Average Long-Lived Memoryless Consensus: The Three-Value Case

Eric Rémila

Université de Lyon
LIP (umr 5668 CNRS - Université Lyon 1- ENS de Lyon)
Université de St-Etienne (IUT Roanne)

Joint work with

Ivan Rapaport

(DIM, CMM (umi 2807 CNRS), Universidad de Chile)

Partially supported by Programs Fondecyt 1090156, Basal-CMM, Instituto Milenio ICDB, Ecos C09E04 and IXXI (Institut des Systèmes Complexes - Complex System Institute, Lyon)

The sponsor page

The consensus problem: informal approach

What qualities are requested for a collective decision, called *consensus*?

- *Representativity*: the consensus corresponds to a sufficient number of individual opinions,
- *Stability*: the consensus is robust to small individual opinion variations.

Problem: the two qualities above are often incompatible. What can we do in in the real life ?

The consensus problem: formalization

General framework.

- We have *n individuals*,
- Each individual is given a set of k possible *opinions* in the set $\mathbb{Z}_k = \{0, 1, 2, \ldots .k-1\}$,
- A global *input state* s is therefore an n-uple of $V = (\mathbb{Z}_k)^n$; and i_s denotes the occurrence number of i in s.
- The *input graph* (V, E), is the unordered graph where two states are neighbors when they only differ in a unique individual.
- A *memoryless consensus function* f is a function from V to $\mathbb{Z}_k$.
- The function f is *anonymous* if $f(s)$ only depends on $0_s, 1_s, \ldots., (k-1)_s$, and not on the place of values.

Representativity:

- We fix a threshold t such that if $f(s) = i$, then $i_s > t$.

  Remark: we need to have:

$$n > k\,t,$$

  in order to be able to satisfy the threshold condition in any case.

The consensus problem: formalization

Stability: criterion for a consensus function f:

- we use a uniform random walk $S_0, S_1, S_2 \ldots$ on S. we define:

$$X_p = card\{j \in \mathbb{N}, 0 \leq j < p \wedge f(S_j) \neq f(S_{j+1})\}$$

$$instability(f) = lim_{p \to \infty}(\mathbb{E}\,(\frac{X_p}{p}))$$

Problem: find a consensus function satisfying representativity, with the lowest instability.

The consensus problem: first simplification

Proposition: Let E_f be the set of unordered pairs of neighboring states $\{s, s'\}$ such that $f(s) \neq f(s')$.

For any memoryless consensus function f, *instability*(f) is well defined, and we have:

$$instability(f) = \frac{card(E_f)}{card(E)} = \frac{2\, card(E_f)}{n(k-1)k^n}$$

Remark: To minimize instability, it suffices to minimize $card(E_f)$!!!

The consensus problem: second simplification

We assume that f is anonymous. In this case,

- States which are equal, up to permutation, can be merged.
- Edges follow the merging process. Each new edge e' receives a weight w(e'), corresponding to the number of merged previous edges.

Let V', E' and E'_f be the respective images of V, E and E_f obtained by merging. We have:

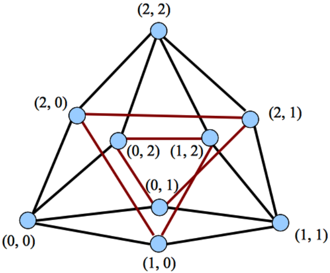
$$instability(f) = \frac{\sum_{e' \in E'_f} w(e')}{\sum_{e' \in E'} w(e')} = \frac{2 \sum_{e' \in E'_f} w(e')}{n(k-1)k^n}$$

Remark: To minimize instability, it suffices to minimize

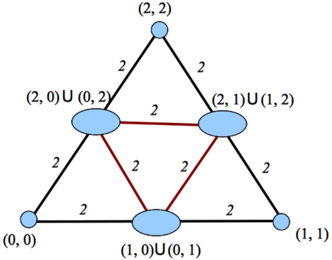
$$\sum_{e' \in E'_f} w(e').$$

Useful since (V', E') is simpler than (V, E).

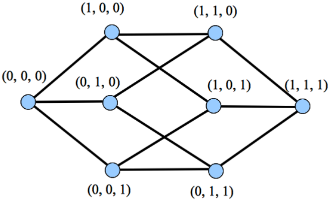
Examples



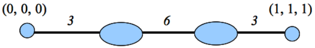
(2, 2)
(2, 0)
(2, 1)
(0, 2)
(1, 2)
(0, 1)
(0, 0)
(1, 1)
(1, 0)



(2, 2)
(2, 0) U (0, 2)
(2, 1) U (1, 2)
(0, 0)
(1, 0) U (0, 1)
(1, 1)
2



(1, 0, 0)
(1, 1, 0)
(0, 0, 0)
(0, 1, 0)
(1, 0, 1)
(1, 1, 1)
(0, 0, 1)
(0, 1, 1)



(0, 0, 0)
3
6
3
(1, 1, 1)

The merged graph (V', E') is a line. Up to symmetry, there exists a unique optimal anonymous strategies.

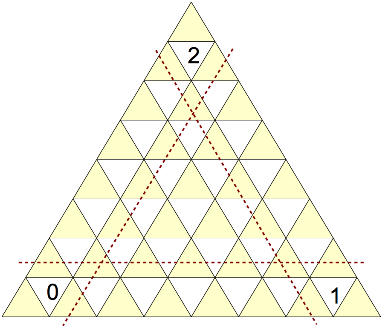
Proposition: Up to symmetry, there exists a unique optimal anonymous strategy, defined by: .

- f(s) $= 0$ if $0_s > t$,
- f(s) $= 1$ 0therwise.

Proposition: (Alea 2009, Sirocco 2008) This strategy is optimal, even among non anonymous functions.

The case when $k = 3$

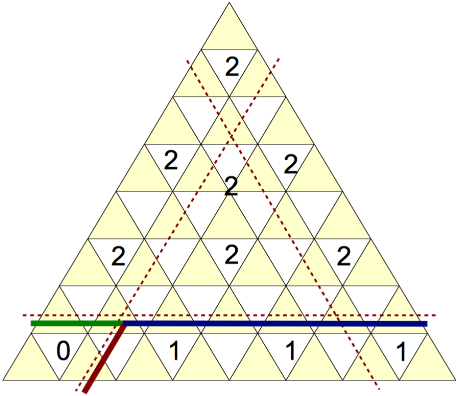
After merging, the graph is a "triangle of triangles".



Weight properties:

- the edge weights of a same small yellow triangle are equal,
- following a straight line from the boundary to the center, weights are increasing.

The main result

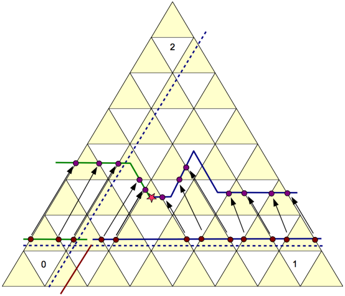


Theorem: up to symmetry, the unique optimal strategy is:

- $f(s) = 2$ if $2_s > t$,
- $f(s) = 1$ if $2_s \leq t$ and $1_s > t$,
- $f(s) = 0$ otherwise.

Proof: the easy minoration

We first compare the strategy f with any other solution f' forming three connected domains.

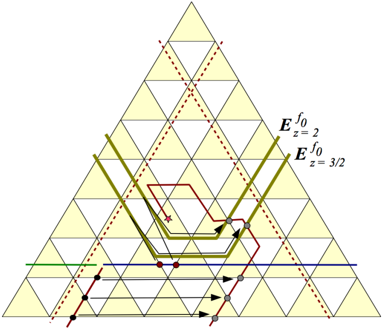


Lemma 1: We can construct a "increasing weight" injective mapping between bicolored edges of f and bicolored edges of f'.

UNIVERSITÉ DE LYON

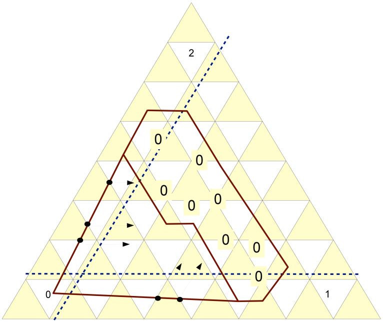
Proof: the complex minoration

We first compare the strategy f with any other solution f' only forming three connected domains.



Proof: bubble deletion

Lemma 2: for any function f'', there exists a function solution f' only forming three connected domains.



Open problems

- Is the strategy optimal among (potentially non anonymous) memoryless function?
- Can the result be extended when there is more than 3 states?
- What happens if a memory is added?
  We hope that we have the optimal strategy with memory (not yet checked and written. May be next year ?).

This is the end

Questions ?