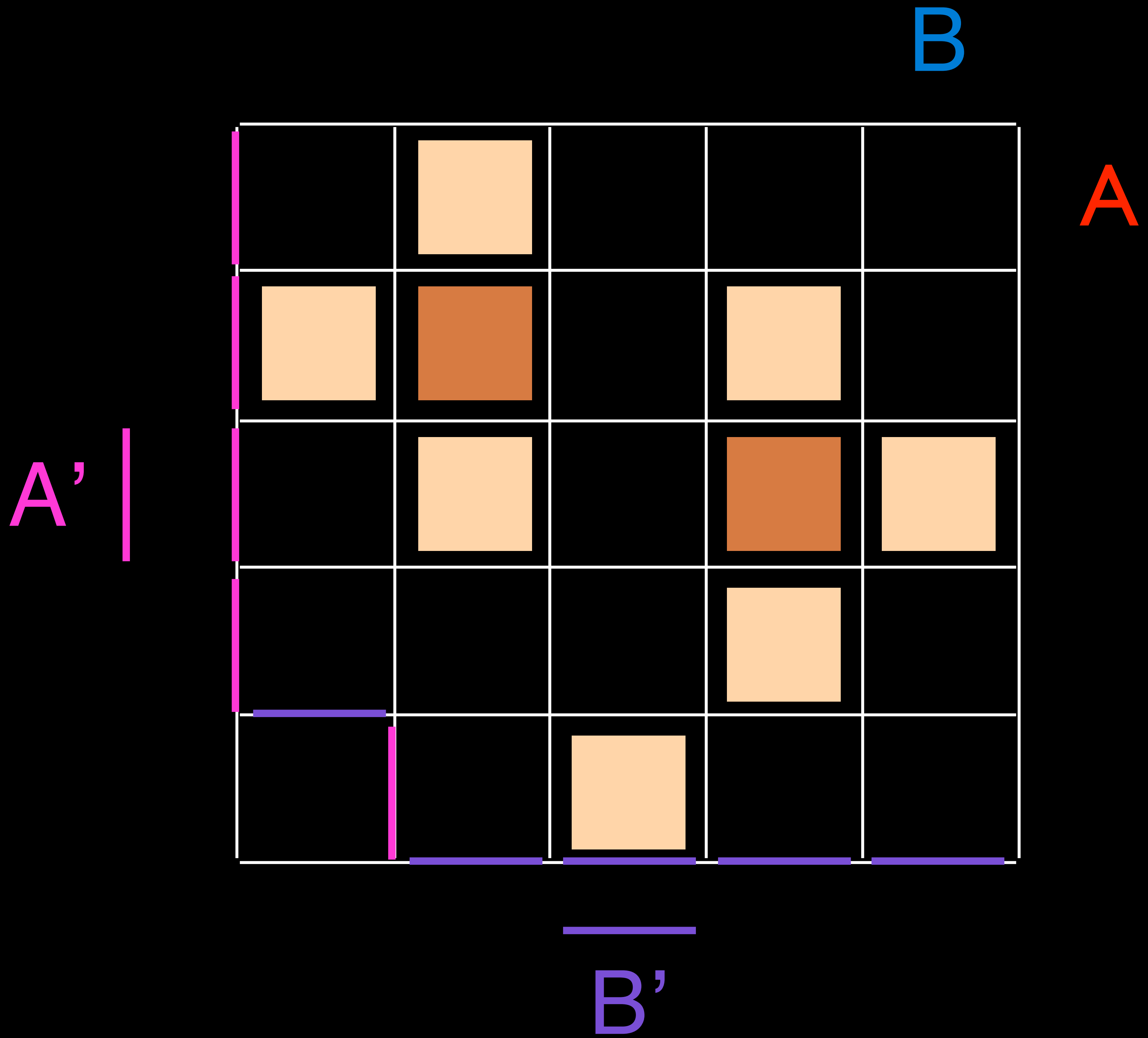
B
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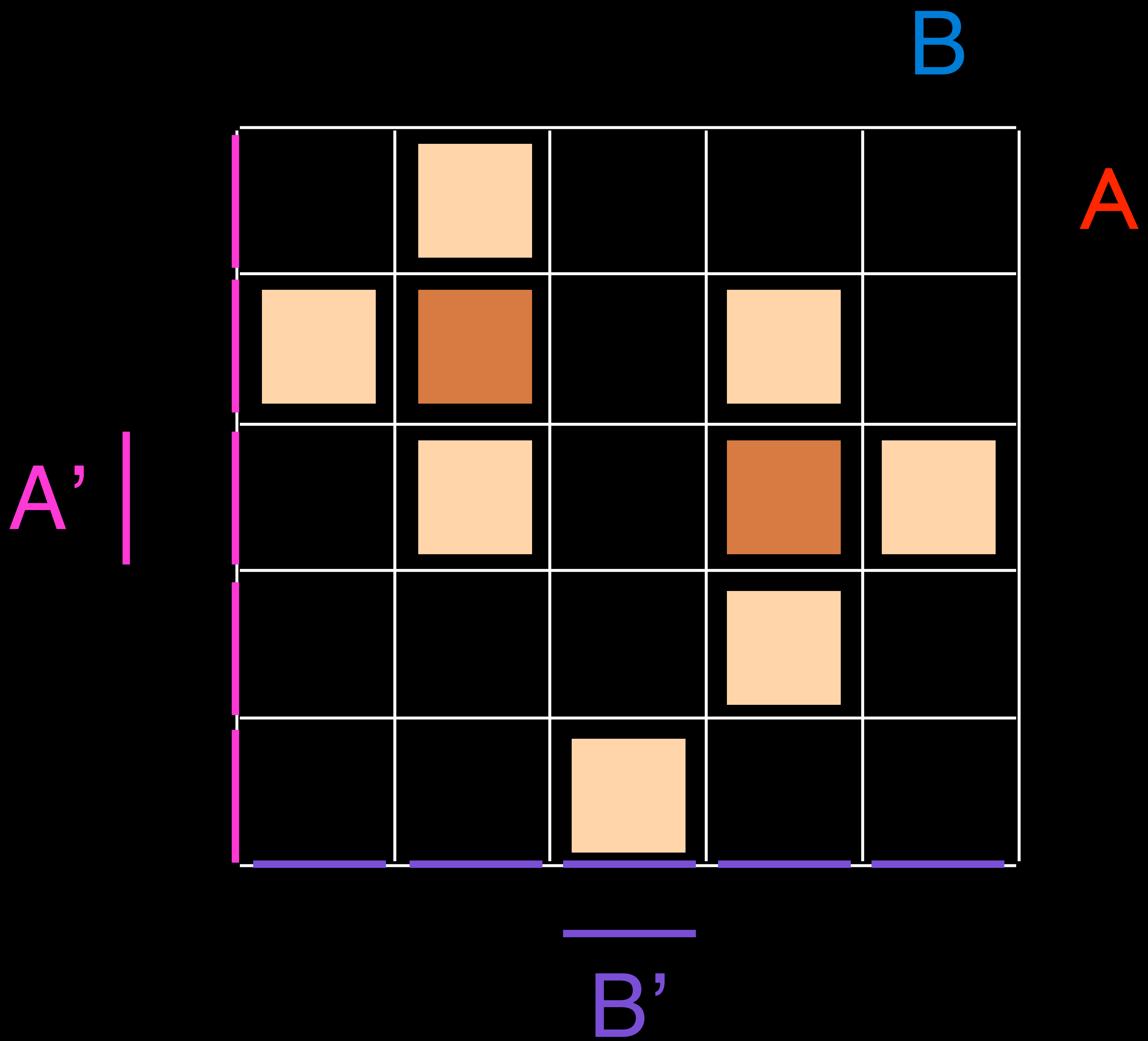
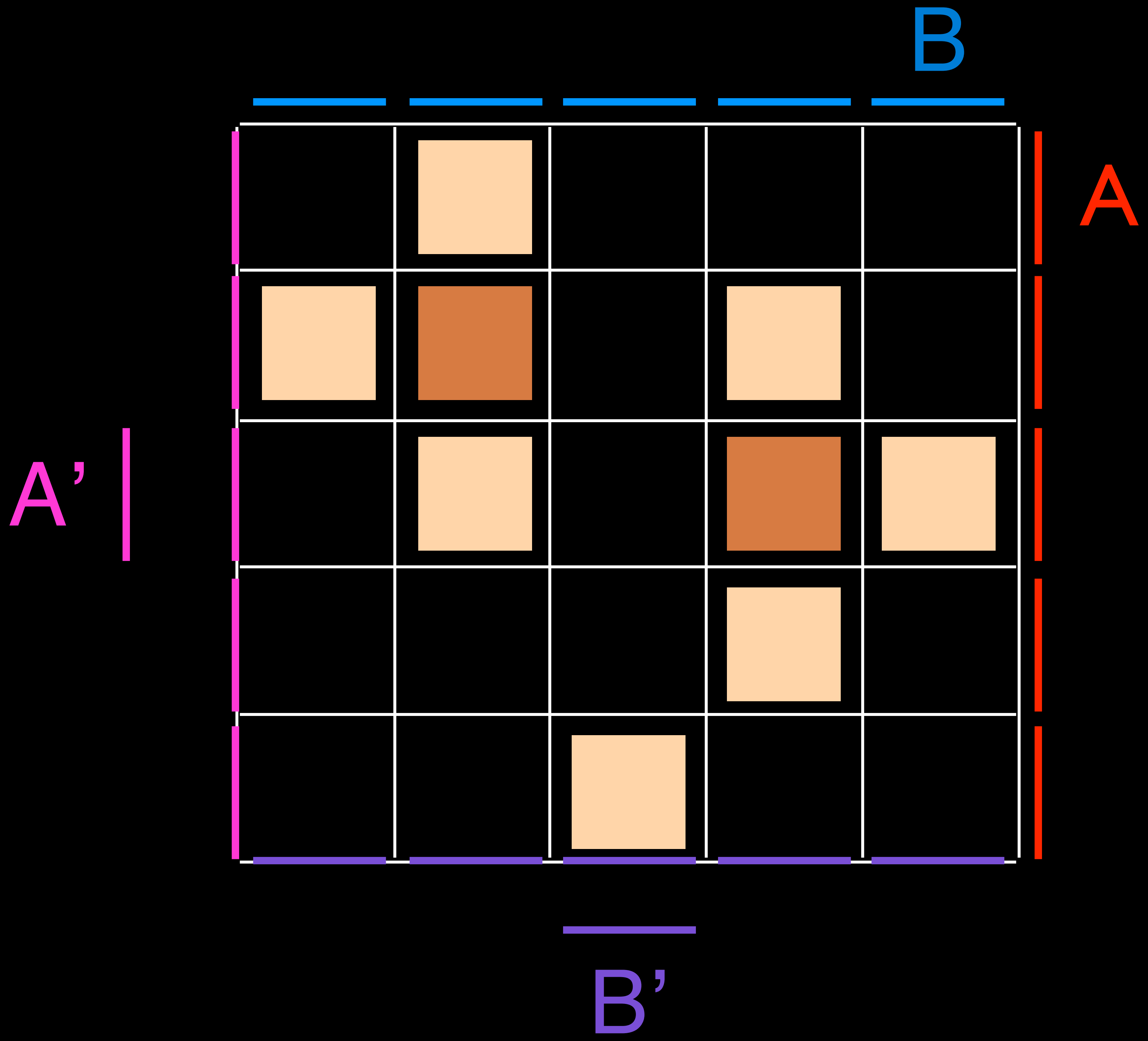


B
A
A’
B’


B
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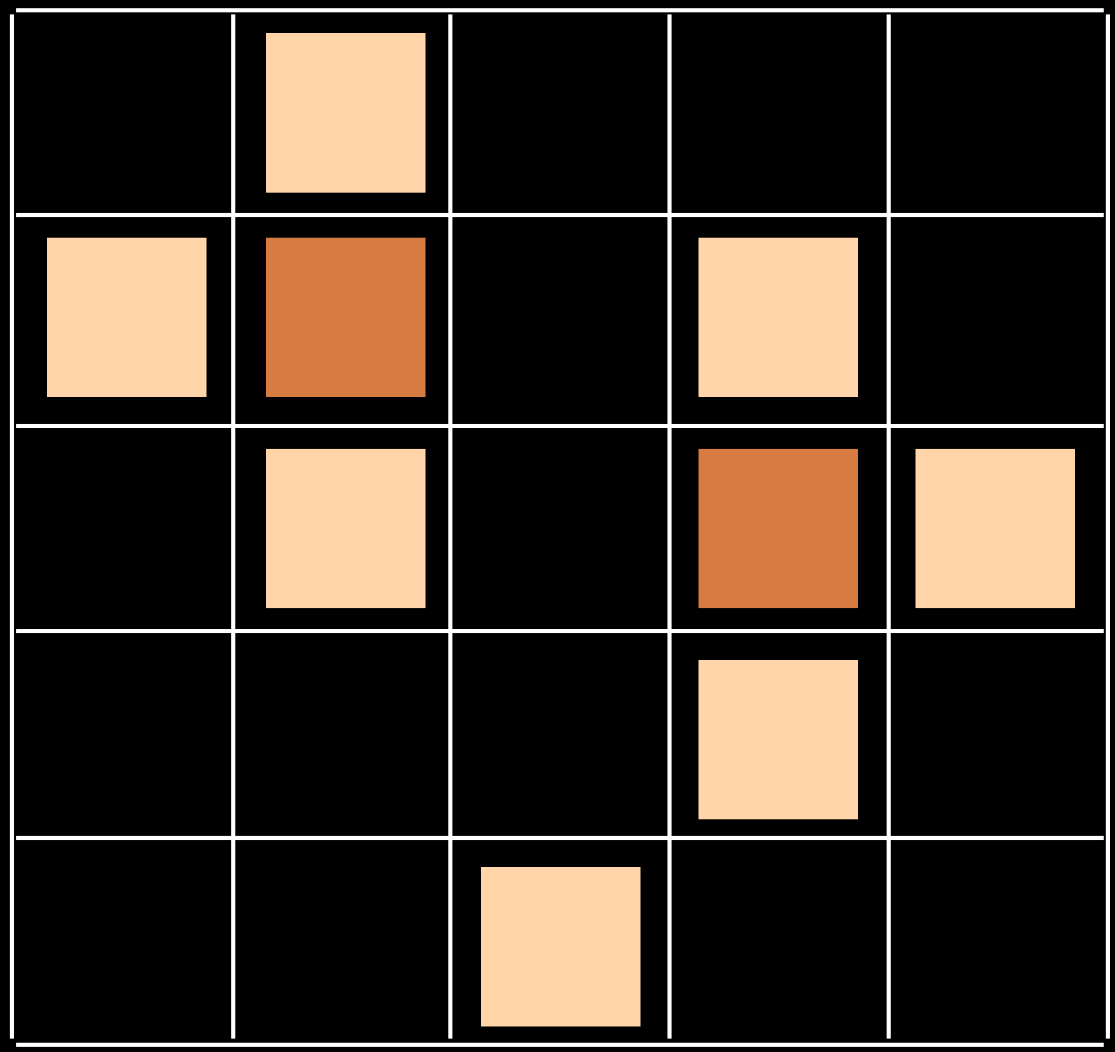


Questions.

- find a "combinatorial representation" for operators A, A', B, B'.

- analogue of RSK (Robinson-Schensted-Knuth) for ASM ?

- analogue of "local rules" (Fomin)

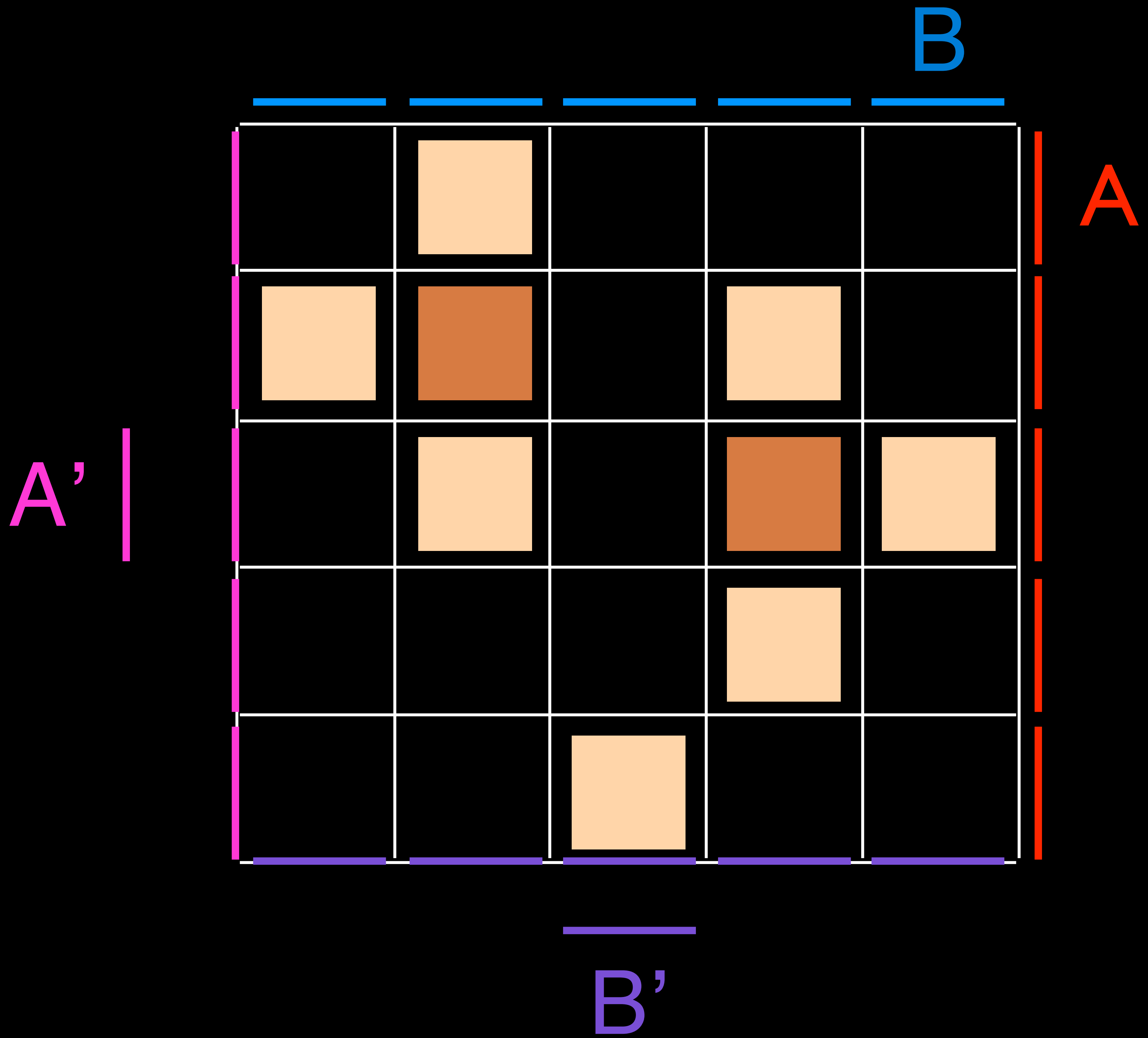
- direct proof of the formula

$$A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!}$$

(nb of ASM of size n)

$= 1, 2, 7, 42, 429, \ldots$

?



B
A
A'
B'

Q-tableaux

Quadratic algebra $\mathbb{Q}$

generators $\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

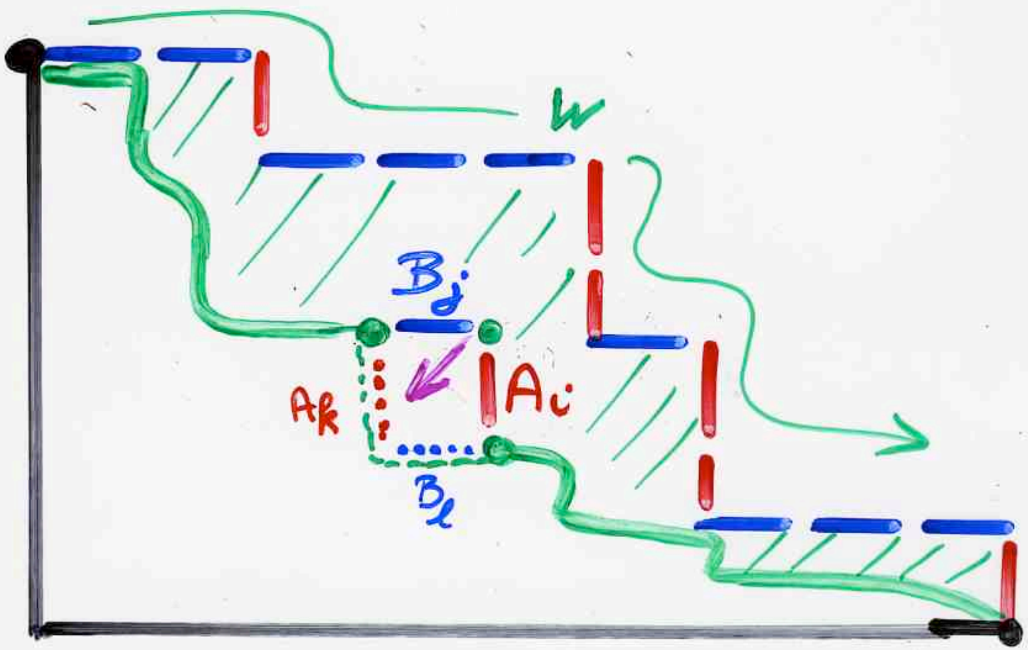
commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \qquad \begin{matrix} i \in I \\ j \in J \end{matrix}$$

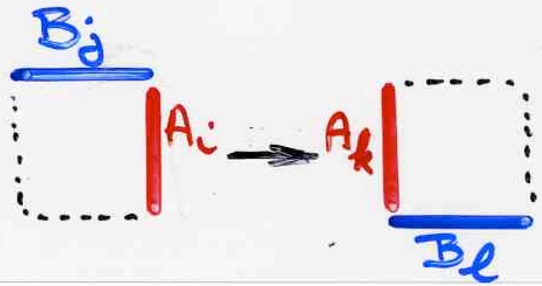
Lemma. In $\mathbb{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

Proof:



"planar" rewriting



S set of labels

$$\varphi : \left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\} = R \longrightarrow S$$

set of rewriting rules

$$B_j A_i \rightarrow C_{ij}^{kl} A_k B_l$$

Def Q-tableau

"image" by φ of a "complete Q-tableau"

Planar automaton

Def. planar automaton $\mathcal{P}$

- 3 finite sets
  - $\mathcal{B}$ horizontal alphabet
  - $\mathcal{A}$ vertical alphabet
  - S planar labels (state)

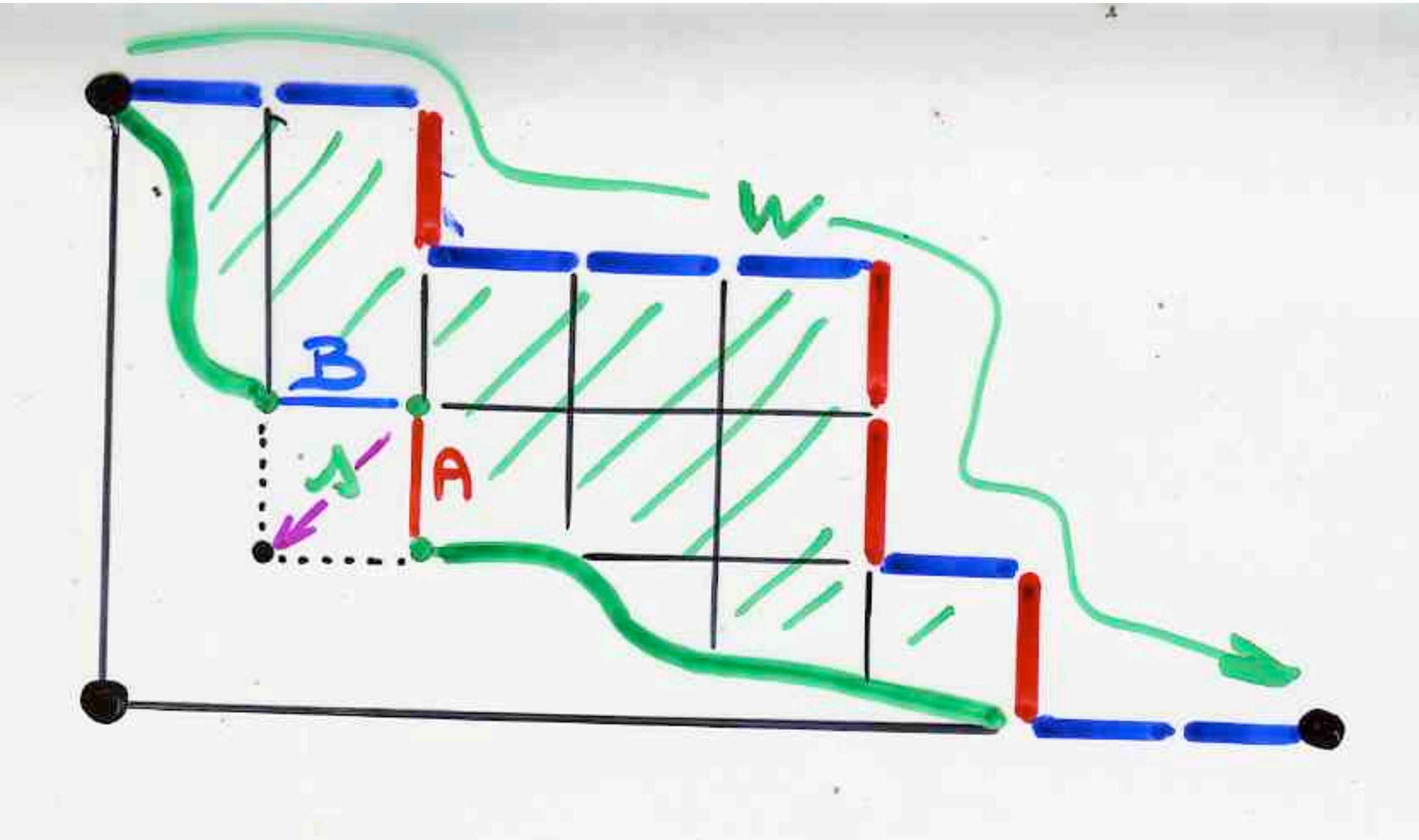
- θ (partial) transition function

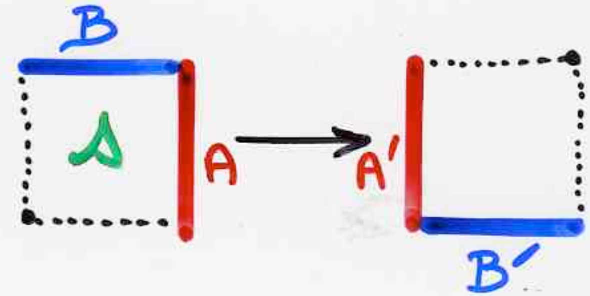
$$(s, B, A) \xrightarrow{\theta} (B', A') \quad \text{or} \quad \emptyset$$

$s \in S$; $B, B' \in \mathcal{B}$; $A, A' \in \mathcal{A}$

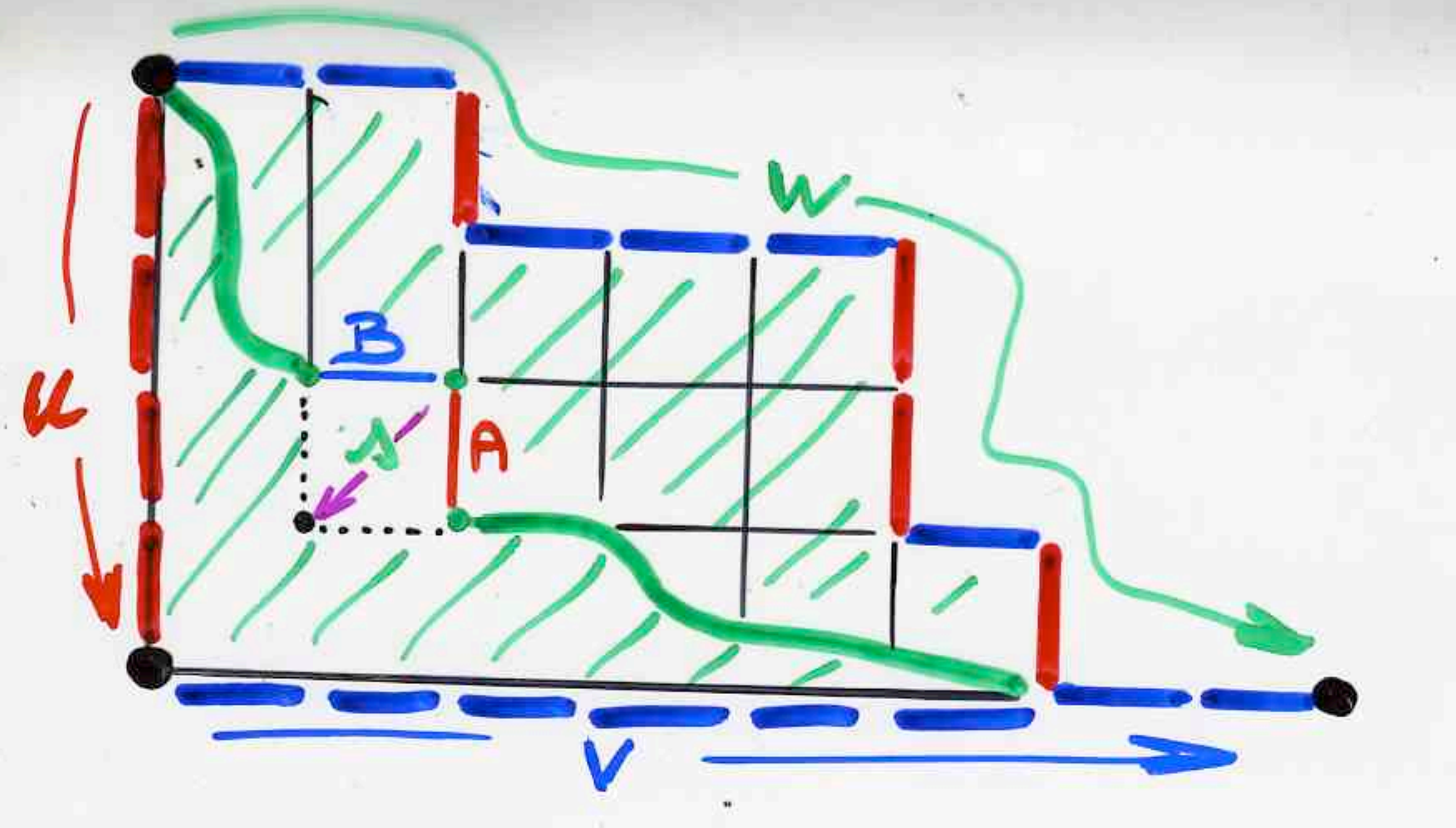
- $w \in (\mathcal{A} \cup \mathcal{B})^*$ initial word
- uv, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$ final word

Def. tableau T accepted by a planar automaton $P = (S, B, A, \theta, w, uv)$

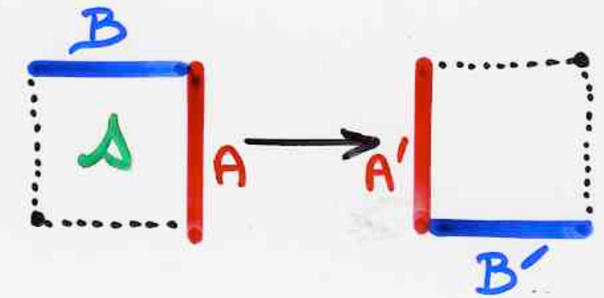




Def. tableau T accepted by a planar automaton P = (S, B, A, θ, w, uv)



planar rewriting



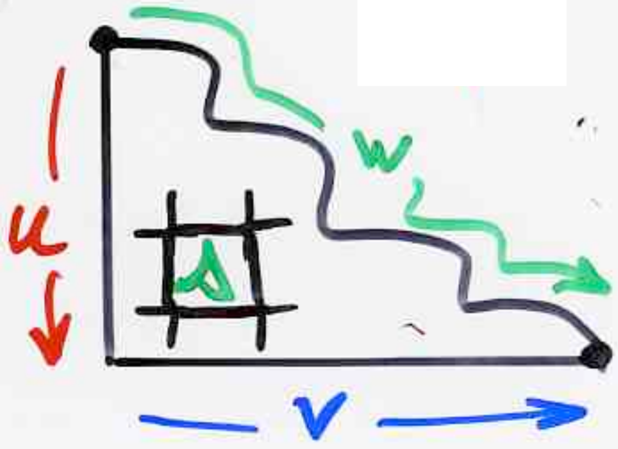
equivalence

Q-tableaux ⟷ tableaux accepted by a planar automaton

Q quadratic algebra

$\mathcal{P} = (S, \beta, \alpha, \theta, w, uv)$

φ



$$BA = \sum_{s \in S} A'B' \qquad (B', A') = \theta(s, B, A)$$

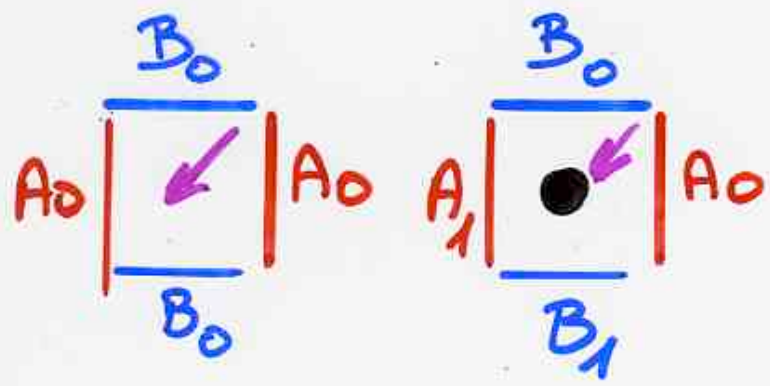
The "RSK planar automaton"

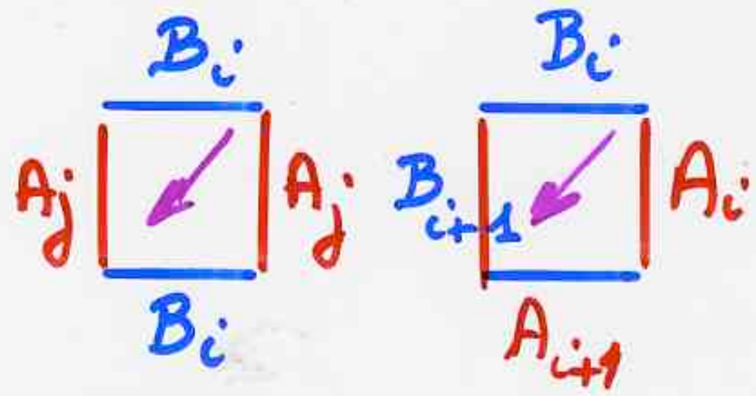
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \quad A_1 \quad A_k\}$

$w \in \{B_0, A_0\}^*$

$S = \{\square, \boxdot\}$





$i \neq j$

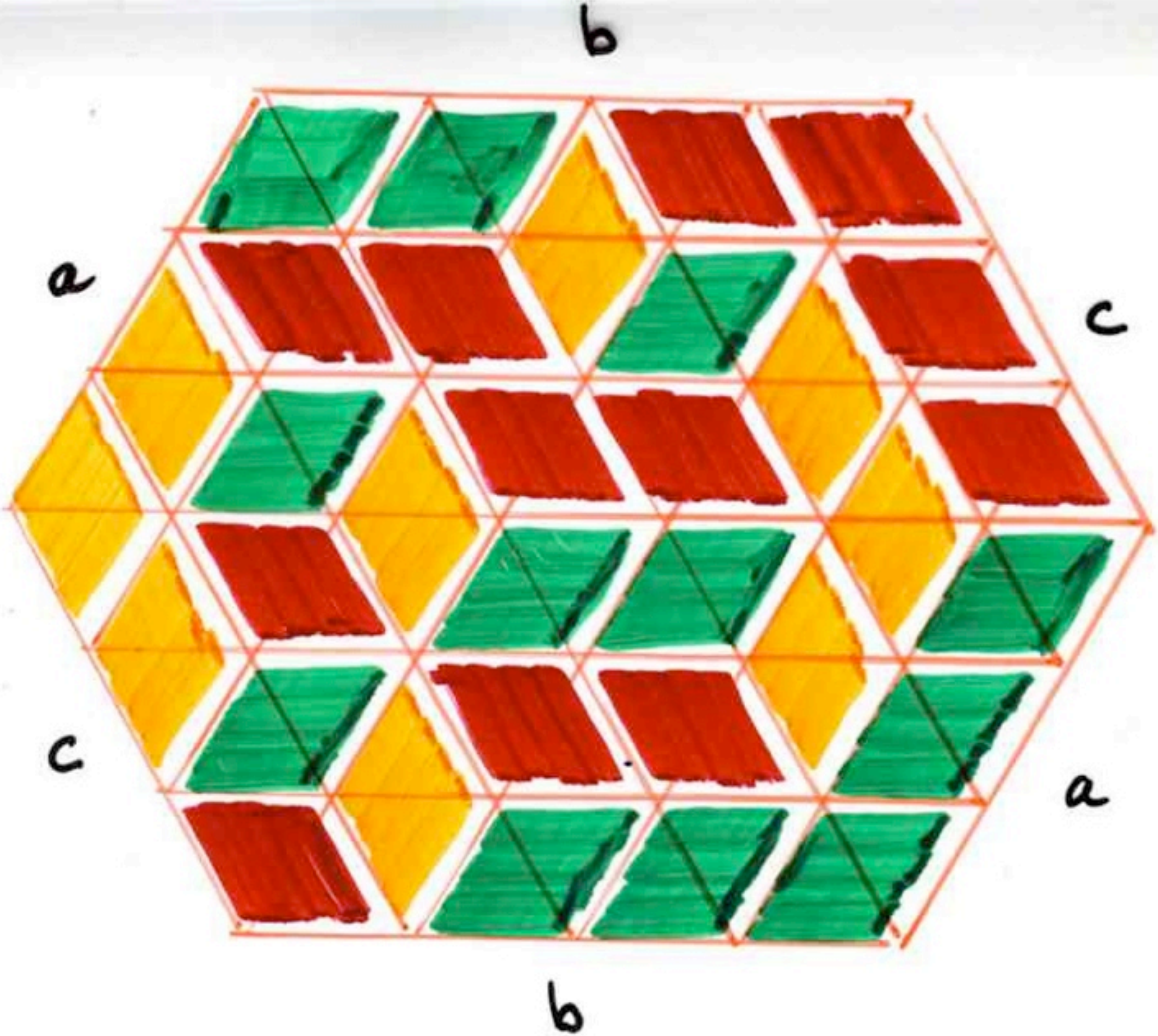
The 8-vertex algebra
(or Z - algebra)

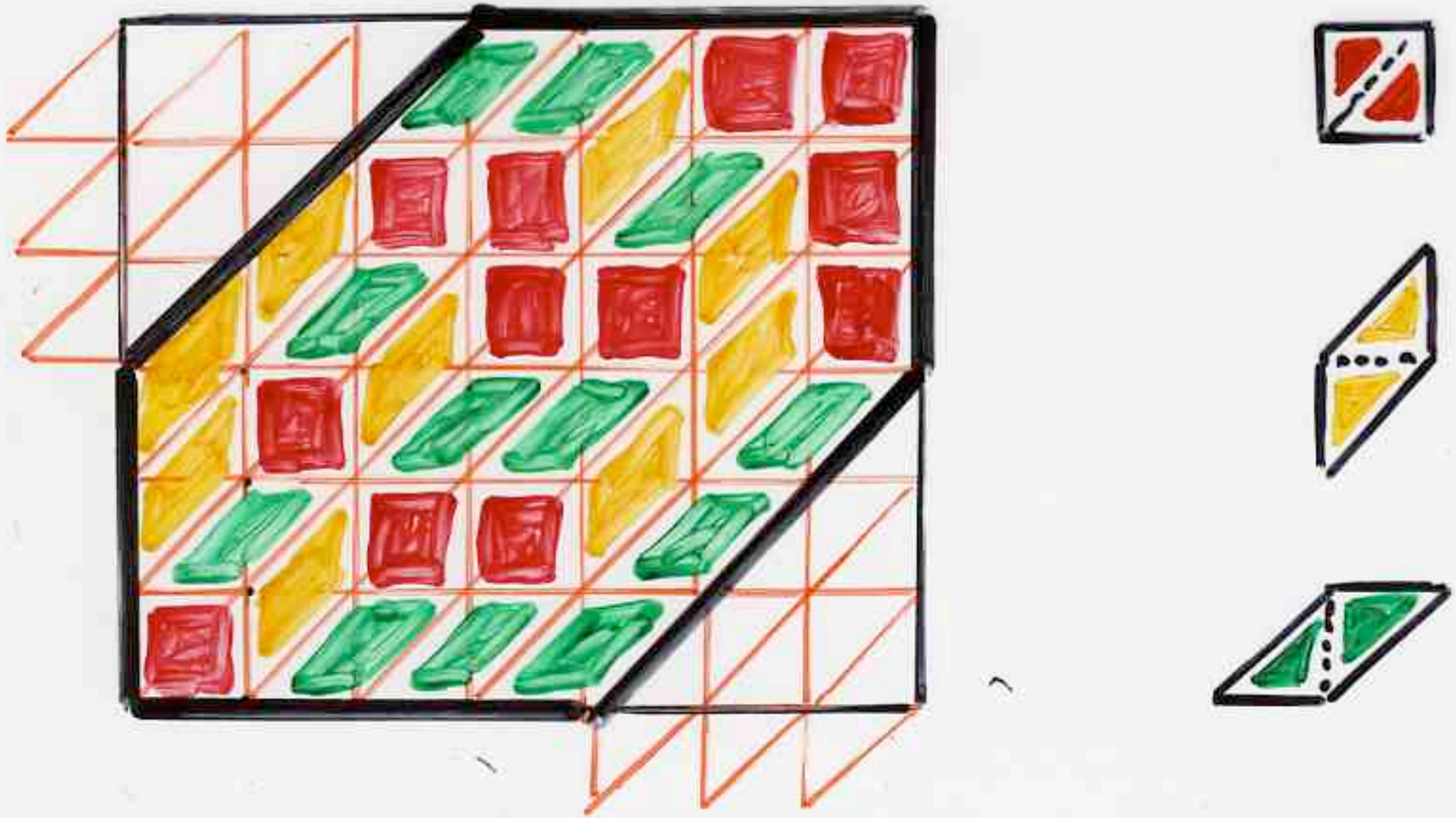
The quadratic algebra Z

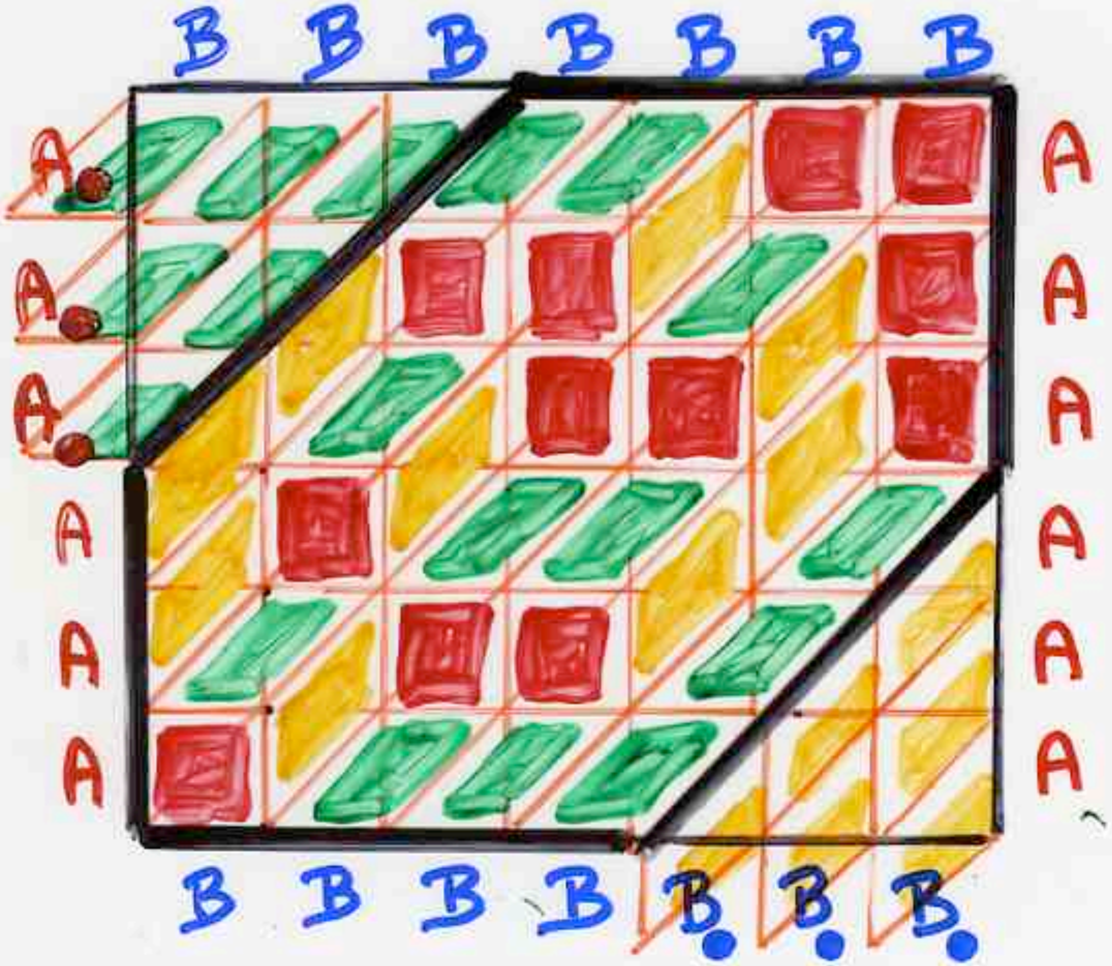
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$
\begin{cases}
B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\
B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\
B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet
\end{cases}
$$

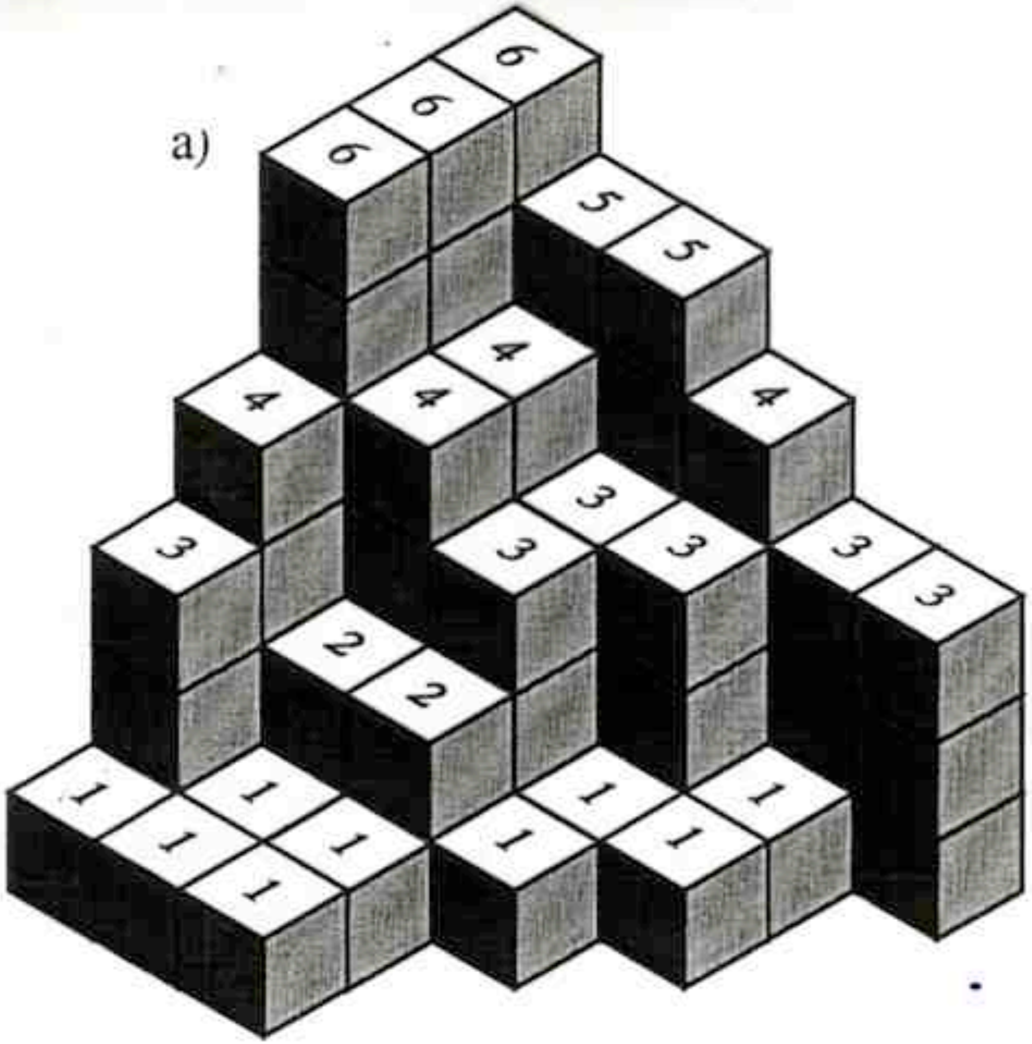

b
a
c
c
a
b





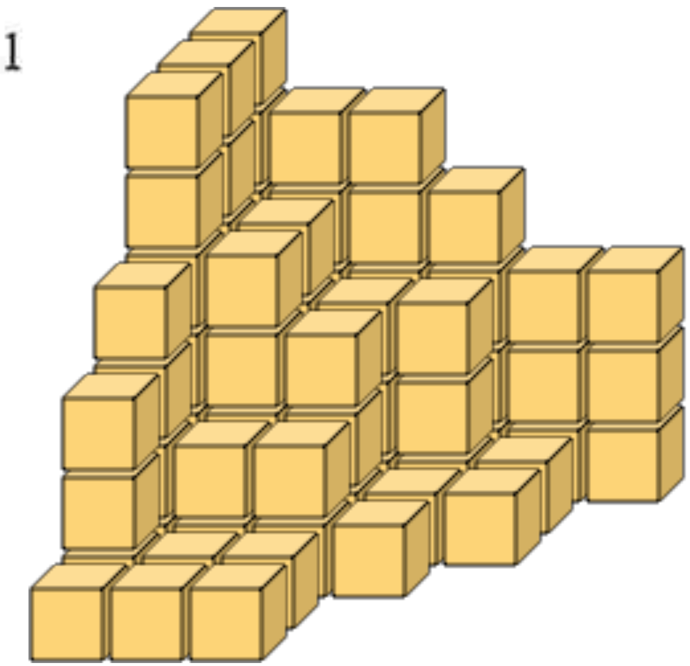
B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

a)

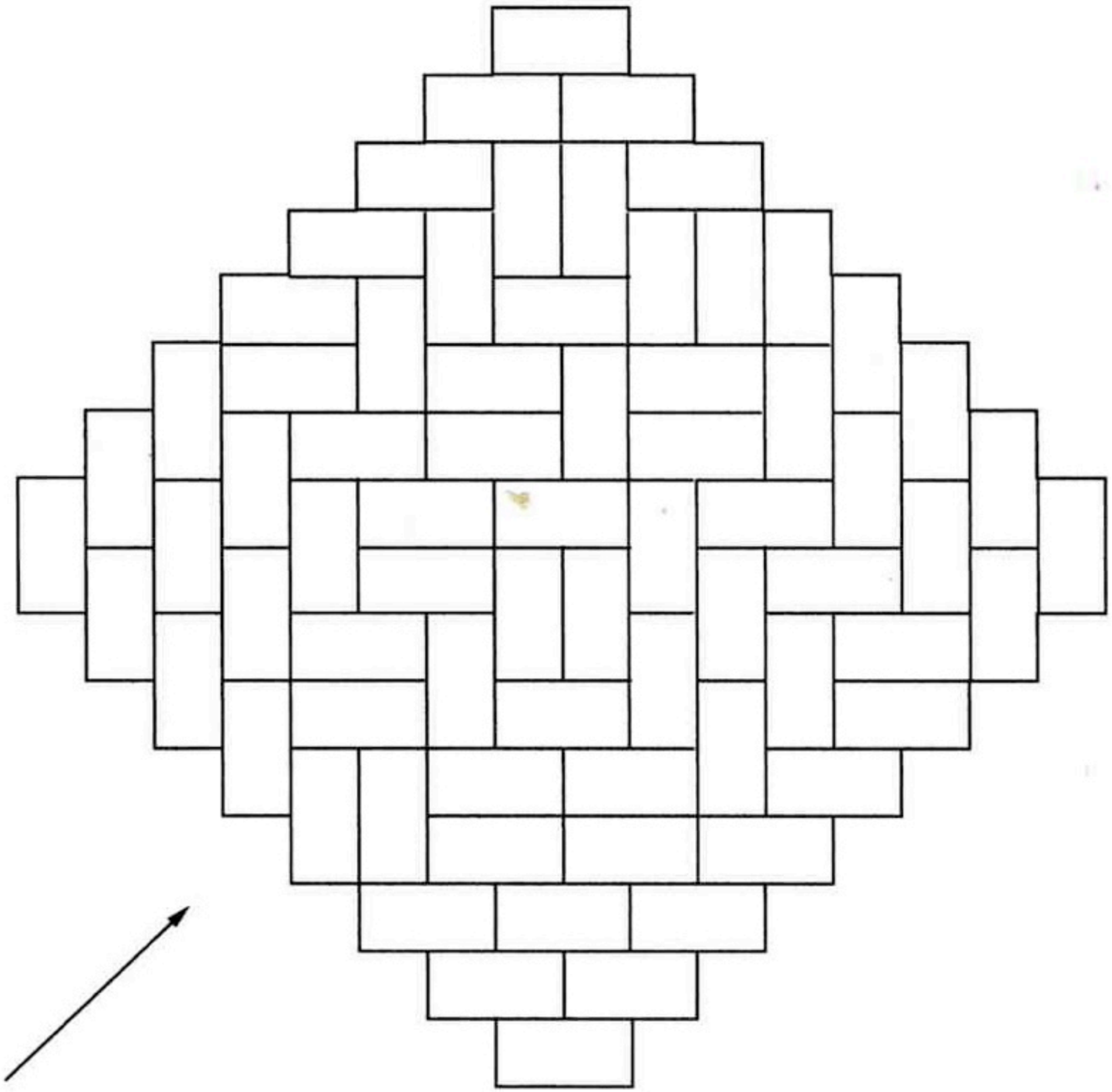


b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

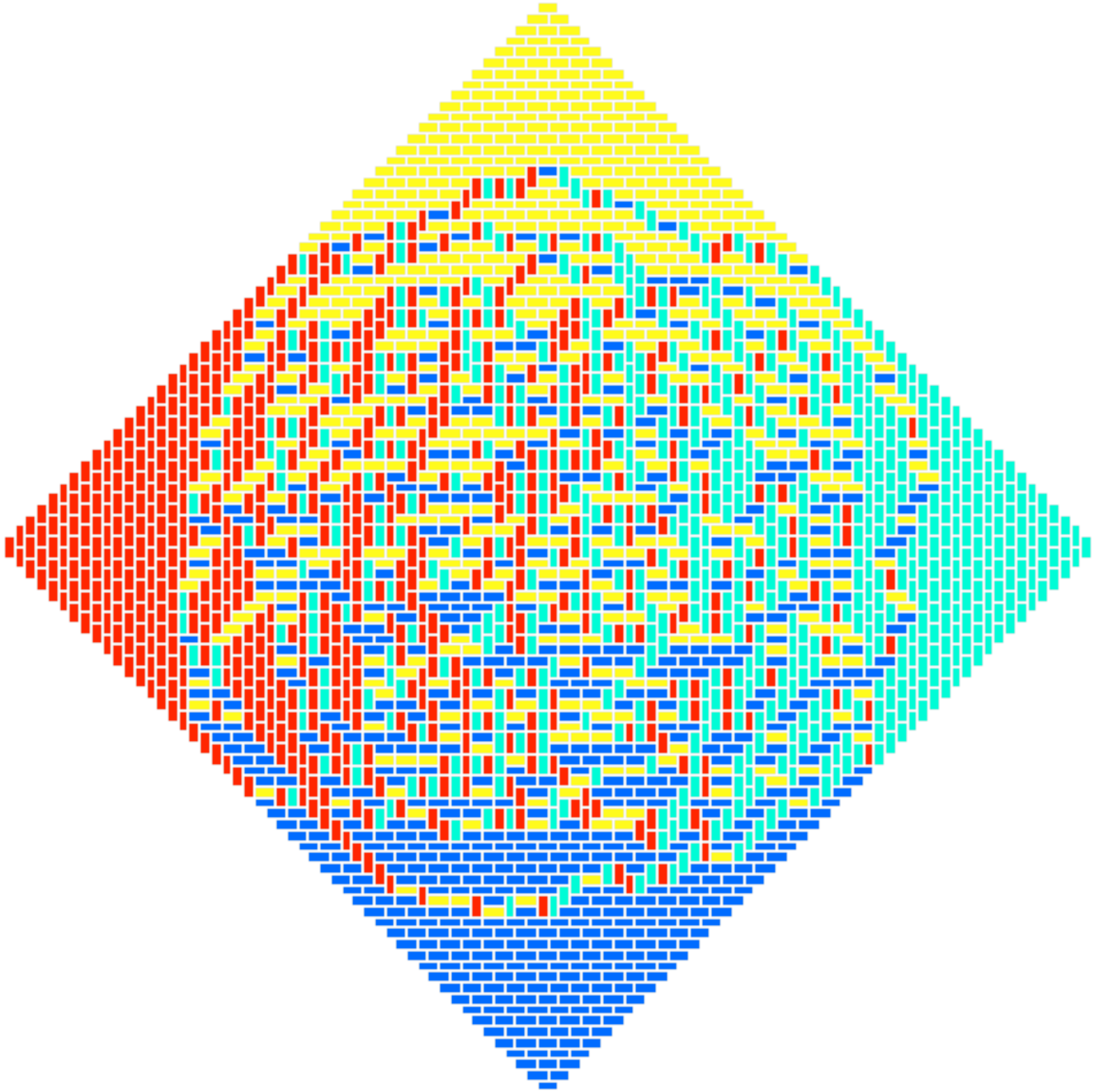


$2^{n(n-1)/2}$

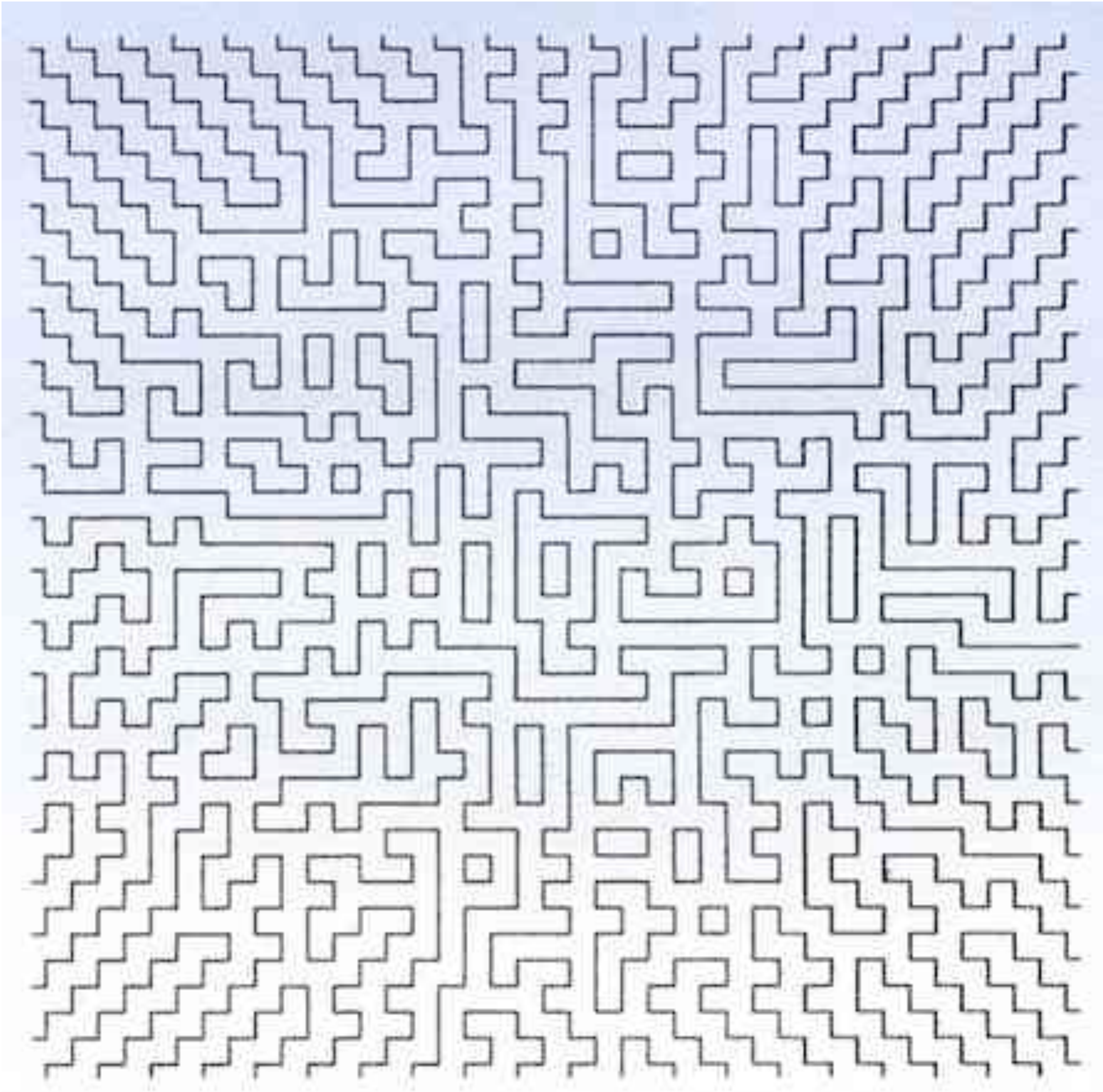


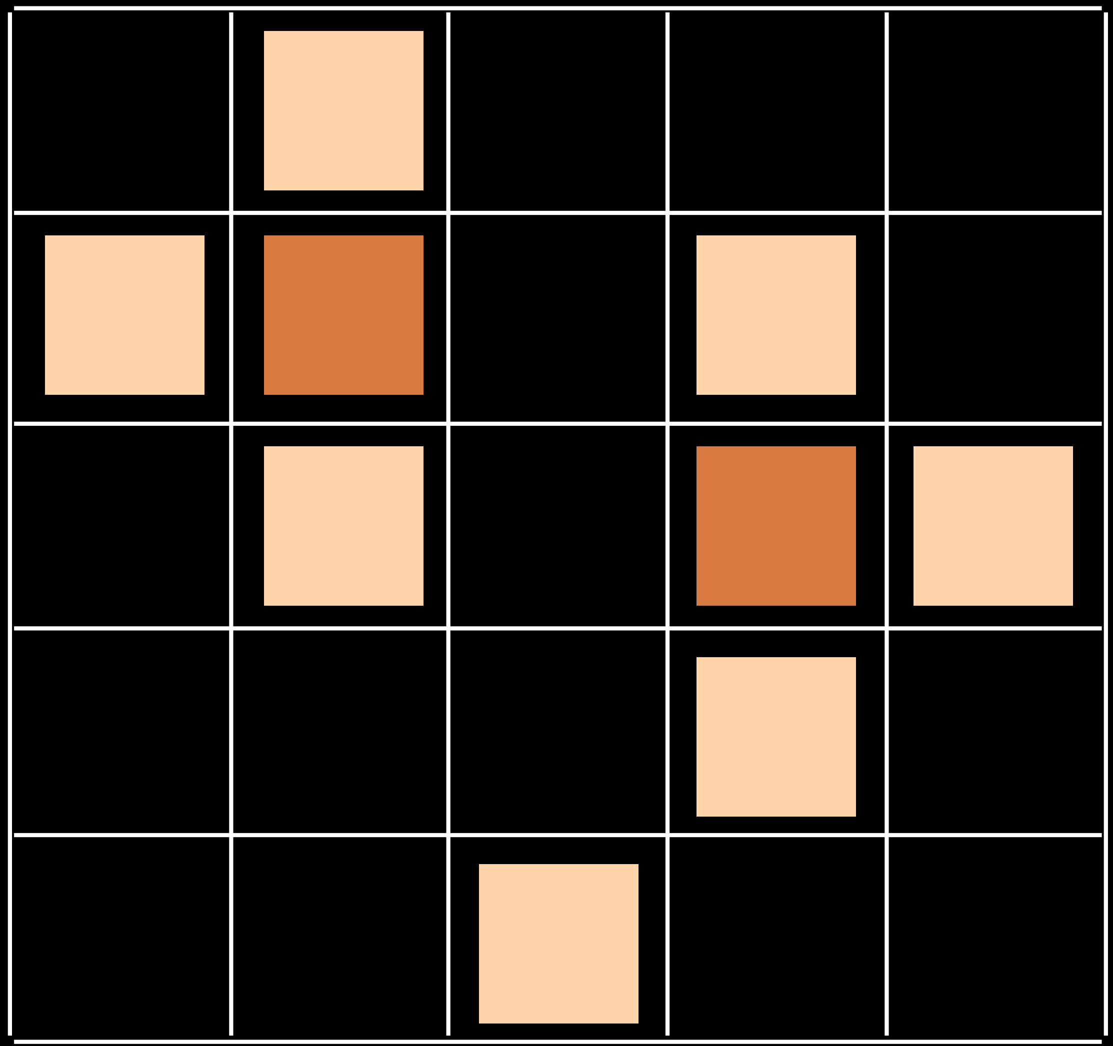
Elkies,
Kuperberg,
Larsen,
Propp
(1992)

random
Aztec
tilings



random
FPL





and beyond ....

cellular Ansatz 3:
from the quadratic algebra Q
how to guess representations

"L'Ansatz cellulaire"

objets combinatoires planarisés

représentation par opérateurs

Physique

histoires de fichiers

bijections

polynômes orthogonaux

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

placements de tours

permutations

tableaux alternatifs

RSK

↔ paires Tableaux Young

↔ permutations

histoires de Laguerre

DE = qED + E + D

PASEP

algèbre quadratique Q

Q-tableaux

ex: ASM, FPL

pavages, 8-vertex

?

commutations

réécritures

planarisation

automates planaires

----- Cours I ----- Cours II

pour plus de détails
voir les diaporamas du cours donné à Talca:

Cours XGV, Universidad de Talca
(December 2010 - January 2011)
Combinatorics and interactions (with physics) (24h)
«The Cellular Ansatz»

accessible sur les sites:

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici

Vulgarisation scientifique voir la page de l'association Cont'Science

http://web.me.com/xgviennot/Xavier_Viennot/

http://web.me.com/xgviennot/Xavier_Viennot2/

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