Limit laws for planar maps

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Planar maps

A planar map is a connected plane graph
(possibly with loops and multiple edges)

It can be encoded combinatorially using rotation systems

A rooted map has a distinguished half-edge

$\mathcal{M}_n$ = rooted maps with n edges

$$M_n = |\mathcal{M}_n| = \frac{2(2n)!}{(n+2)!n!} 3^n$$

Parameters of planar maps

- Number of vertices
- Number of loops, bridges, blocks equal to H
- Size of largest block
- Degree of the root face (or vertex)
- Maximum degree
- Diameter (Eric Fusy)

- Maps in surfaces of positive genus
- Graphs instead of maps

Enumeration (Tutte)

$$M(z,u) = \sum_{n\geq 0, k\geq 0} M_{n,k} u^k z^n$$

z marks edges, u marks degree of the root face

$$M(z,u) = 1 + zu^2 M(z,u)^2 + z \sum_{n\geq 0, k\geq 0} M_{n,k} z^n (u + u^2 + \cdots + u^{k+1})$$

$$M(z,u) = 1 + zu^2 M(z,u)^2 + zu \frac{M(z,1) - uM(z,u)}{1-u}$$

The quadratic method

$$M(z,u) = 1 + zu^2 M(z,u)^2 + zu\frac{M(z,1) - uM(z,u)}{1-u}$$

$$A(z,u)M(z,u)^2 + B(z,u)M(z,u) + C(z,u) = 0$$

$$(2AM + B)^2 = B^2 - 4AC$$

Assume there is a function $u(z)$ such that

$$2A(z,u(z))M(z,u(z)) + B(z,u(z)) = 0$$

Then

$$B(z,u(z))^2 - 4A(z,u(z))C(z,u(z)) = 0$$

$$\frac{\partial}{\partial u}\left[B(z,u(z))^2 - 4A(z,u(z))C(z,u(z))\right] = 0$$

Eliminate $M(z,1)$ from above, obtain $u(z)$ and then find $M(z,1)$

$$M(z,1) = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2}$$

From this it follows

$$M_n = [z^n]M(z,1) = \frac{2(2n)!}{(n+2)!n!}3^n$$

$$M_n \sim \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n$$

And an explicit expression for

$$M(z,u) = \sum_{n\geq 0, k\geq 0} M_{n,k} u^k z^n$$

Number of vertices

Let x mark vertices

$$M(z,u,x) = x + zu^2 M(z,u,x)^2 + zu\frac{M(z,1,x) - uM(z,u,x)}{1-u}$$

Again quadratic method and we find $u(z,x)$ such that

$$M(z,x) = M(z,1,x) = \mathrm{Rat}(z,x,u(z,x))$$

The singularity $\rho(x)$ of $M(z,x)$ satisfies $P(\rho(x),x) = 0$

$$[z^n]M(z,x) \sim c(x) n^{-5/2} \rho(x)^{-n}$$

By the quasi-powers theorem we get

Theorem

$X_n :=$ number of vertices in random planar maps n edges
Then X_n is asymp normal and

$$E(X_n) \sim n/2 \qquad \sigma^2(X_n) \sim 5n/32$$

Euler's formula

$$v + f = n + 2$$

By duality we get

$$2E(X_n) = n + 2$$

$$E(X_n) = \frac{n}{2} + 1$$

2-connected (non-separable) maps

Core C of a map M: block containing the root

M is obtained by attaching a map at each corner of C

B_n = number of 2-connected maps with n edges

$$M(z) = \sum B_k z^k M(z)^{2k} = B(zM(z)^2)$$

Set $x = zM(z)^2$ and then

$$B(x) = M(z(x))$$

$$B_n = \frac{2(3n-3)!}{(2n-1)!n!}$$

Number of blocks

w marks blocks in a map

$$M(z,w) = wB(zM(z,w)^2) - w + 1$$

We know $B(z)$ and can compute

$$M(z,w) = 1 + 2wz + (w + 8w^2)z^2 + \cdots$$

Singularity at

$$\rho(w) = \frac{4}{3(w^2 + 6w + 9)}$$

Theorem

$X_n :=$ number of blocks in random planar maps n edges
Then X_n is asymptotically normal and

$$E(X_n) \sim n/2 \qquad \sigma^2(X_n) \sim 3n/8$$

Given C fixed (rooted) 2-connected map

$X_n^C :=$ number of blocks in a random map equal to C

Theorem

X_n^C is asymptotically normal and

$$E(X_n^C) \sim \frac{3}{2}\left(\frac{4}{27}\right)^{|C|} \cdot n$$

Corollary

Number of loops (or bridges) is normal with expectation $\sim 2n/9$

Structure of blocks

How large are the blocks of a random map M?

Gao Wormald 1999
The size X_n of the largest block is of order $n/3$ and deviations are of order at most $n^{2/3}$

$$\mathbf{P}(|X_n - n/3| < w(n)n^{2/3}|) \to 1 \qquad \text{as } n \to \infty$$

where $w(n) \to \infty$

Core C of a map M: block containing the root

M is obtained by attaching a map at each corner of C

Banderier Flajolet Schaeffer Soria 2001

The size of C_n has a bimodal distribution

- $|C_n|$ is $O(1)$ with prob p
- $|C_n|$ is large with prob $1-p$

$$\mathbf{P}(|C_n| = n/3 + xn^{2/3}) \to n^{-2/3} g(x)$$

Where $g(x)$ is a density (of a stable law of index 3/2)

There is a unique largest block w.h.p.

This also applies to 3-connected components, triangulations ...

Degree of the root

$M(z,u) = \sum M_{n,k} u^k z^n$

$$p_n(u) = \frac{[z^n]M(z,u)}{[z^n]M(z,1)} \qquad \text{PGF}$$

Singularity analysis

$$M(z,1) = M_0 + M_2(1 - z/12) + M_3(1 - z/12)^{3/2} + \cdots$$

Explicit quadratic for $M(z,u)$ in terms of $M(z,1)$
Singularity of $M(z,u)$ does not change for $u \sim 1$

$$M(z,u) = M_0(u) + M_2(u)(1 - z/12) + M_3(u)(1 - z/12)^{3/2} + \cdots$$

$$p_n(u) \to p(u) = \frac{M_3(u)}{M_3} = \frac{u\sqrt{3}}{\sqrt{(2+u)(6-5u)^3}}$$

$$= \frac{1}{12}u + \frac{1}{12}u^2 + \frac{13}{144}u^3 + \cdots$$

$$p(u) = \sum p_k u^k = \frac{u\sqrt{3}}{\sqrt{(2+u)\,(6-5\,u)^3}}$$

By singularity analysis (with respect to u)

$$p_k \sim c \cdot k^{1/2} \left(\frac{5}{6}\right)^k$$

Liskovets (1999) observed that for many classes of planar maps the pattern is the same

2-connected, 3-connected, Eulerian, triangulations, ...

We have tried to give a rationale for this fact
with Michael Drmota

Universality of tail estimates

$\mathcal{M}$ class of planar maps

$M(z,u)$ generating function u marks degree of the root

Assume

$$(g_1 M(z,u) + g_2)^2 = g_3$$

$$g_j = G_j(z,u,y(z)) \qquad y(z) = M(z,1)$$

z_0 singularity of $M(z,u)$ for $u \sim 1$

$y_0 = y(z_0)$

$$G_3(z_0,y_0,u_0) = 0, \qquad \frac{\partial G_3}{\partial u} G_3(z_0,y_0,u_0) = 0$$

Theorem [Drmota MN]

If $\partial^2 G_3/\partial u^2 G_3(z_0,y_0,u_0) = 0$ then

$$\frac{[z^n u^k]M(z,u)}{[z^n]y(z)} \to p_k \qquad \text{as } n \to \infty$$

$$p_k \sim c \cdot k^{1/2} u_0^{-k}$$

Planar maps

$$(g_1 M(z,u) + g_2)^2 = g_3$$

$$G_3 = 1 - 2\,u - 2\,u^2 z + u^2 + 6\,u^3 z + u^4 z^2 - 4\,u^4 z + \left(4u^4 z^2 - 4u^3 z^2\right) y(z)$$

$$z_0 = 1/12 \qquad y_0 = 4/3 \qquad u_0 = 6/5$$

$$\frac{\partial^2 G_3}{\partial u^2} G_3(1/12, 4/3, 6/5) = 0$$

Corollary

The limit distribution (p_k) of the degree of the root face has tail

$$p_k \sim c \cdot k^{1/2} (5/6)^k$$

2-connected maps

$$M(z) = \sum B_k z^k M(z)^{2k} = B(zM(z)^2)$$

$$M(z,u) = \sum_{n,k} B_{n,k} u^k M(z,u)^k M(z)^{2n-k} = B\left(zM(z)^2, \frac{uM(z,u)}{M(z)}\right)$$

Set $x = zM(z)^2, \quad w = uM(z,u)/M(z)$
Elimination gives a quadratic equation for $B(x,w)$

$$G_3 = 3w^2x^2 - x - 3wyx + y^2/2$$

$$x_0 = 4/27 \qquad y_0 = 4/3 \qquad w_0 = 3/2$$

$$\frac{\partial^2 G_3}{\partial u^2} G_3(4/27, 4/3, 3/2) = 0$$

Bipartite maps

Very similar to general maps for degree of root face
By duality: degree of root vertex for Eulerian maps

Degree of root vertex in bipartite maps has the same distribution as degree of root face:

$$|\text{Bipartite maps of size } n \text{ and deg(root face)} = 2k|$$

$$= |\text{Bipartite maps of size } n \text{ and deg(root vertex)} = k|$$

Take a bipartite map M colored B and W
Add a red (R) vertex inside each face, and make it adjacent to all the vertices in the face
Obtain an Eulerian triangulation T, which is 3-colorable

Idea of the proof

$$G(z, u(z), y(z)) = 0 \qquad G_u(z, u(z), y(z)) = 0$$

Consider $G_u(z, u(z), y(z)) = 0$ with z, u as independent variables and $y = Y(z, u)$ unknown function

Then solve $G(z, u, Y(z, u)) = 0$

$$u(z) = g_1(z) + g_2(z)\sqrt{1 - z/z_0}$$

$$y(z) = h_1(z) + h_2(z)(1 - z/z_0)^{3/2}$$

This explains the usual estimates for maps

$$[z^n]y(z) \sim c \cdot n^{-5/2} z_0^{-n}$$

From

$$(g_1 M(z,u) + g_2)^2 = g_3$$

We get

$$M(z,u) = \frac{\sqrt{G_3(z,u,y(z))} - G_2(z,u,y(z))}{G_1(z,u,y(z))}$$

Essential contribution comes from $\sqrt{G_3(z,u,y(z))})$
Expanding G_3 and estimating coefficients using Cauchy's formula

$$[z^n u^k]M(z,u) \sim d \cdot n^{-5/2} z_0^{-n} k^{1/2} u_0^{-k}$$

uniformly for $k \leq C \log n$

Maximum degree

$$p_k = \mathbf{P}(\deg(\text{root}) = k)$$

Expected number of vertices of degree k is $\sim p_k n$

Since $p_k \asymp q^k \qquad q < 1$
it is natural to expect that $p_k n$ is negligible when

$$k = \frac{\log n}{\log 1/q}$$

Δ_n = maximum degree in maps of size n
Gao Wormald 2000

$$E(\Delta_n) \sim \frac{\log n}{\log(6/5)}$$

Precise limit distribution for Δ_n

Maps on surfaces

S_g orientable surface of genus g

Maps in S_g must be 2-cellular

$$v - n + f = 2 - 2g$$

The number of maps is (Bender Canfield 1986)

$$M_n(g) \sim t_g \, n^{5(g-1)/2} 12^n$$

Gao Richmond 1994
Distribution of the degree of the root independent of genus

Planar graphs and graphs on surfaces

Graphs not embedded and labelled at vertices

- Generating functions not algebraic
- $G(x,y) = \sum G_{n,k} y^k \frac{x^n}{n!}$ x vertices, y edges
- Necessary to go from rooted graphs to unrooted graphs
- If $g > 0$ further complications because 3-connected graphs do not have a unique embedding in Sg

Giménez Noy Rué 2009
Main structural parameters for planar graphs as for planar maps

Chapuy Fusy Giménez Mohar MN 2010
Main structural parameters for graphs of genus g do not depend on the genus