

Séminaire de Combinatoire Philippe Flajolet

Algèbres quadratiques,
Robinson-Schensted,
matrices à signes alternants
et au delà ...

26 Mai 2011
IHP, Paris

XGV
LaBRI, Bordeaux

Heisenberg
operators
 U, D

$$UD = DU + I$$

$$UD = DU + I$$

Lemme - Tout mot $w \in \{U, D\}^*$
s'écrit

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} D^i U^i$$

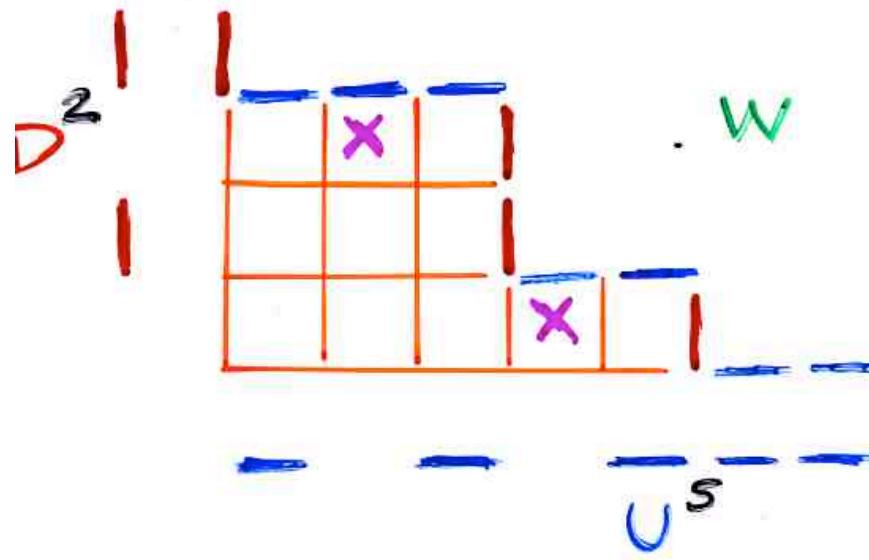
normal ordering

$$c_{n,0} = n!$$

notation

$$w \rightarrow F_w$$

diagram
Ferrers



Prop- $c_{i,j}(w) =$ nb de "placement" tours sur F

avec $i = |w|_D - k$

$j = |w|_U - k$

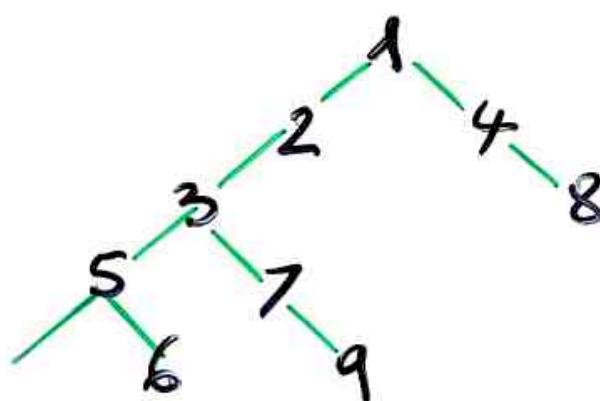
Prop. $w = (UD)^n$

$$c_{k,k}(w) = S_{n+1, k+1} \quad \text{Stirling}$$

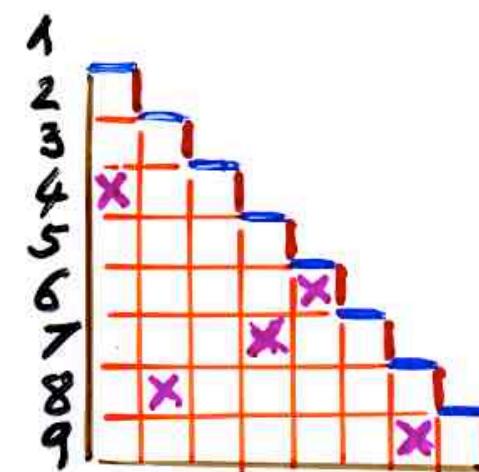
nombre de partitions

de $\{1, 2, \dots, n+1\}$

en $(k+1)$ blocs



- {1, 4, 8}
- {2}
- {3, 7, 9}
- {5, 6}



K. Penson, I. Solomon

P. Blasiak, A. Horzela

G. Duchamp

P. Blasiak, P. Flajolet

2010

A

product par x

S

$\cdot \frac{d}{dx} ()$

The cellular Ansatz

first part

$$UD = DU + I$$

$$UD \rightarrow DU$$

$$UD \rightarrow I$$

$$UD = DU + \text{Id}$$

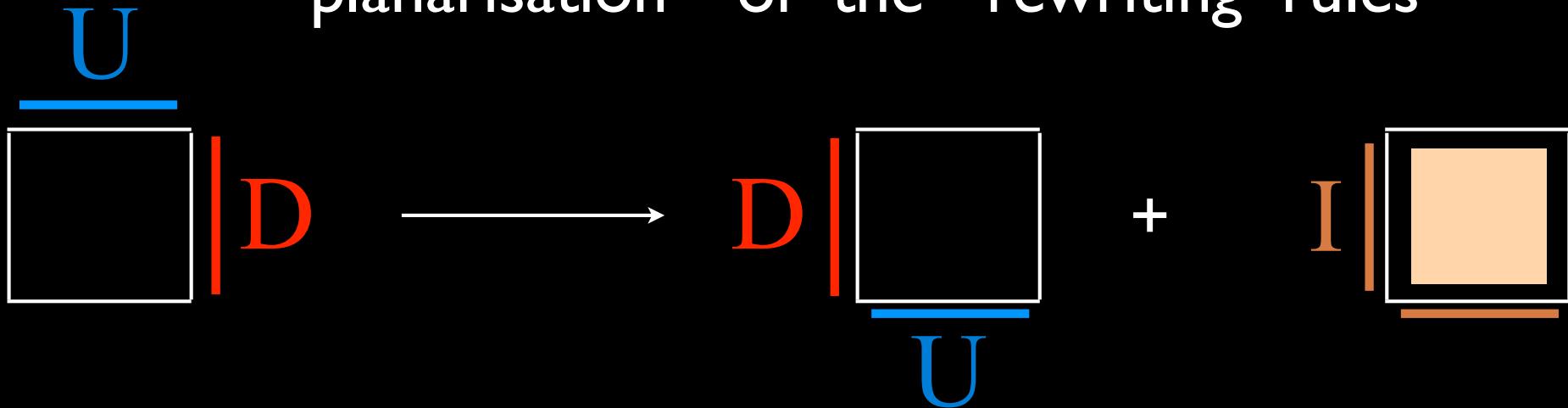
$$U^n D^n = ?$$

$$\begin{aligned} UUUDDDD &= UU(DU + \text{Id})DD \\ &= UUDUDD + UUDD \\ &= UDUUDD + 2 UUDD \\ &= DUVUDD + 3 UUDD \end{aligned}$$

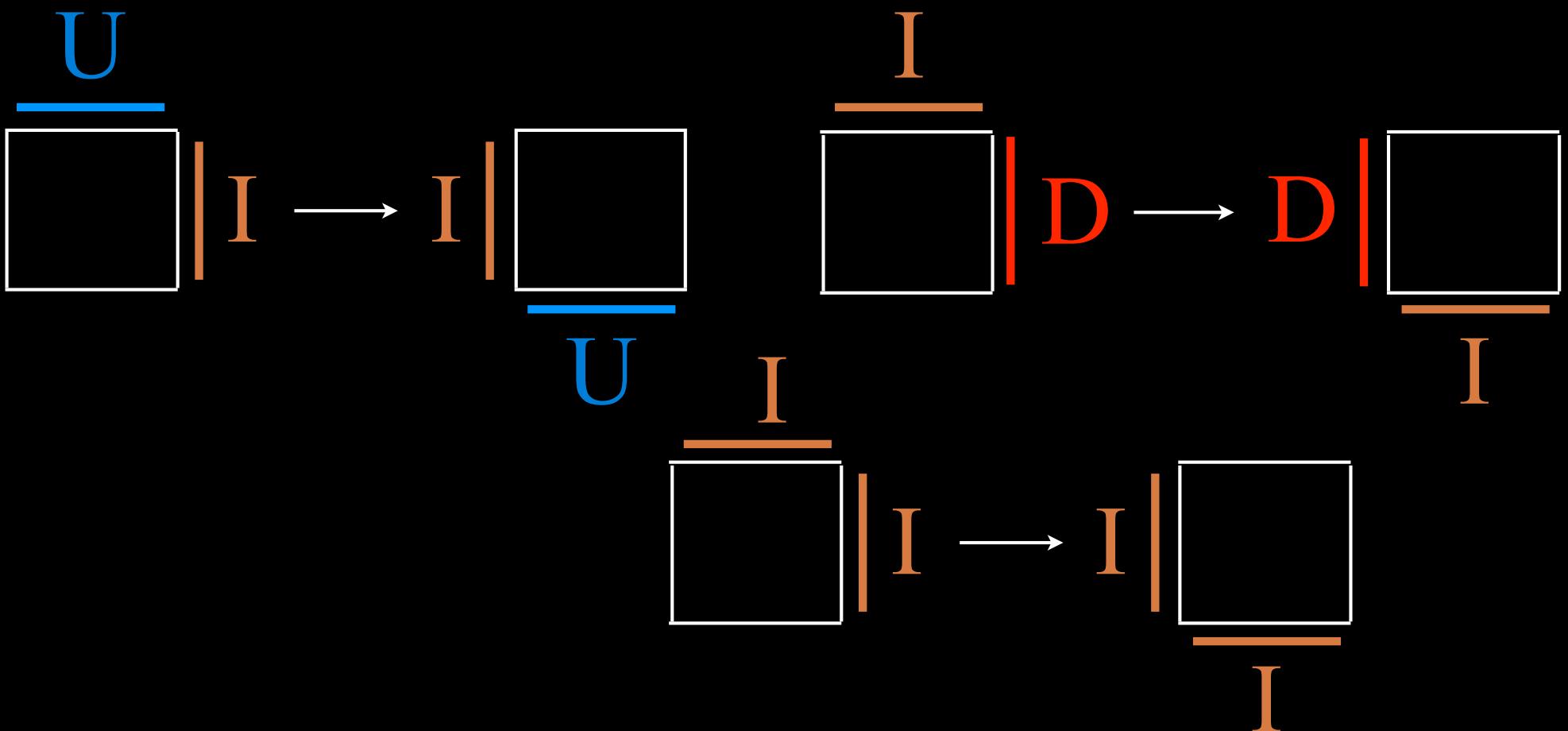
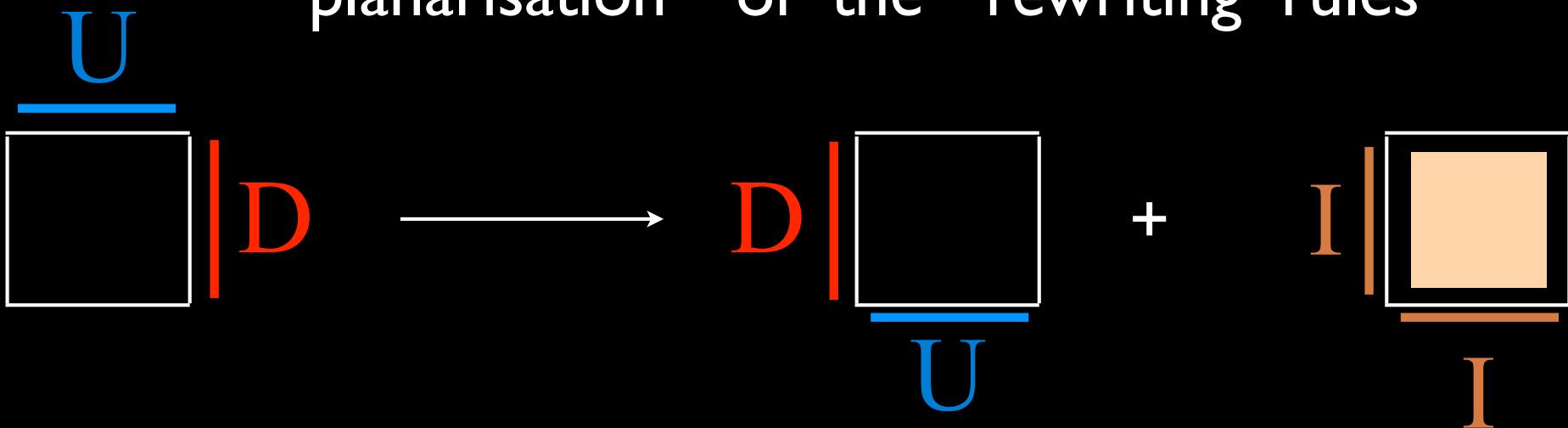
$$\begin{aligned}
 UUDD &= UDUUD + UD \\
 &= \overbrace{DUUDU}^{\text{DUUD}} + 2 \overbrace{UD}^{\text{DU}} \\
 &= \overbrace{DUUDU}^{\text{DUUD}} + \overbrace{DU}^{\text{DU}} + 2(DU + Id) \\
 &= \overbrace{DUUDU}^{\text{DUUD}} + 2 \overbrace{DU}^{\text{DU}} \\
 &= DDUU + 4DU + 2Id
 \end{aligned}$$

$$\begin{aligned}
 U^3 D^3 &= DU(DDUU + 4DU + 2Id) + \\
 &\quad 3(DDUU + 4DU + 2Id) \\
 &= DDUUDU + DDUU \\
 &\quad + 4(DDUU + DU) + 2DU \\
 &\quad + 3 DDUU + 12DU + 6Id \\
 &= D^3 U^3 + 9D^2 U^2 + 18DU + 6Id
 \end{aligned}$$

“planarisation” of the “rewriting rules”



“planarisation” of the “rewriting rules”



$$\left\{ \begin{array}{l} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{array} \right.$$

$$\left\{ \begin{array}{l} UD \rightarrow DU \qquad \qquad \qquad UD \rightarrow I_v I_h \\ U I_v \rightarrow I_v U \\ I_h D \rightarrow D I_h \\ I_h I_v \rightarrow I_v I_h \end{array} \right.$$

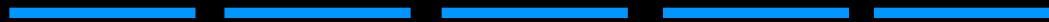
rewriting rules

$$\frac{U}{\overline{U}} | D \longrightarrow D | \frac{\bullet}{\overline{U}} + I | \frac{\square}{\overline{I}}$$

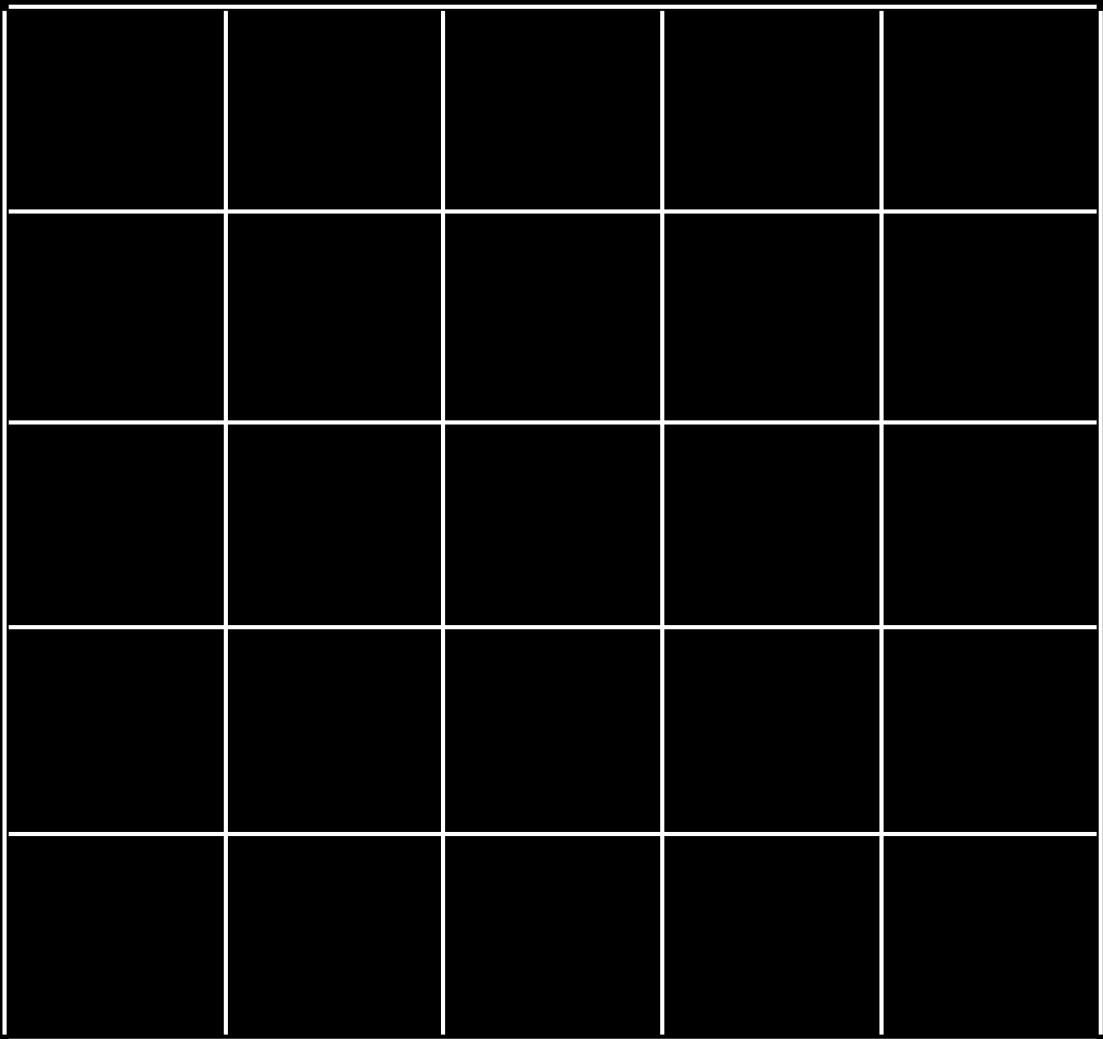
$$\frac{U}{\overline{U}} | I \longrightarrow I | \frac{\square}{\overline{U}} \quad \frac{I}{\overline{I}} | D \longrightarrow D | \frac{\square}{\overline{I}}$$

$$\frac{I}{\overline{I}} | I \longrightarrow I | \frac{\square}{\overline{I}}$$

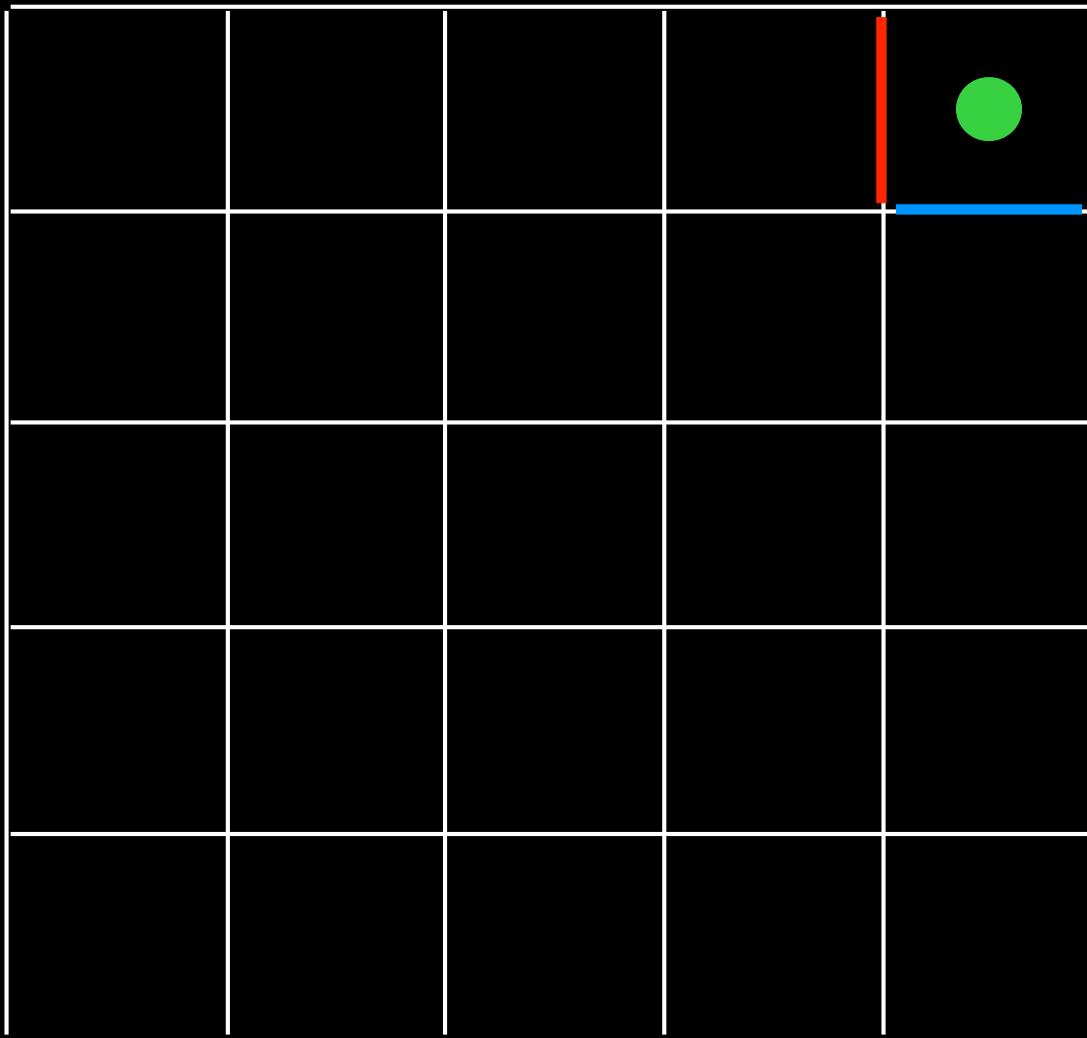
U



D

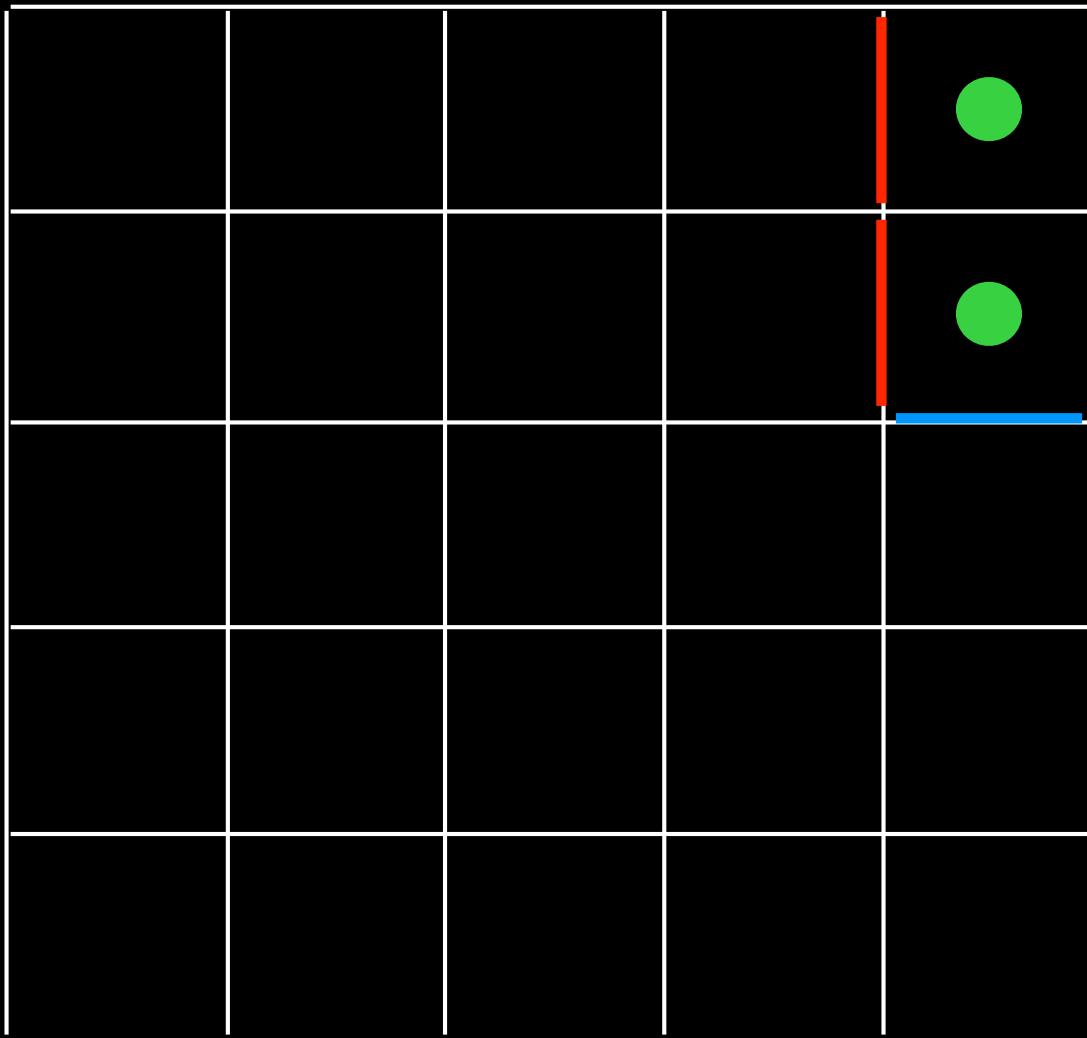


U



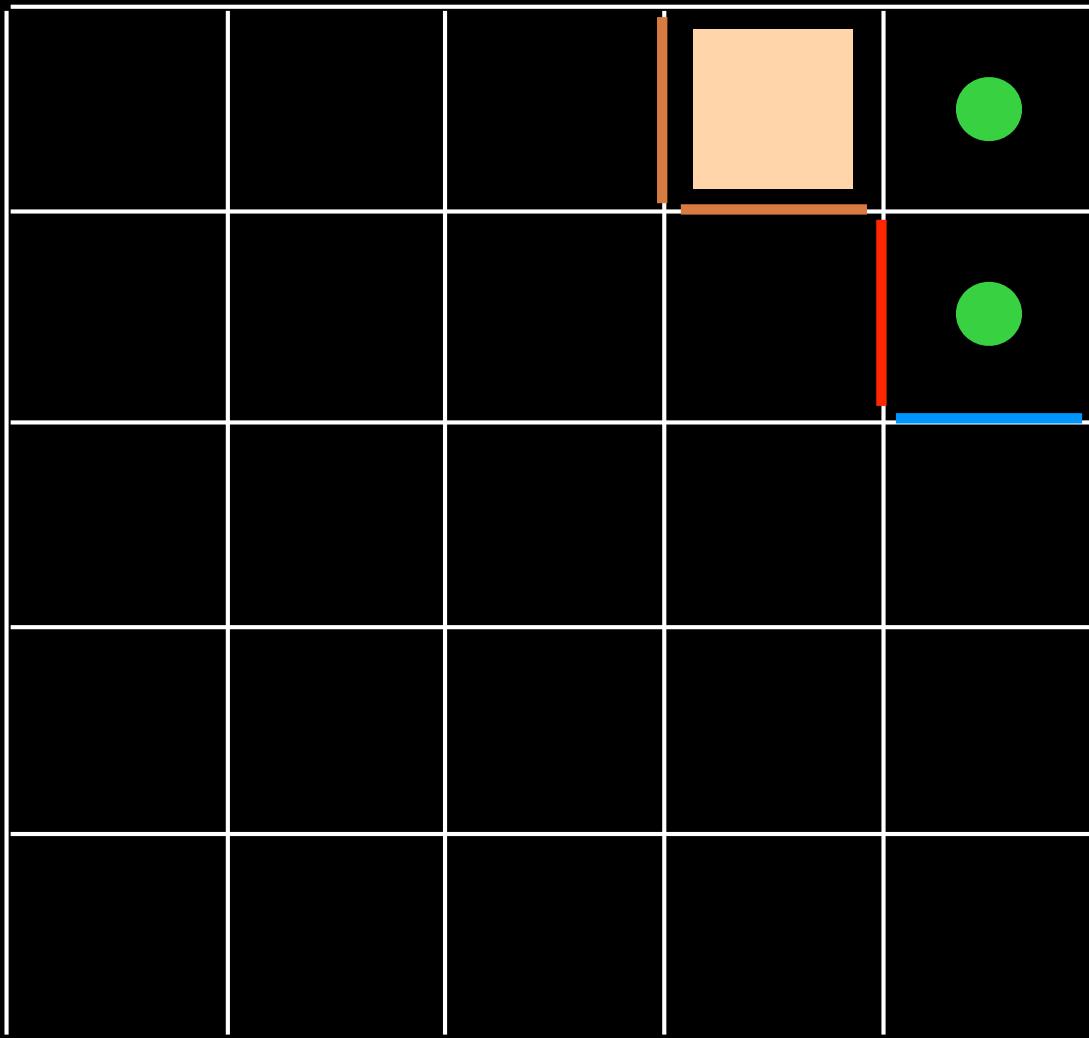
D

U



D

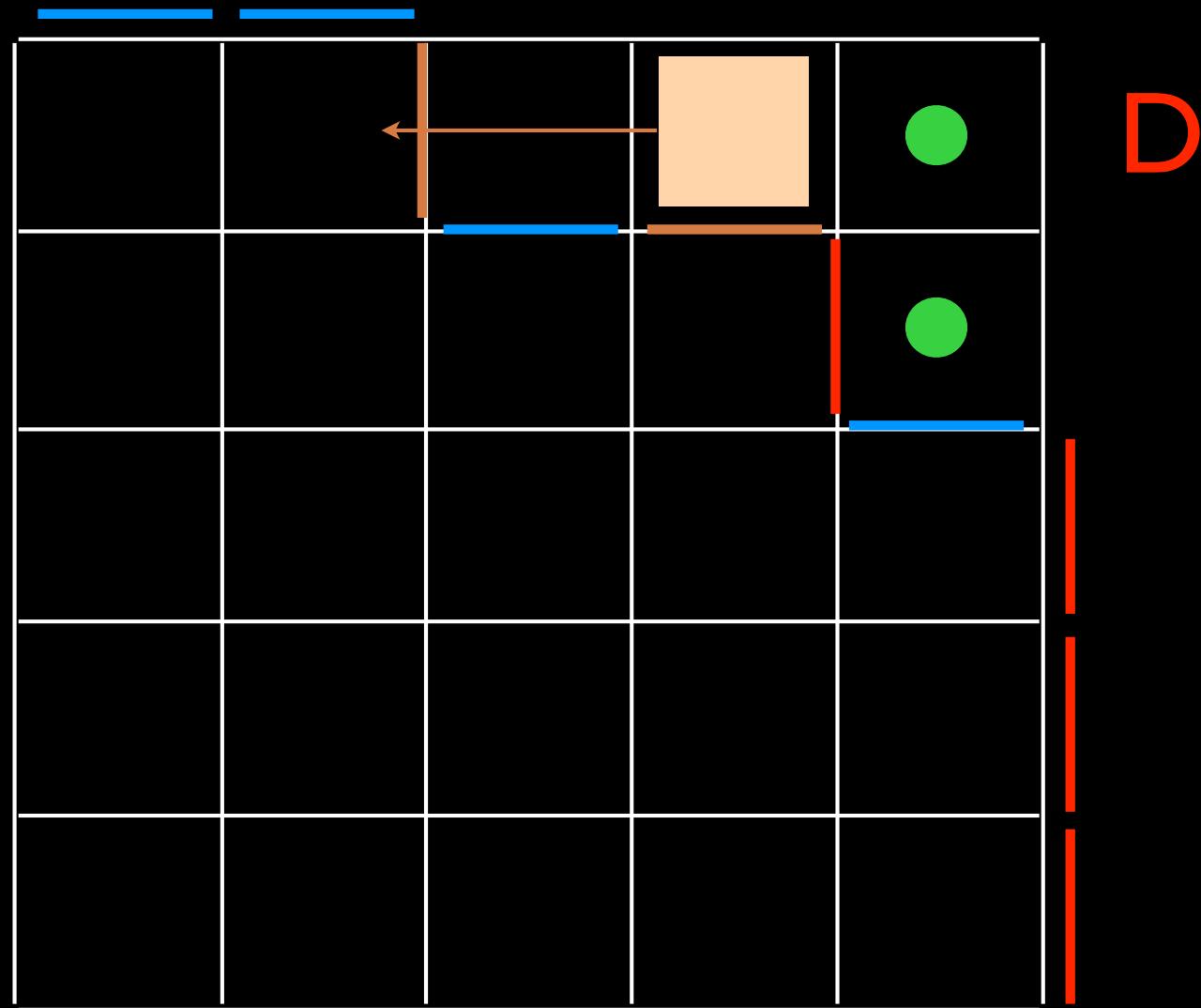
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D

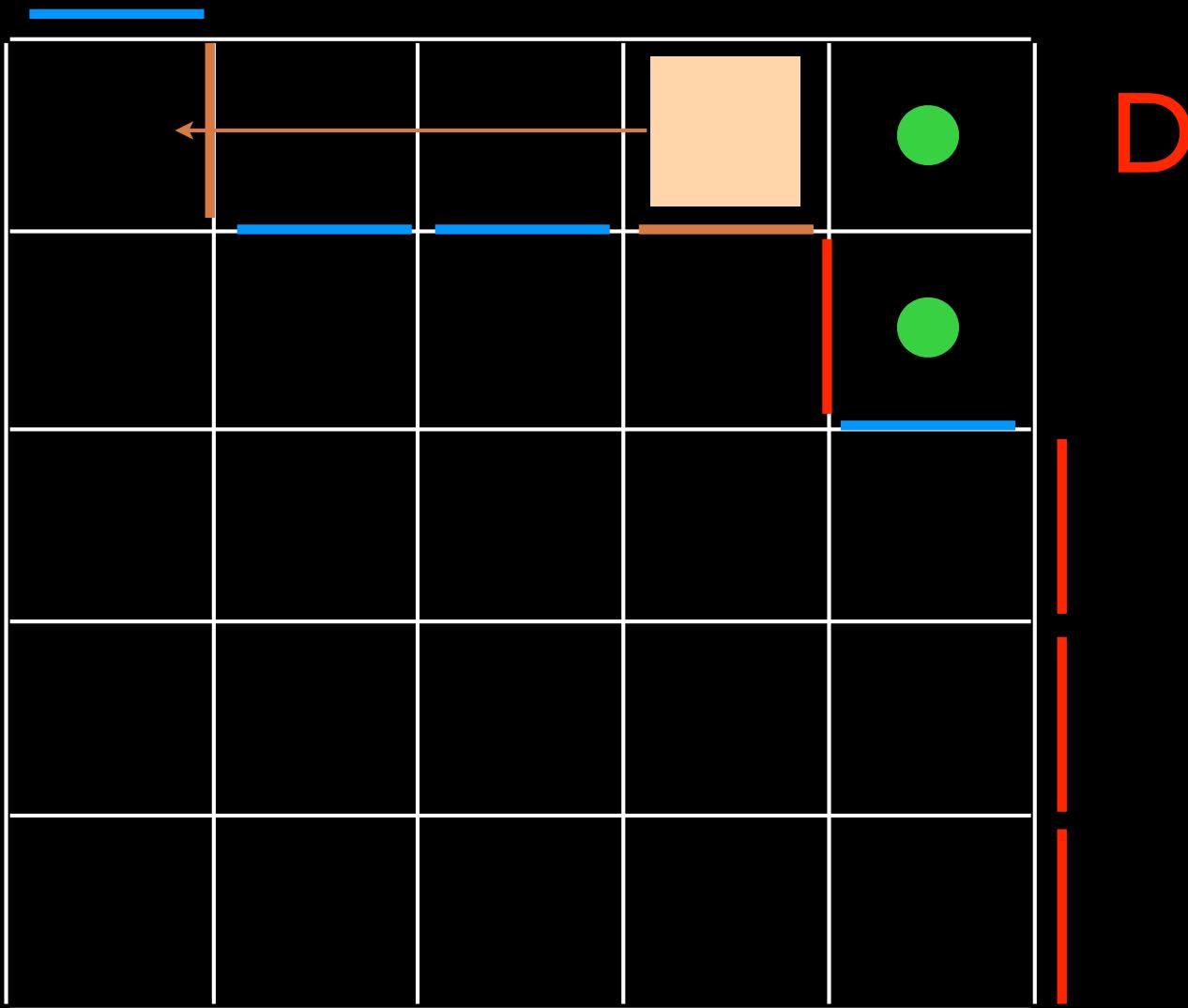


U

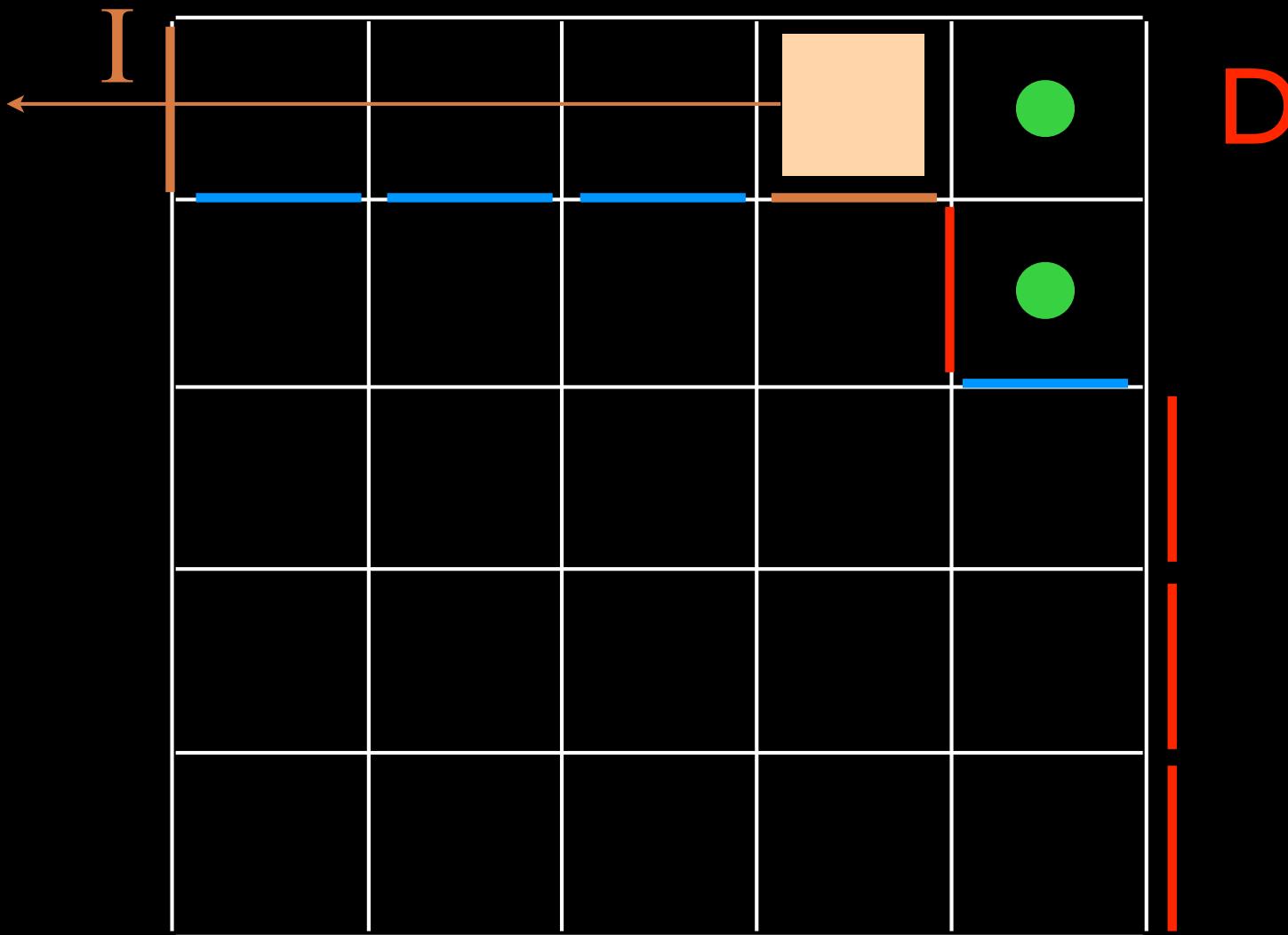


D

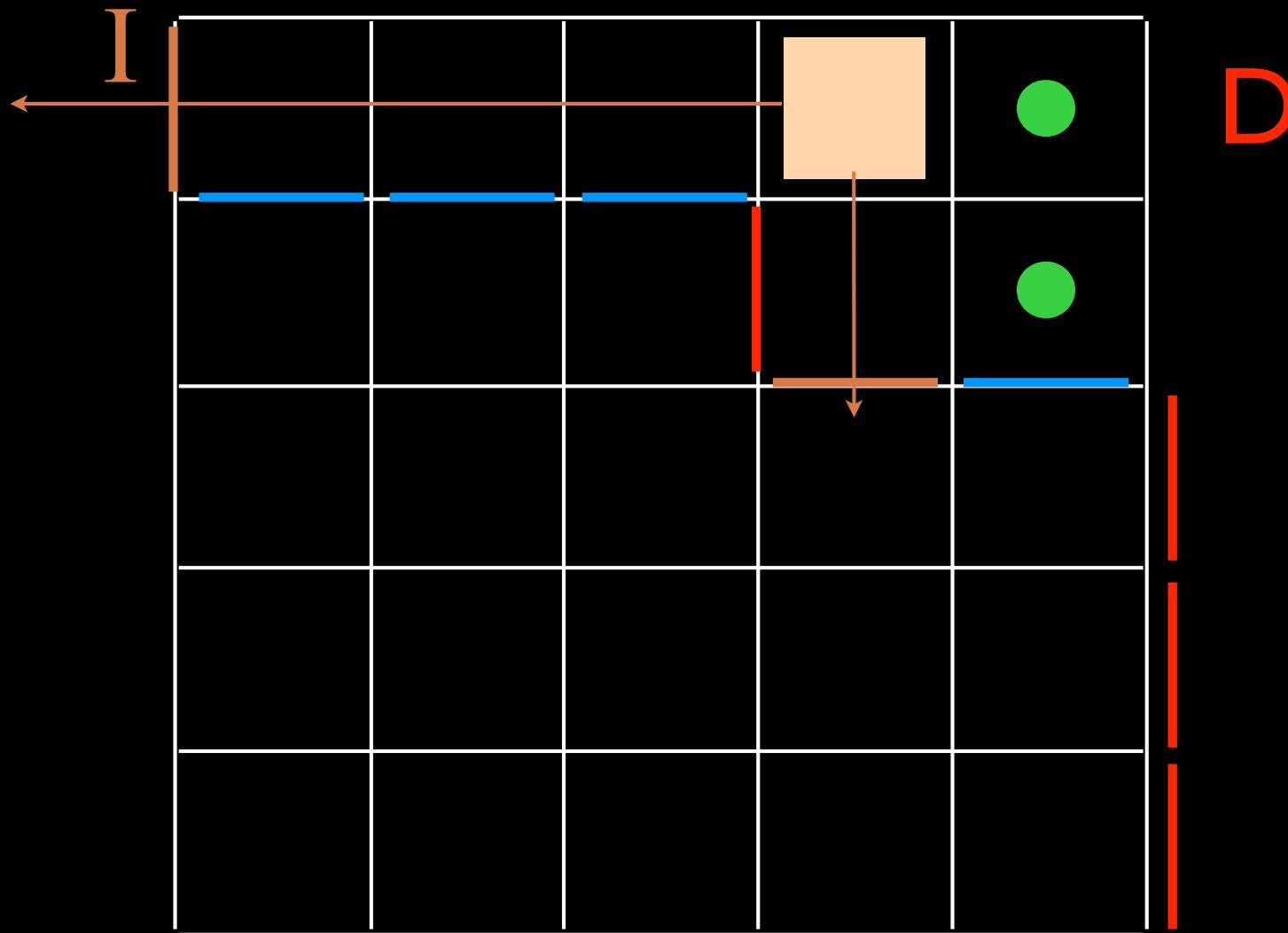
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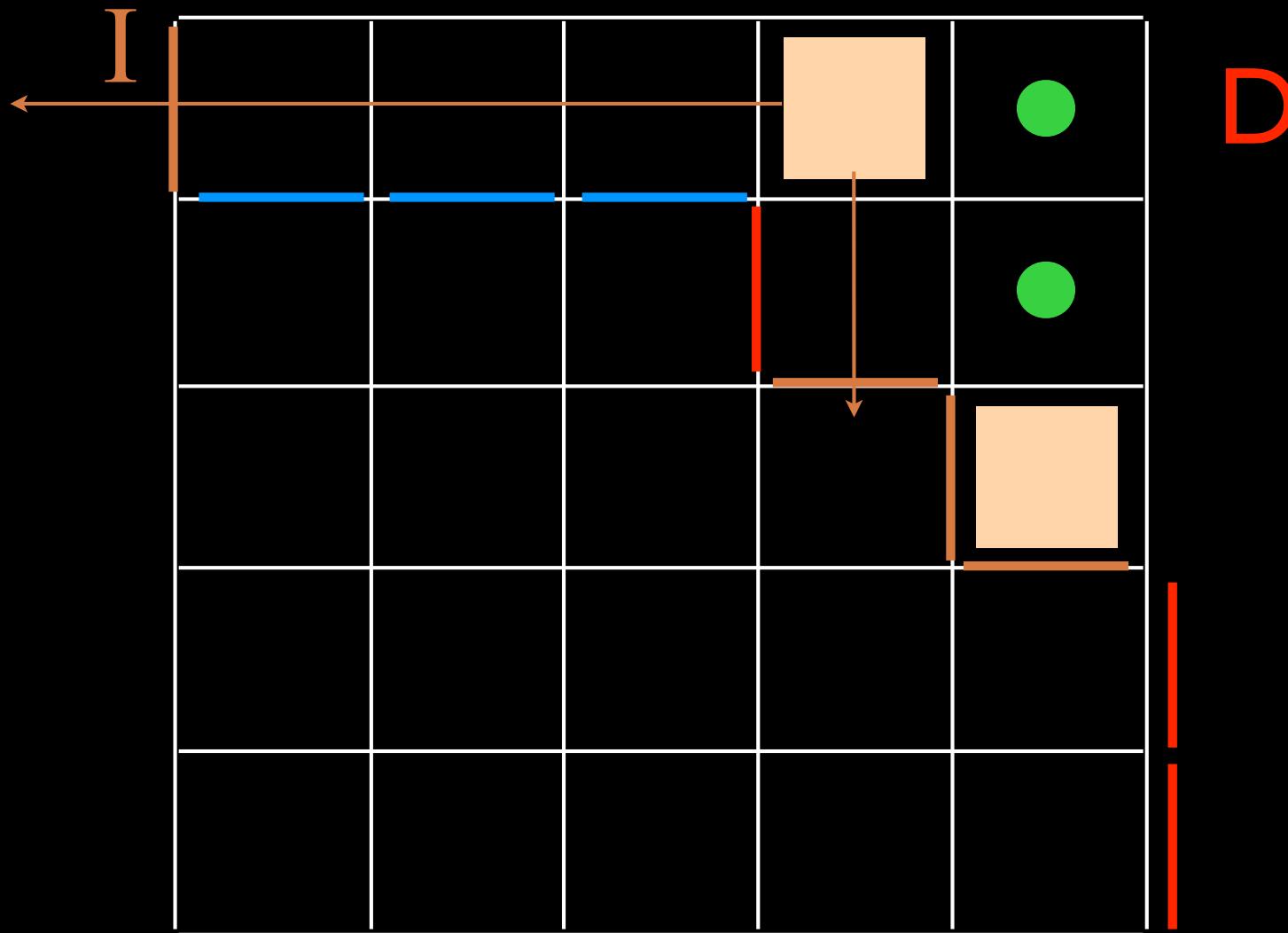
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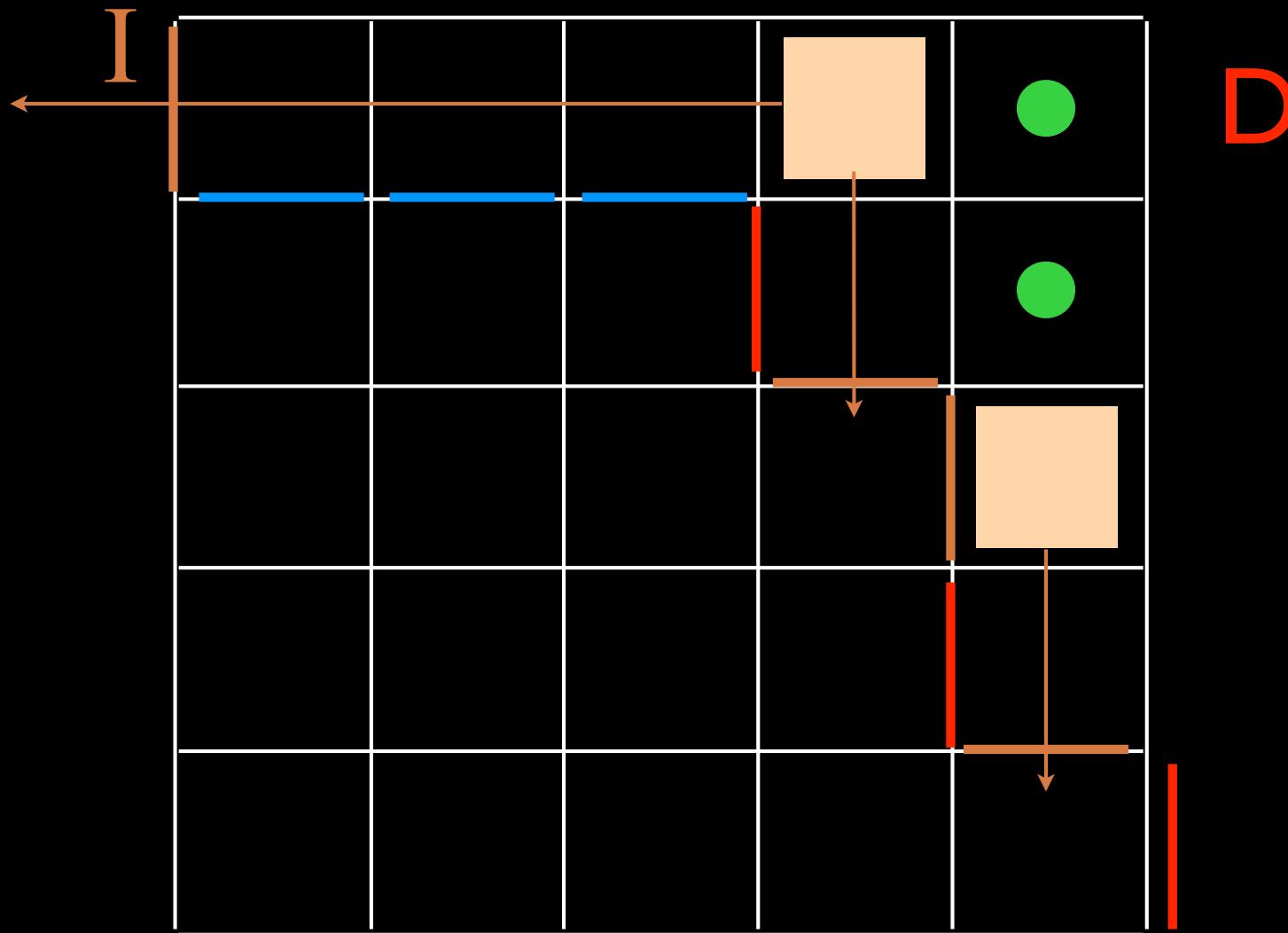
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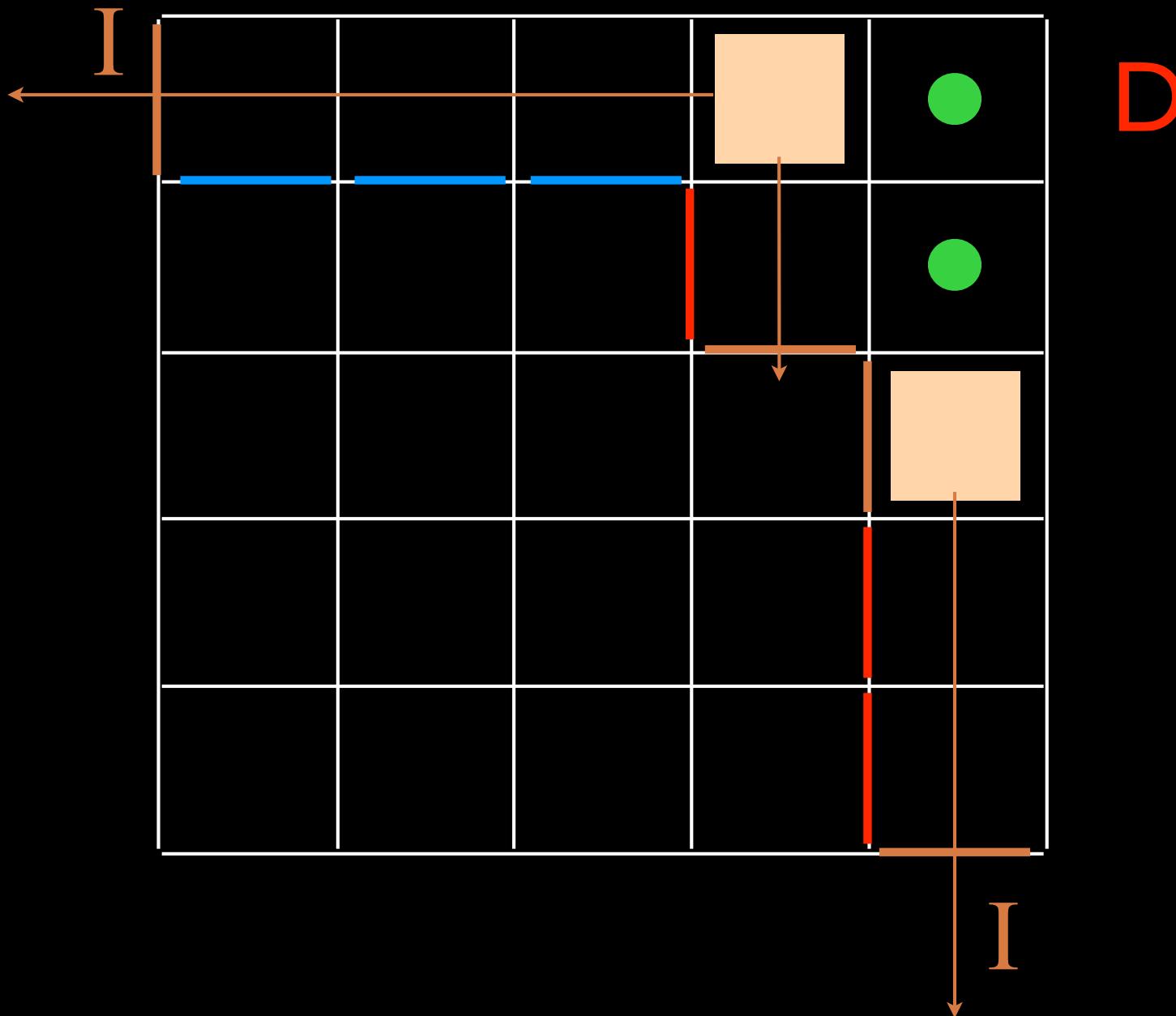
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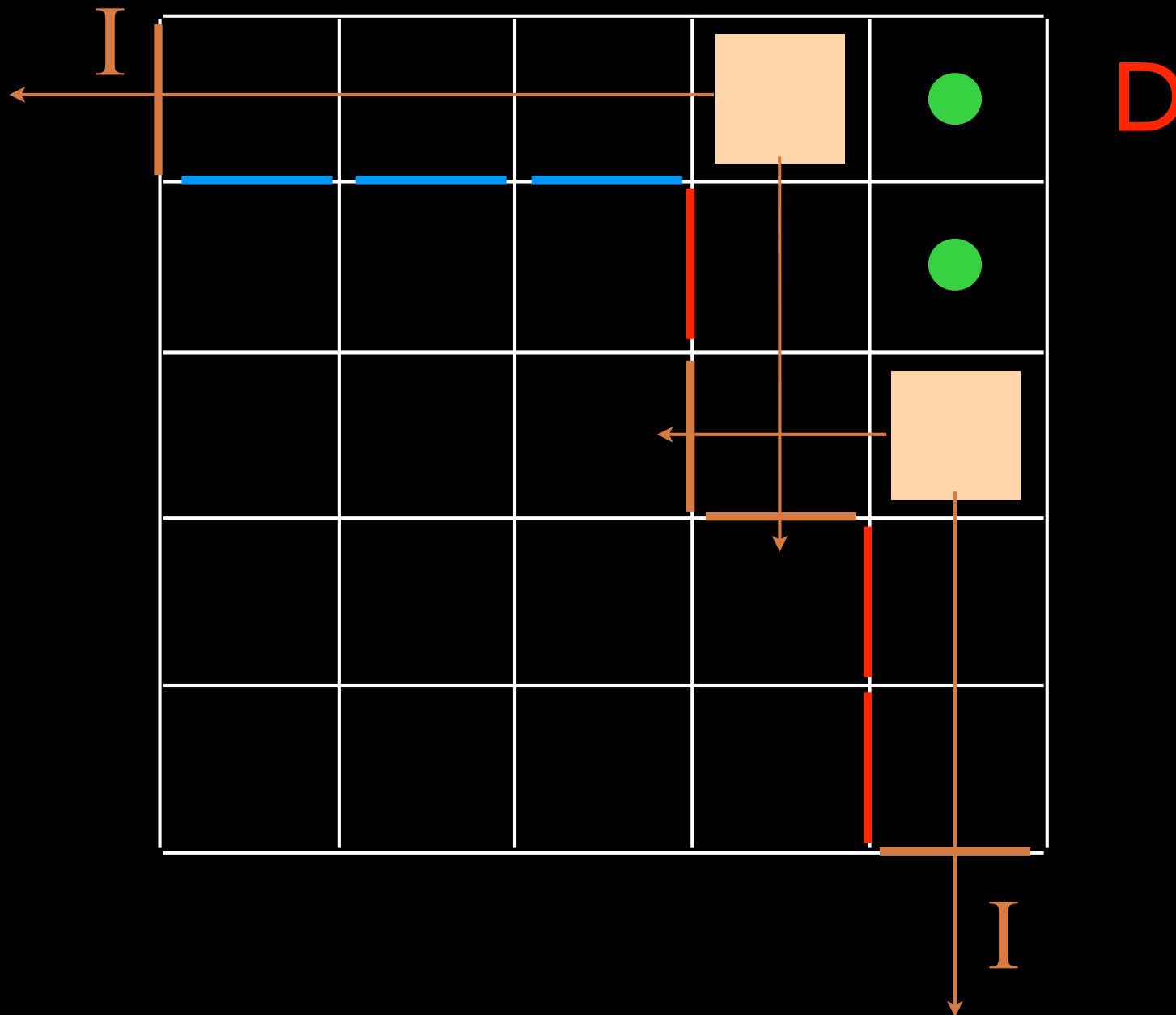
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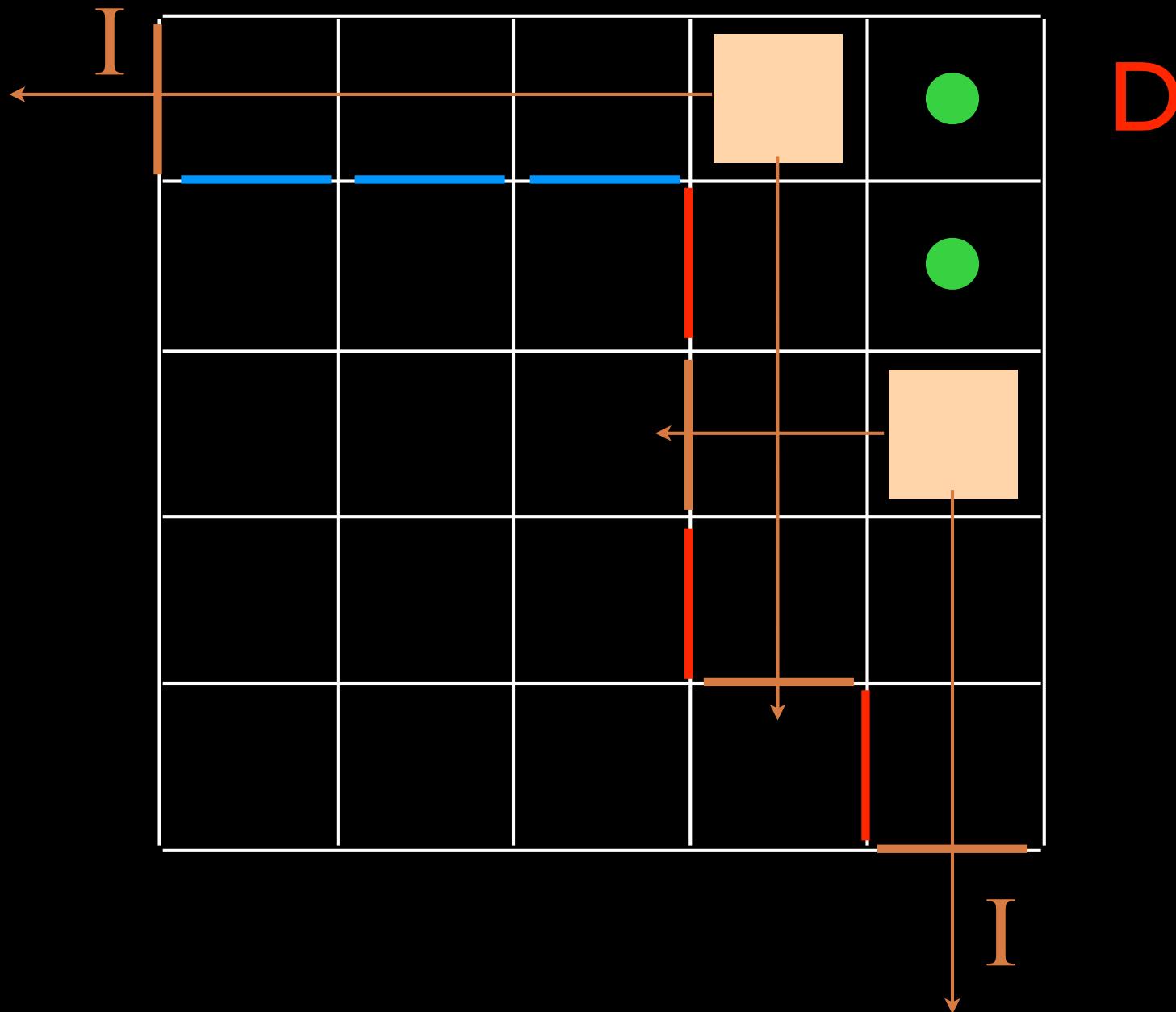
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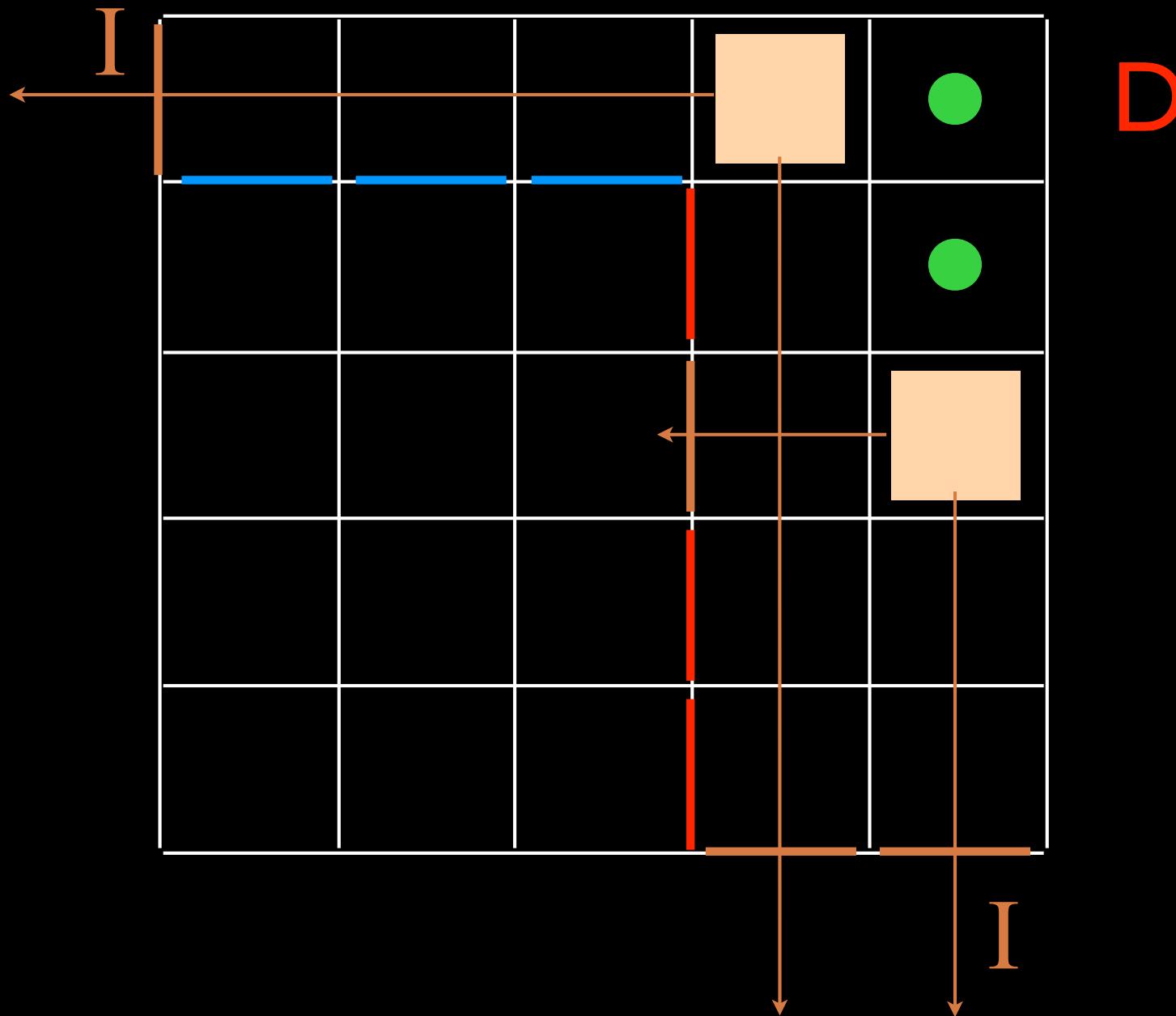
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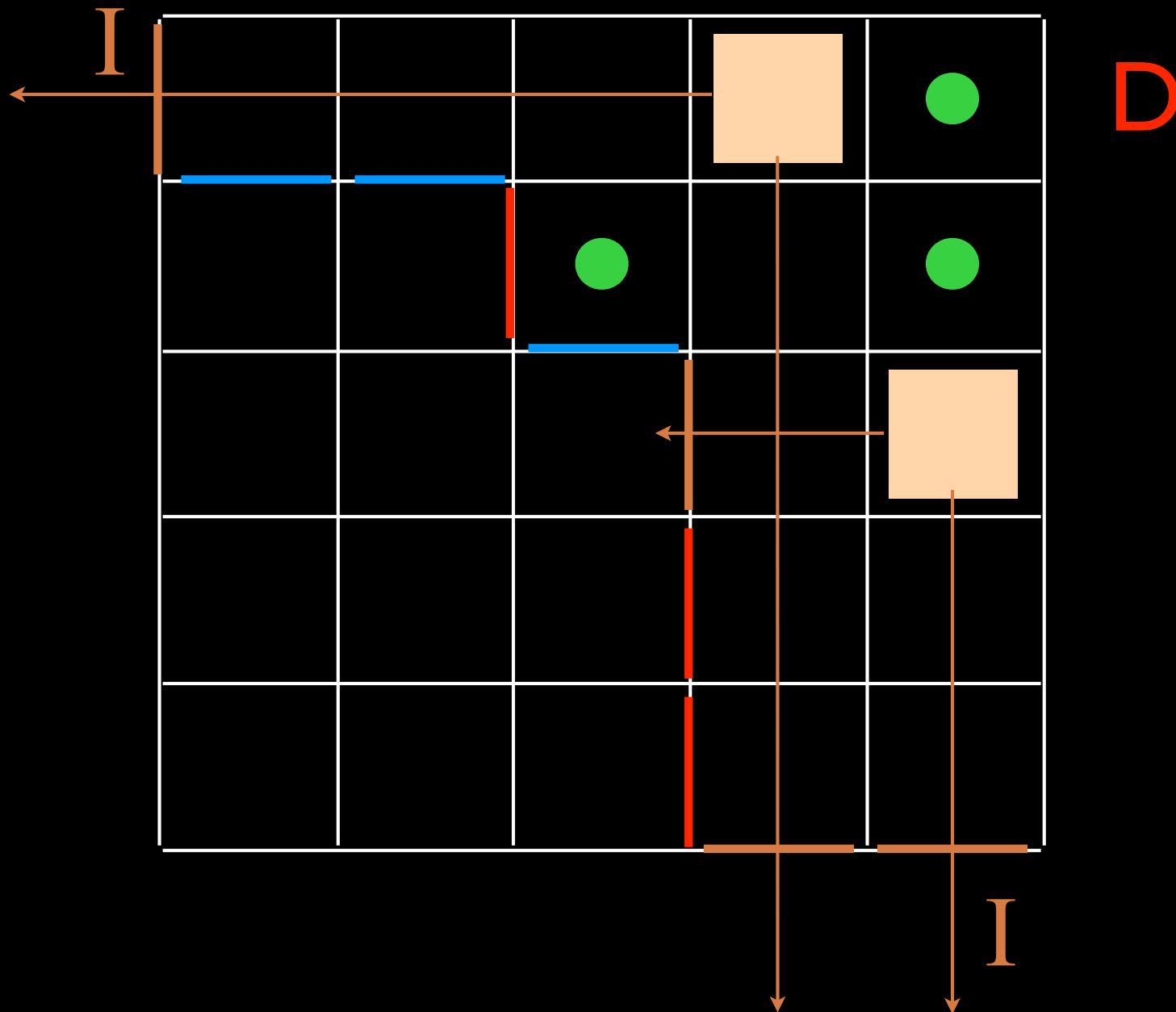
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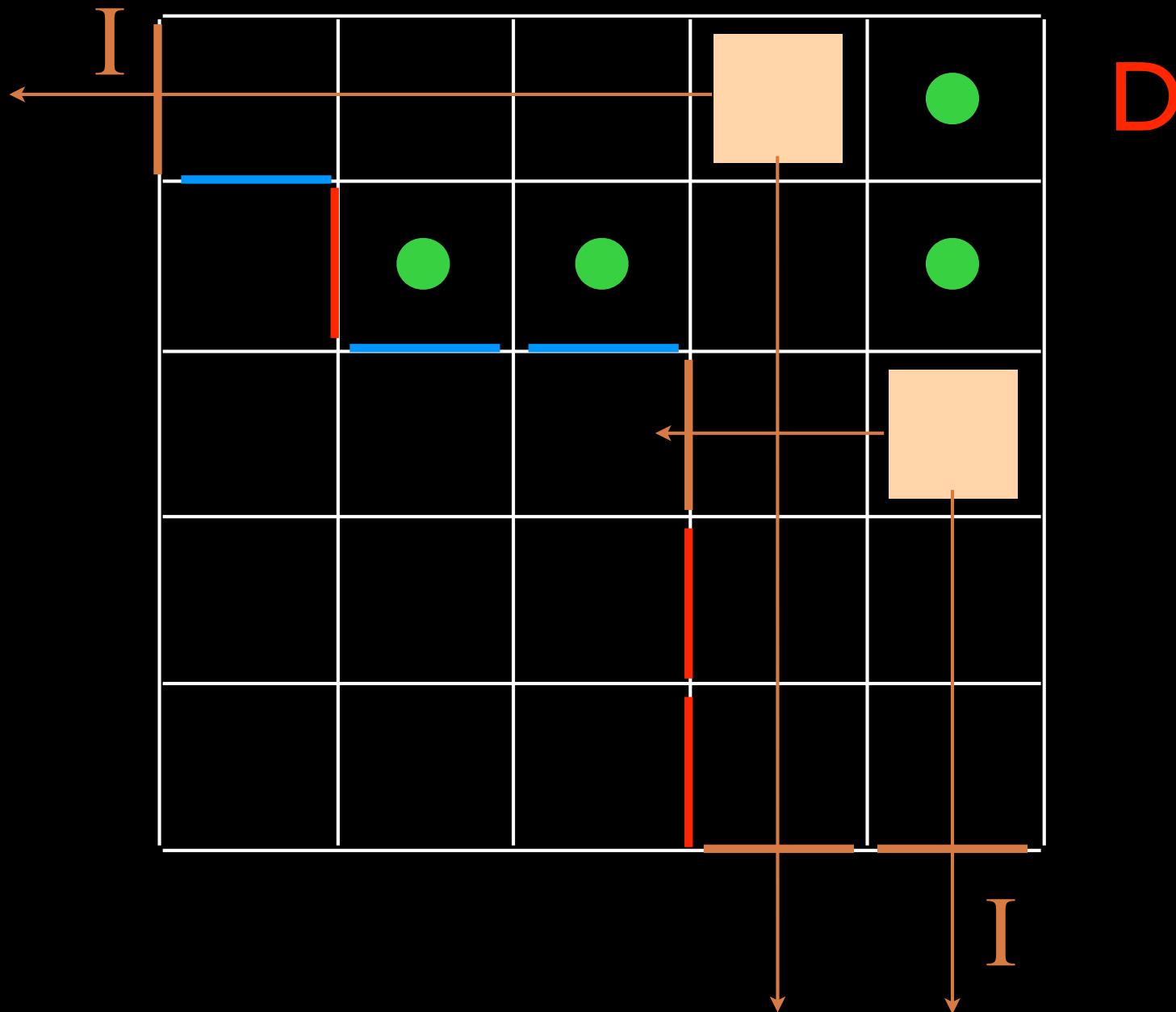
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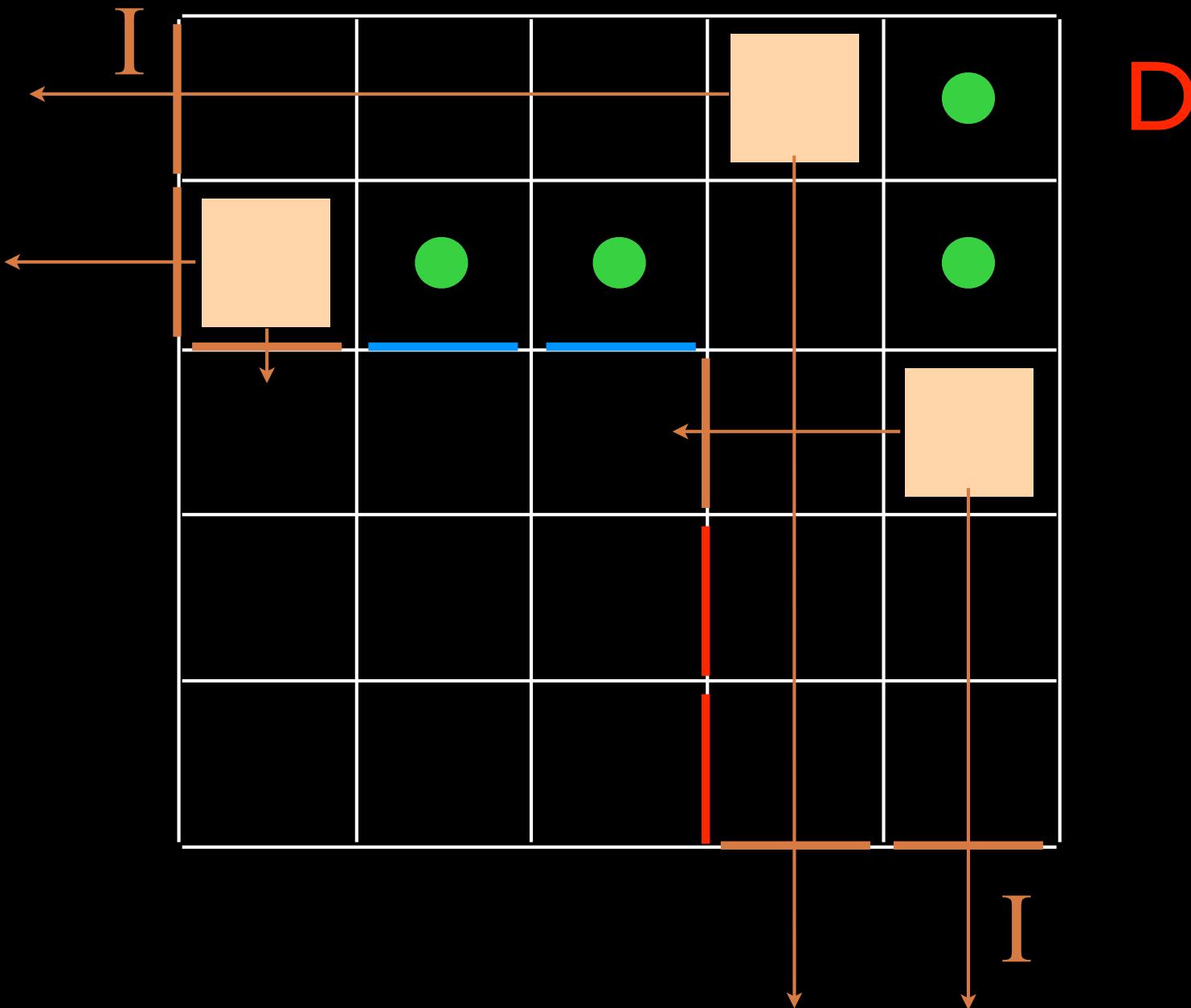
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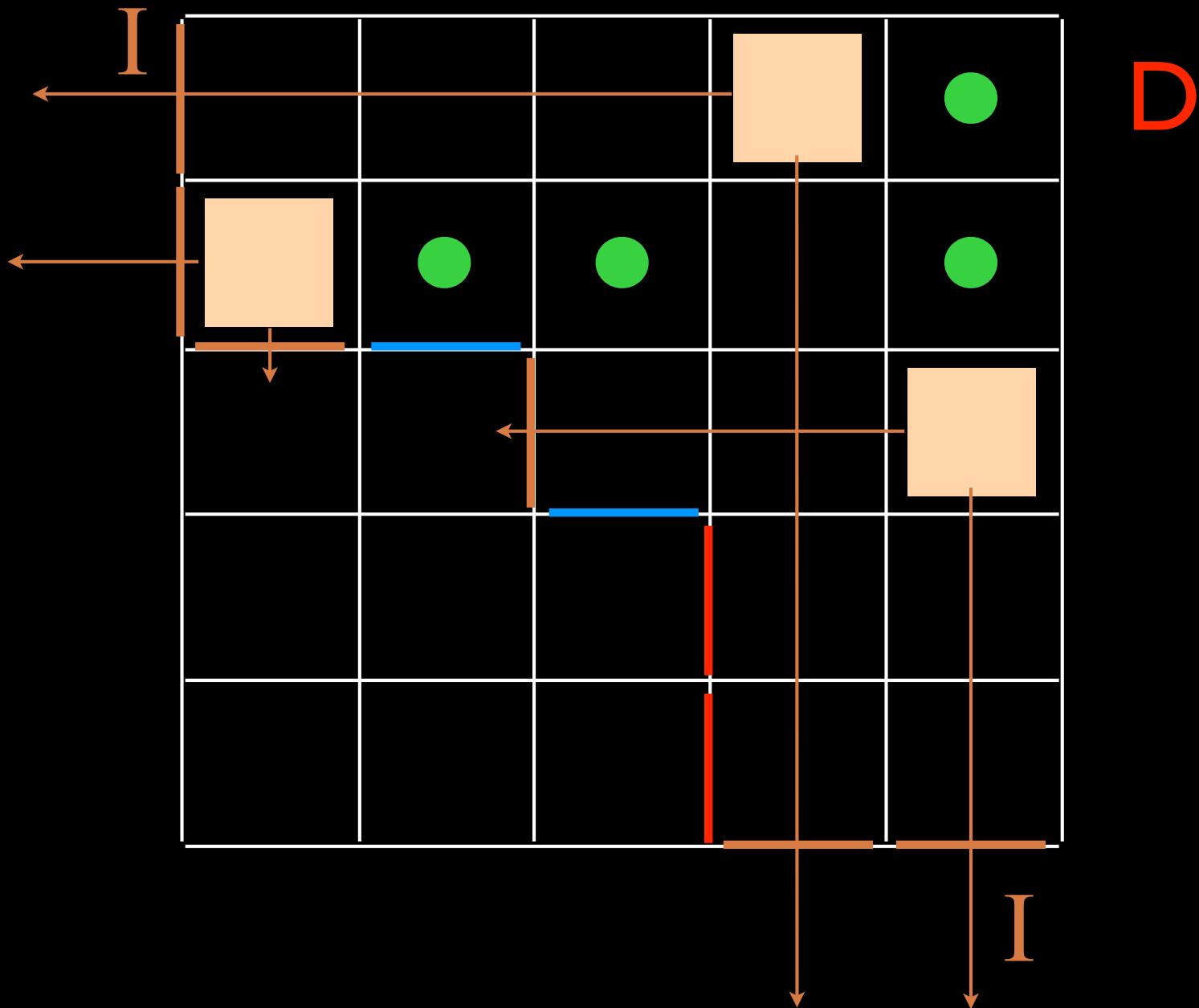
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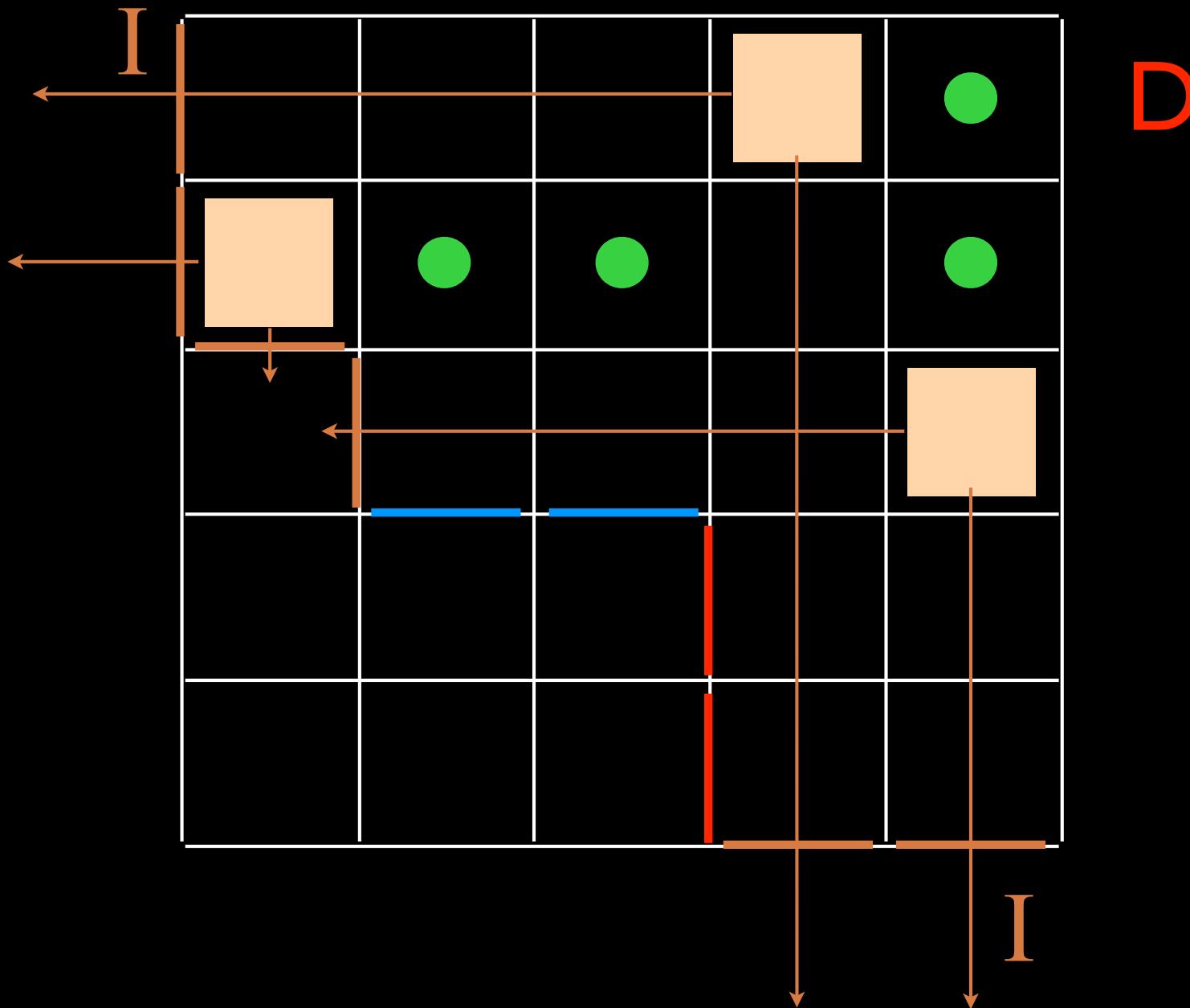
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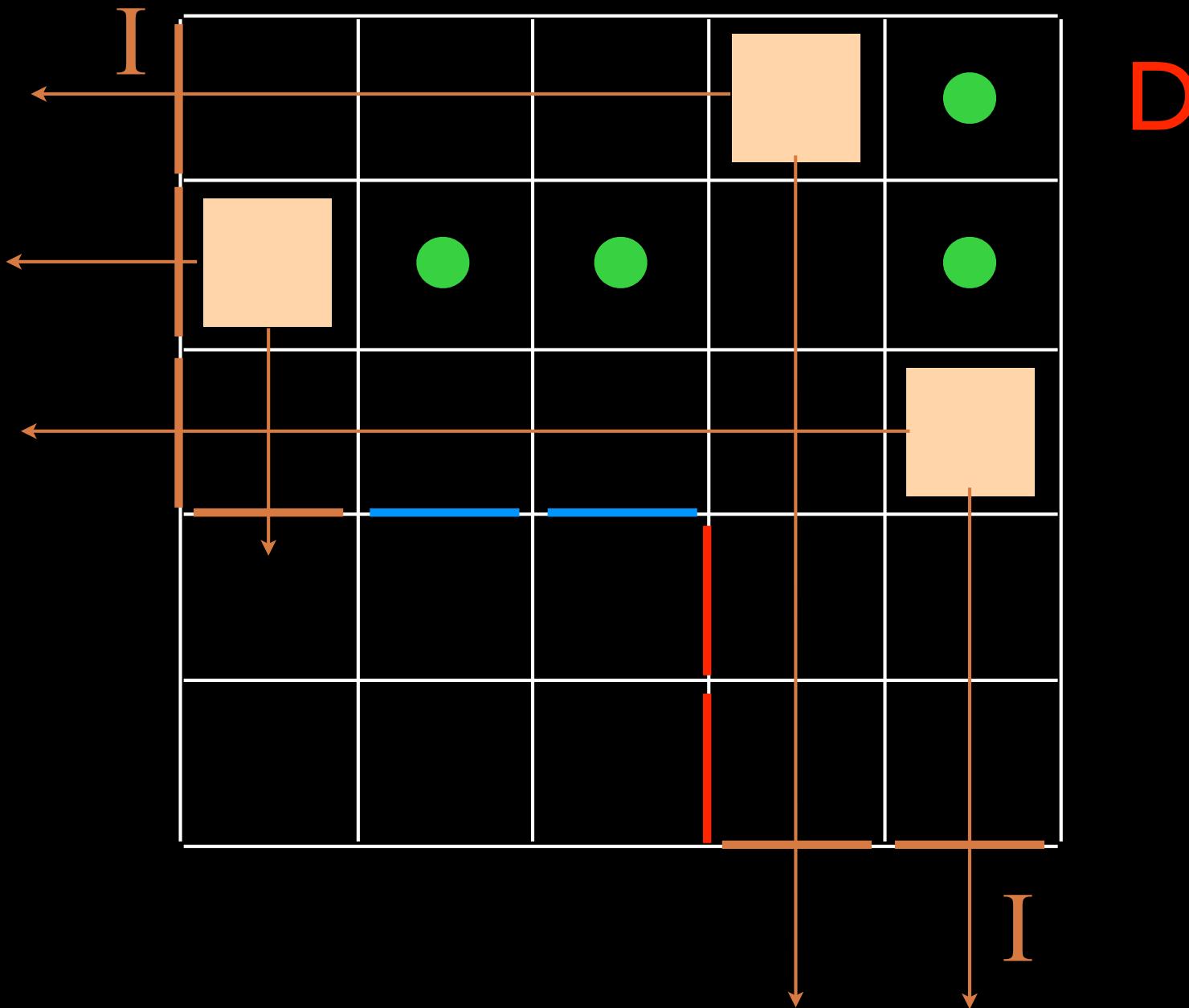
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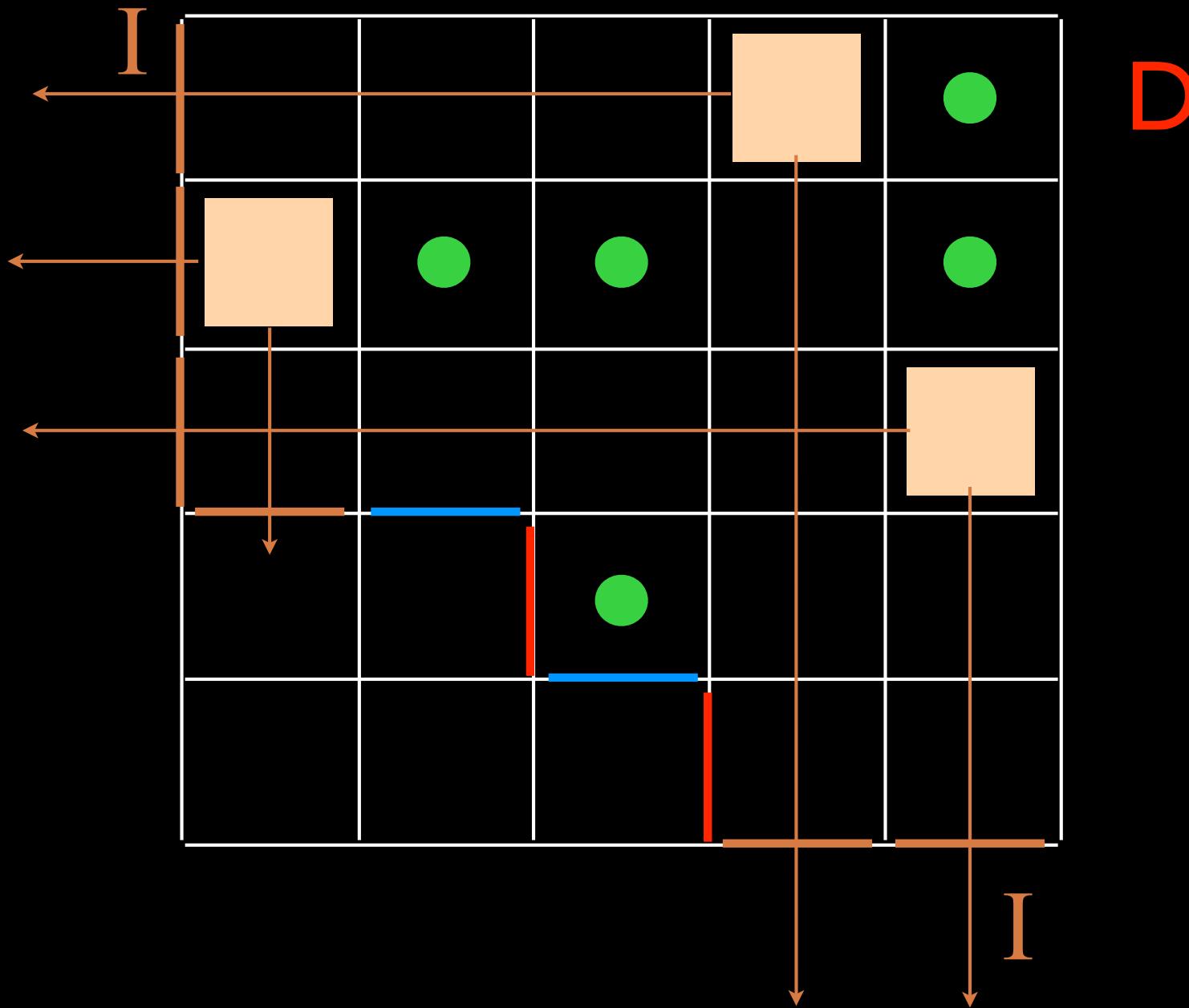
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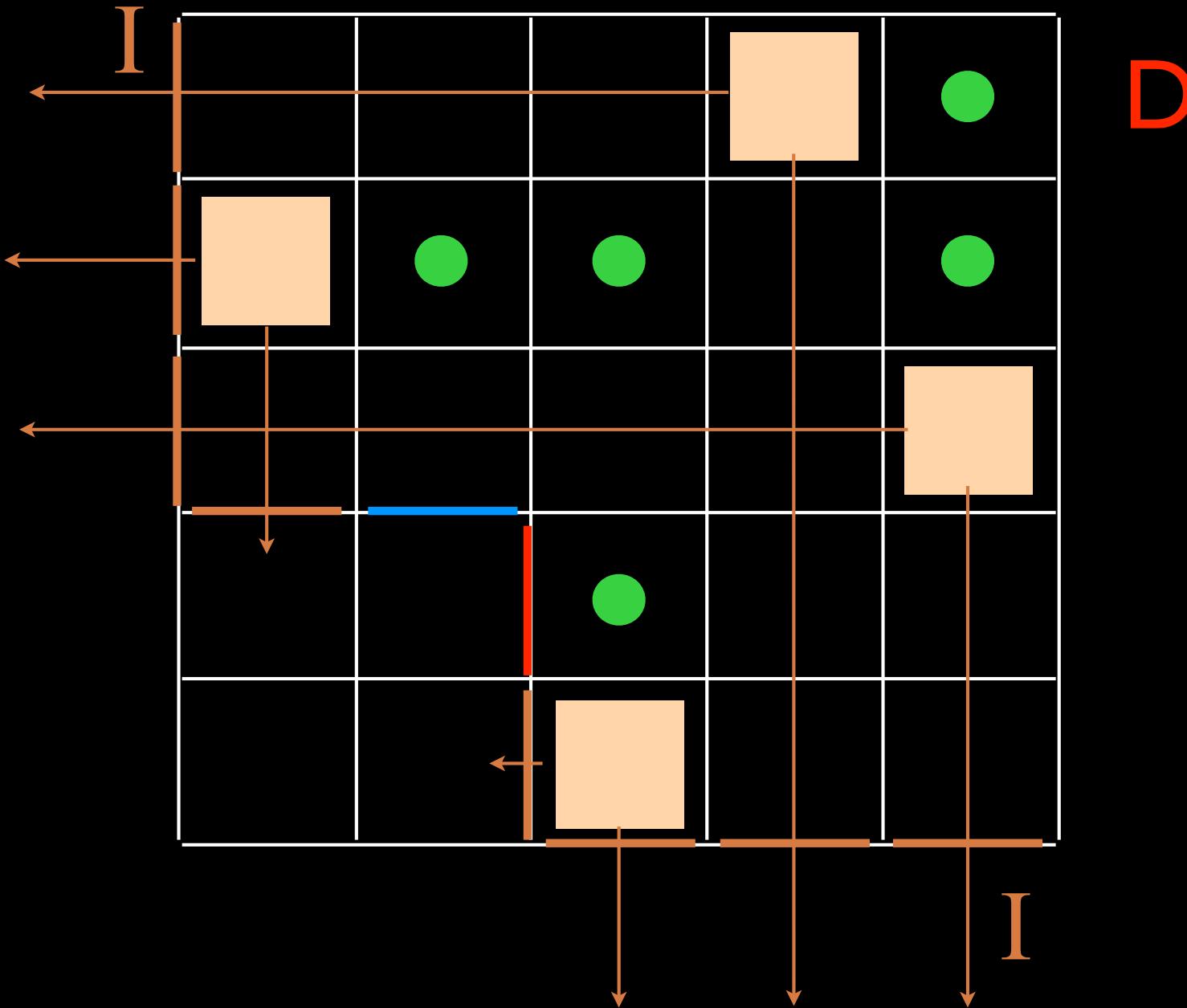
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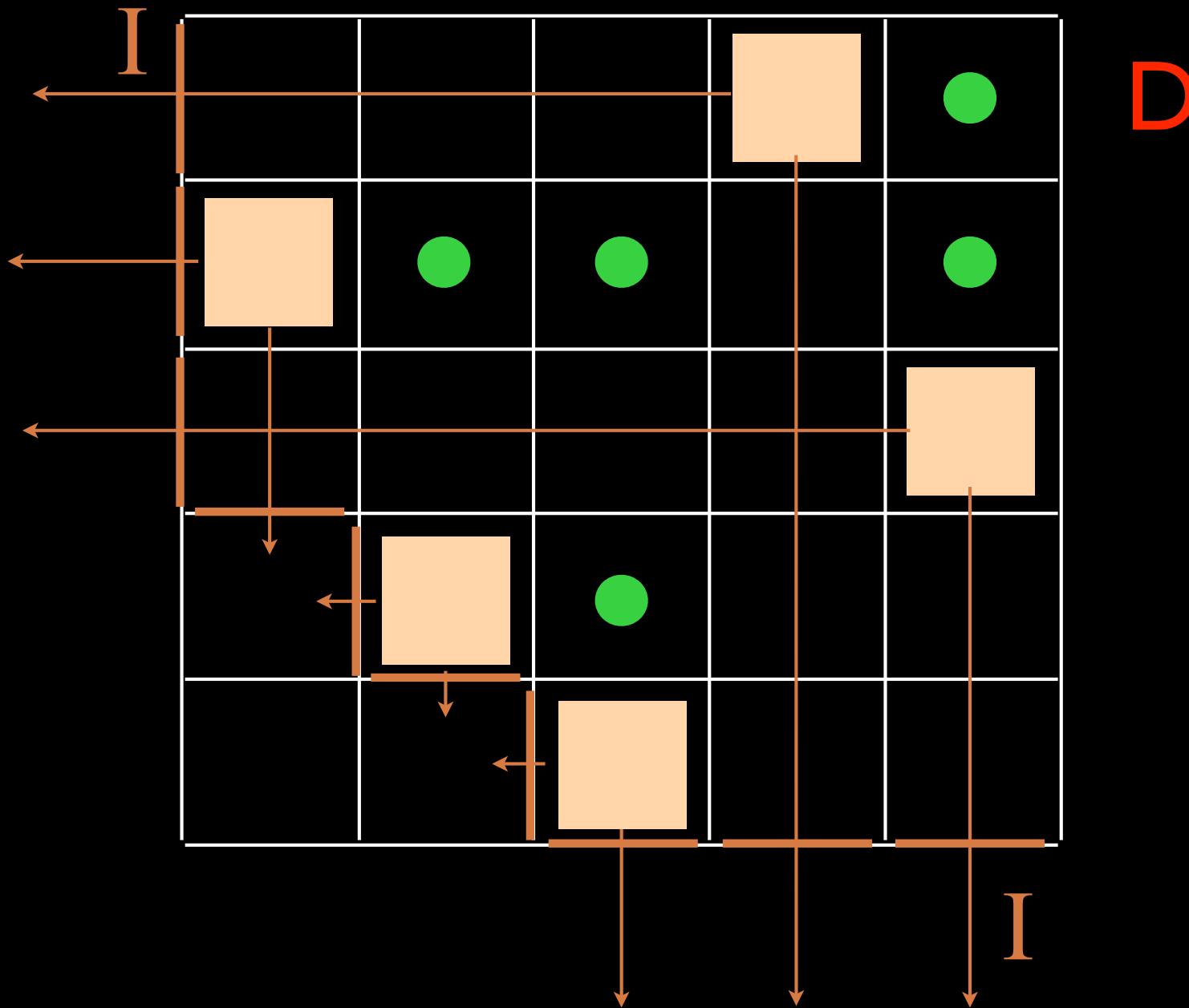
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U



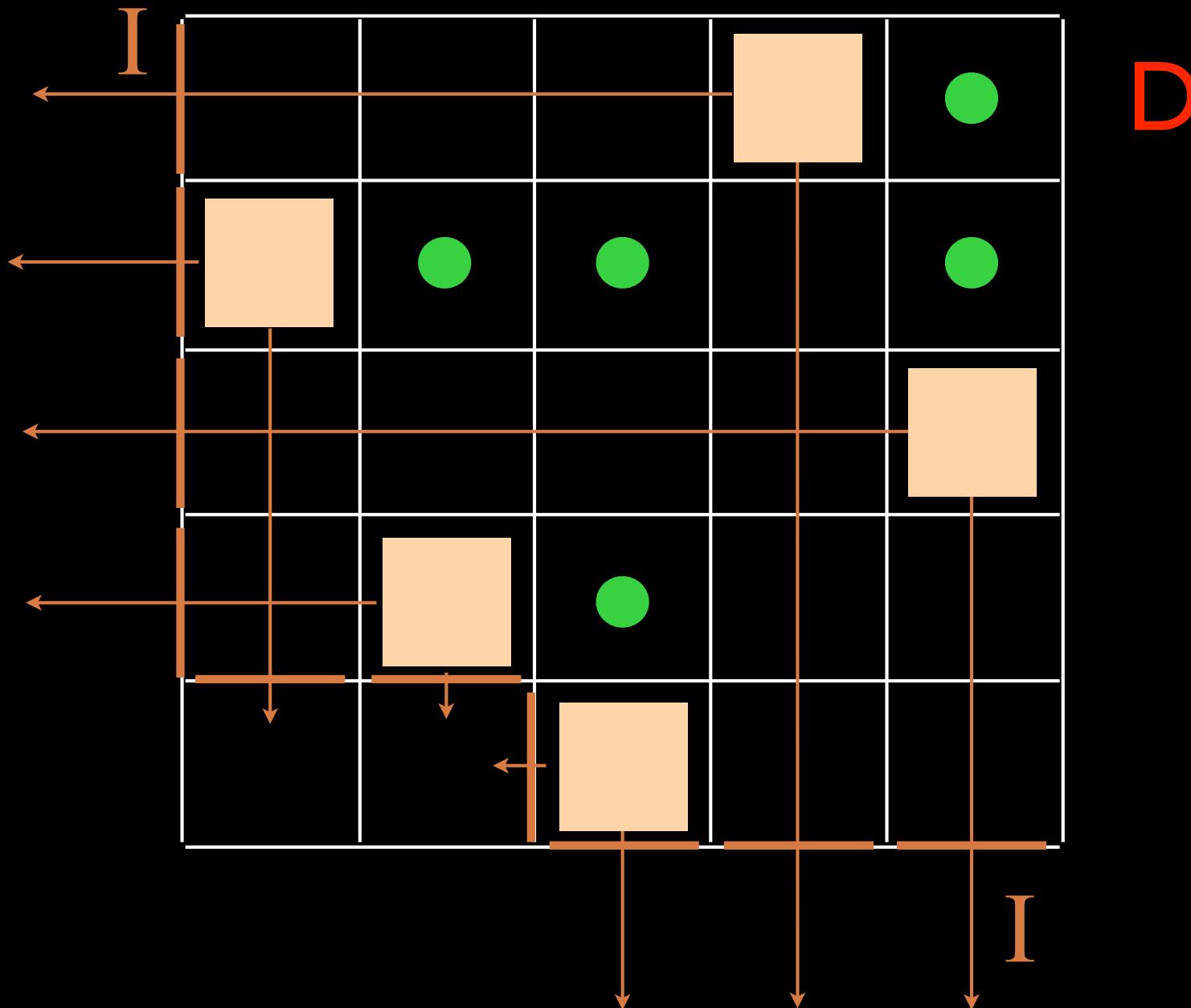
U



D

I

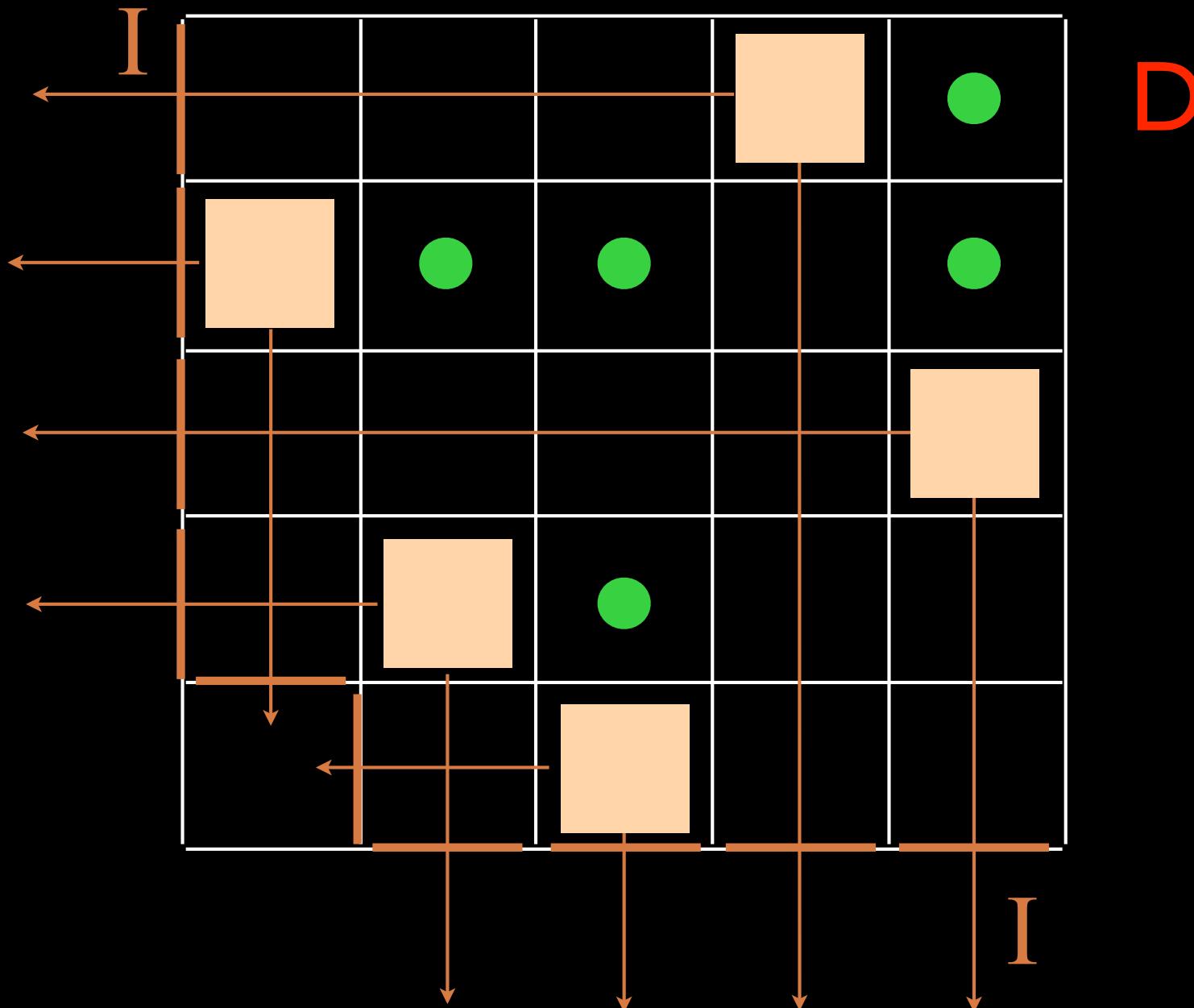
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D

I

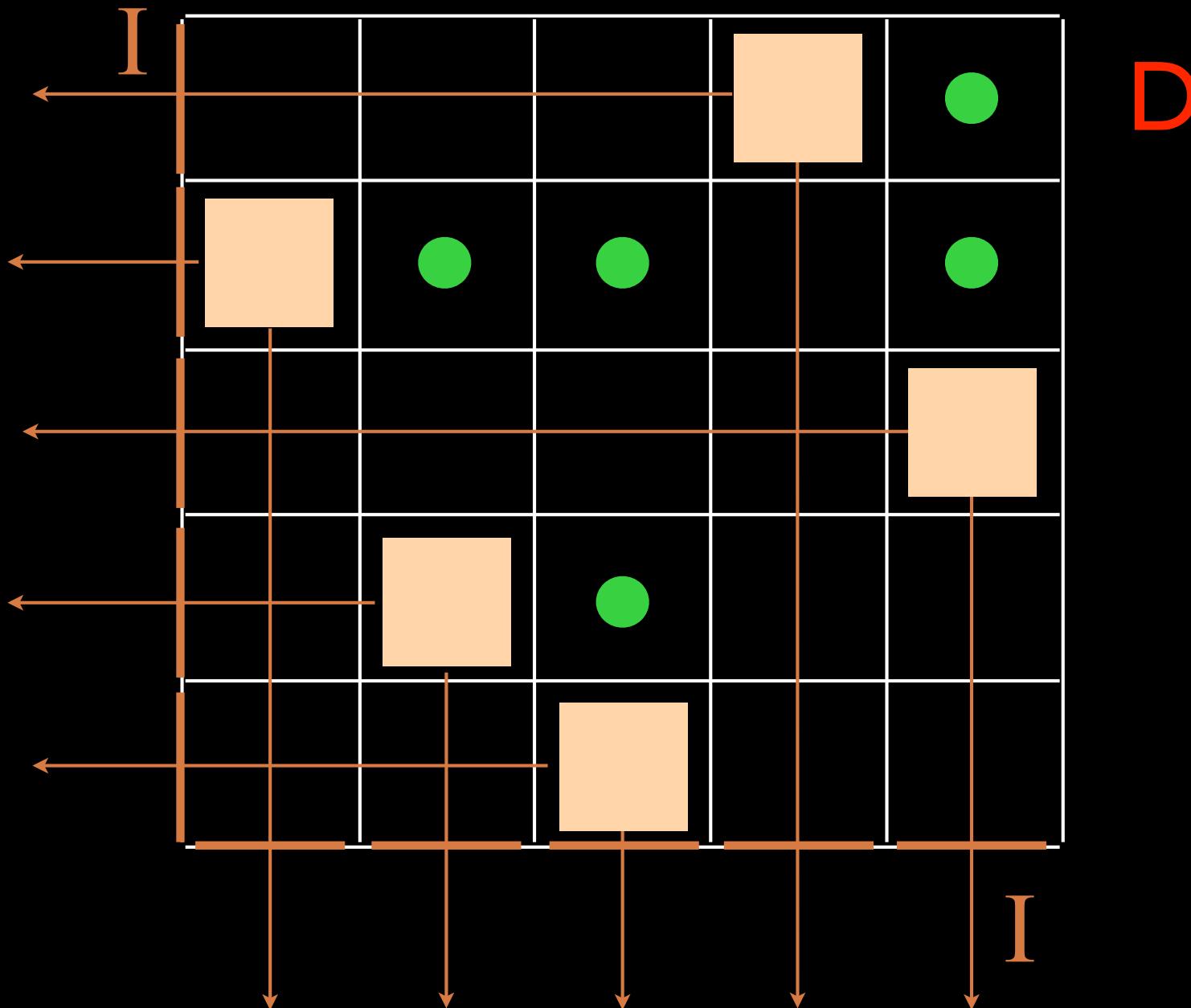
U



D

I

U



$$\left\{ \begin{array}{l} \textcolor{blue}{UD = DU + I_v I_h} \\ \textcolor{blue}{U I_v = I_v U} \\ \textcolor{brown}{I_h D = D I_h} \\ \textcolor{brown}{I_h I_v = I_v I_h} \end{array} \right.$$

Quadratic algebra \mathbb{Q}

5 rewriting rules

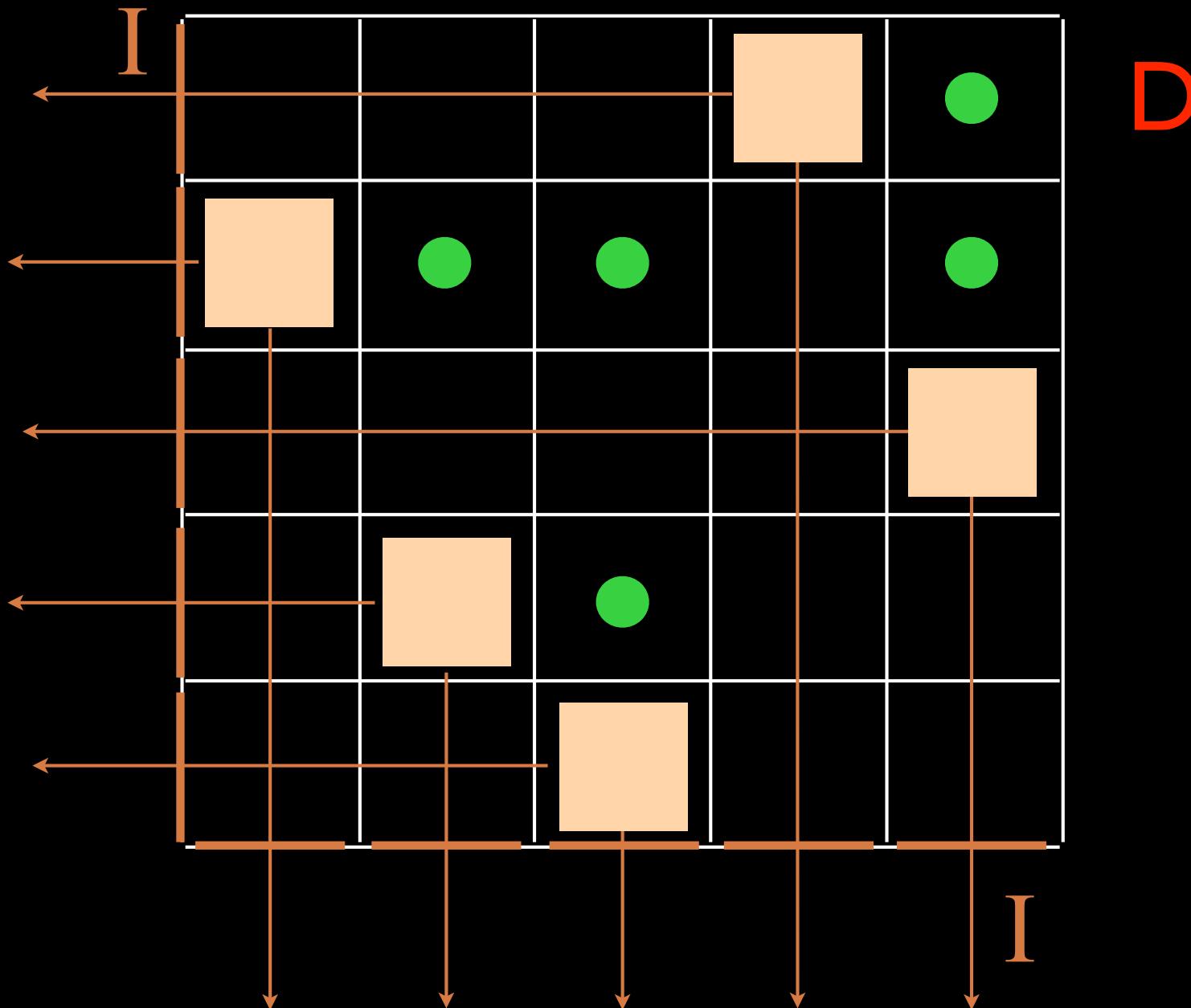
"complete"



\mathbb{Q} -tableau (5 labels)

\mathbb{Q} -tableau (2 labels)

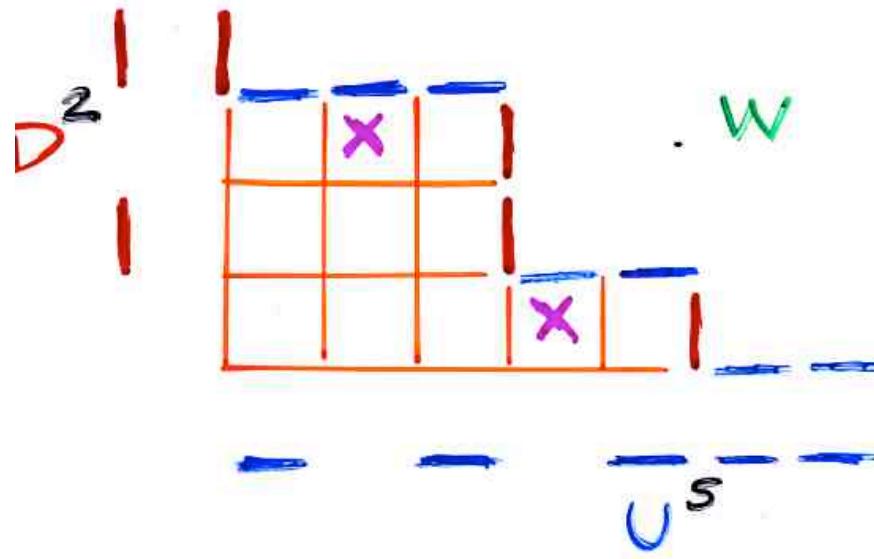
U



notation

$$w \rightarrow F_w$$

diagram
Ferrers



Prop- $c_{i,j}(w) =$ nb de "placement" tours sur F

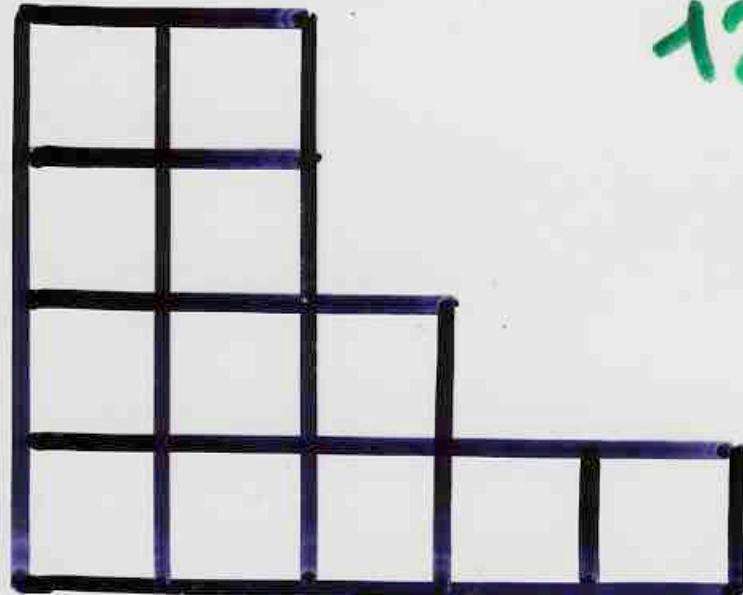
avec $i = |w|_D - k$
 $j = |w|_U - k$

An introduction to RSK

G. de B. Robinson, 1938
C. Schensted, 1961

2
2
3
5

12

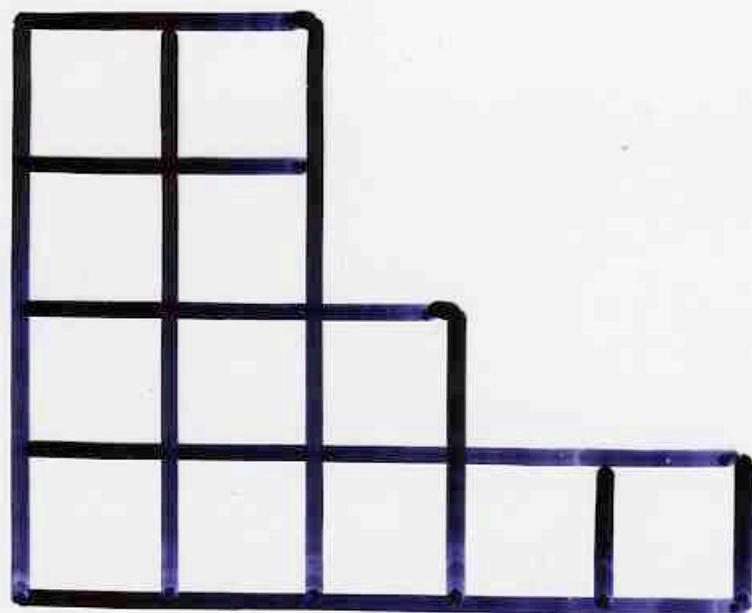


$$12 = n = 5 + 3 + 2 + 2$$

Ferrers

diagram

Partition of n



7	12			
6	10			
3	5	9		
1	2	4	8	11

Young
tableau

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1						

3						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2						
1						

3						
1						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2									
1	3								

3									
1	6								

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2									
1	3	4							

3									
1	6	10							

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2									
1	3	4							

3									
1	6	10							2

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2									
1	3	4							

3					6				
1	2	10							

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5								
1	3	4							

3	6								
1	2	10							

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5								
1	3	4							

3	6								
1	2	10							5

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5								
1	3	4							

3	6								
1	2	5							

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4							

3	6	10							
1	2	5							

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

3	6	10							
1	2	5	8						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

3	6	10							
1	2	5	8					4	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

3	6	10							
1	2	4	8						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

3	6	10							
1	2	4	8						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

				6					
3	5	10							
1	2	4	8						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8									
2	5	6							
1	3	4	7						

6									
3	5	10							
1	2	4	8						

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8									
2	5	6							
1	3	4	7	9					

6									
3	5	10							
1	2	4	8	9					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8									
2	5	6							
1	3	4	7	9					

6									
3	5	10							
1	2	4	8	9				7	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8									
2	5	6							
1	3	4	7	9					

6									
3	5	10							
1	2	4	7	9					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8									
2	5	6							
1	3	4	7	9					

6									
3	5	10							
1	2	4	7	9					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8									
2	5	6							
1	3	4	7	9					

6									10
3	5	8							
1	2	4	7	9					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$

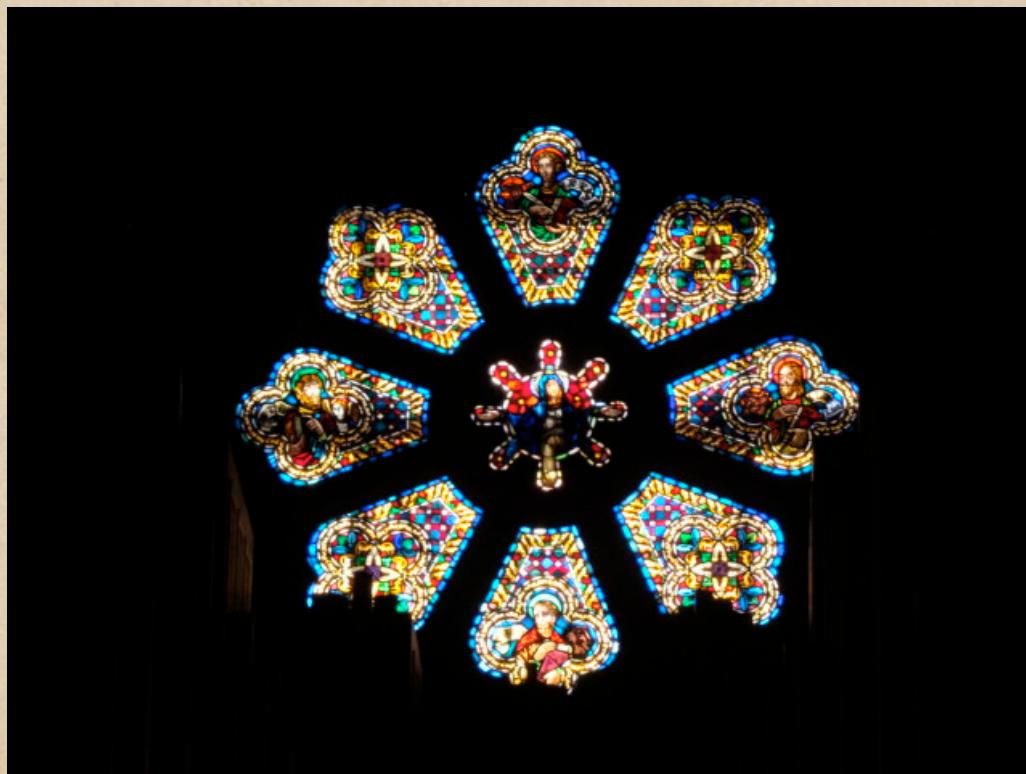
Donald Knuth

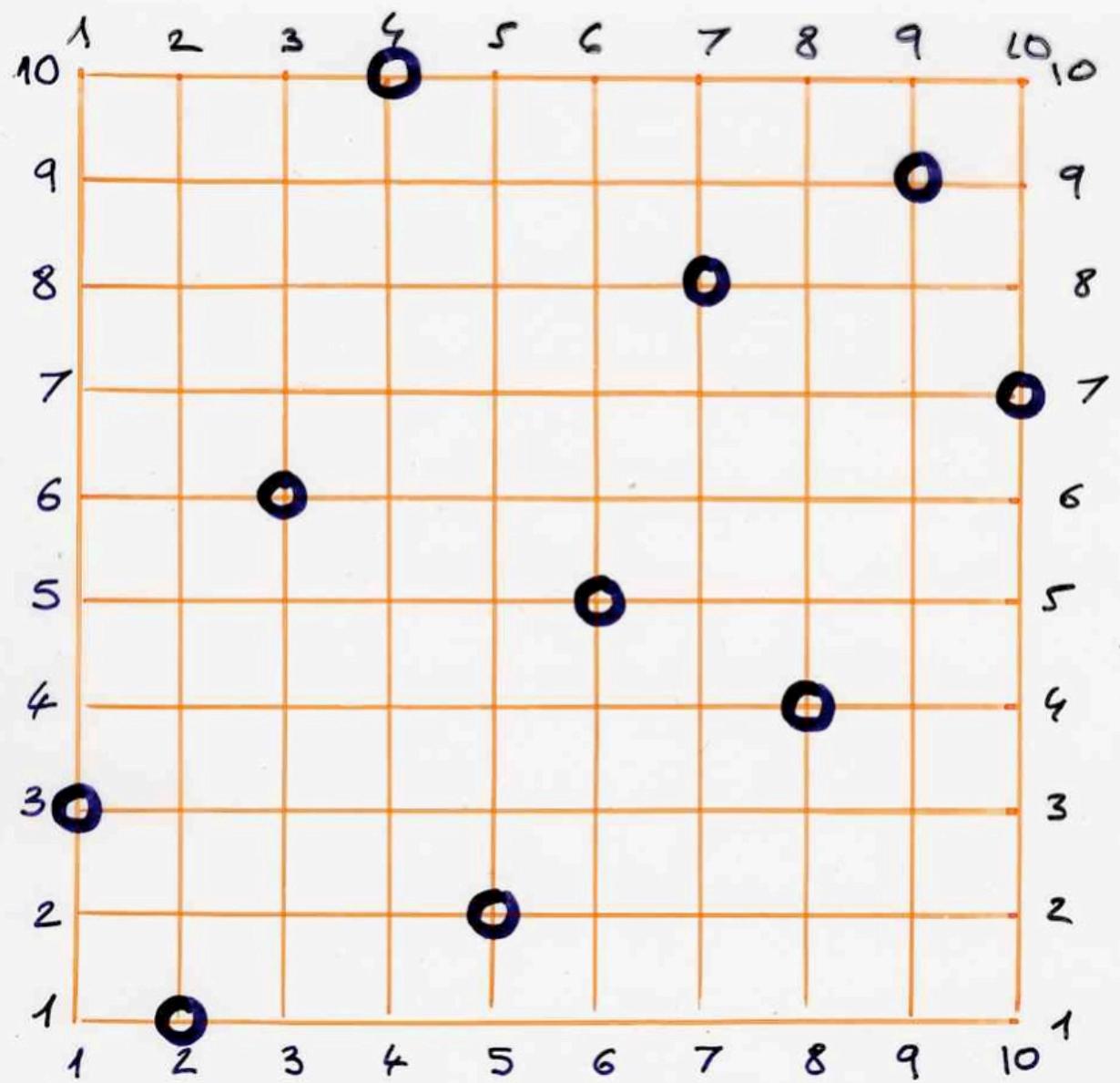
(1972)

"The unusual nature of these coincidences might lead us to suspect that some sort of witchcraft is operating behind the scenes"

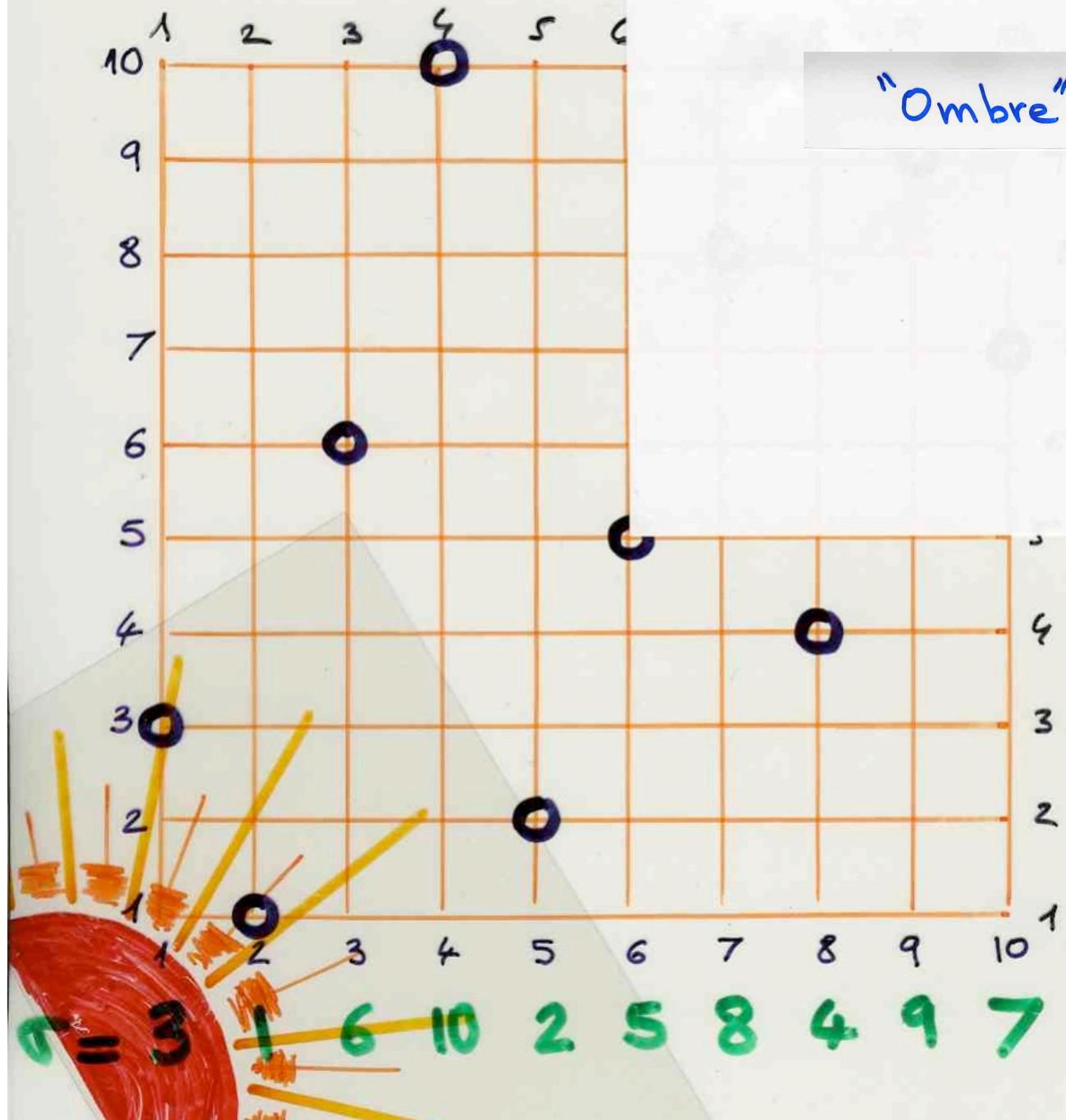
A geometric version of RSK
with “light” and “shadow lines”

xgv, 1976





$$r = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$



"Ombre" d'un point

"Ombre" de la permutation

= union des ombres

10

9

8

7

6

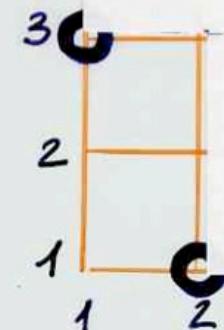
5

4

3

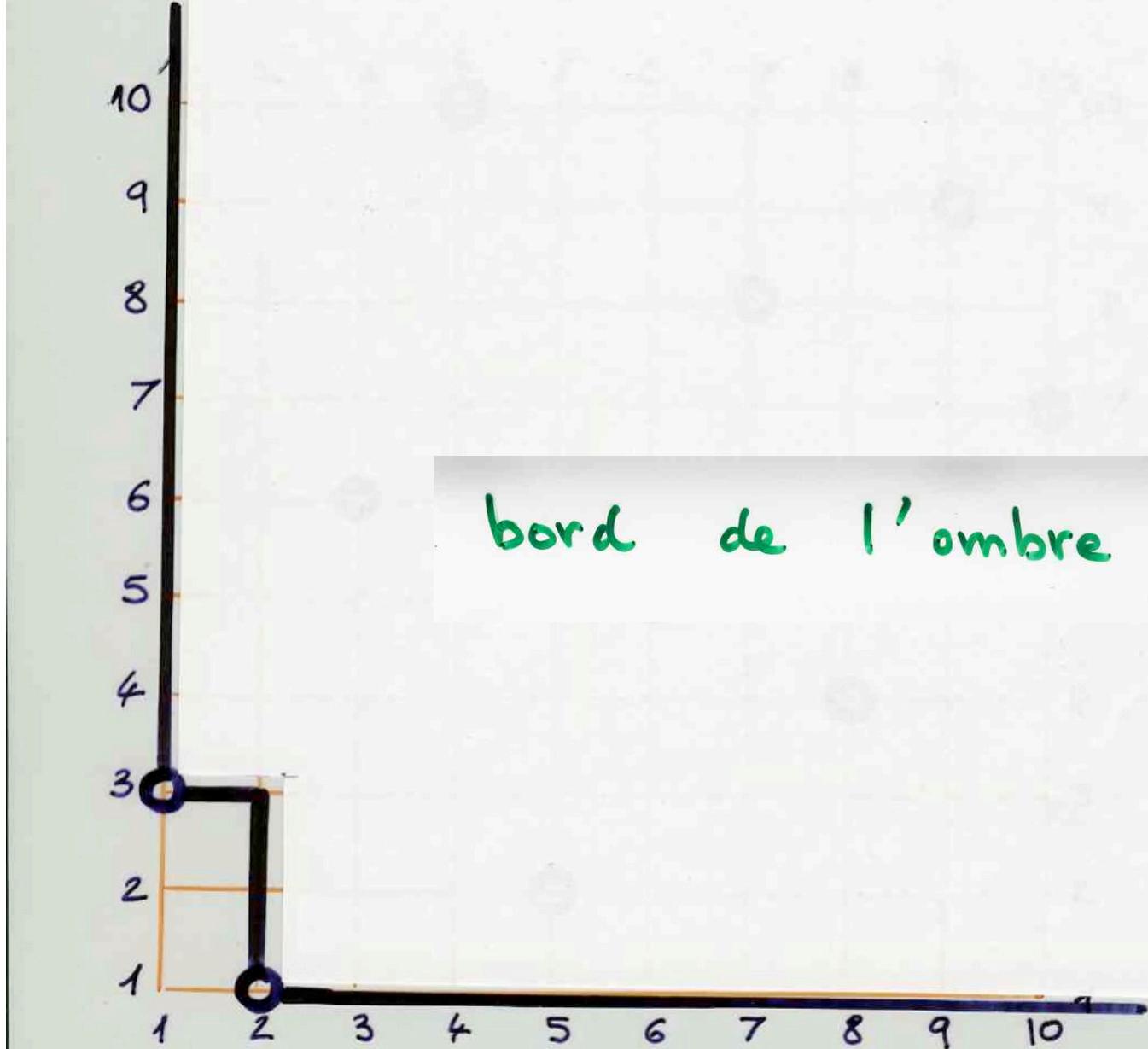
2

1



1 3 4 5 6 7 8 9 10

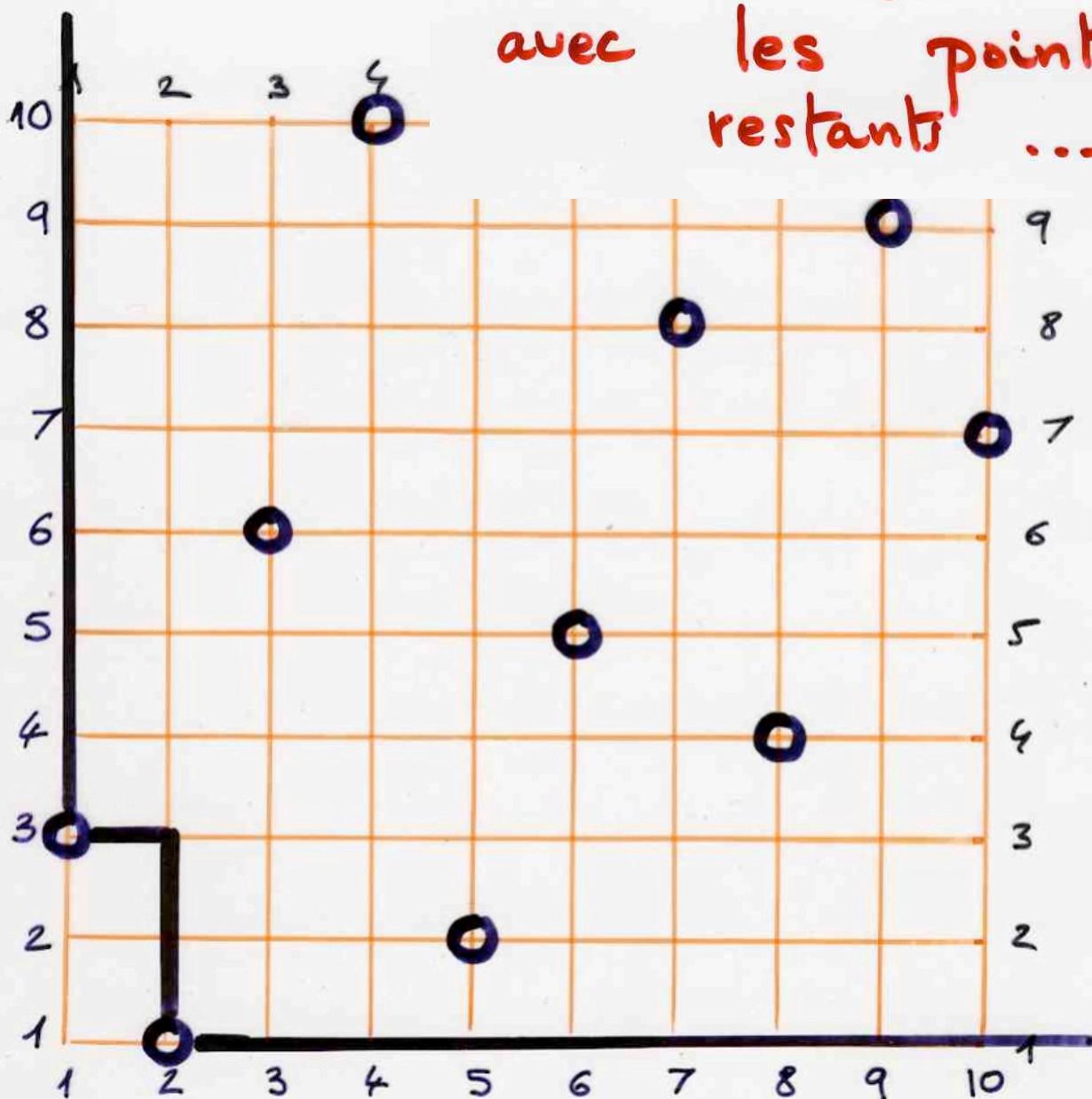
$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$



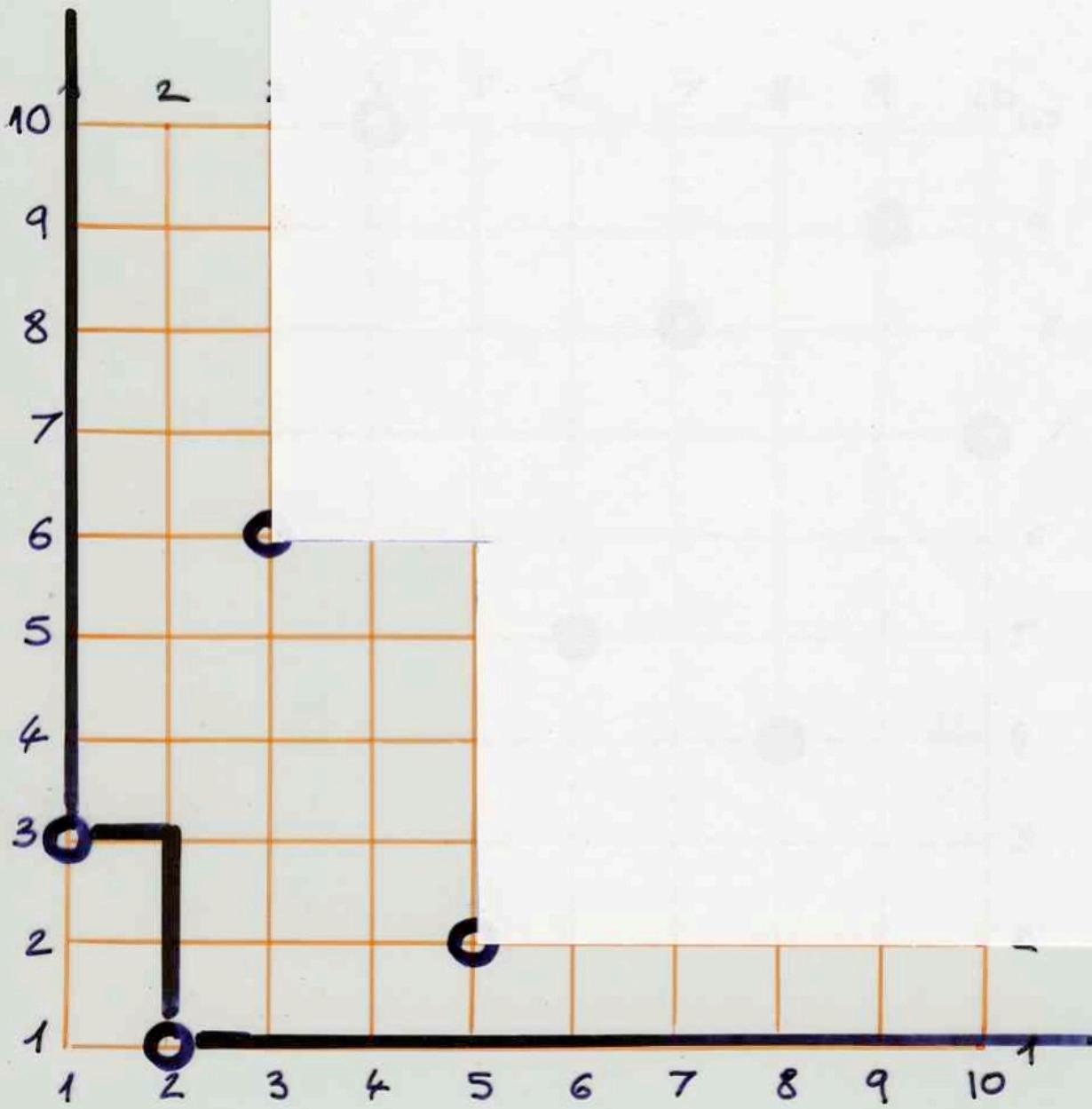
bord de l'ombre

$$f = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$

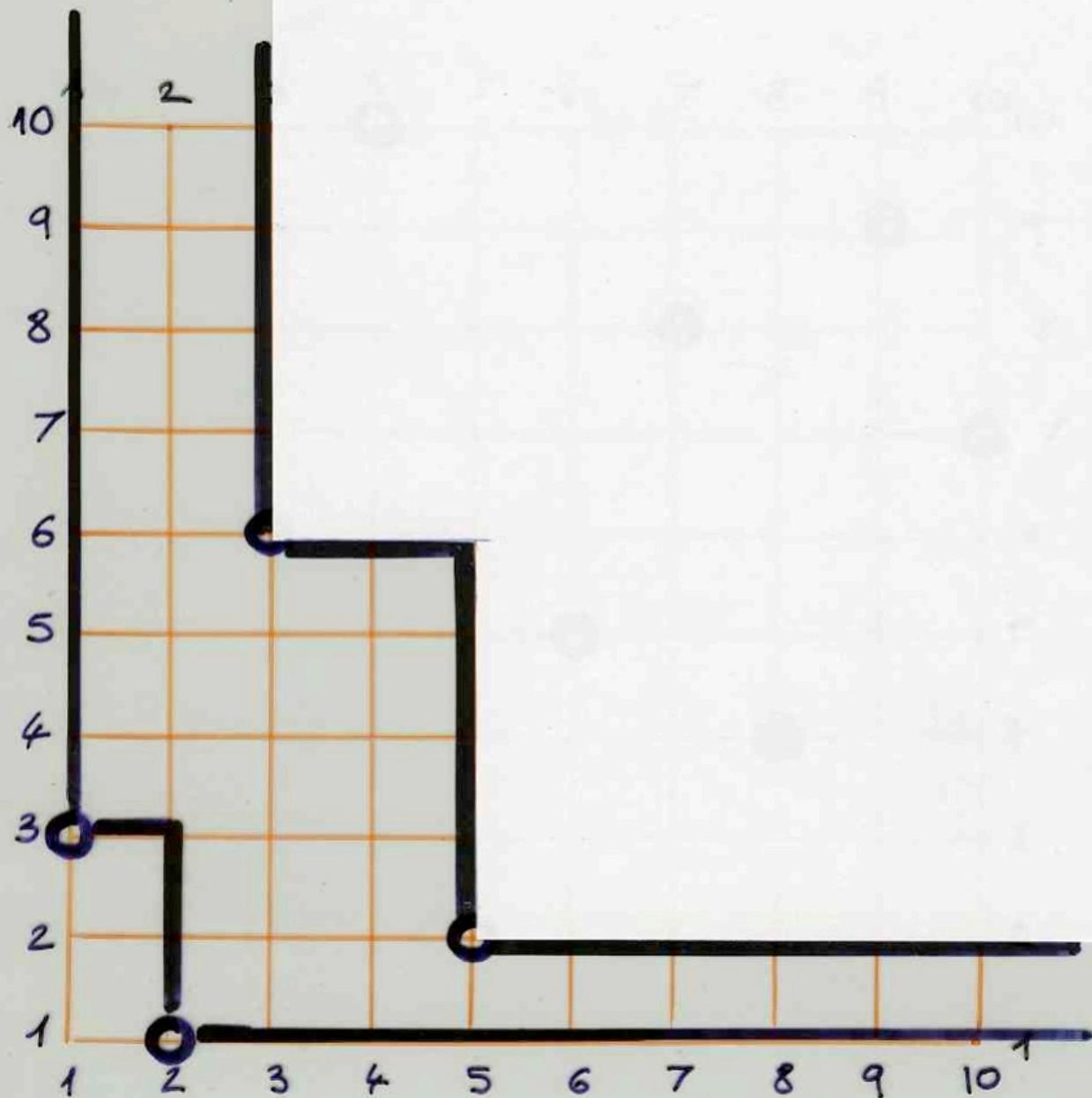
recommençons
avec les points
restants



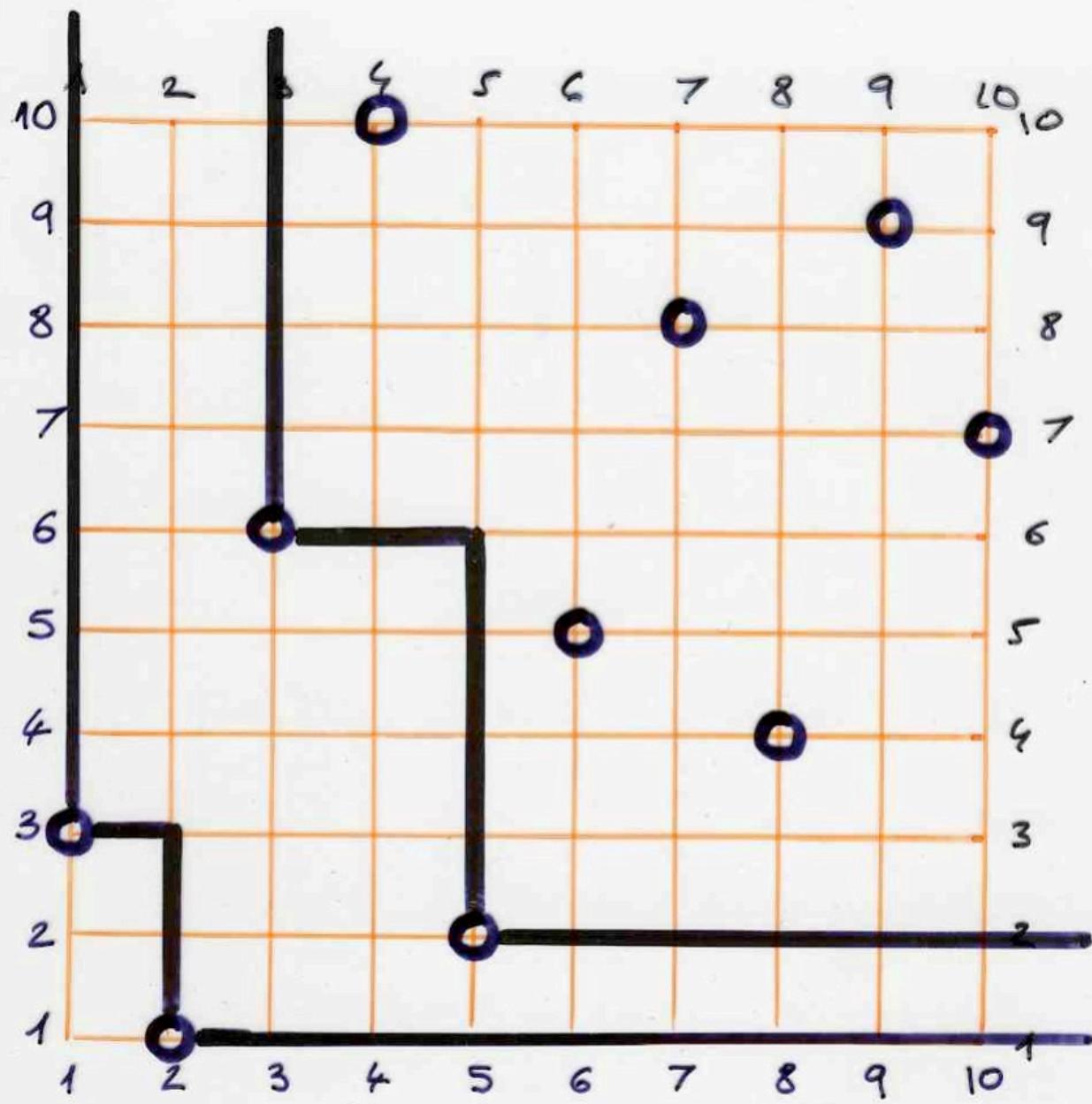
$$P = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$



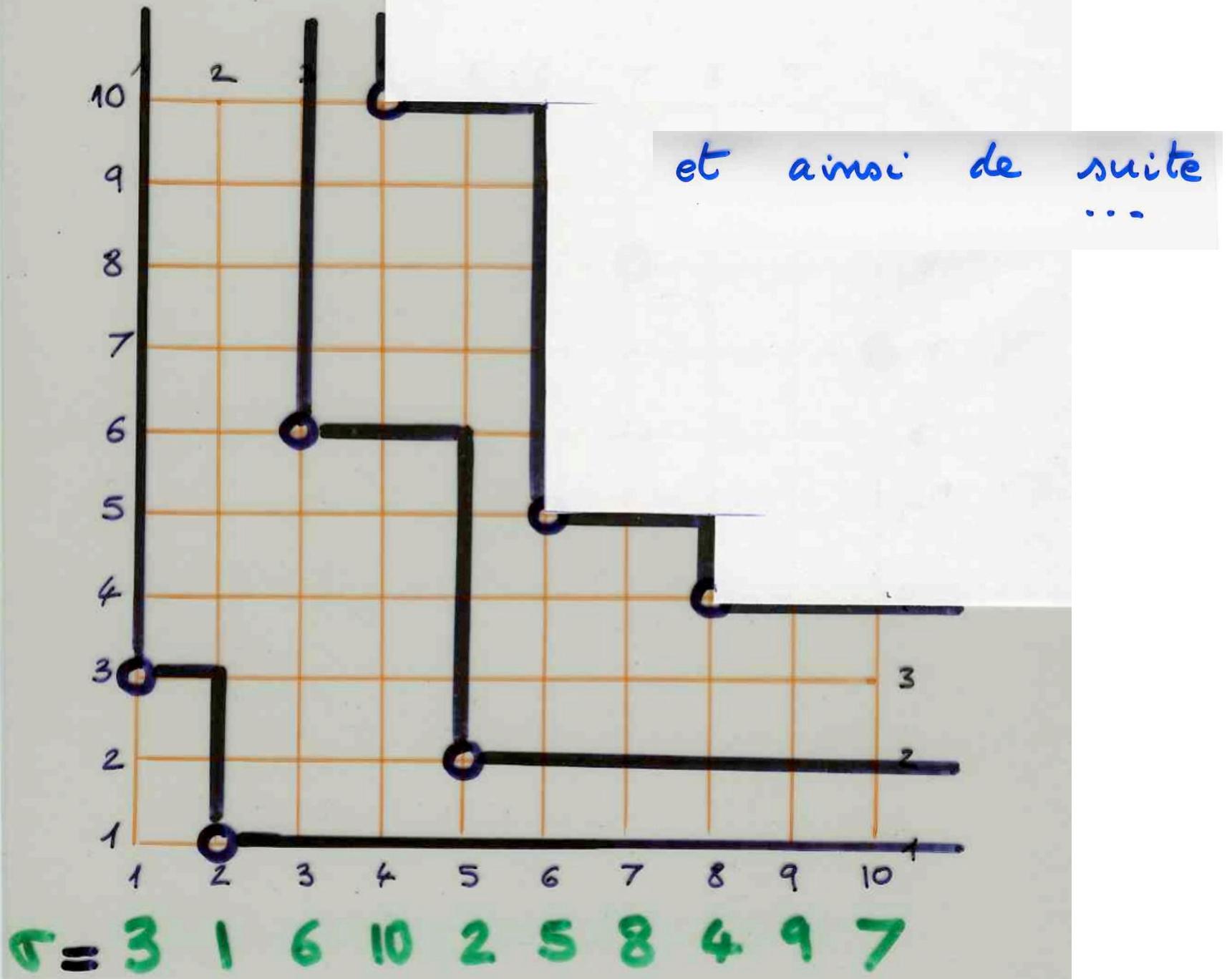
$$r = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$

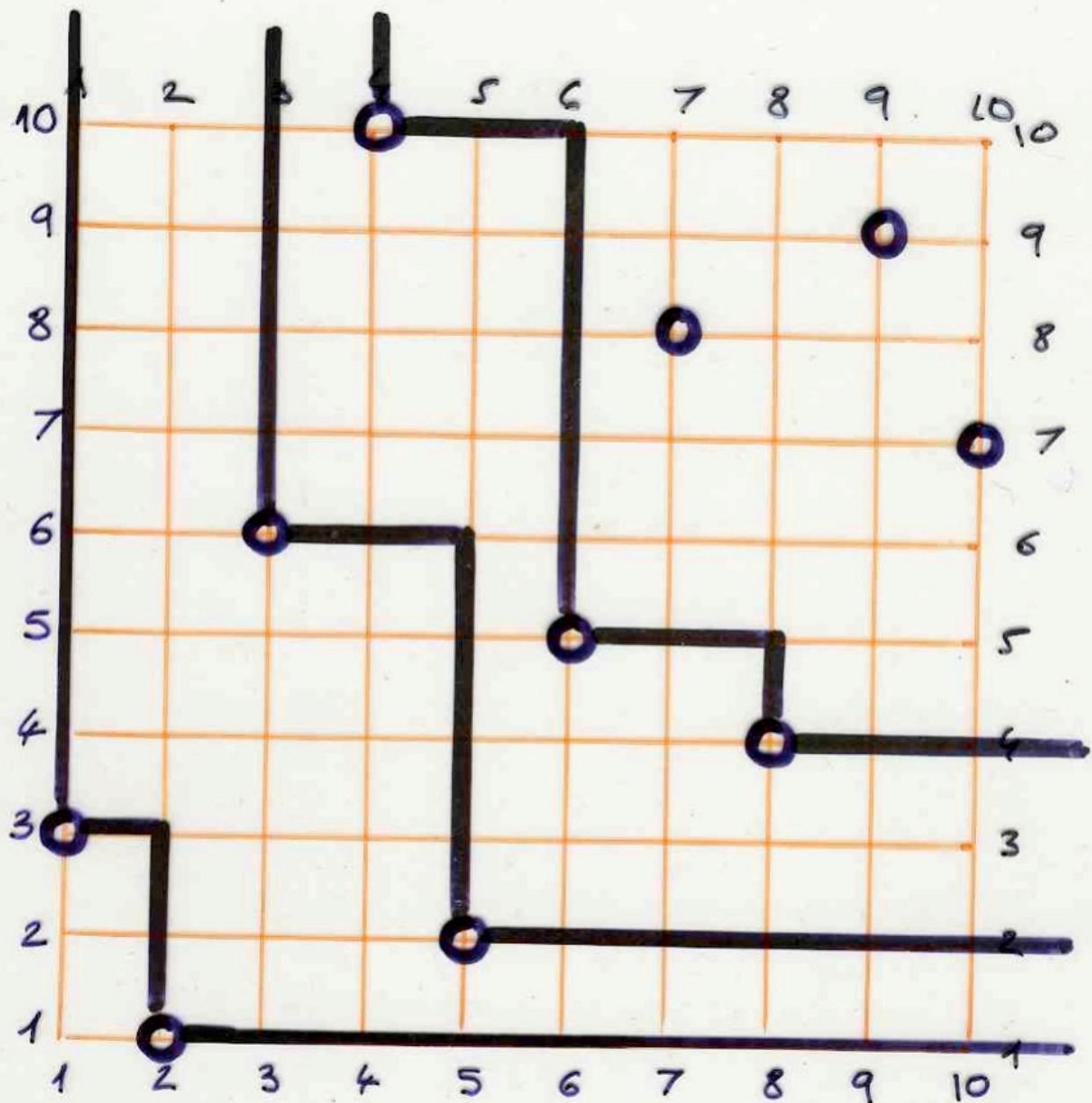


$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

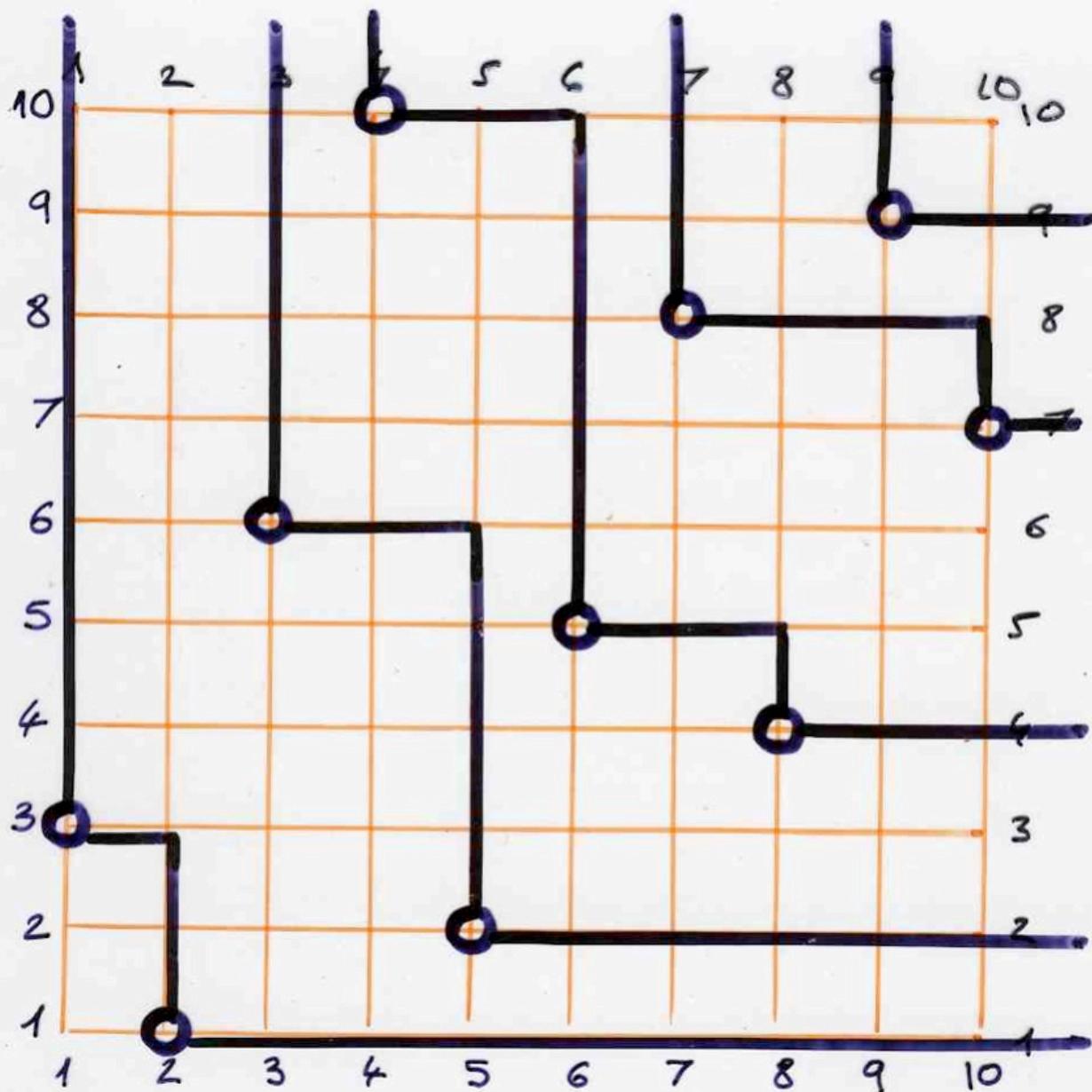


$$r = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$





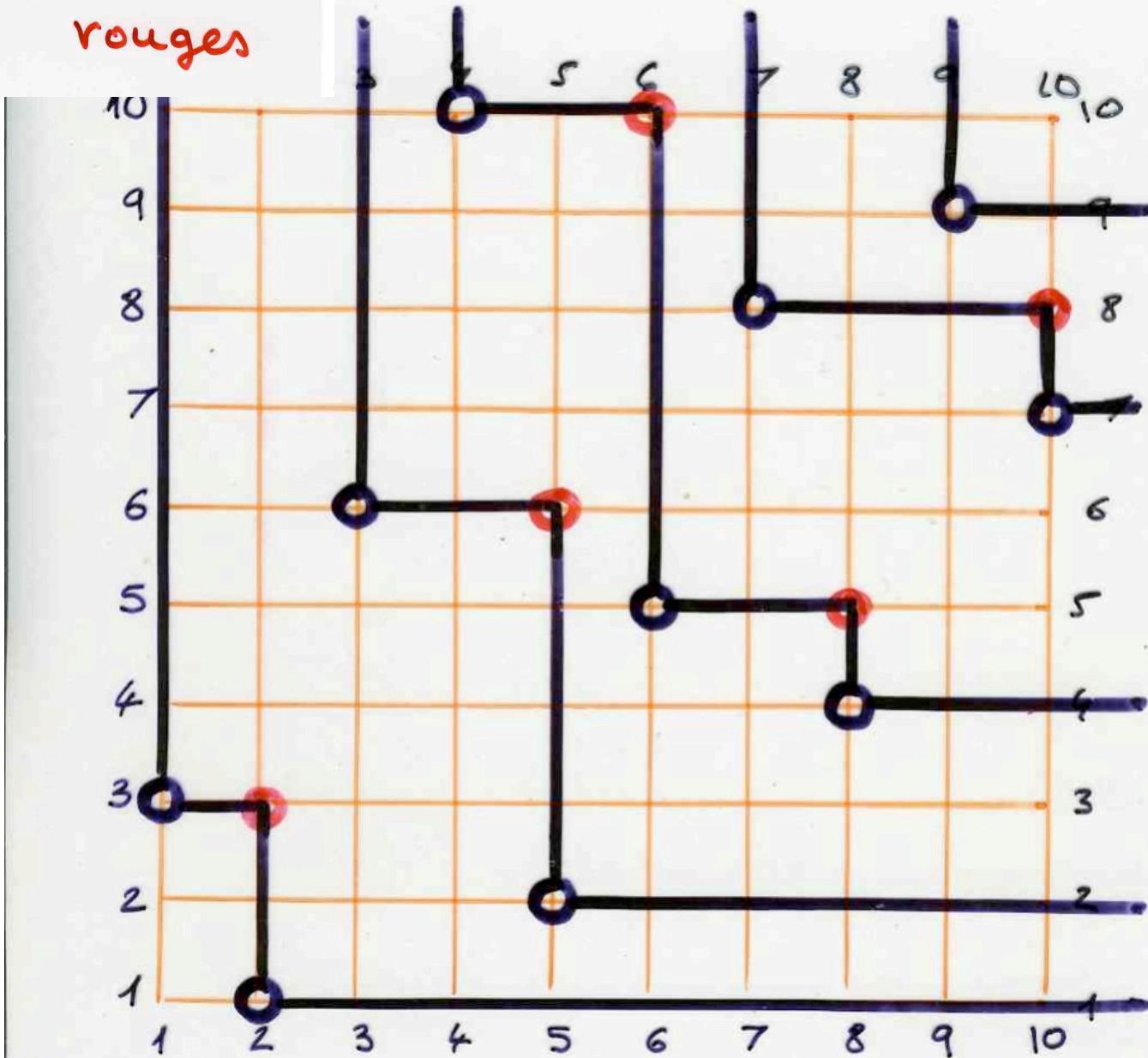
$r = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$



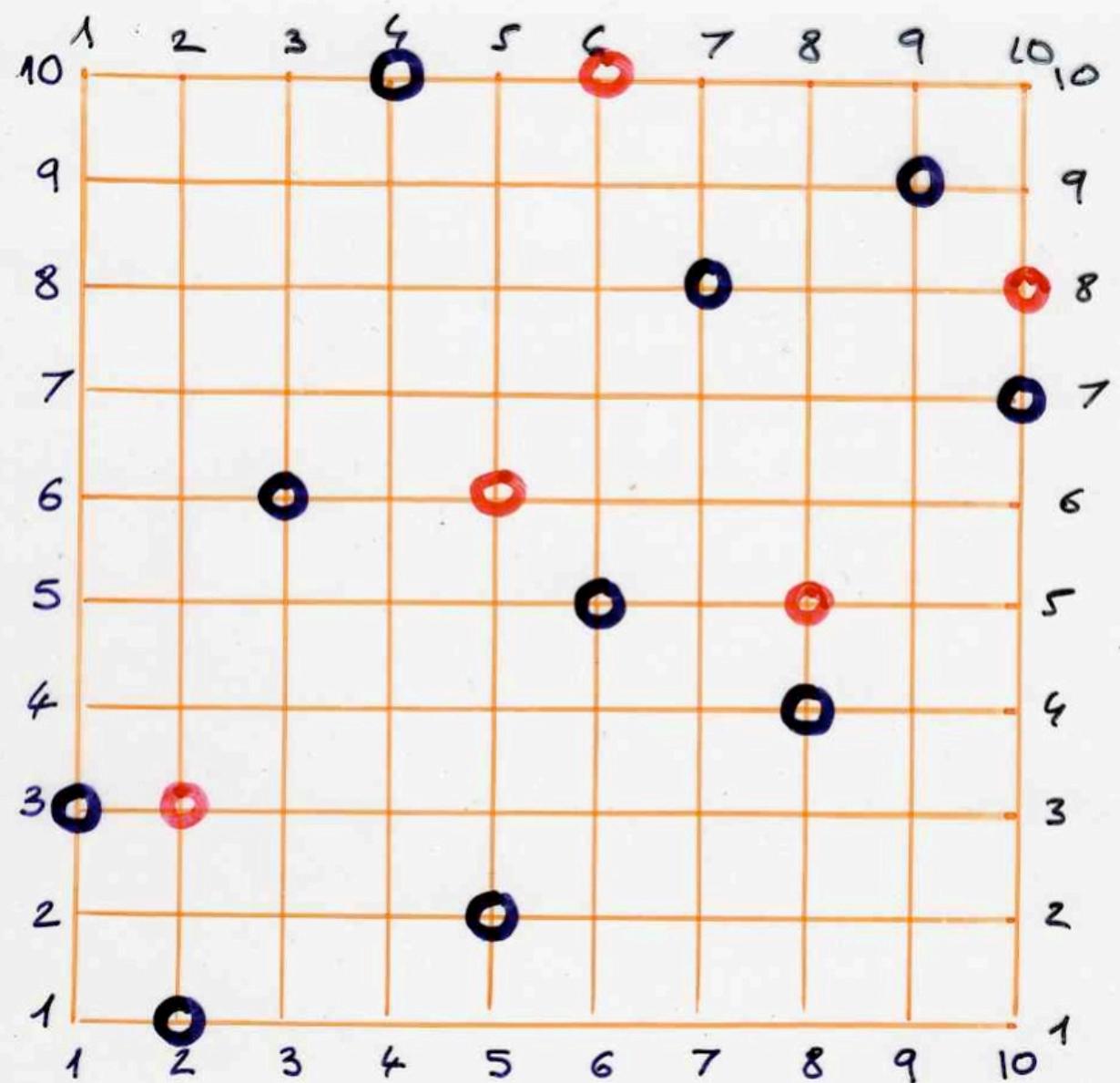
$$\tau = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$

des nouveaux points

les rouges

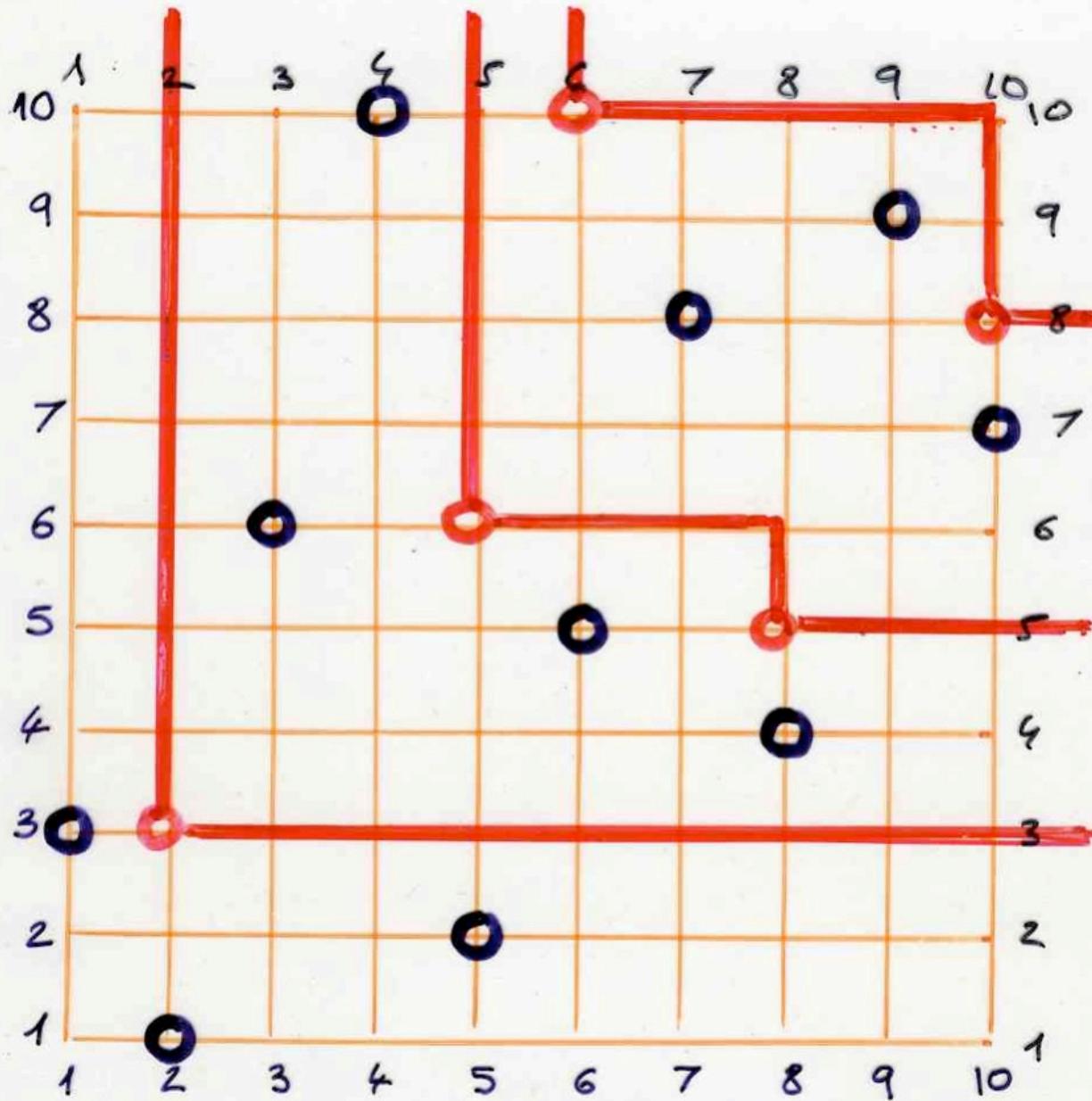


$$\tau = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$

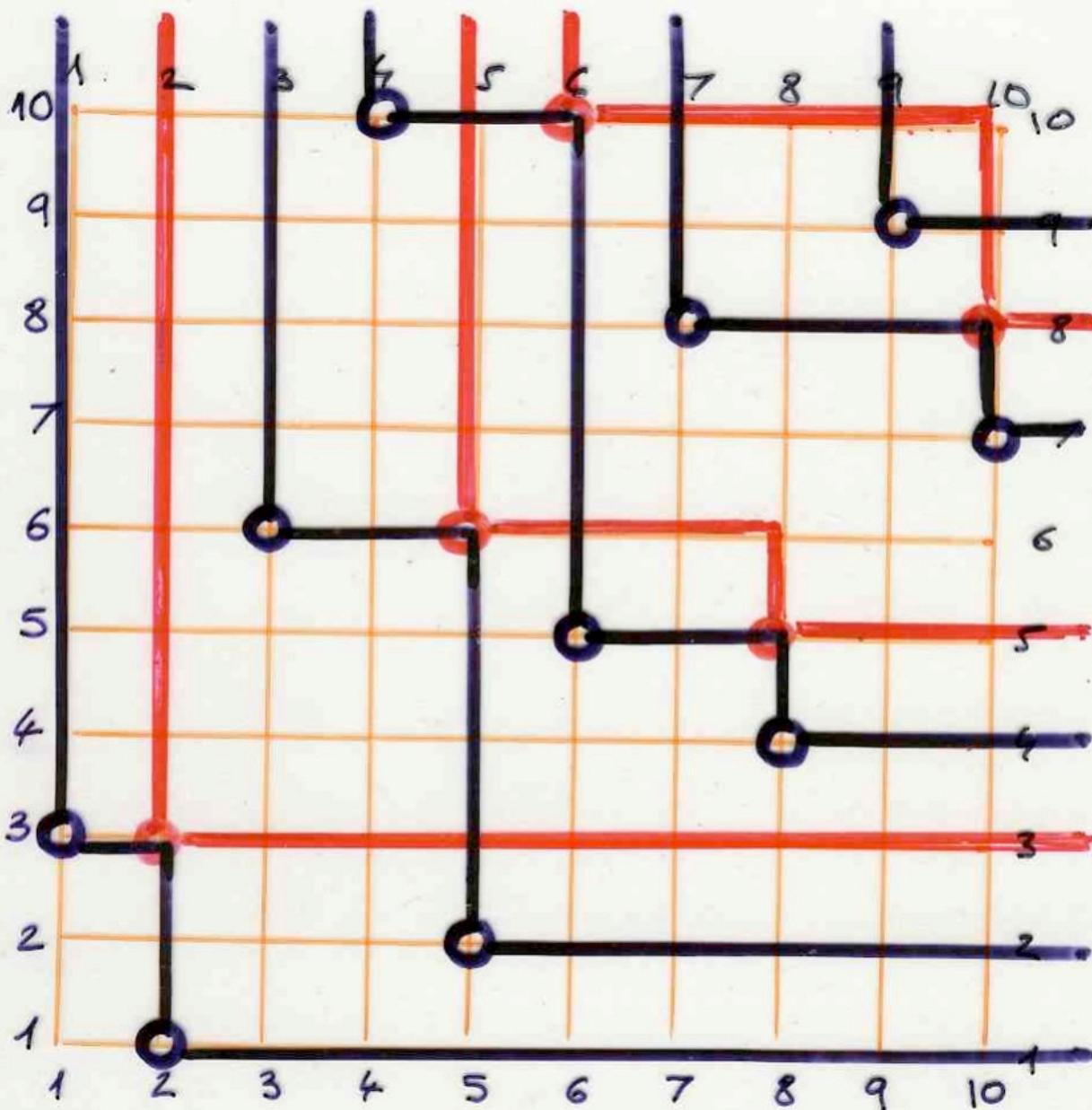


$$r = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$

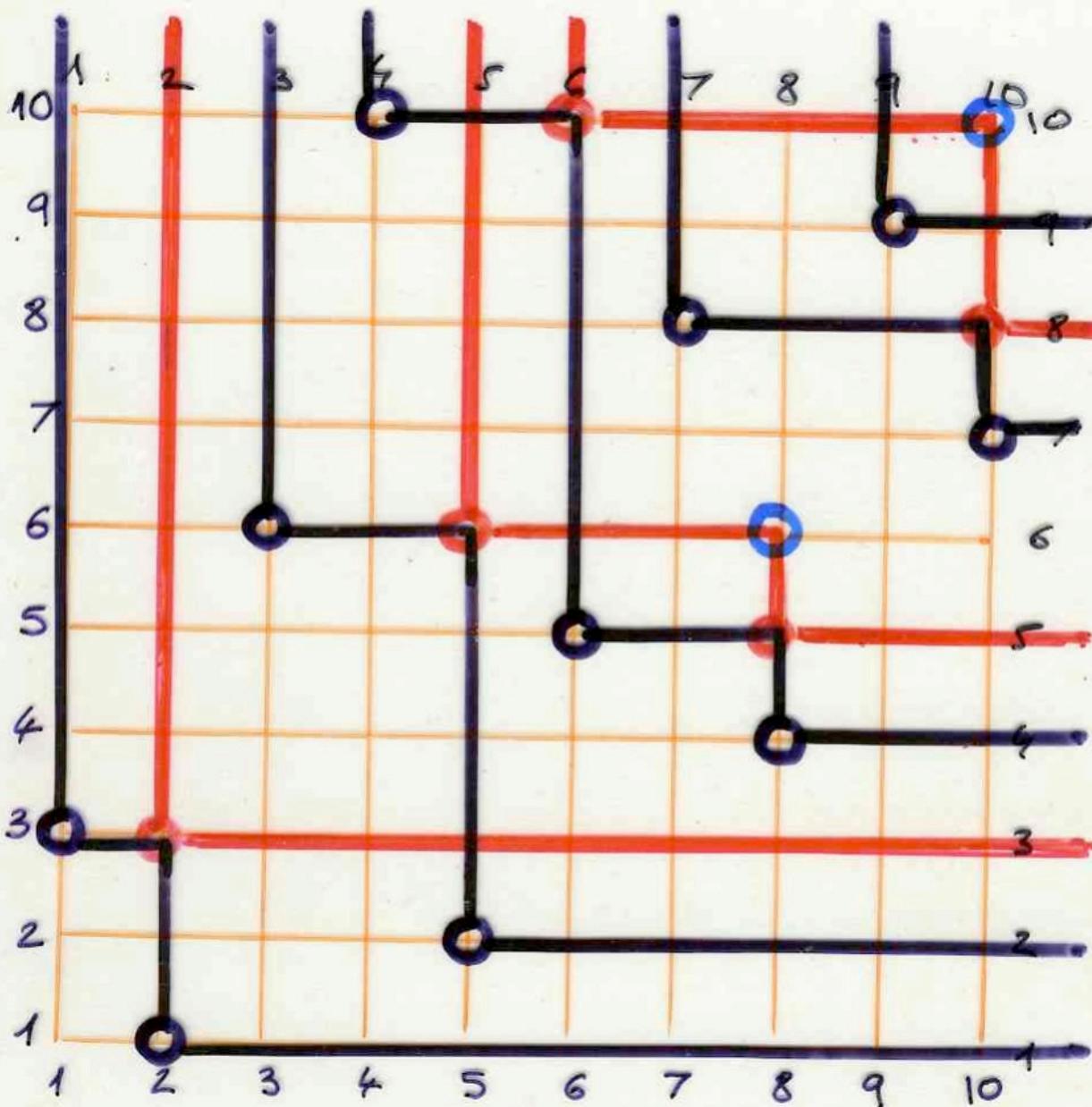
Répétons sur les points
rouges la construction
des bords d'ombres.



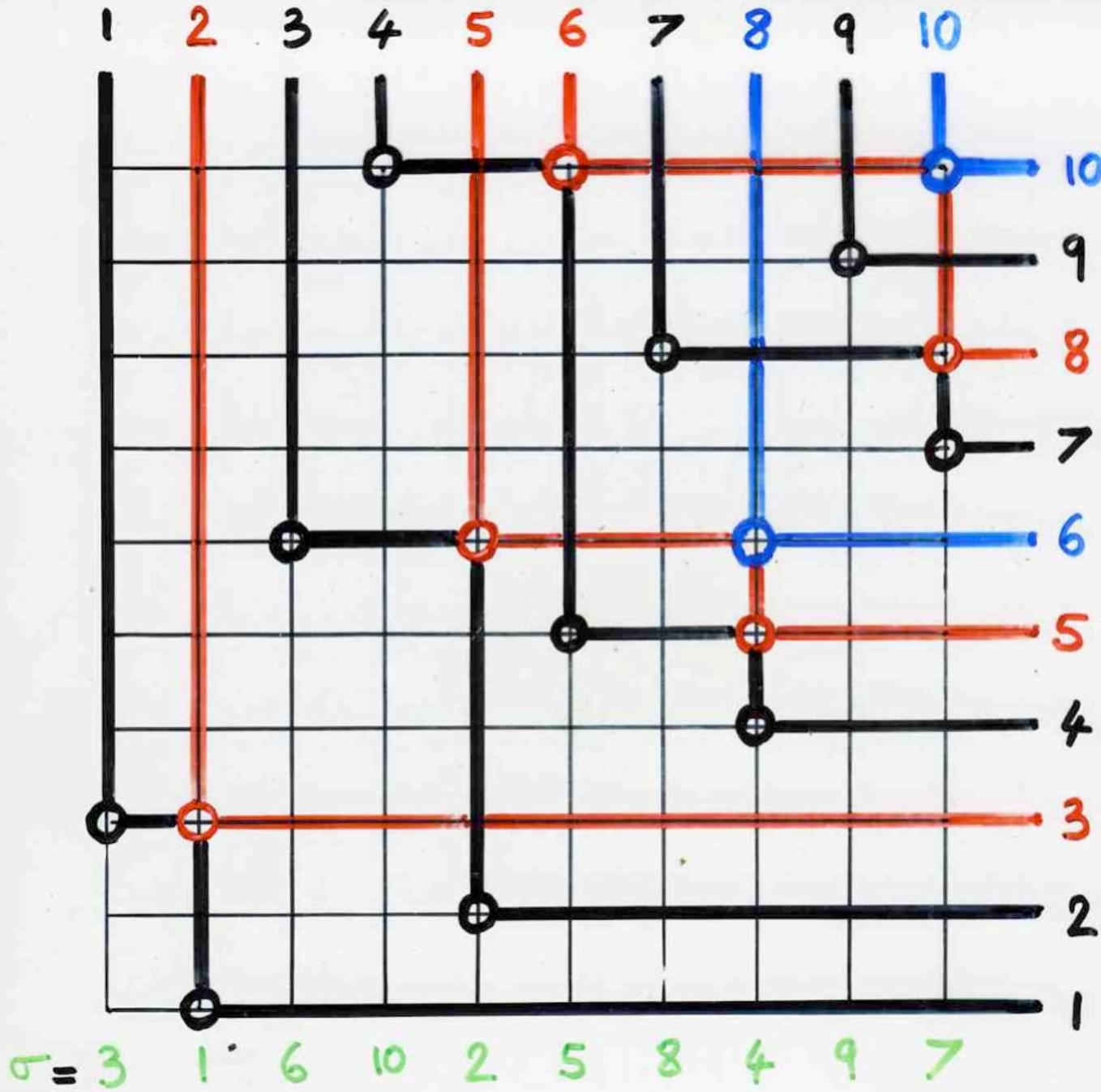
$$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$



$\tau = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$



$$\tau = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$$





9

1 2 3 4 5 6 7 8 9 10

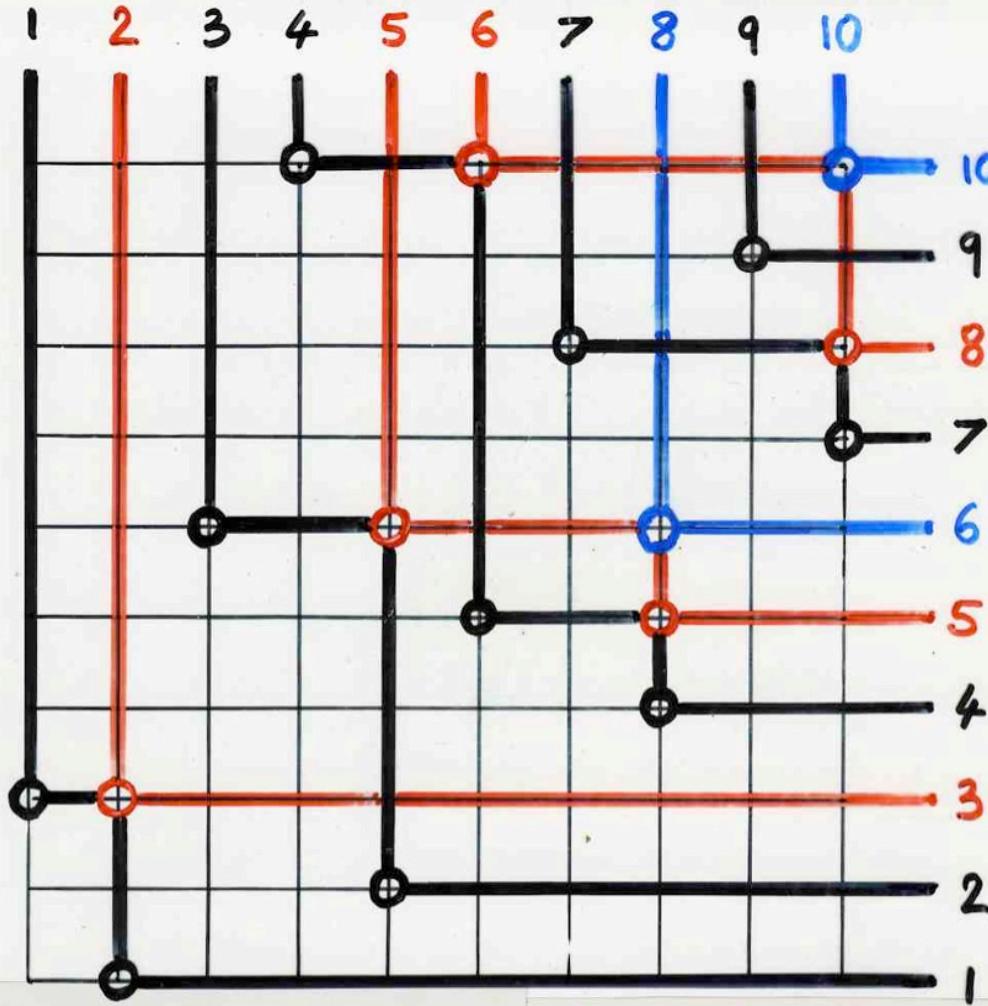
8	10			
2	5	6		
1	3	4	7	9

Q

6	10			
3	5	8		
1	2	4	7	9

P

10
9
8
7
6
5
4
3
2
1



$\sigma = 3 \ 1 \ 6 \ 10 \ 2 \ 5 \ 8 \ 4 \ 9 \ 7$

6	10
3	5
8	
1	2
4	7
9	

P

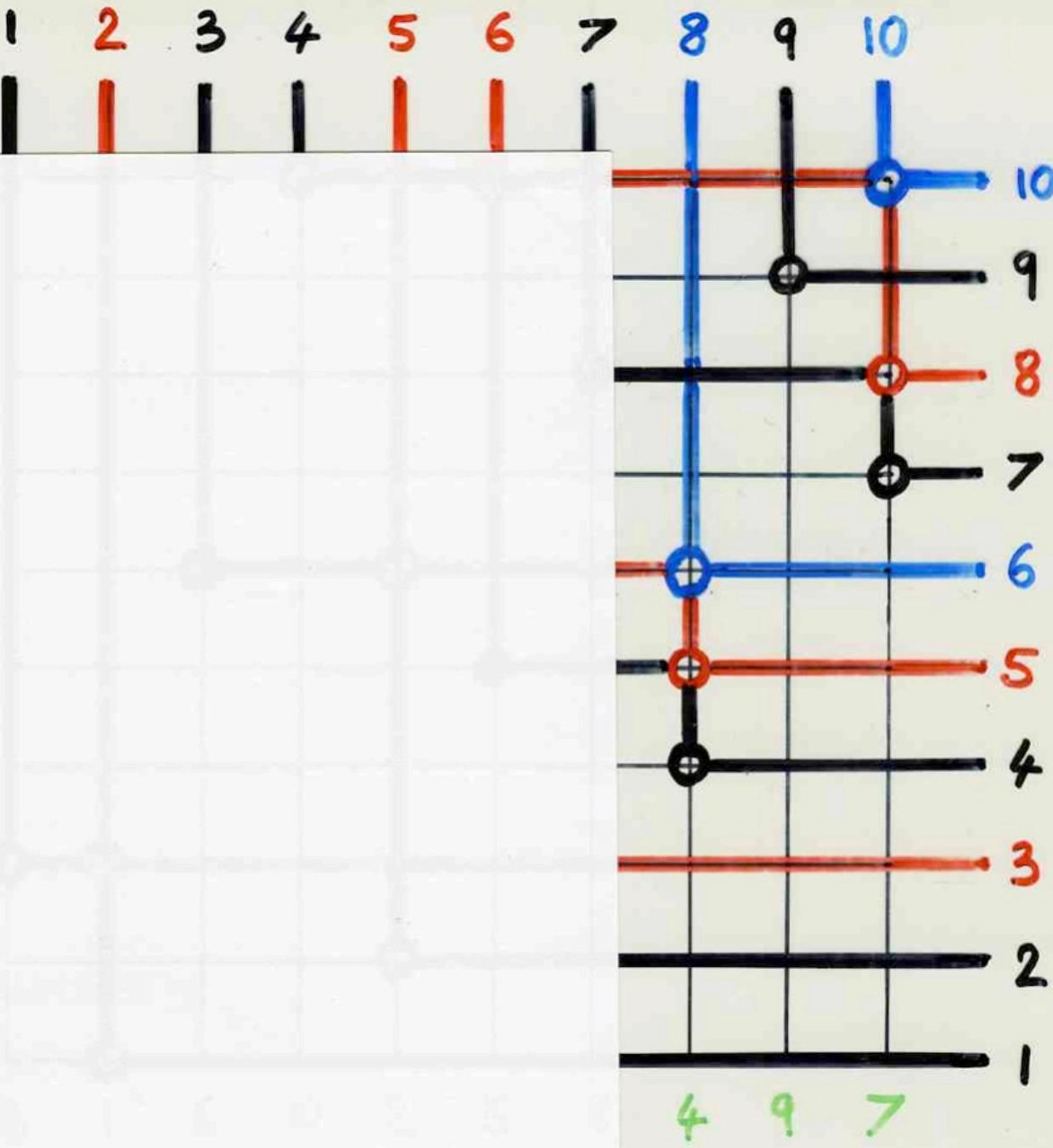
8	10
2	5
6	
1	3
4	7
9	

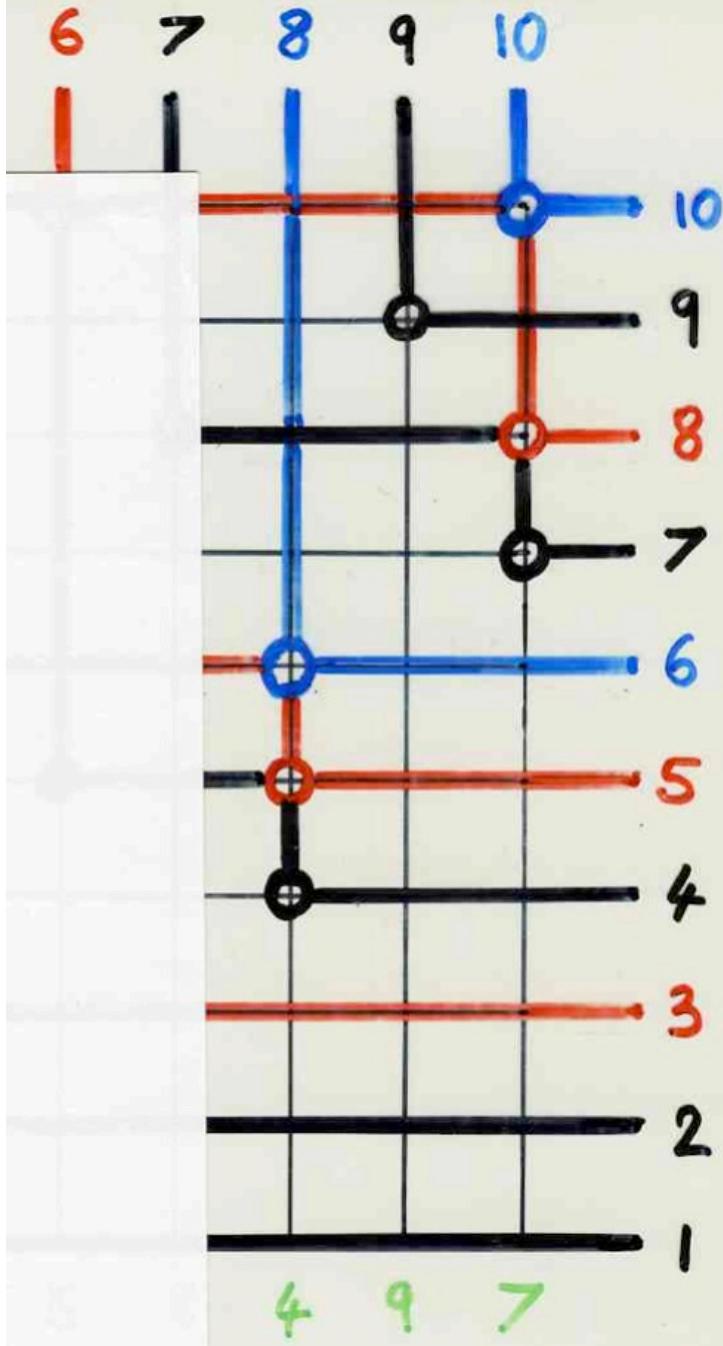
Q

$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$

proof of the equivalence
insertions --- geometric construction

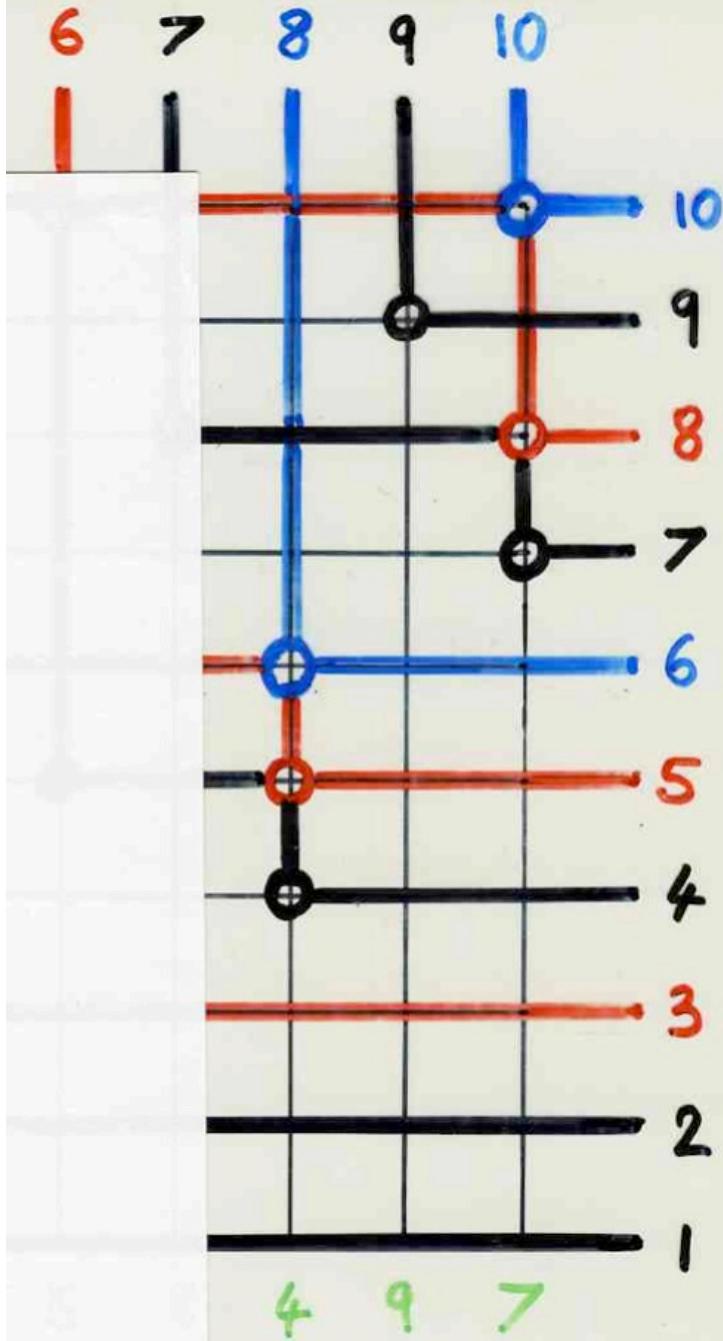




1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

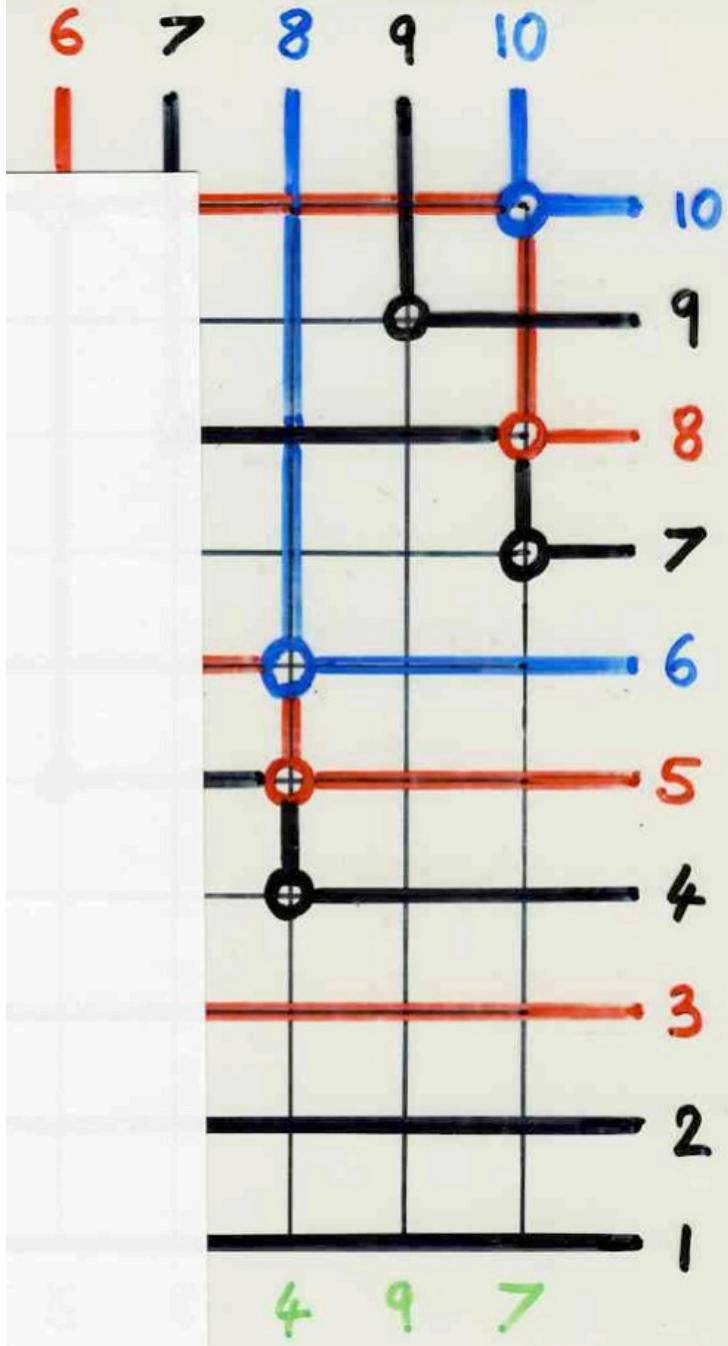
3	6	10							
1	2	5	8						4



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

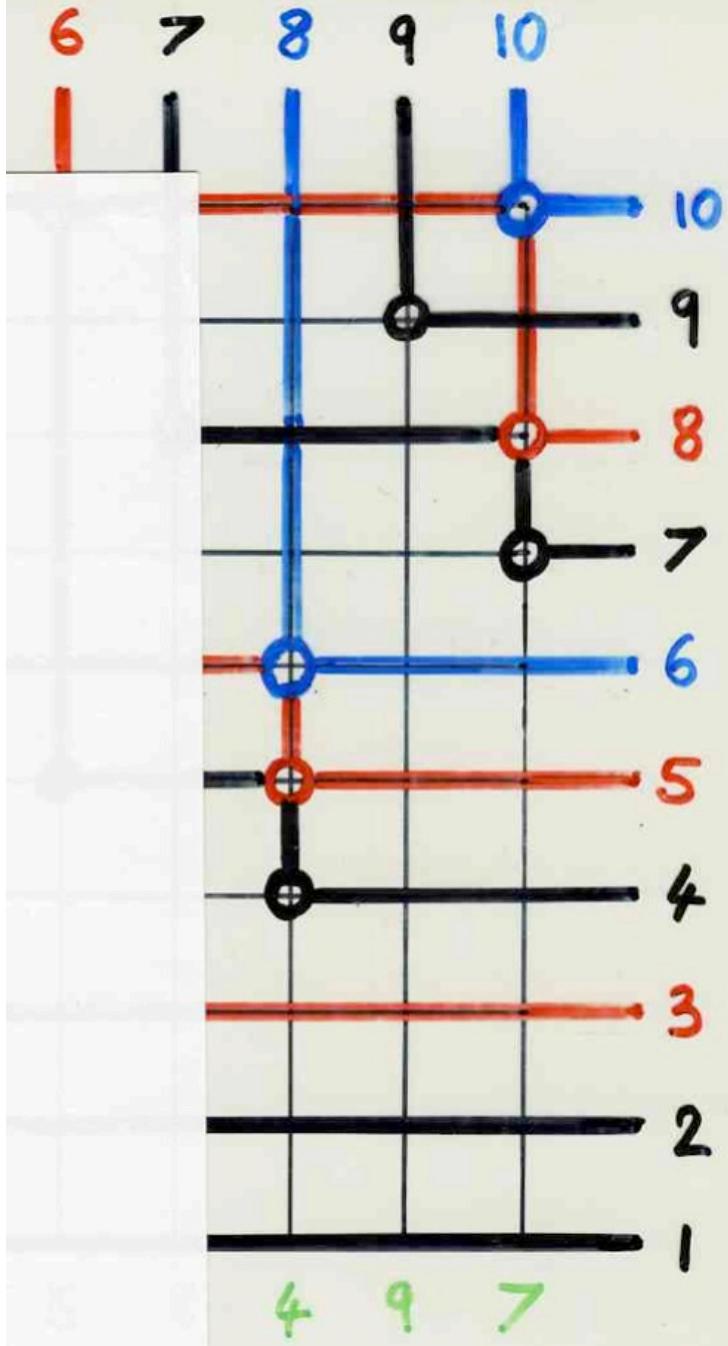
3	6	10							
1	2	4	8						



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

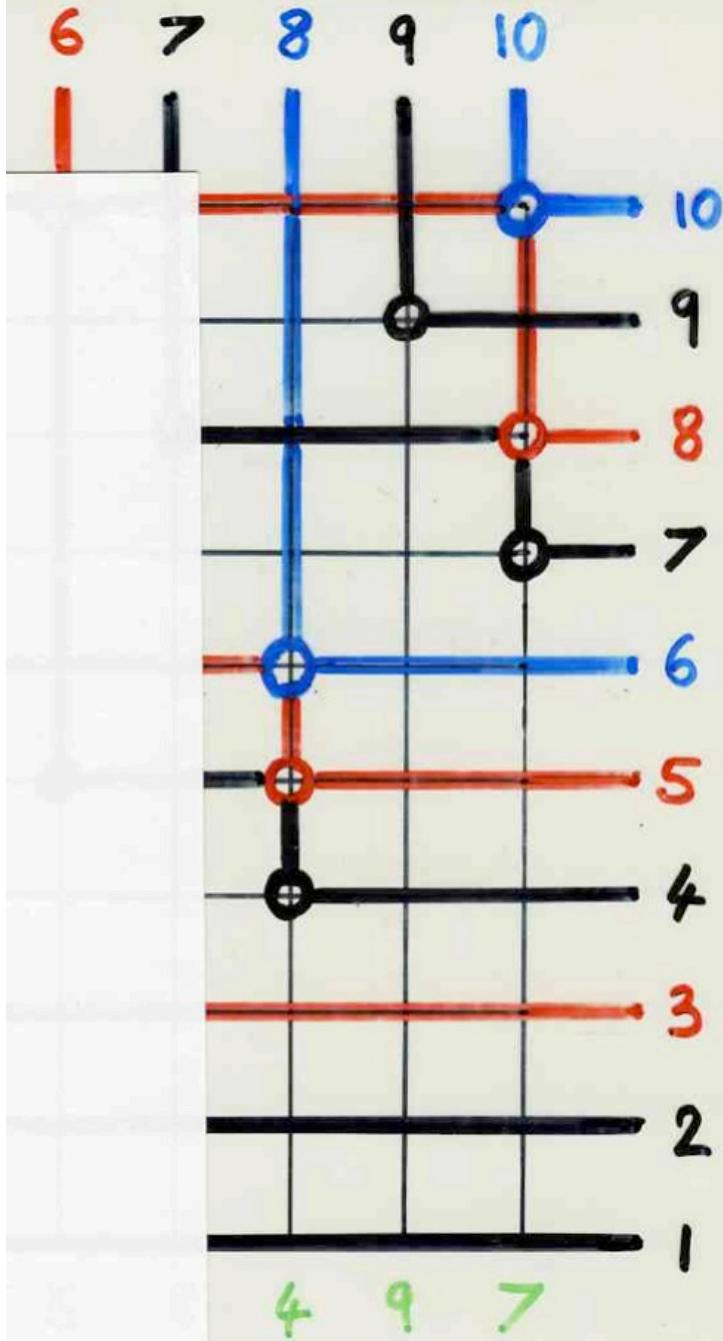
3	6	10							
1	2	4	8						



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6							
1	3	4	7						

			6						
3	5	10							
1	2	4	8						



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8				
2	5	6		
1	3	4	7	

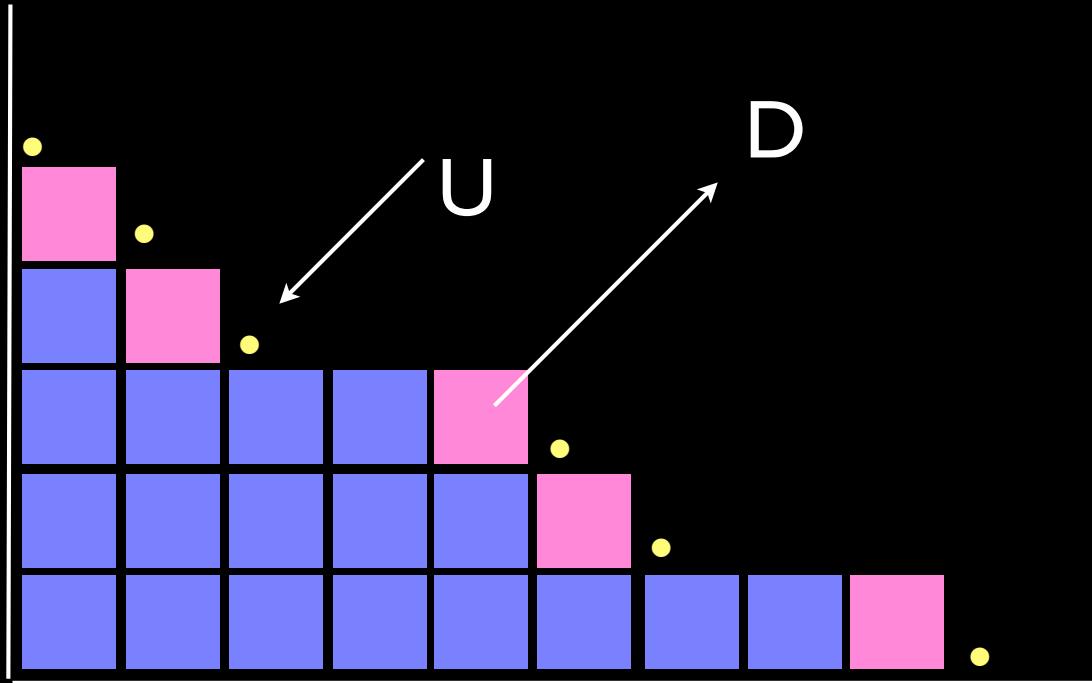
6				
3	5	10		
1	2	4	8	

representation of the operators U, D



Sergey Fomin
(with C. K.)

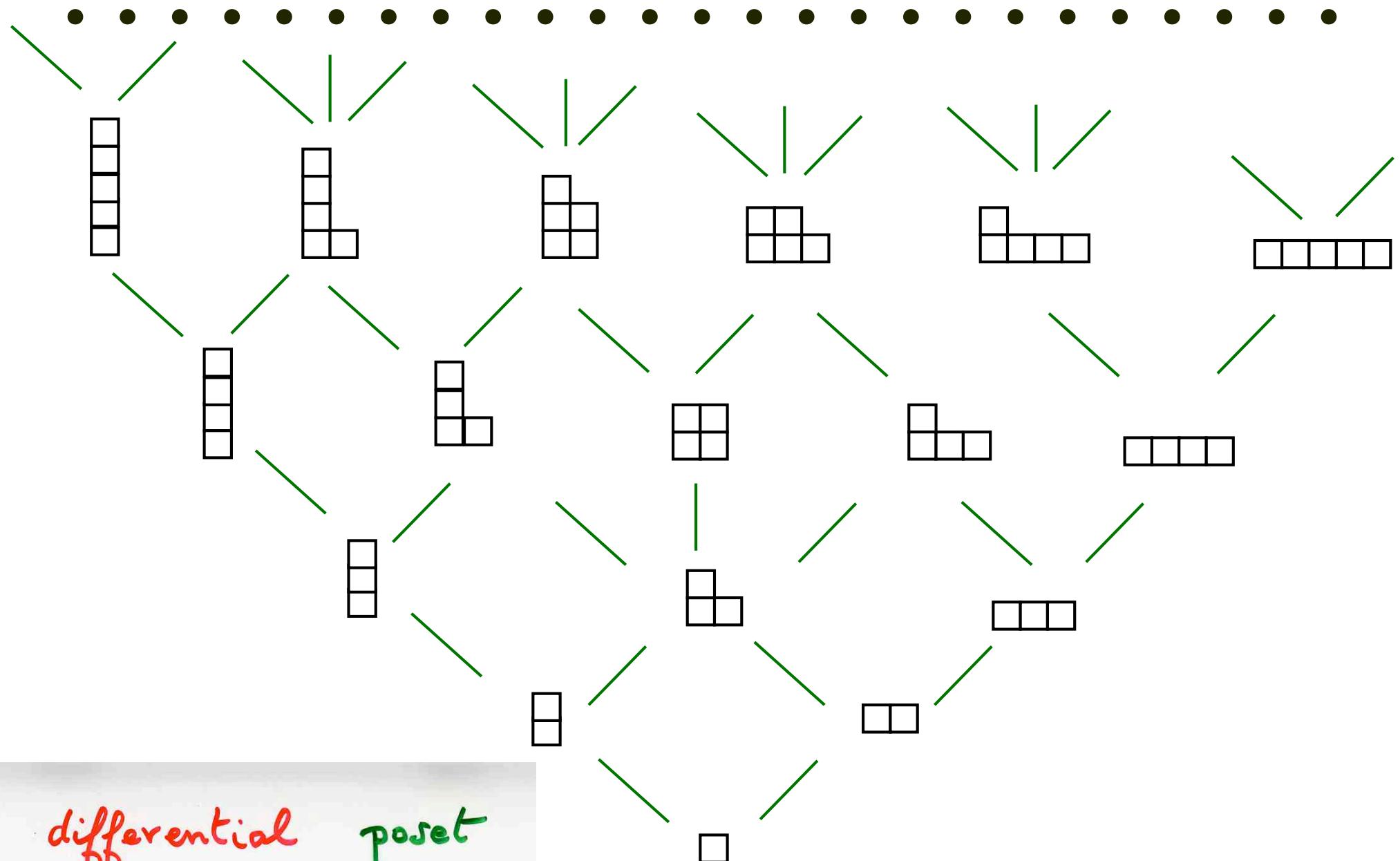
Operators U and D



adding
or deleting
a cell in
a Ferrers
diagram

Young lattice

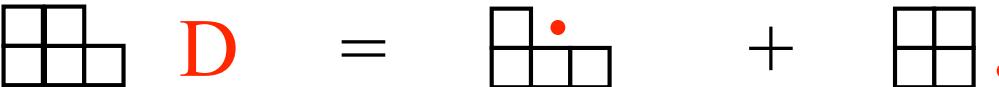
Young lattice



differential poset

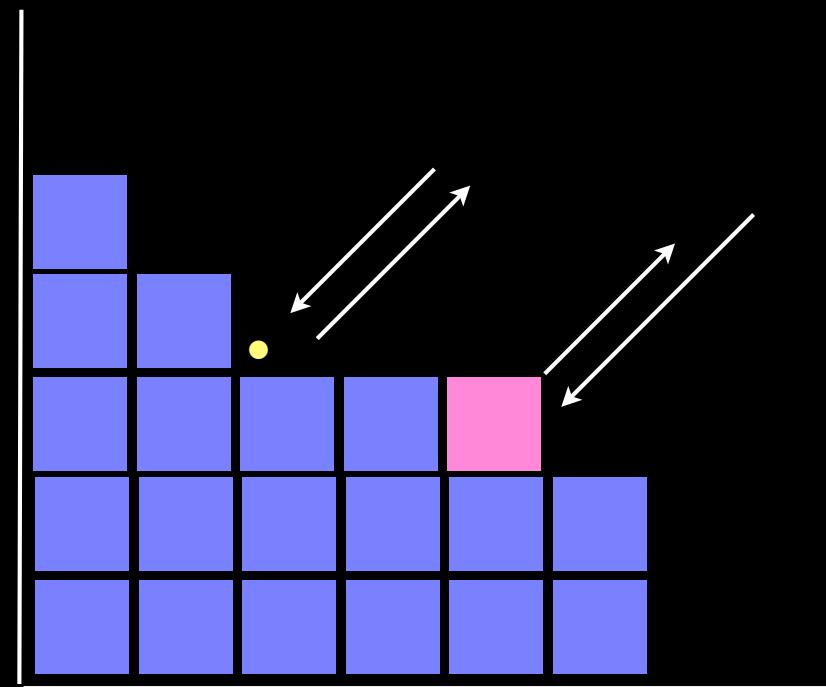
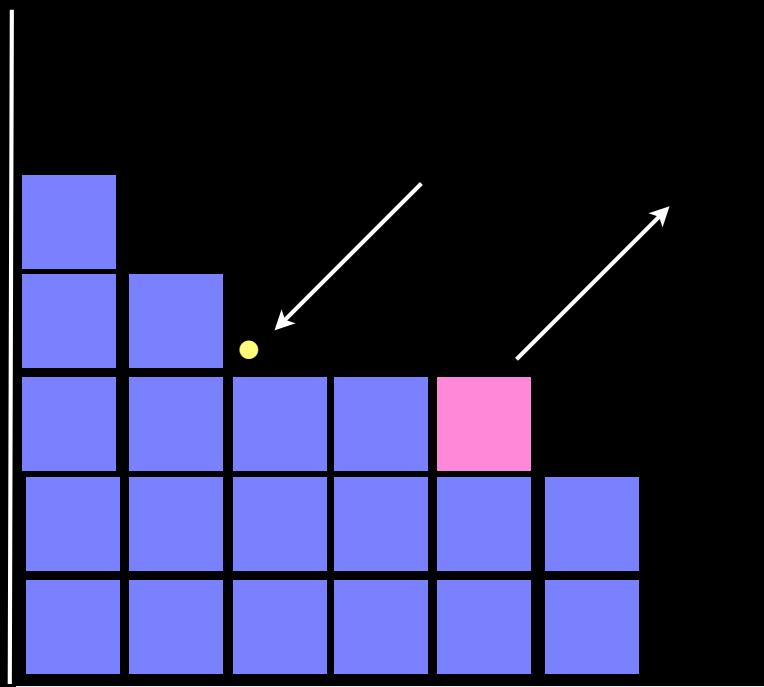
Fomin, Stanley

$$\begin{array}{c} \text{ }\end{array} \quad \text{U} \quad = \quad \begin{array}{c} \text{ }\end{array} + \quad \begin{array}{c} \text{ }\end{array} + \quad \begin{array}{c} \text{ }\end{array}$$


$$\begin{array}{c} \text{ }\end{array} \quad \text{D} \quad = \quad \begin{array}{c} \text{ }\end{array} + \quad \begin{array}{c} \text{ }\end{array} .$$


Heisenberg commutation relation

$$UD = DU + I$$



$$\begin{array}{c} \begin{array}{ccccc} \begin{array}{c} \text{U} \\ = \end{array} & \begin{array}{c} \text{D} \\ = \end{array} & \begin{array}{c} \text{UD} \\ = \end{array} \end{array} \quad \begin{array}{ccccccc} \begin{array}{c} \text{U} \\ = \end{array} & \begin{array}{c} \text{D} \\ = \end{array} & \begin{array}{c} \text{UD} \\ = \end{array} \end{array} \end{array}$$

Diagram illustrating the decomposition of a 3x3 matrix into its upper (U), lower (D), and transpose (UD) components.

The matrices are represented as sets of 3x3 grids:

- U**: Top-left grid (3x3).
- D**: Bottom-right grid (3x3).
- UD**: Transpose grid (3x3).

The decomposition equations are:

$$\begin{aligned} U &= \begin{array}{c} \text{U}_1 \\ \vdots \\ \text{U}_5 \end{array} + \begin{array}{c} \text{U}_6 \\ \vdots \\ \text{U}_{10} \end{array} + \begin{array}{c} \text{U}_{11} \\ \vdots \\ \text{U}_{15} \end{array} \\ D &= \begin{array}{c} \text{D}_1 \\ \vdots \\ \text{D}_5 \end{array} + \begin{array}{c} \text{D}_6 \\ \vdots \\ \text{D}_{10} \end{array} + \begin{array}{c} \text{D}_{11} \\ \vdots \\ \text{D}_{15} \end{array} \\ UD &= \begin{array}{c} \text{UD}_1 \\ \vdots \\ \text{UD}_5 \end{array} + \begin{array}{c} \text{UD}_6 \\ \vdots \\ \text{UD}_{10} \end{array} + \begin{array}{c} \text{UD}_{11} \\ \vdots \\ \text{UD}_{15} \end{array} + \begin{array}{c} \text{UD}_{16} \\ \vdots \\ \text{UD}_{20} \end{array} + \begin{array}{c} \text{UD}_{21} \\ \vdots \\ \text{UD}_{25} \end{array} \end{aligned}$$

Red arrows indicate the mapping from the U and D components to the UD components. Specifically, the first five terms in each row map to the first five terms in the UD row, and the remaining ten terms map to the remaining ten terms in the UD row.

$$\begin{array}{l}
 \begin{array}{c} \text{U} \\ = \end{array} \quad \begin{array}{c} \text{U} \\ + \end{array} \quad \begin{array}{c} \text{U} \\ + \end{array} \quad \begin{array}{c} \text{U} \\ + \end{array} \\
 \begin{array}{c} \text{D} \\ = \end{array} \quad \begin{array}{c} \text{D} \\ + \end{array} \\
 \begin{array}{c} \text{UD} \\ = \end{array} \quad \begin{array}{c} \text{UD} \\ + \end{array} \\
 \begin{array}{c} \text{DU} \\ = \end{array} \quad \begin{array}{c} \text{DU} \\ + \end{array}
 \end{array}$$

Diagram illustrating the decomposition of a 3x3 matrix into its row and column components. The matrices are represented as sets of 3x3 grids.

- U:** The first row shows the matrix U decomposed into four components. Red arrows point from the first three components to the second row.
- D:** The second row shows the matrix D decomposed into four components. Blue arrows point from the first three components to the third row.
- UD:** The third row shows the product UD decomposed into seven components. Blue arrows point from the first three components to the fourth row.
- DU:** The fourth row shows the product DU decomposed into six components. Blue arrows point from the first three components to the fourth row.

The components are represented by 3x3 grids where some cells are filled (black) and some are empty (white). The components for U and D are identical in structure but differ in their arrangement.

$$\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \text{U} = \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \text{D} = \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \text{UD} = \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \quad \text{DU} = \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} + \quad \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ \text{U} = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the top-left square missing]} \end{array}$$

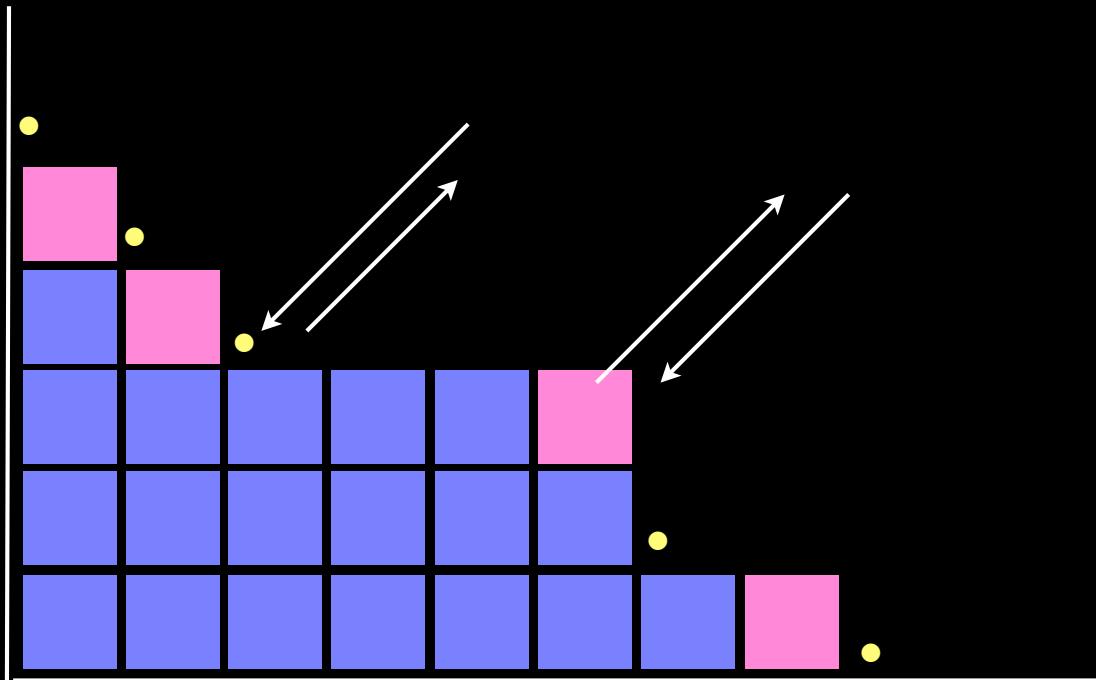
$$\begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ \text{D} = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid]} \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ \text{UD} = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ \text{DU} = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ + \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \end{array}$$

$$\begin{array}{c} \text{[Diagram of a 3x3 grid with the bottom-right square missing]} \\ (\text{UD-DU}) = \end{array} \begin{array}{c} \text{[Diagram of a 3x3 grid]} \end{array}$$

$$UD = DU + I$$



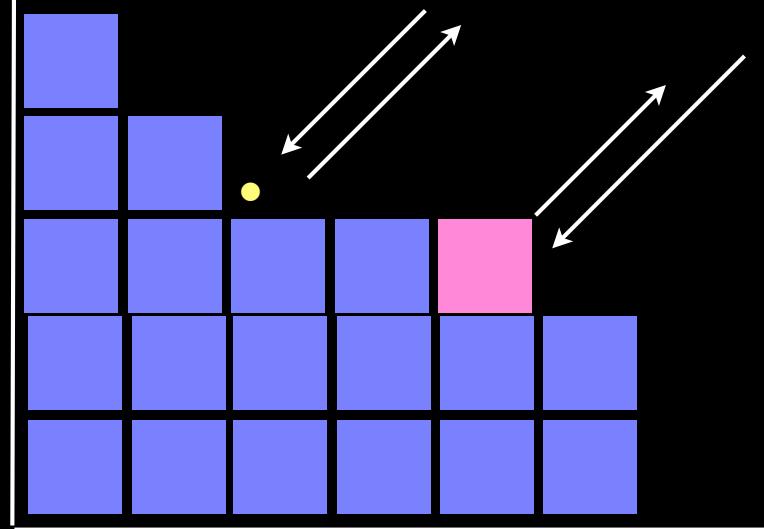
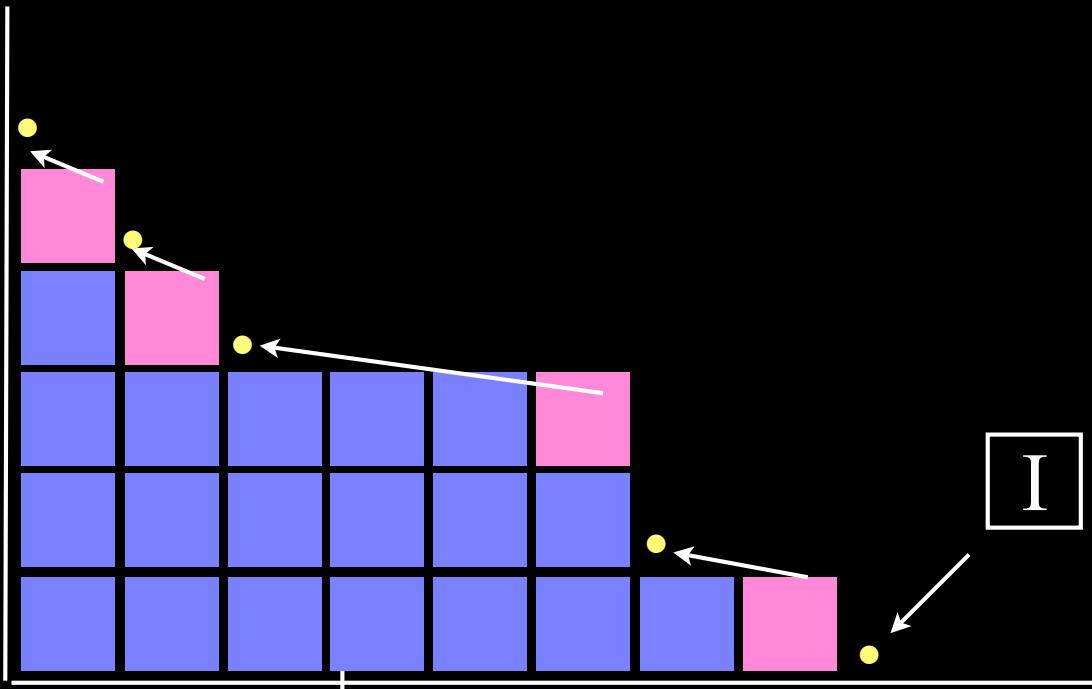
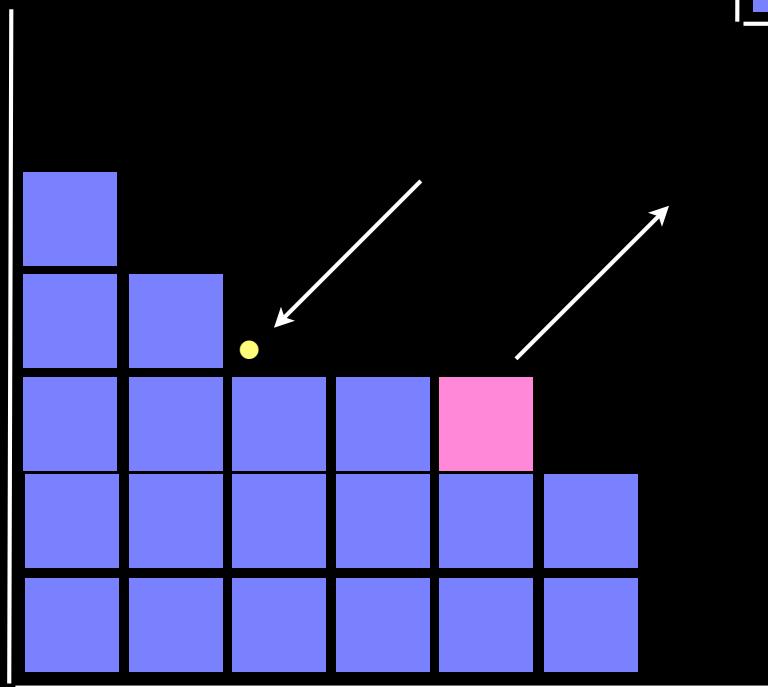
The cellular Ansatz
second part:

guided construction
of a bijection

(from the representation of U and D)

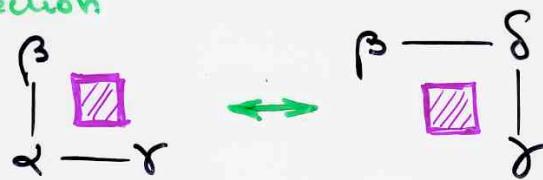
combinatorial “representation” of the
commutation relation $UD = DU + I$

$$UD = DU + I$$



Commutation diagrams

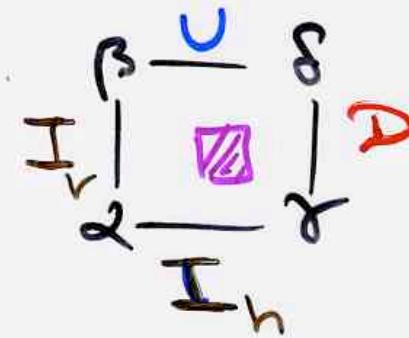
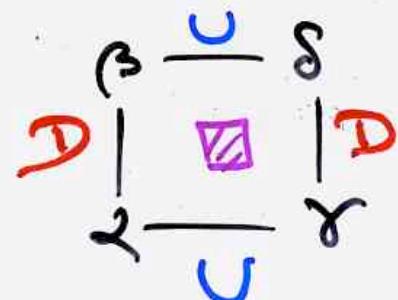
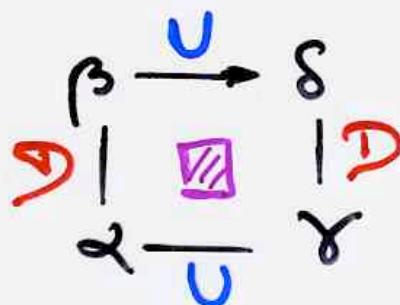
bijection



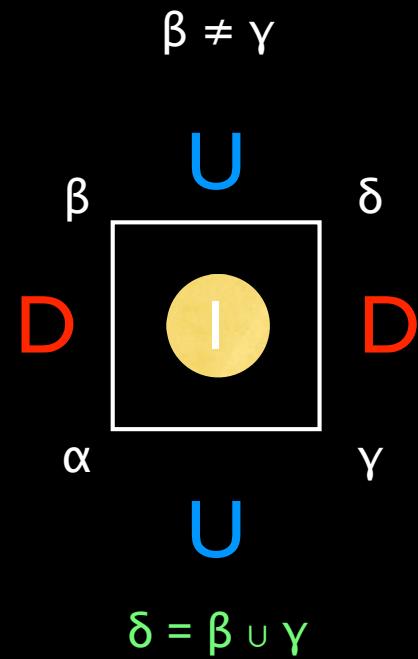
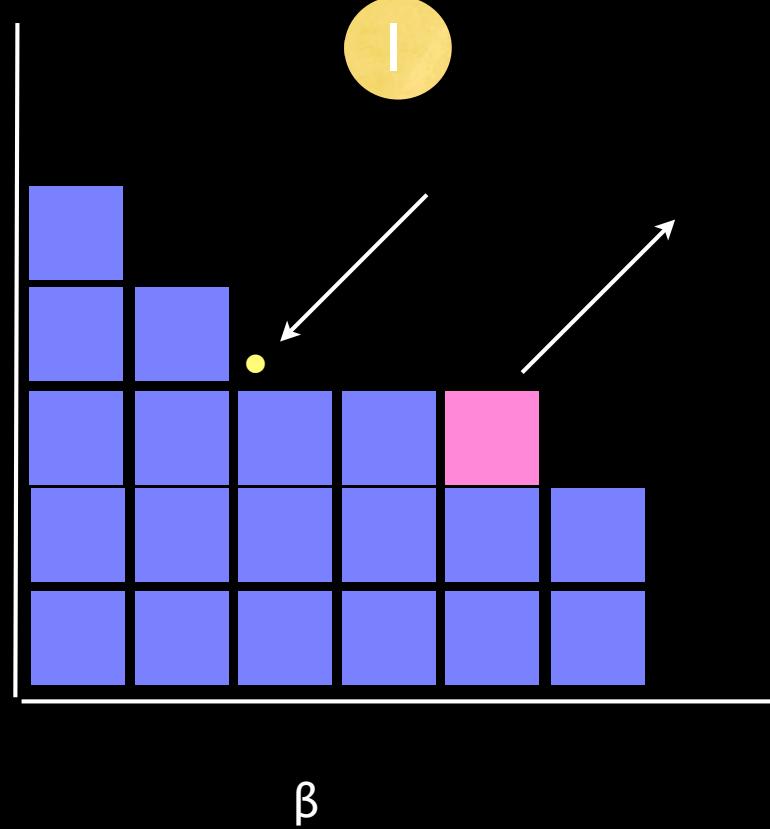
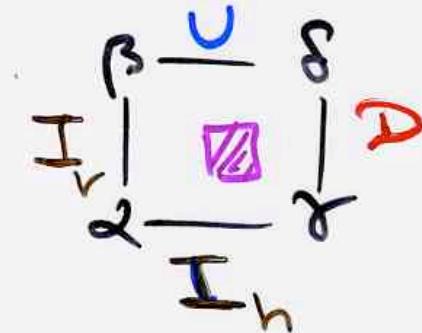
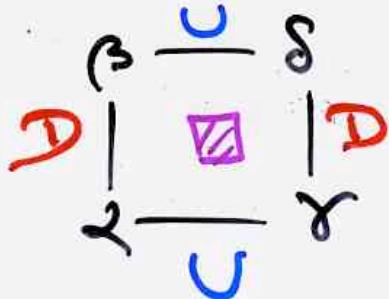
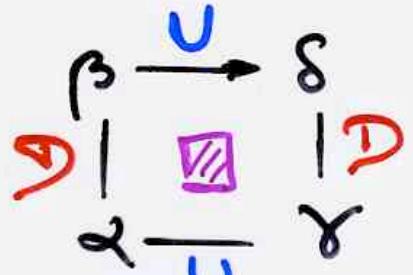
$\alpha, \beta, \gamma, \delta$ Ferrers
diagrams

label
of the
rewriting rule

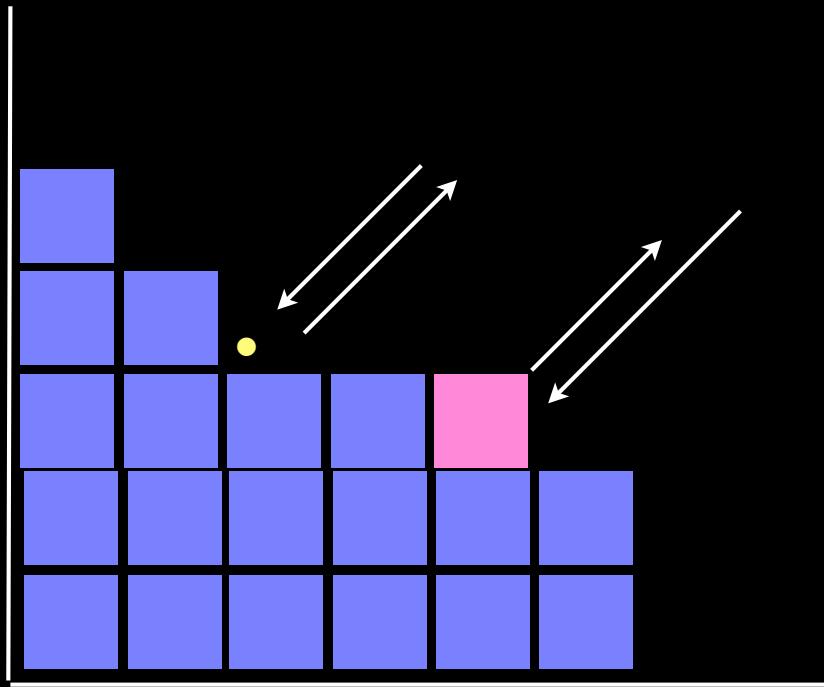
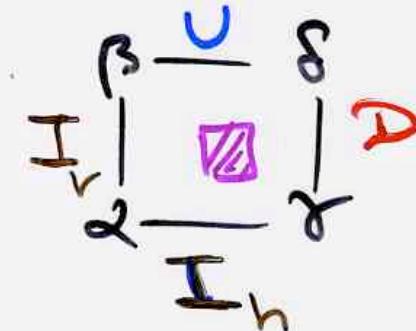
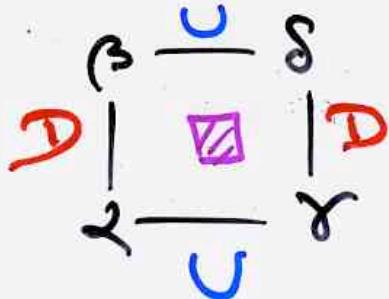
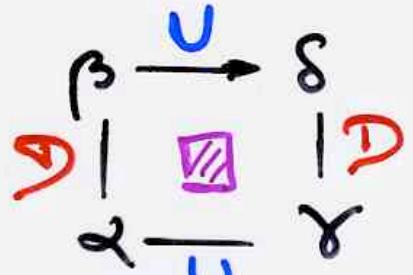
$$UD = DU + I_v I_h$$



$$UD = DU + I_v I_h$$

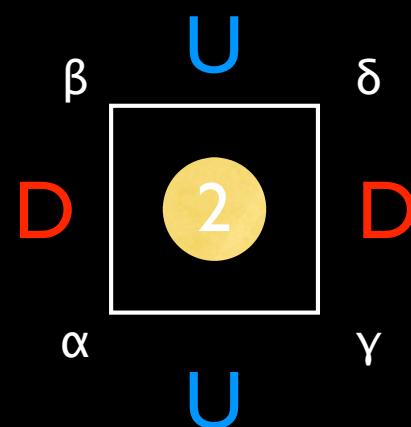


$$UD = DU + I_v I_h$$



$$\beta = \gamma$$

$$\beta = \gamma$$



$$\beta = \gamma = \alpha + (i)$$

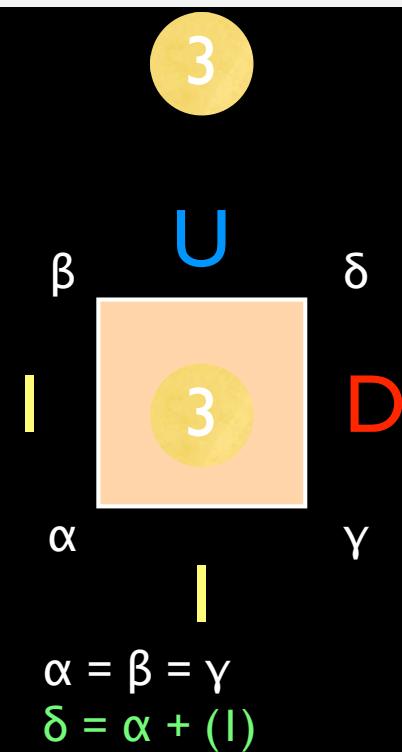
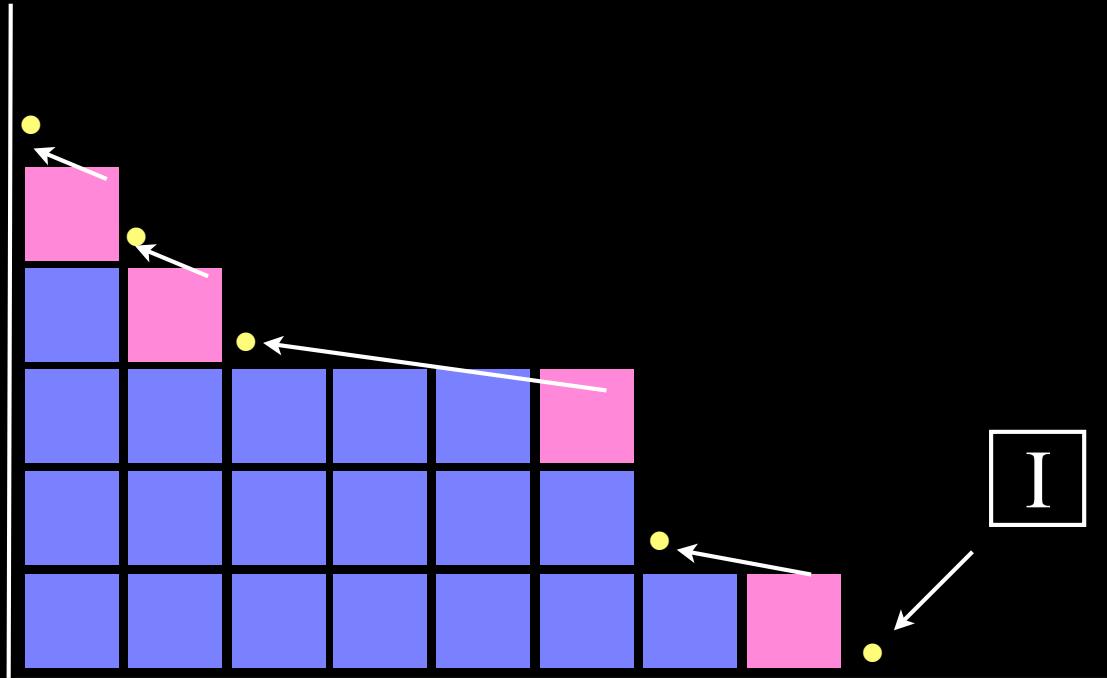
$$\delta = \beta + (i+1)$$

$$UD = DU + I_v I_h$$

$$\begin{array}{c} \beta \xrightarrow{U} \delta \\ D \downarrow \quad \quad | \\ \alpha \xrightarrow{U} \gamma \end{array}$$

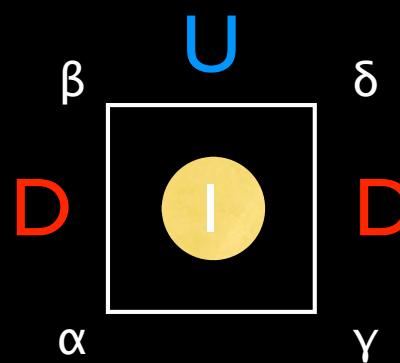
$$\begin{array}{c} \beta \xrightarrow{U} \delta \\ D \downarrow \quad \quad | \\ \alpha \xrightarrow{U} \gamma \end{array}$$

$$\begin{array}{c} \beta \xrightarrow{U} \delta \\ D \downarrow \quad \quad | \\ I_v \quad \quad | \\ \alpha \xrightarrow{U} \gamma \\ I_h \end{array}$$



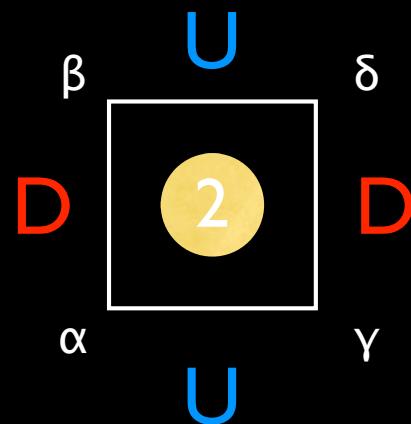
$$\left\{ \begin{array}{l} \textcolor{blue}{UD} = \textcolor{red}{D}U + I_v I_h \\ \textcolor{blue}{U} I_v = I_v \textcolor{blue}{U} \\ \textcolor{brown}{I}_h \textcolor{red}{D} = \textcolor{red}{D} \textcolor{brown}{I}_h \\ \textcolor{brown}{I}_h I_v = I_v I_h \end{array} \right.$$

$\beta \neq \gamma$

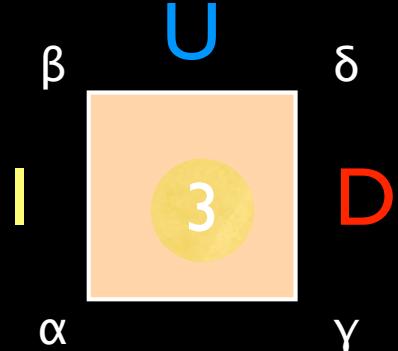


$$\delta = \beta \cup \gamma$$

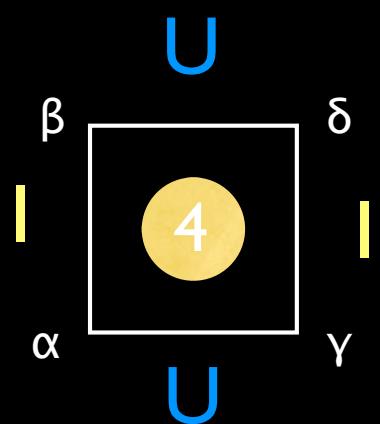
$\beta = \gamma$



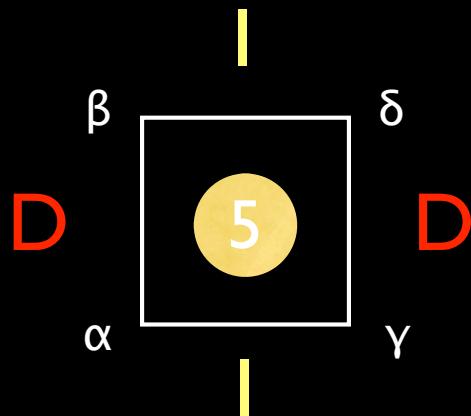
$$\begin{aligned}\beta &= \gamma = \alpha + (i) \\ \delta &= \beta + (i+1)\end{aligned}$$



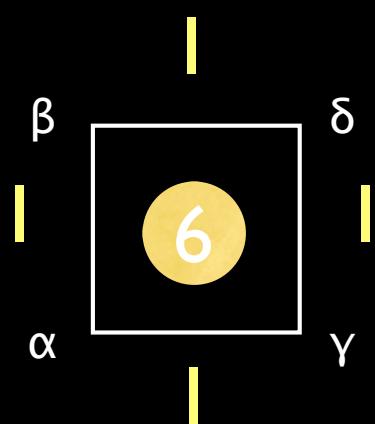
$$\begin{aligned}\alpha &= \beta = \gamma \\ \delta &= \alpha + (I)\end{aligned}$$



$$\begin{aligned}\alpha &= \beta \\ \delta &= \gamma = \beta + (i)\end{aligned}$$

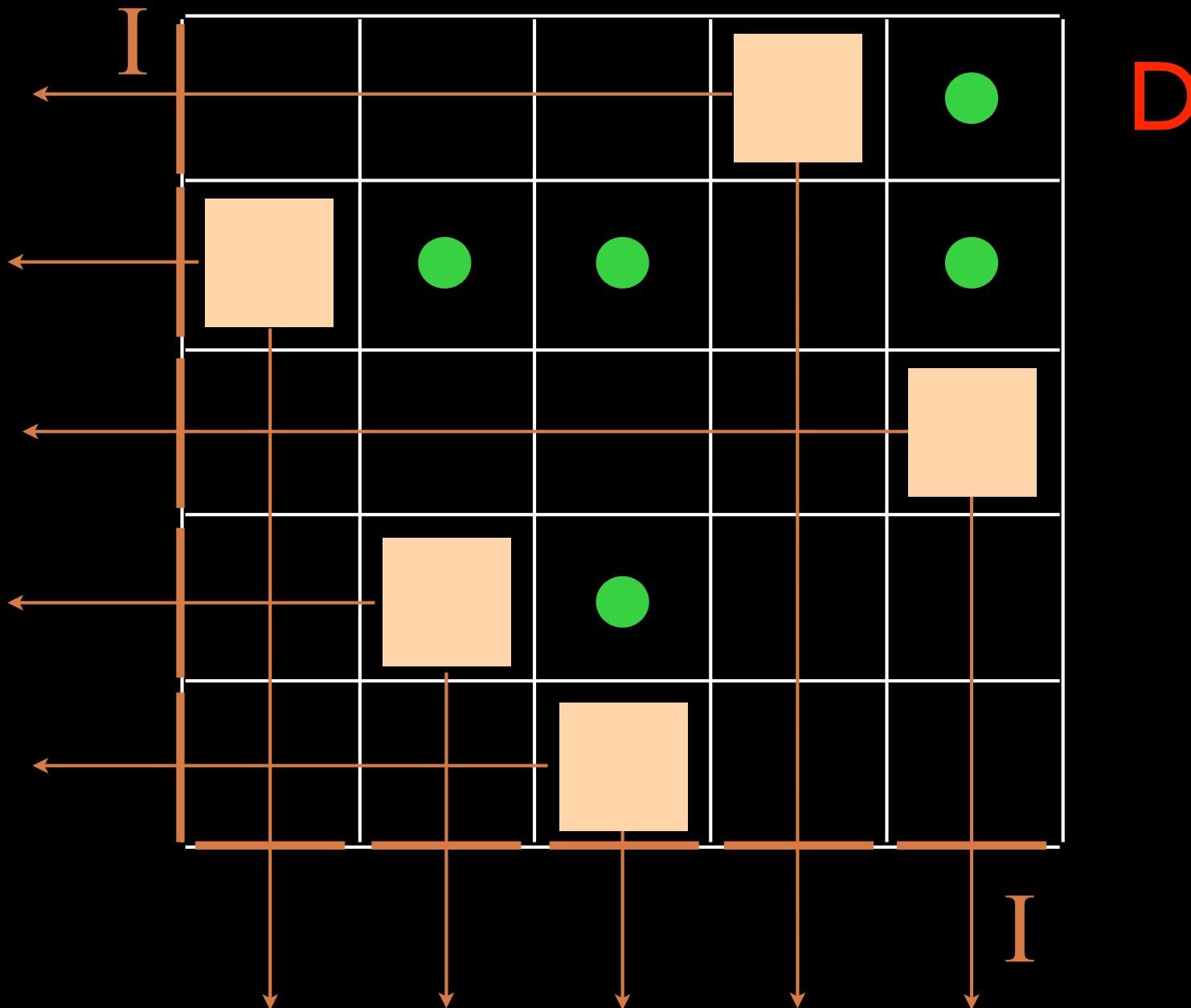


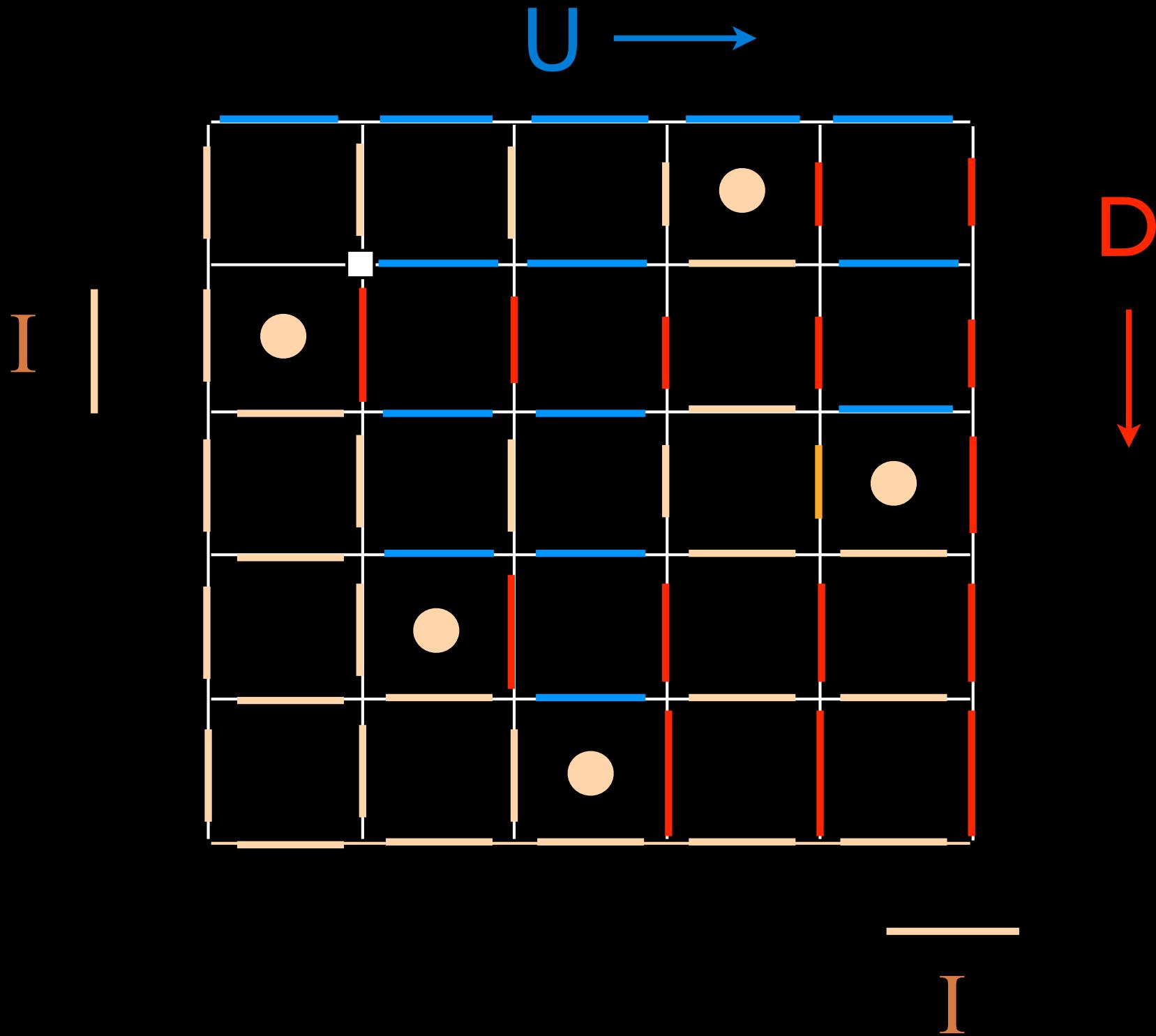
$$\begin{aligned}\alpha &= \gamma \\ \delta &= \beta = \alpha + (i)\end{aligned}$$

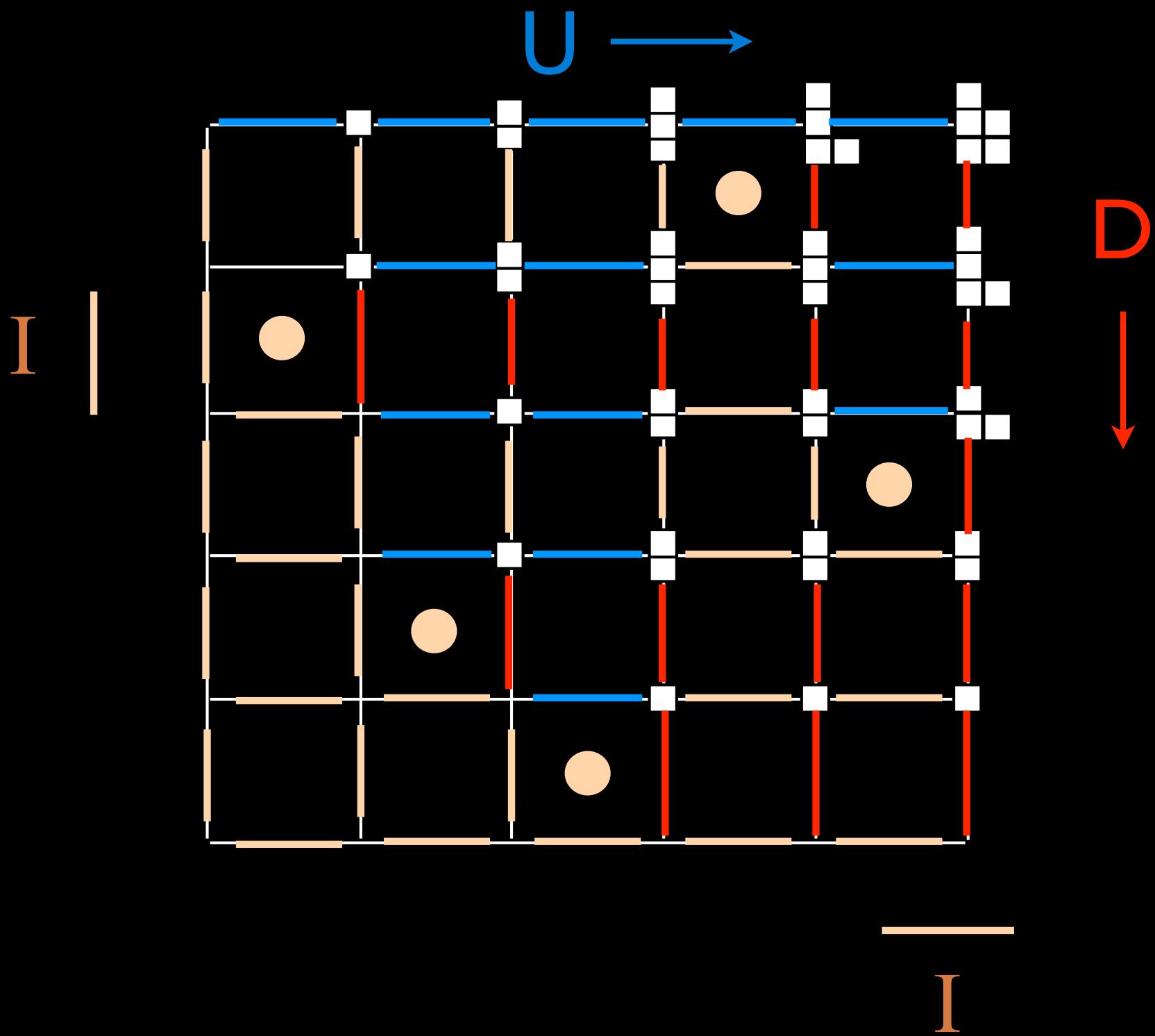


$$\delta = \alpha = \beta = \gamma$$

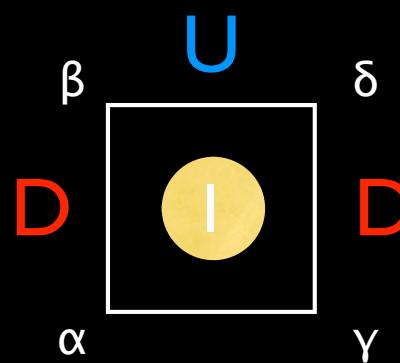
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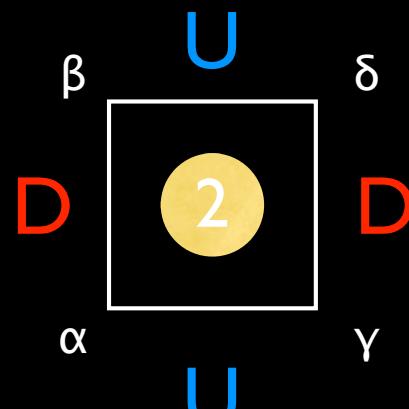


$\beta \neq \gamma$

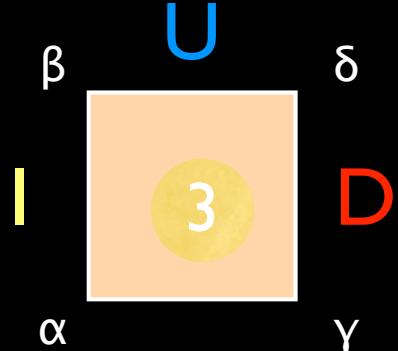


$$\delta = \beta \cup \gamma$$

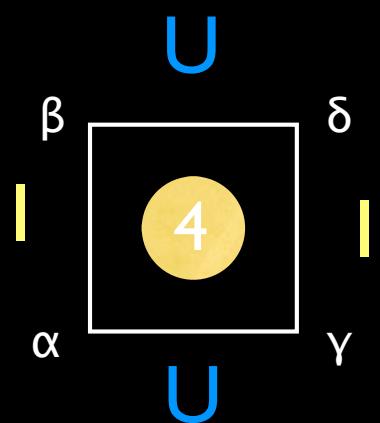
$\beta = \gamma$



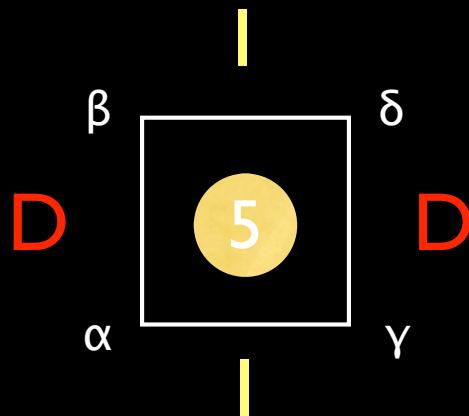
$$\begin{aligned}\beta &= \gamma = \alpha + (i) \\ \delta &= \beta + (i+1)\end{aligned}$$



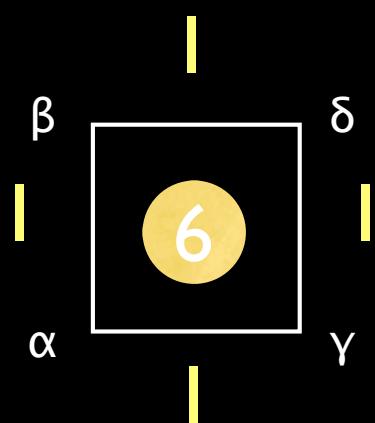
$$\begin{aligned}\alpha &= \beta = \gamma \\ \delta &= \alpha + (I)\end{aligned}$$



$$\begin{aligned}\alpha &= \beta \\ \delta &= \gamma = \beta + (i)\end{aligned}$$

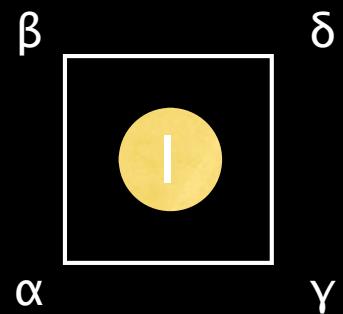


$$\begin{aligned}\alpha &= \gamma \\ \delta &= \beta = \alpha + (i)\end{aligned}$$

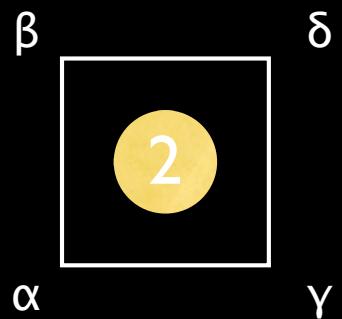


$$\delta = \alpha = \beta = \gamma$$

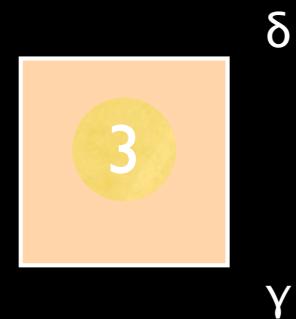
$\beta \neq \gamma$



$\beta = \gamma$
 $\alpha \neq \beta$



$\alpha = \beta = \gamma$



$\delta = \beta \cup \gamma$

$\beta = \gamma = \alpha + (i)$
 $\delta = \beta + (i+1)$

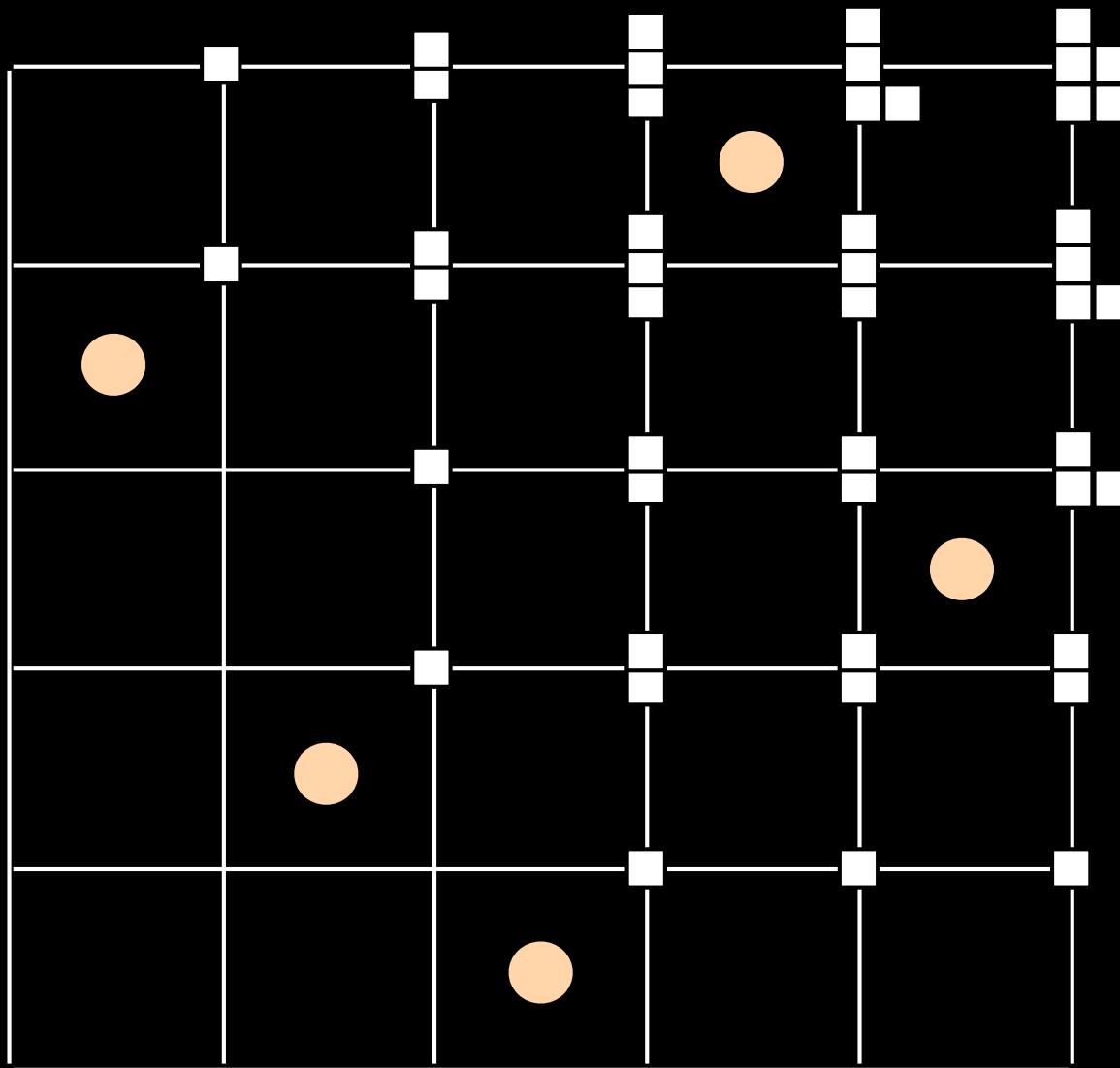
$\delta = \alpha + (l)$



$\alpha = \beta = \gamma$



$\delta = \alpha = \beta = \gamma$



RSK with
Fomin's
“local rules”

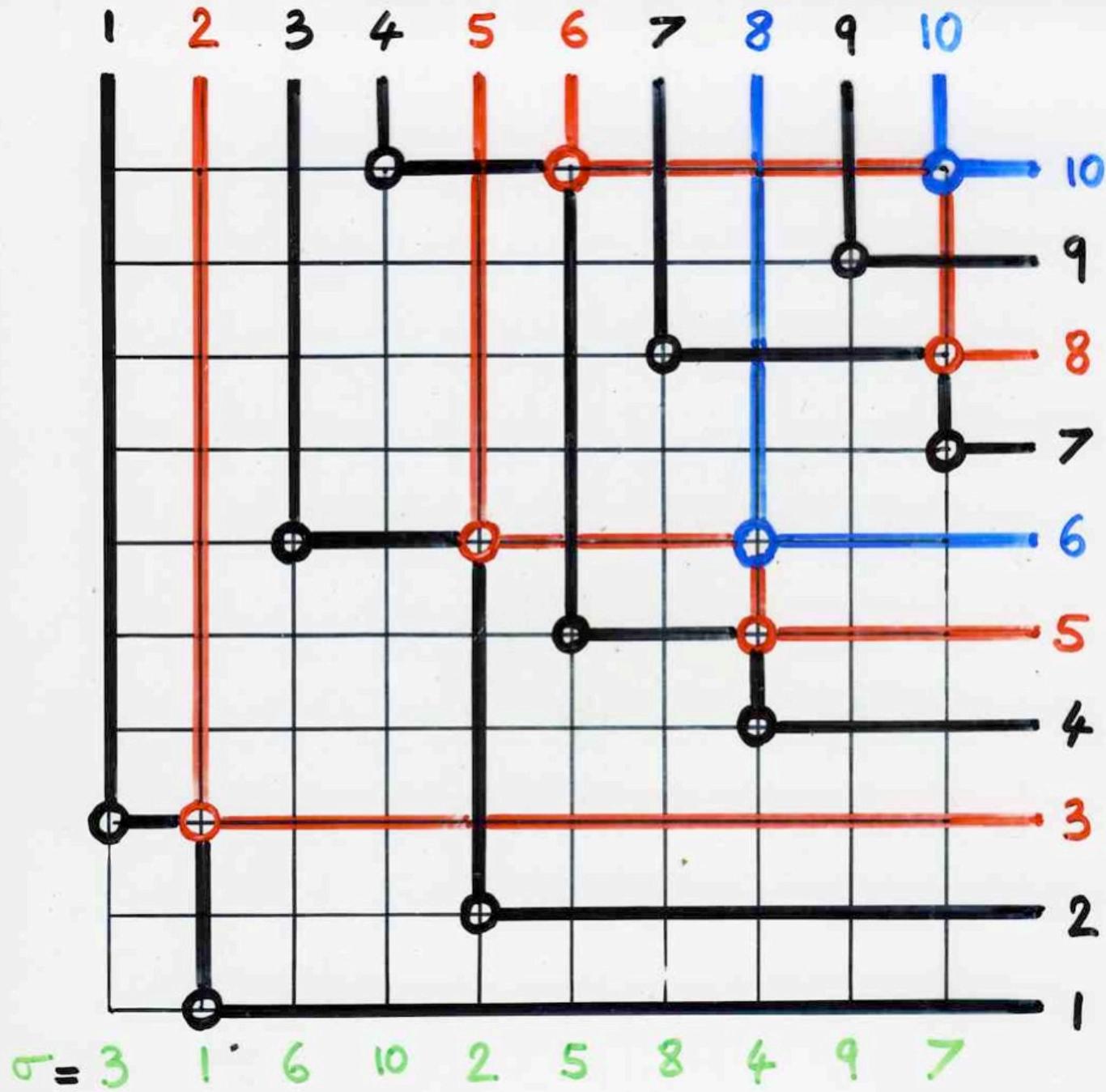
$$UD = q DU + I$$

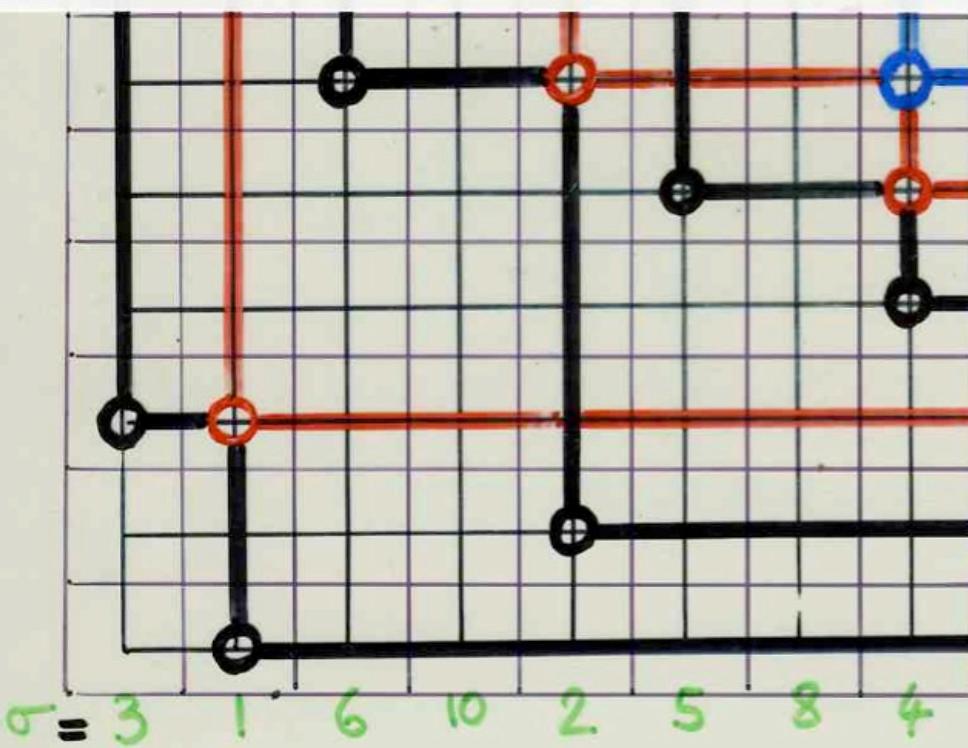
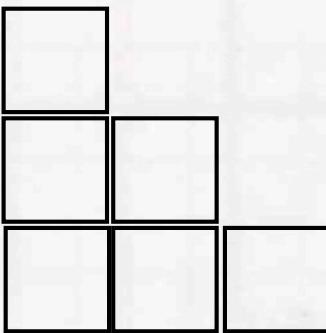


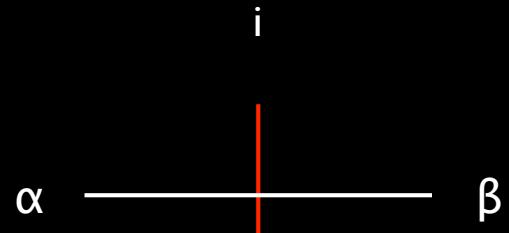
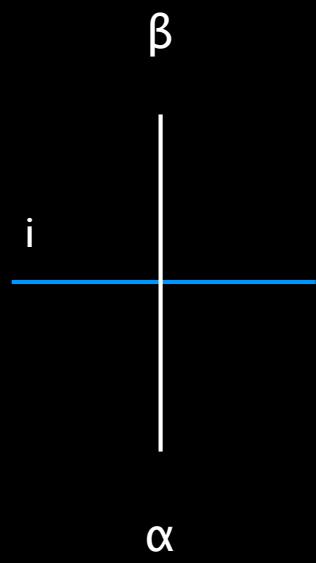
Sergey Fomin
(with C. K.)

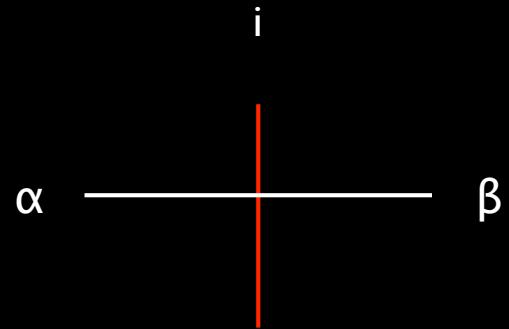
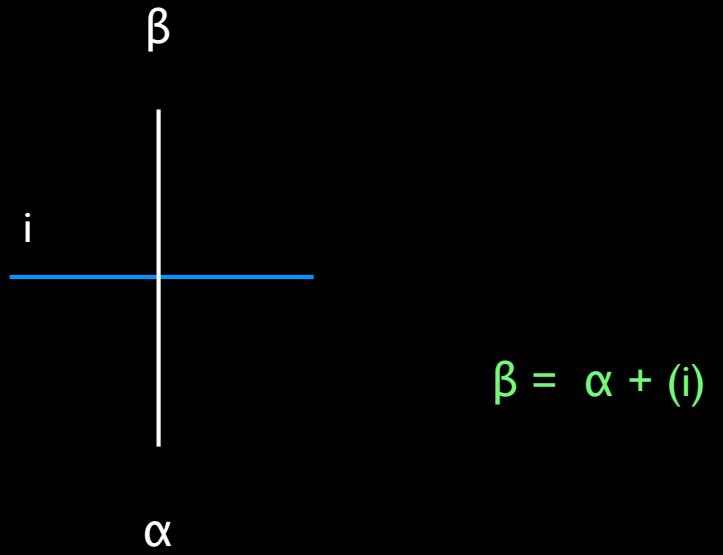
local RSK and geometric RSK

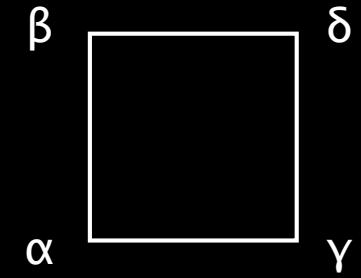
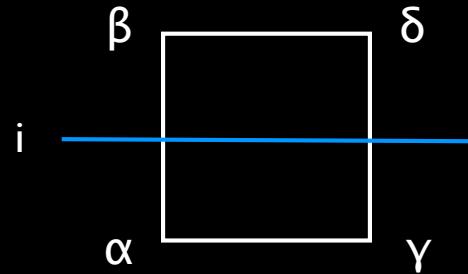
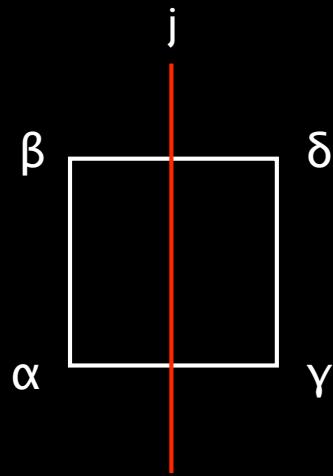
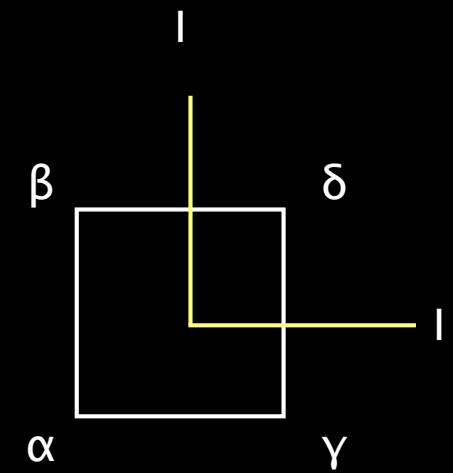
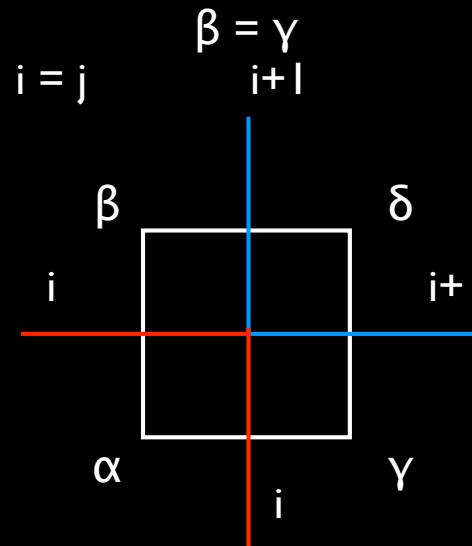
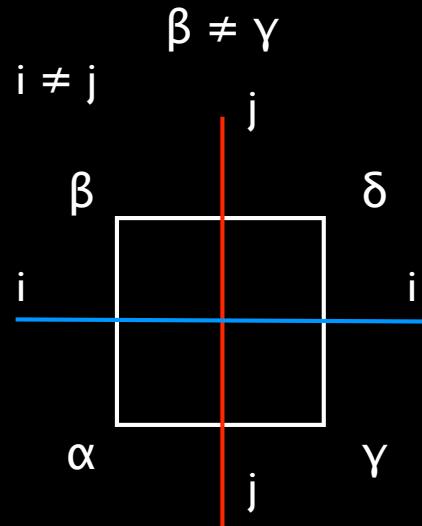
(the geometric construction with “light” and “shadow” for RSK
leads to a simple proof of the fact that RSK and the “local rules”
give the same bijection)

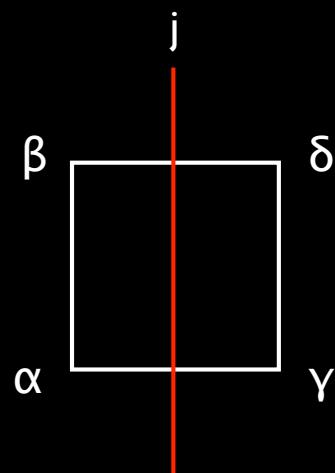
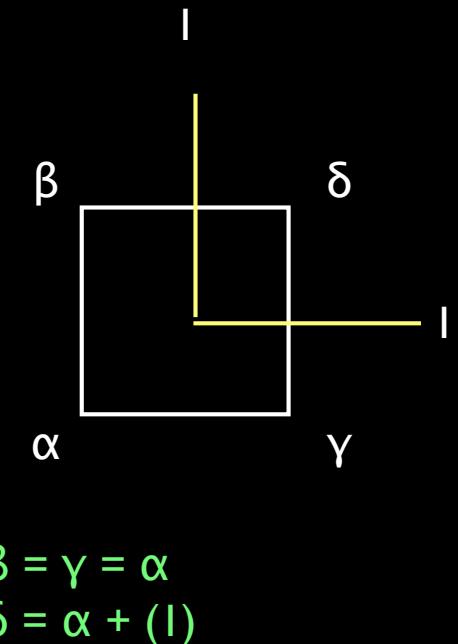
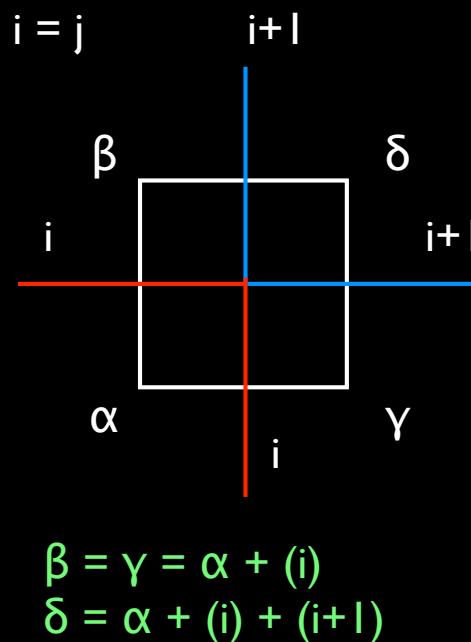
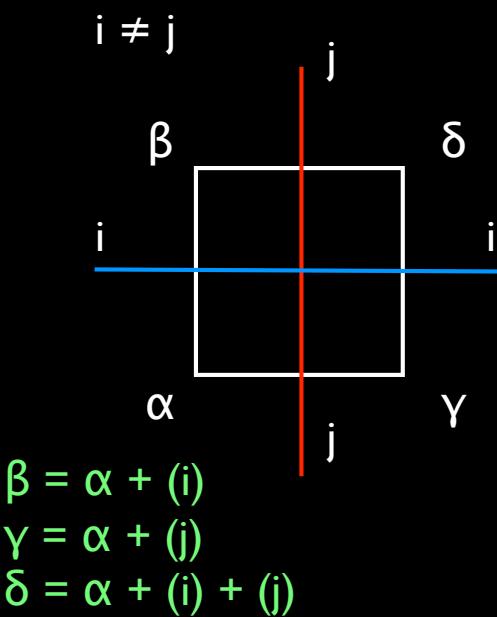




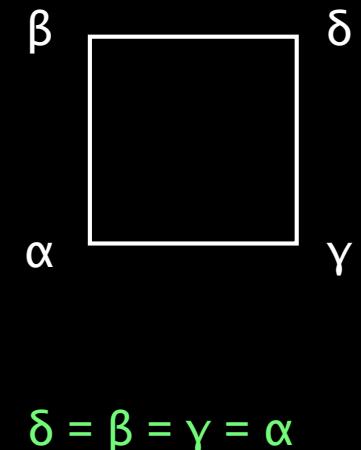
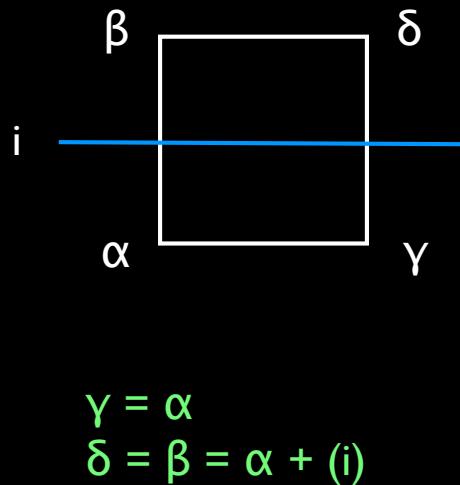


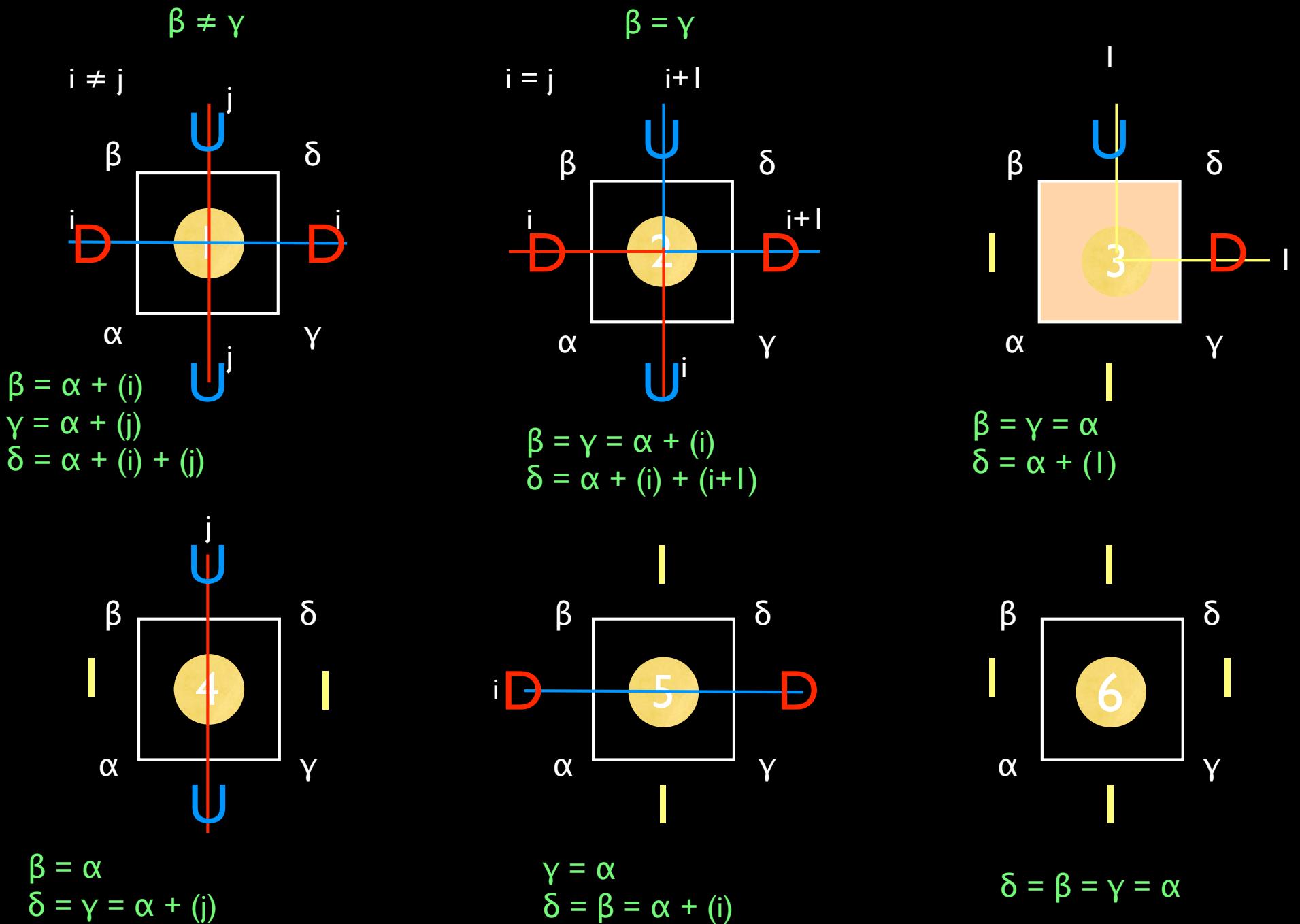




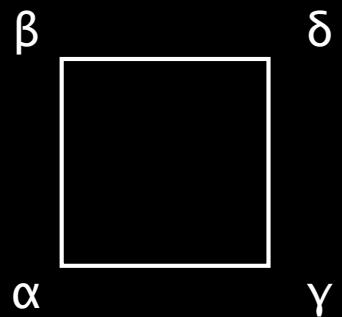


$$\beta = \alpha
\delta = \gamma = \alpha + (j)$$



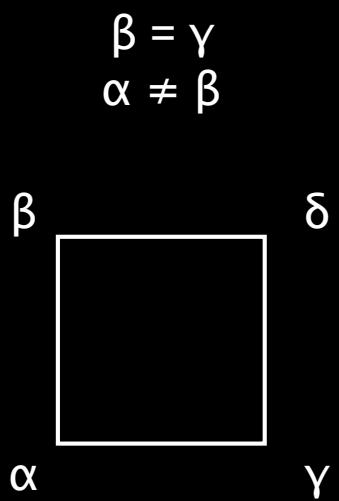


$\beta \neq \gamma$

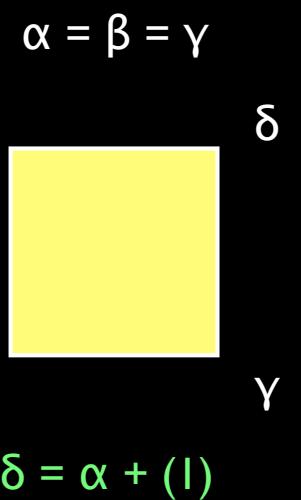


$$\delta = \beta \cup \gamma$$

$\beta = \gamma$

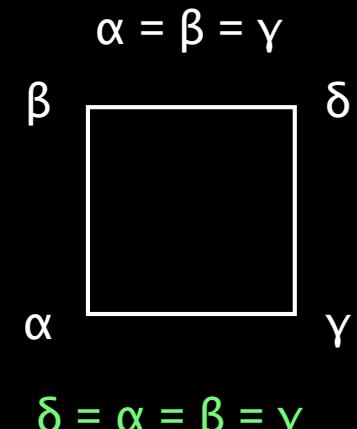


$$\begin{aligned}\beta &= \gamma \\ \alpha &\neq \beta\end{aligned}$$

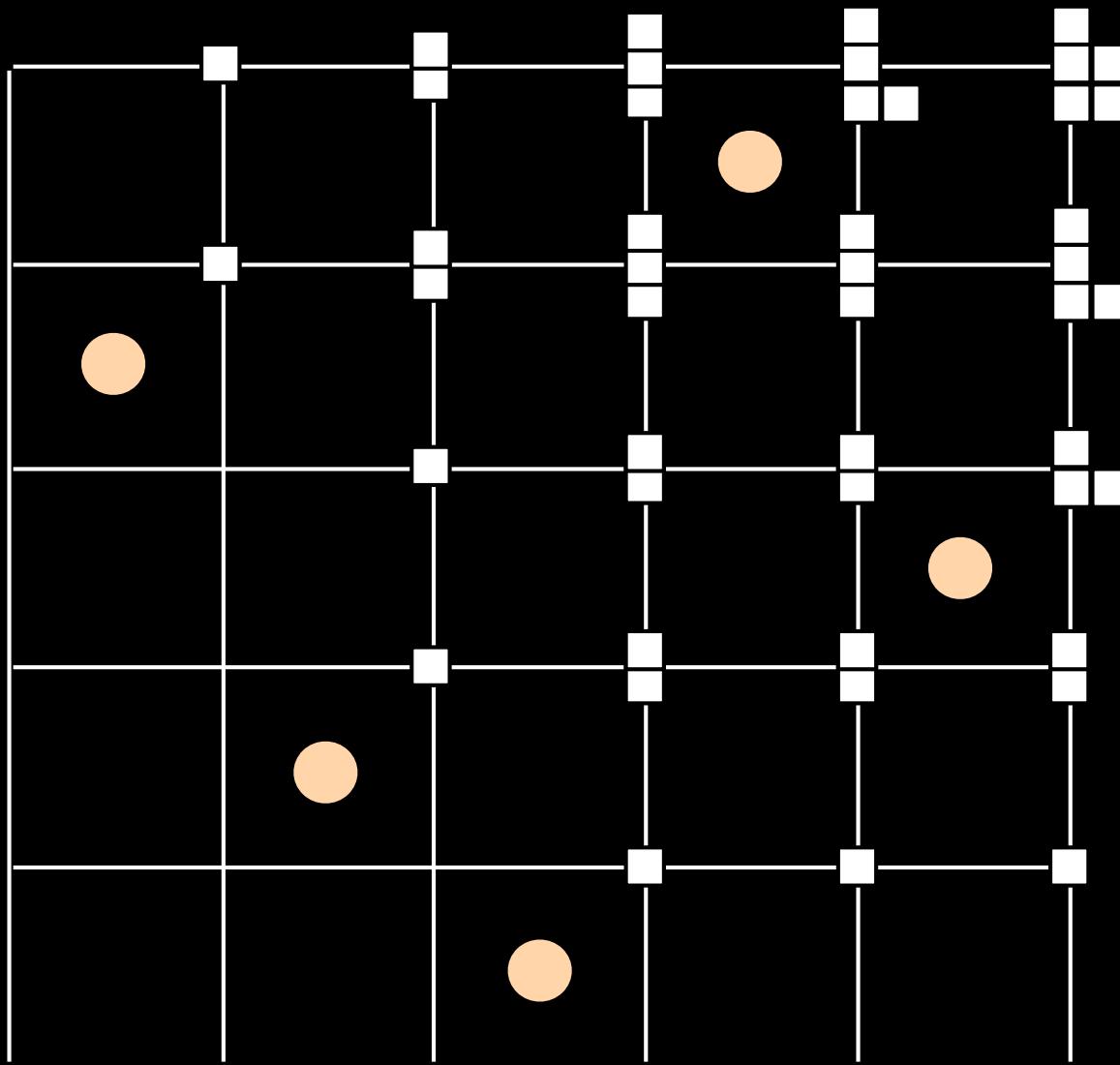


$$\delta = \alpha + (l)$$

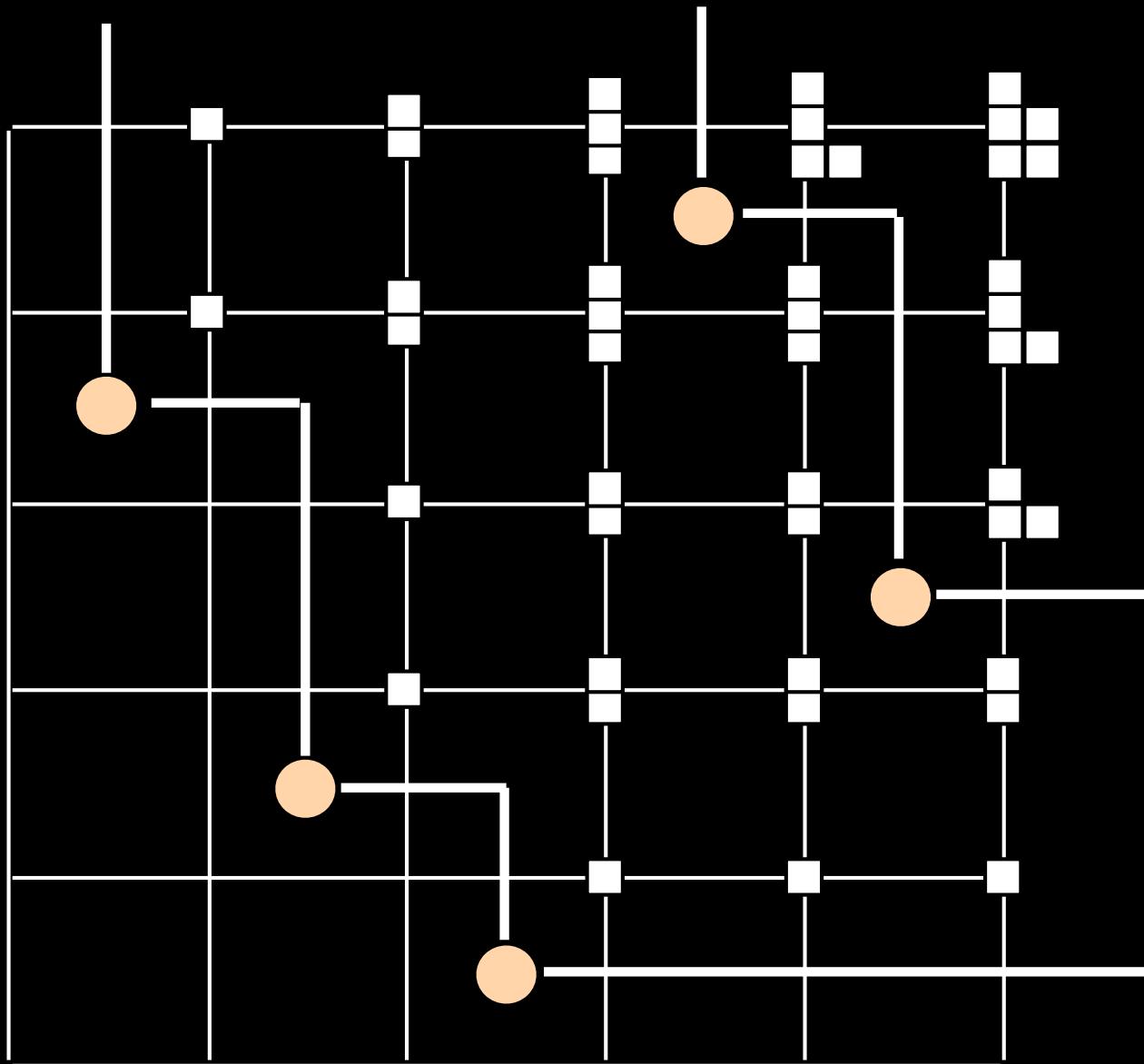
$$\begin{aligned}\beta &= \gamma = \alpha + (i) \\ \delta &= \beta + (i+l)\end{aligned}$$

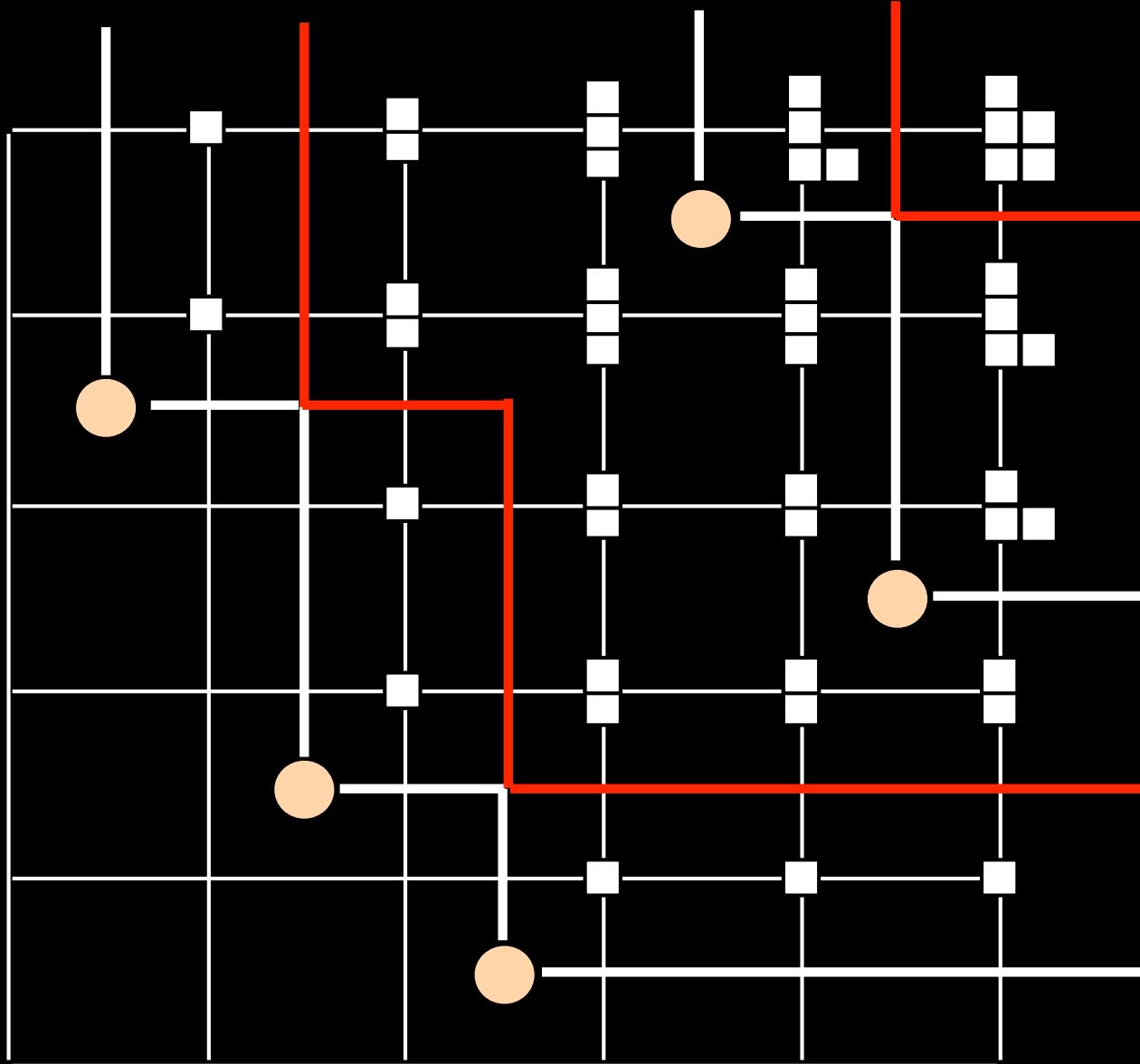


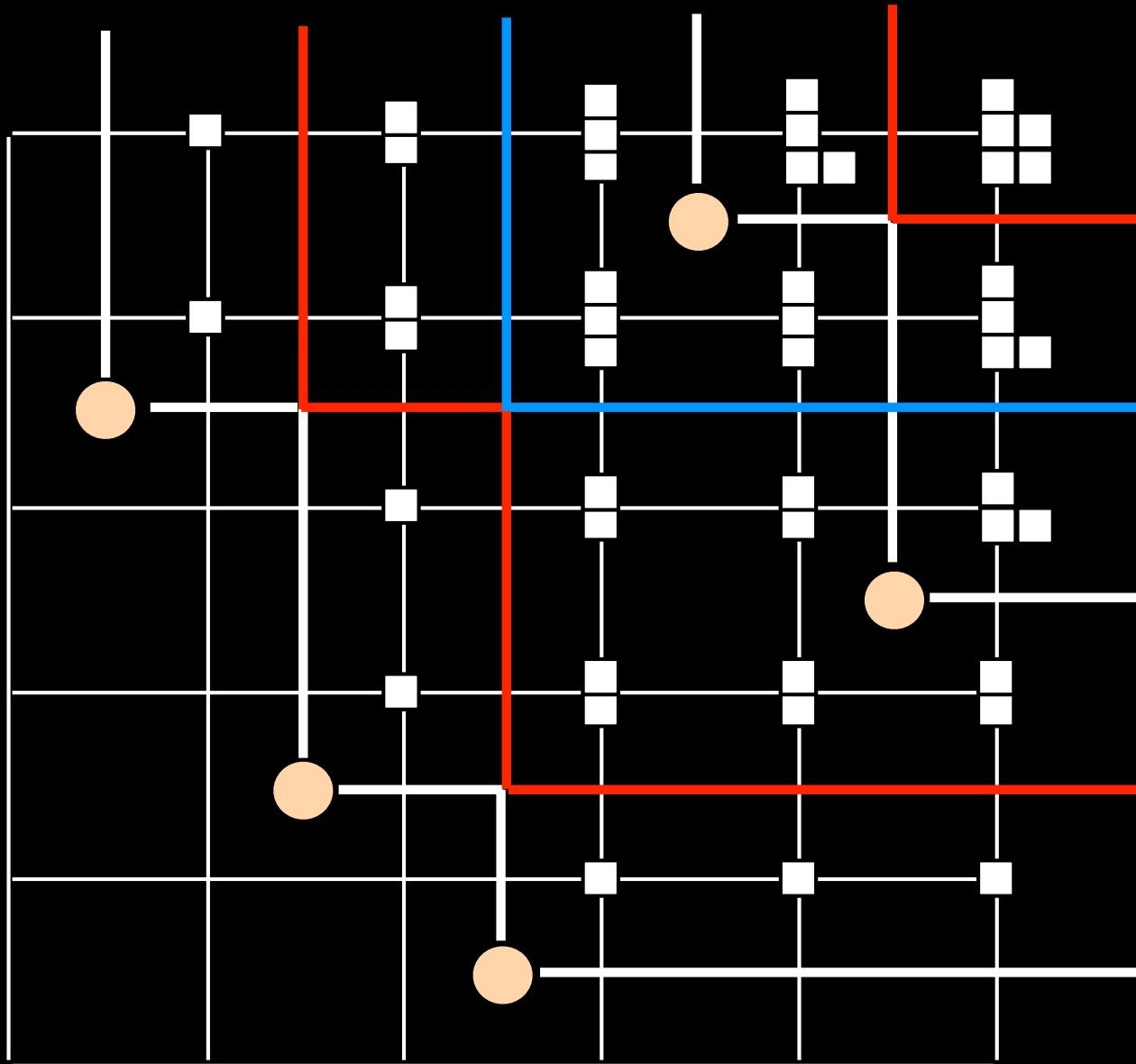
$$\delta = \alpha = \beta = \gamma$$

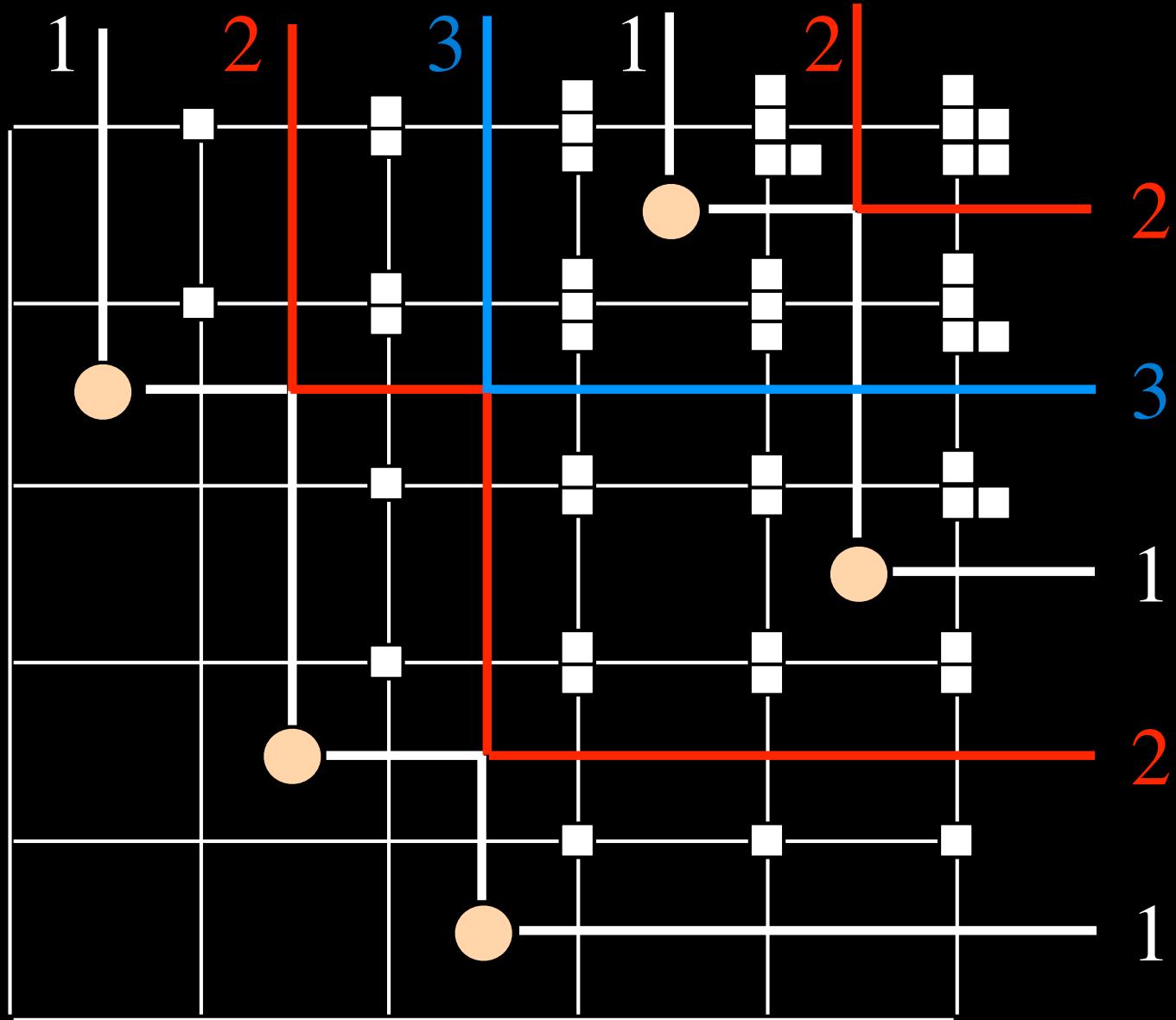


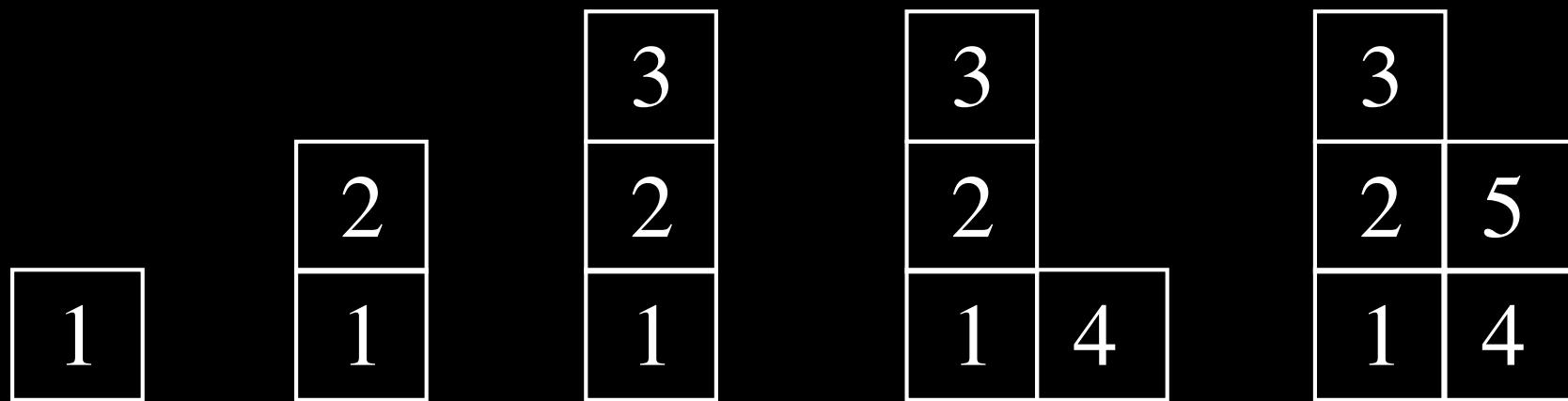
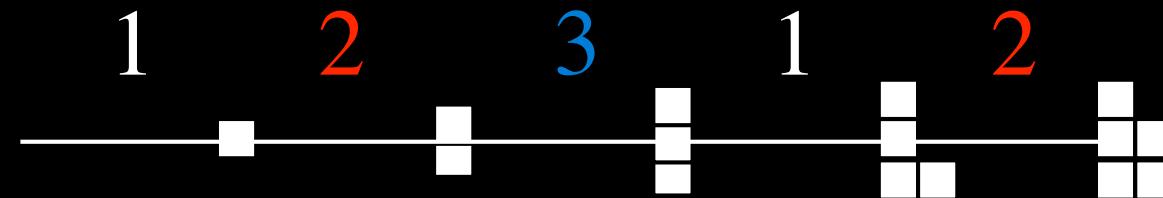
4 2 1 5 3











w = 1 2 3 1 2

Yamanuchi word

	3
2	5
1	4

1

2

3

1

2

2

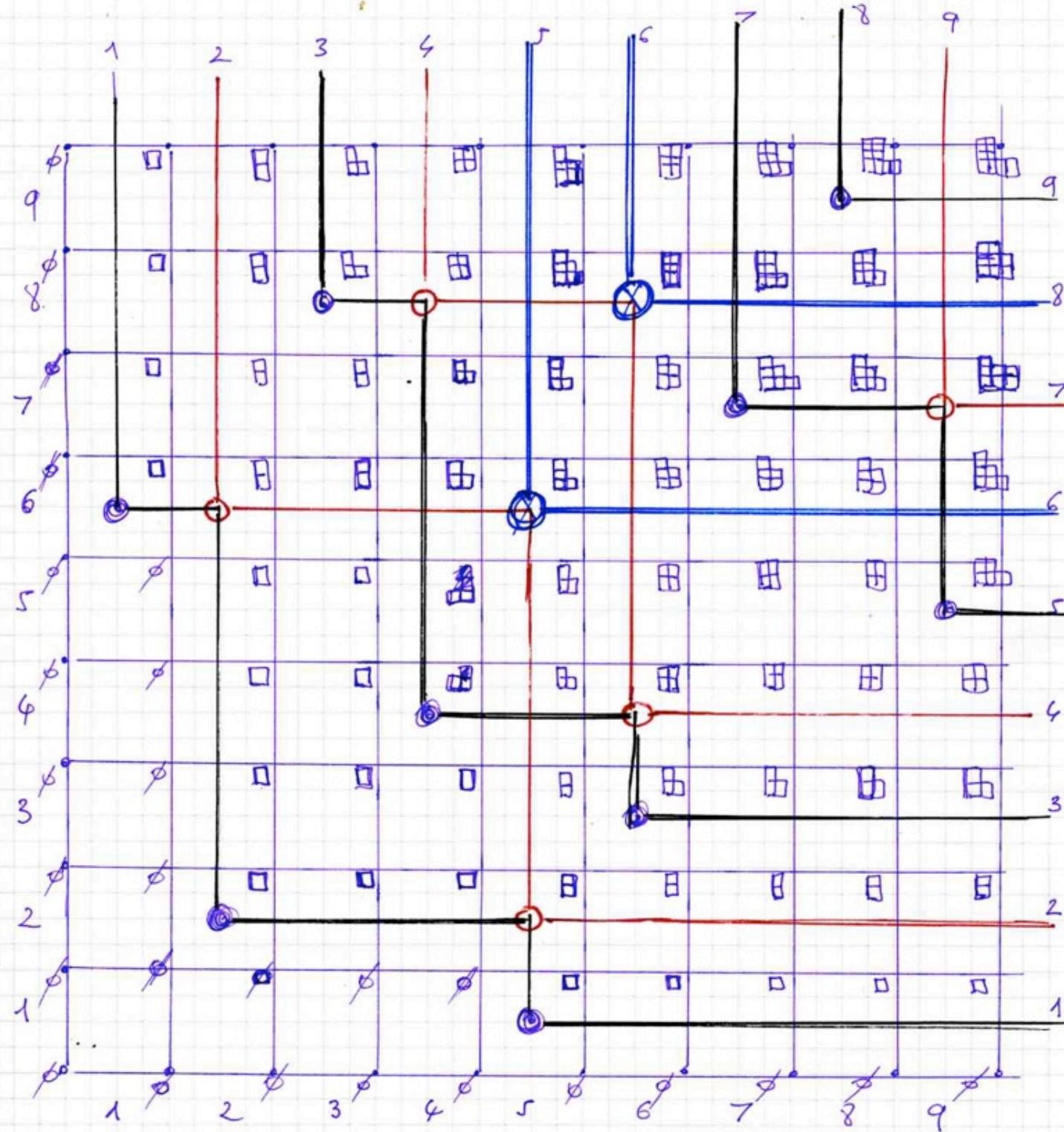
3

1

2

	4
2	5
1	3

	2
4	1
5	3



5	6		
2	4	9	
1	3	7	8

Q

another example
with
6 2 8 4 1 3 7 9 5

6	8		
2	4	7	
1	3	5	9

P

Sergey Fomin

- Schur operators and Knuth correspondences, *Journal of Combinatorial Theory, Ser.A* **72** (1995), 277-292.
- Duality of graded graphs, *Journal of Algebraic Combinatorics* **3** (1994), 357-404.
- Schensted algorithms for dual graded graphs, *Journal of Algebraic Combinatorics* **4** (1995), 5-45.
- Dual graphs and Schensted correspondences, *Series formelles et combinatoire algébrique*, P.Leroux and C.Reutenauer, Ed., Montreal, LACIM, UQAM, 1992, 221-236.
- Finite posets and Ferrers shapes (with T.Britz, 41 pages) *Advances in Mathematics* **158** (2000), 86-127.

A survey on the Greene-Kleitman correspondence; many proofs are new.

- Knuth equivalence, jeu de taquin, and the Littlewood-Richardson rule (30 pages) Appendix 1 to Chapter 7 in: R.P.Stanley, *Enumerative Combinatorics, vol.2*, Cambridge University Press, 1999.

Richard P. Stanley

- Differential posets, *J. Amer. Math. Soc.* **1** (1988), 919-961.
- Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.),

The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145-165.



Xavier Gérard Viennot

- Une forme géométrique de la correspondance de Robinson-Schensted, in “Combinatoire et Représentation du groupe symétrique” (D. Foata ed.) Lecture Notes in Mathematics n° 579, pp 29-68, 1976

Marc van Leeuwen

- The Robinson-Schensted and Schützenberger algorithms, an elementary approach (a 272 Kb dvi file) Electronic Journal of Combinatorics, Foata Festschrift, Vol 3(no.2), R15 (1996)

Guoniu Han

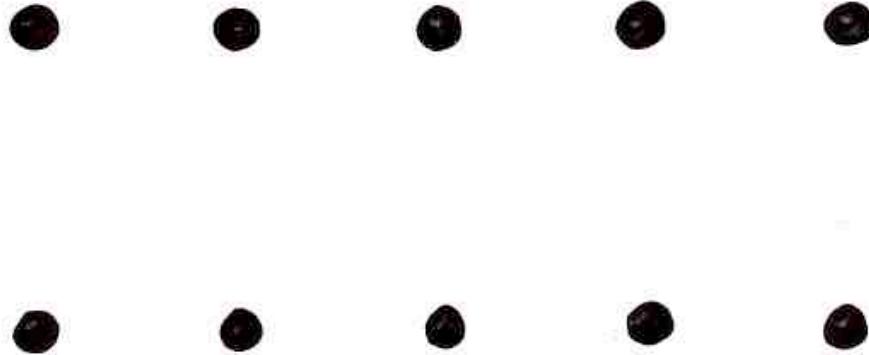
<http://math.u-strasbg.fr/~guoniu/software/rsk/index.html>

Autour de la correspondance de Robinson-Schensted

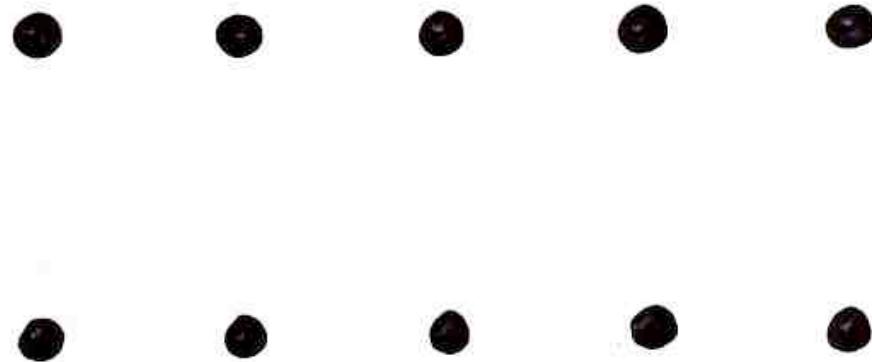
Exposé au SLC 52 et LascouxFest, 29/03/2004

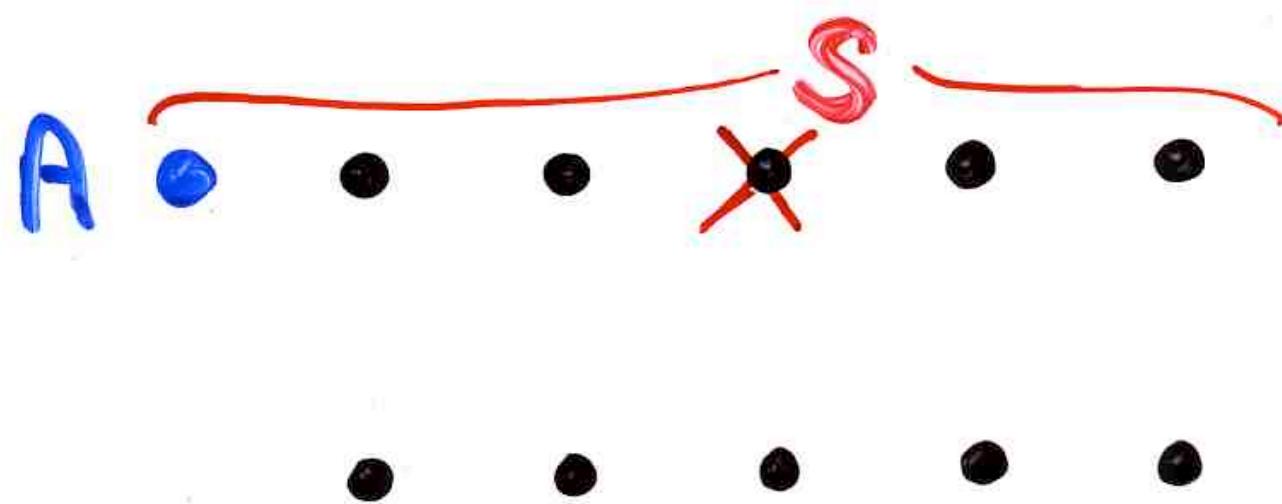


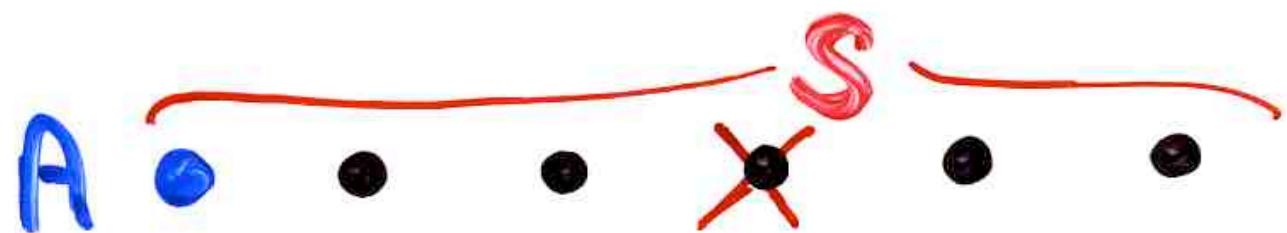
other representations
for $UD = DU + I$

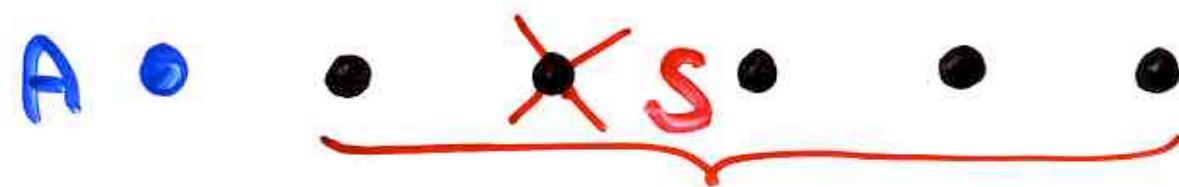
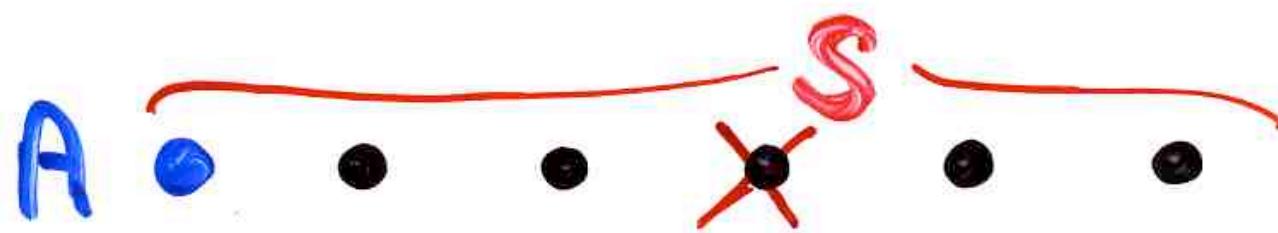


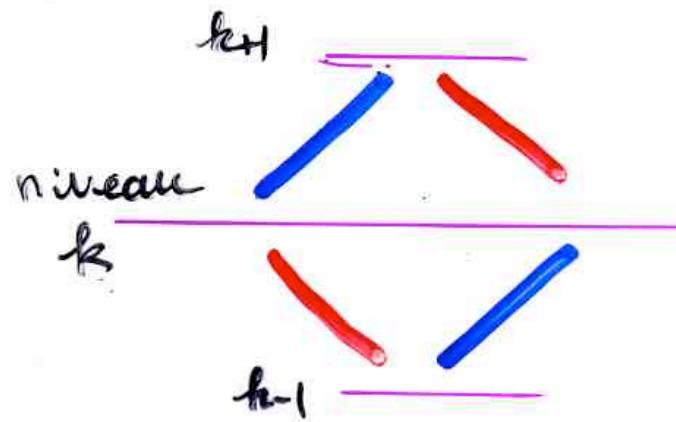
A o











possibilities

$$\begin{array}{lll} AS & 1 \times (k+1) & = k+1 \\ SA & k \times 1 & = k \end{array}$$

$$UD = DU + I$$

Prop- w mot de Dyck

$$c_{0,0}(w) = v_H(w) \quad \begin{matrix} \text{valuation} \\ \text{Hermite} \end{matrix}$$

$$\begin{aligned} & \cancel{x}_k = k \\ & = \text{nb d'histoires d'Hermite} \\ & \text{associées à } w \end{aligned}$$

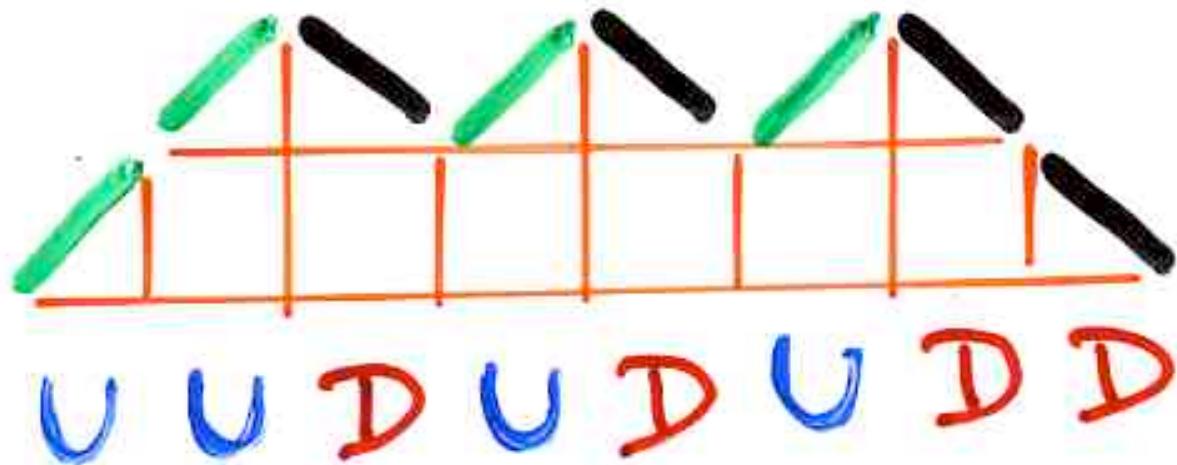
polynôme d'Hermite $H_n(x)$

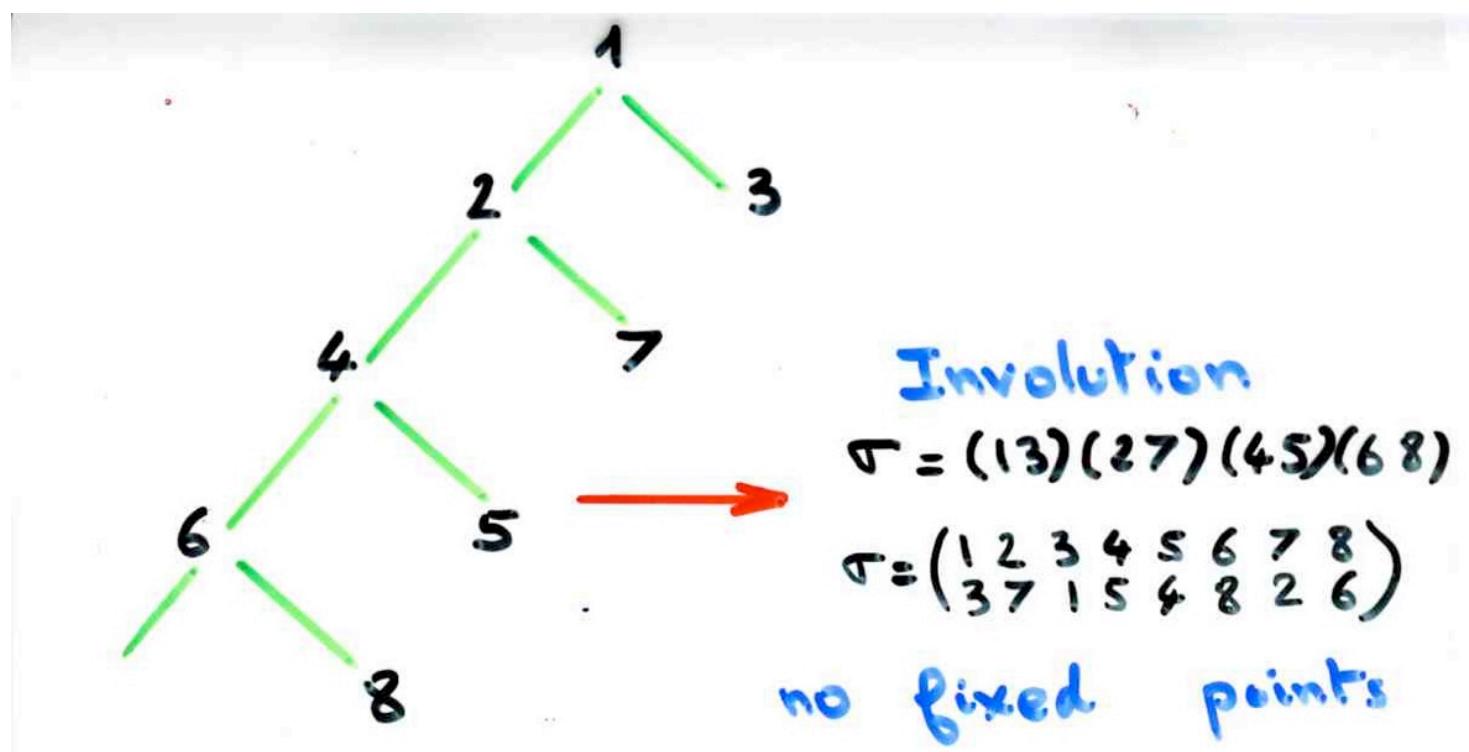
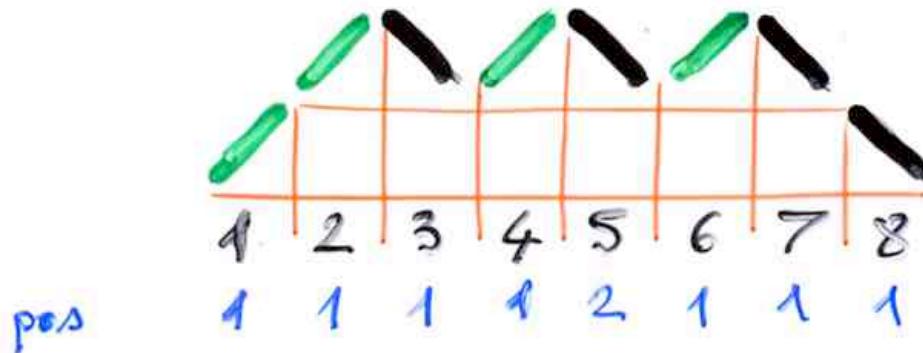
$$a_k = k ; \quad b_k = 0$$
$$(k \geq 1) \quad (k \geq 0)$$

$$a_k = 1 \quad \begin{cases} b'_k = 0 \\ b''_k = 0 \end{cases} \quad c_k = k$$

Histones d'Hermite

ex :





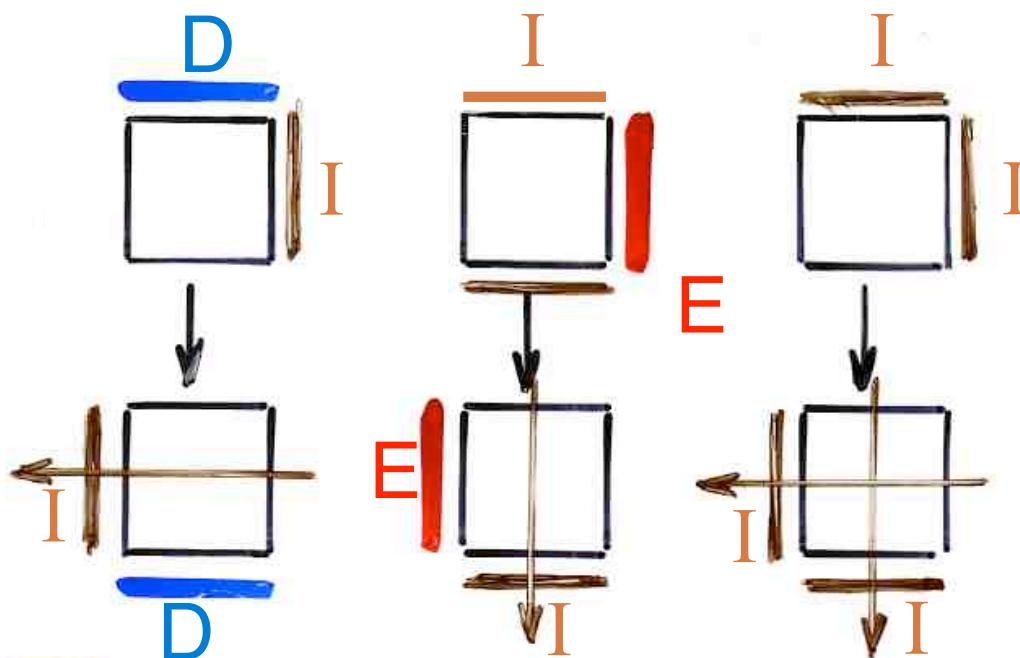
The PASEP algebra

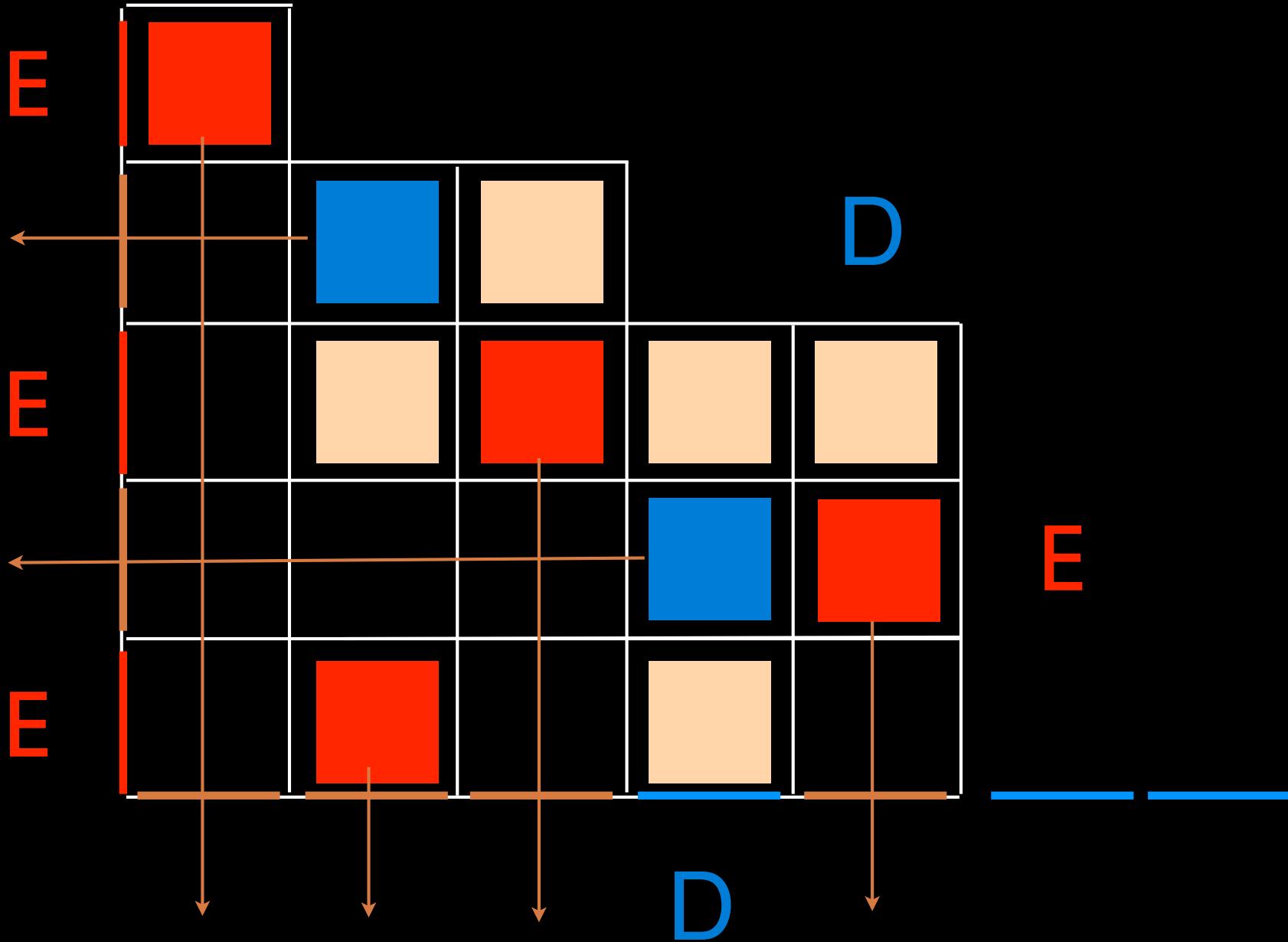
$$DE = qED + E + D$$

Proof: "planarization" of the rewriting rules

$$\boxed{D} \mid E \rightarrow q \boxed{E} \mid \boxed{\cancel{X}} + \boxed{E} \mid \boxed{I} + I \mid \boxed{D}$$

\boxed{I} identity





alternative tableau

A 5x5 grid with the following colored squares:

- Row 1, Column 1: Red square
- Row 2, Column 2: Blue square
- Row 3, Column 3: Red square
- Row 4, Column 4: Blue square
- Row 5, Column 1: Red square

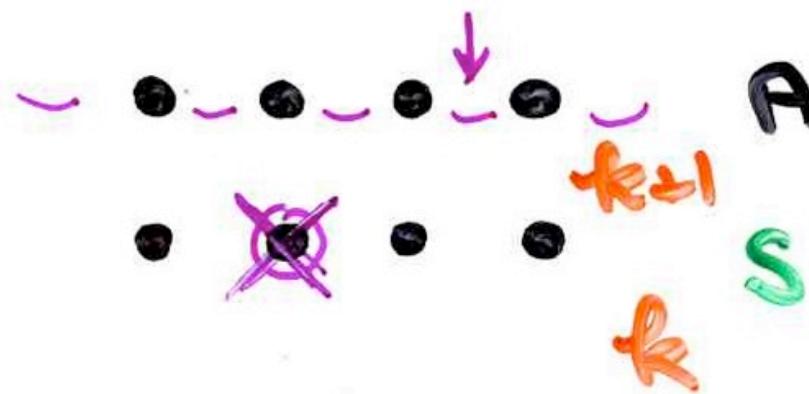
Opérations primitives

A

ajout

S

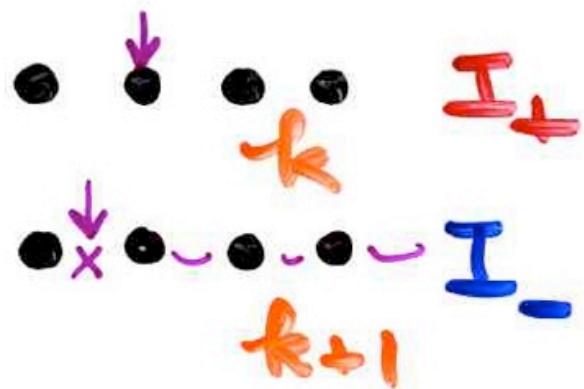
suppression



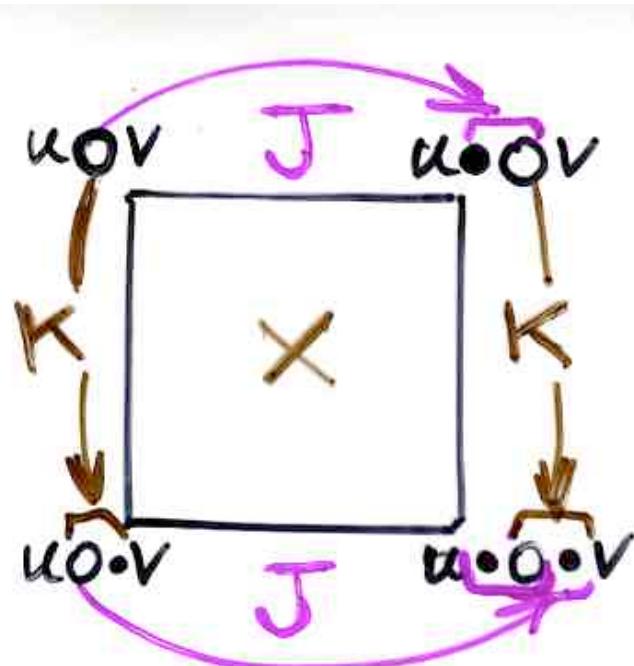
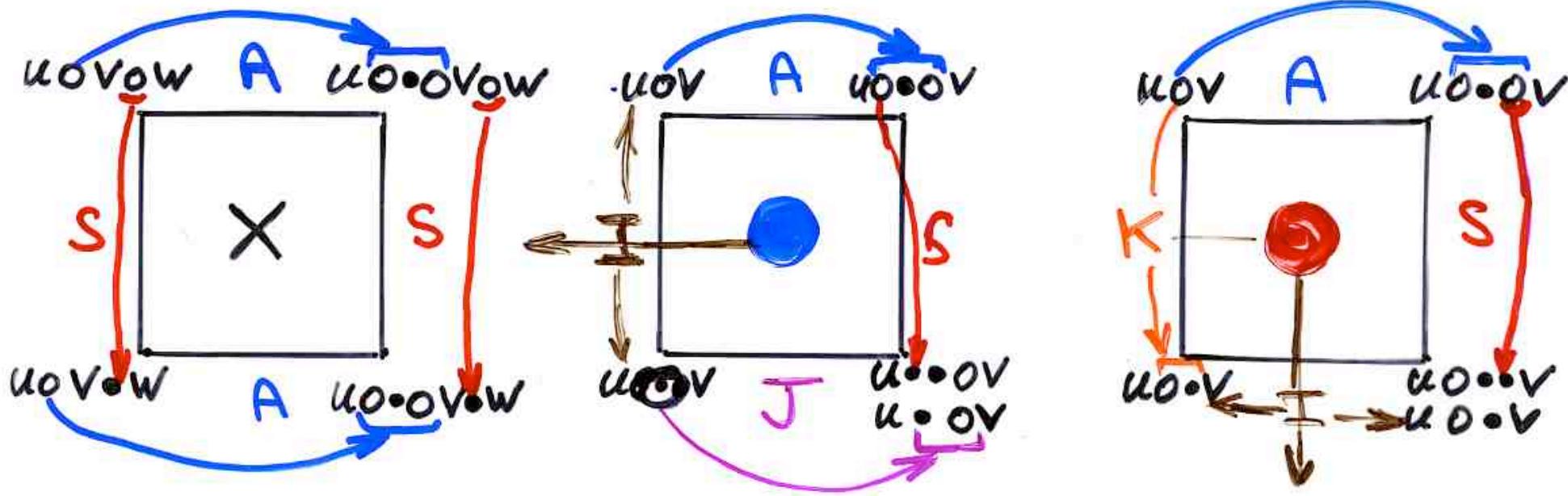
I₊

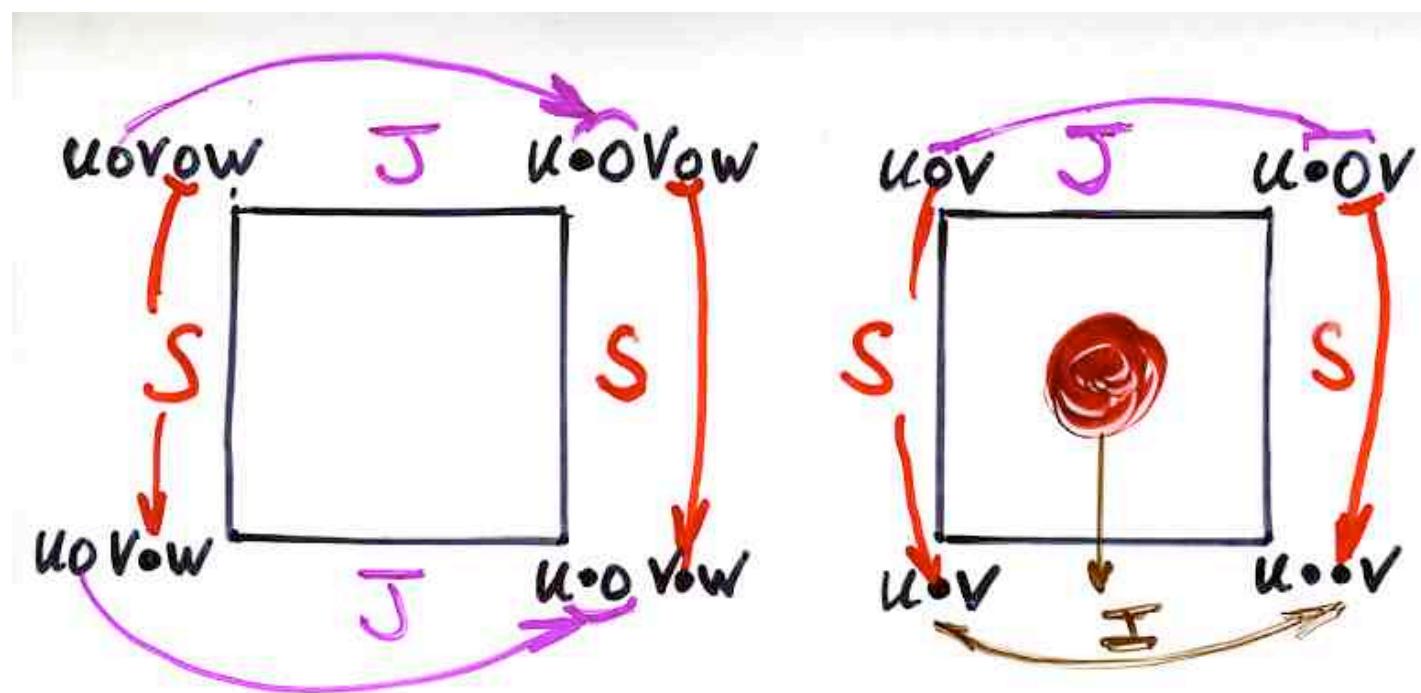
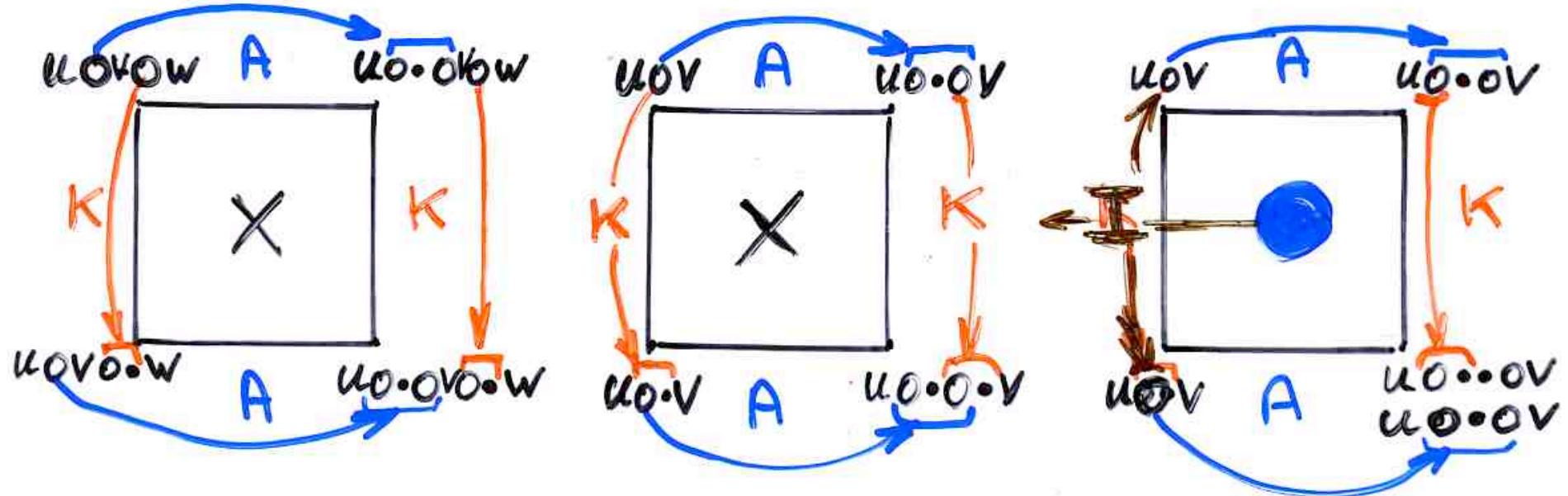
I₋

positive
interrogation
negative



number of choices for each
primitive operations





Lemma.

$$AS = SA + J + K$$

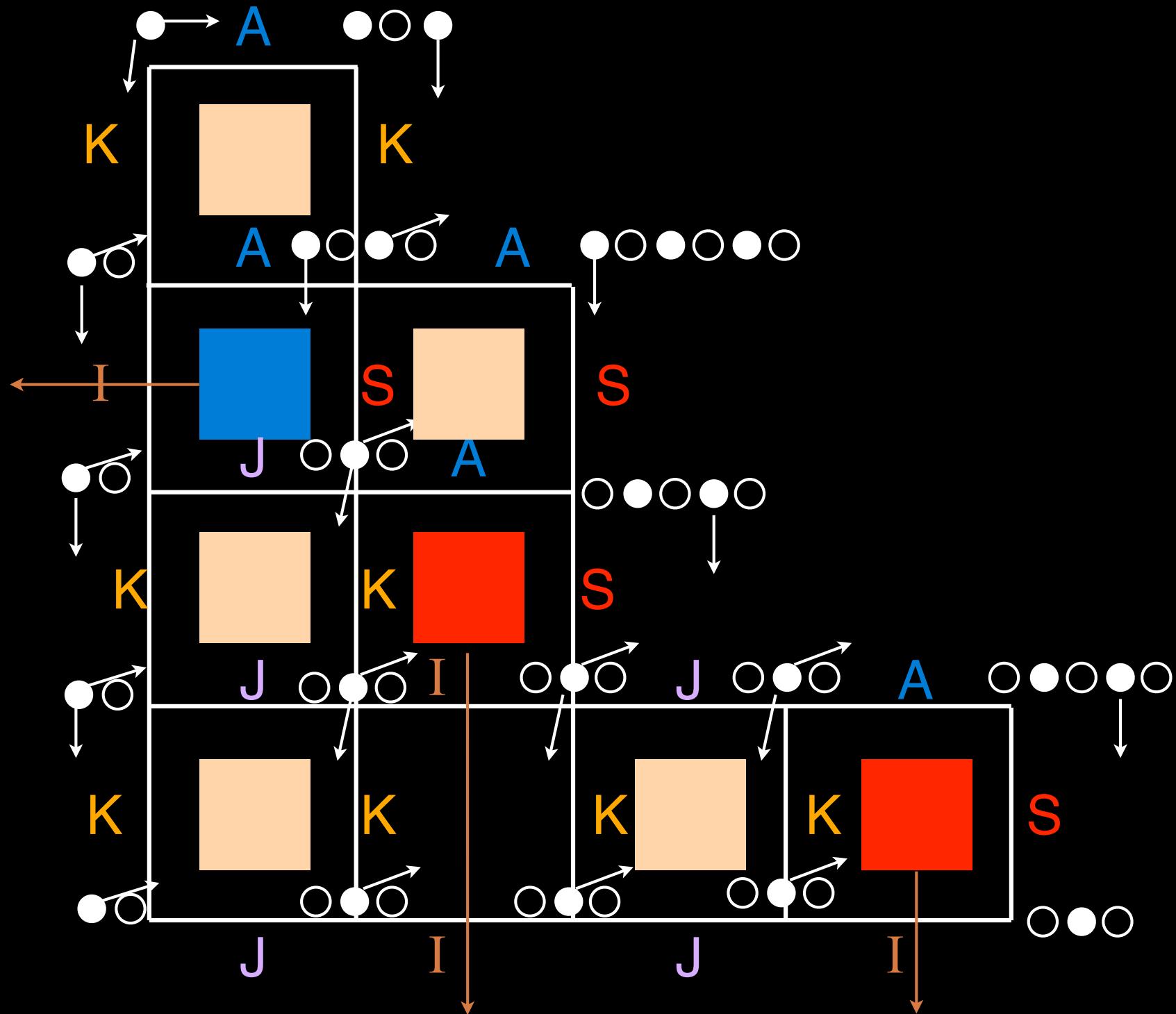
$$AK = KA + A$$

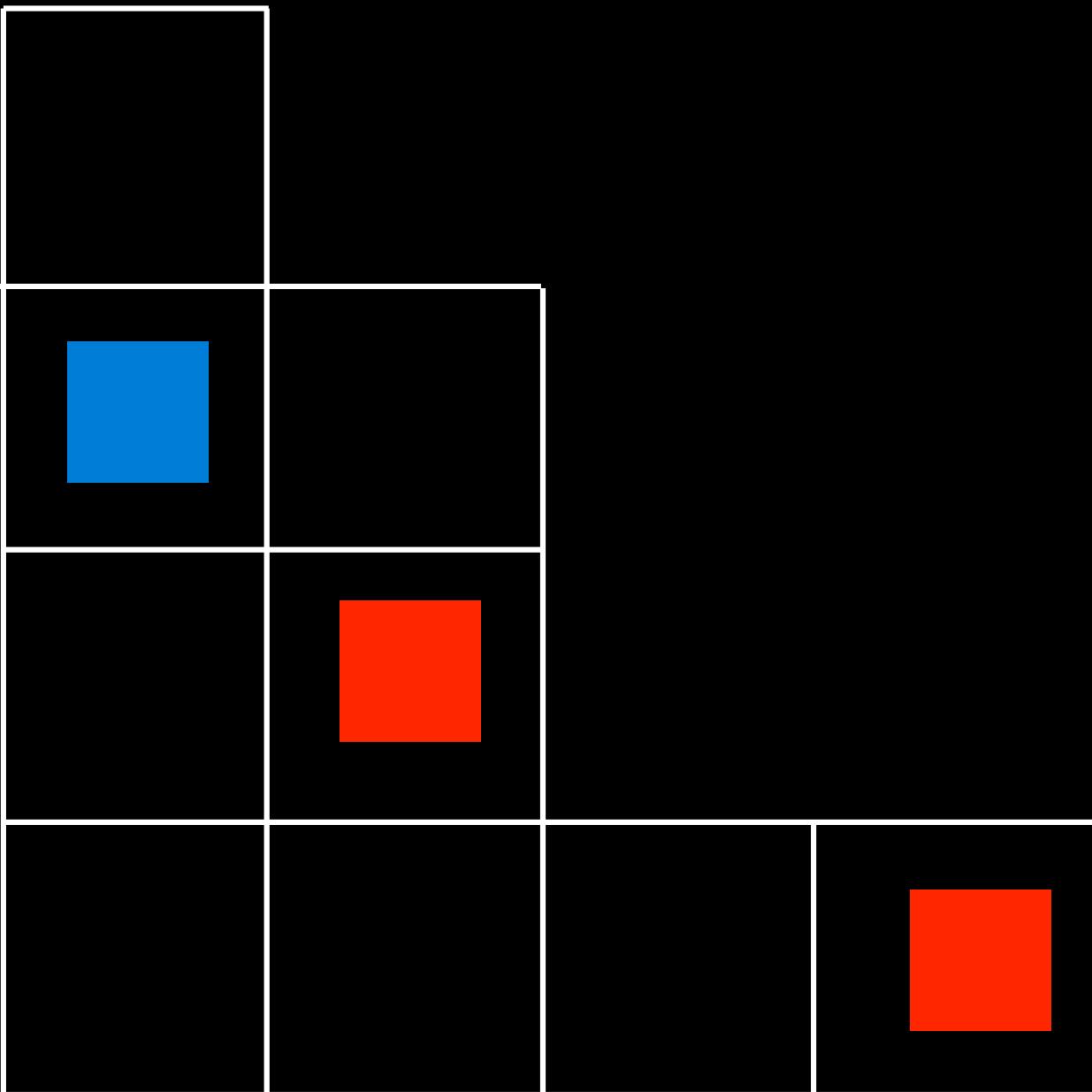
$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$





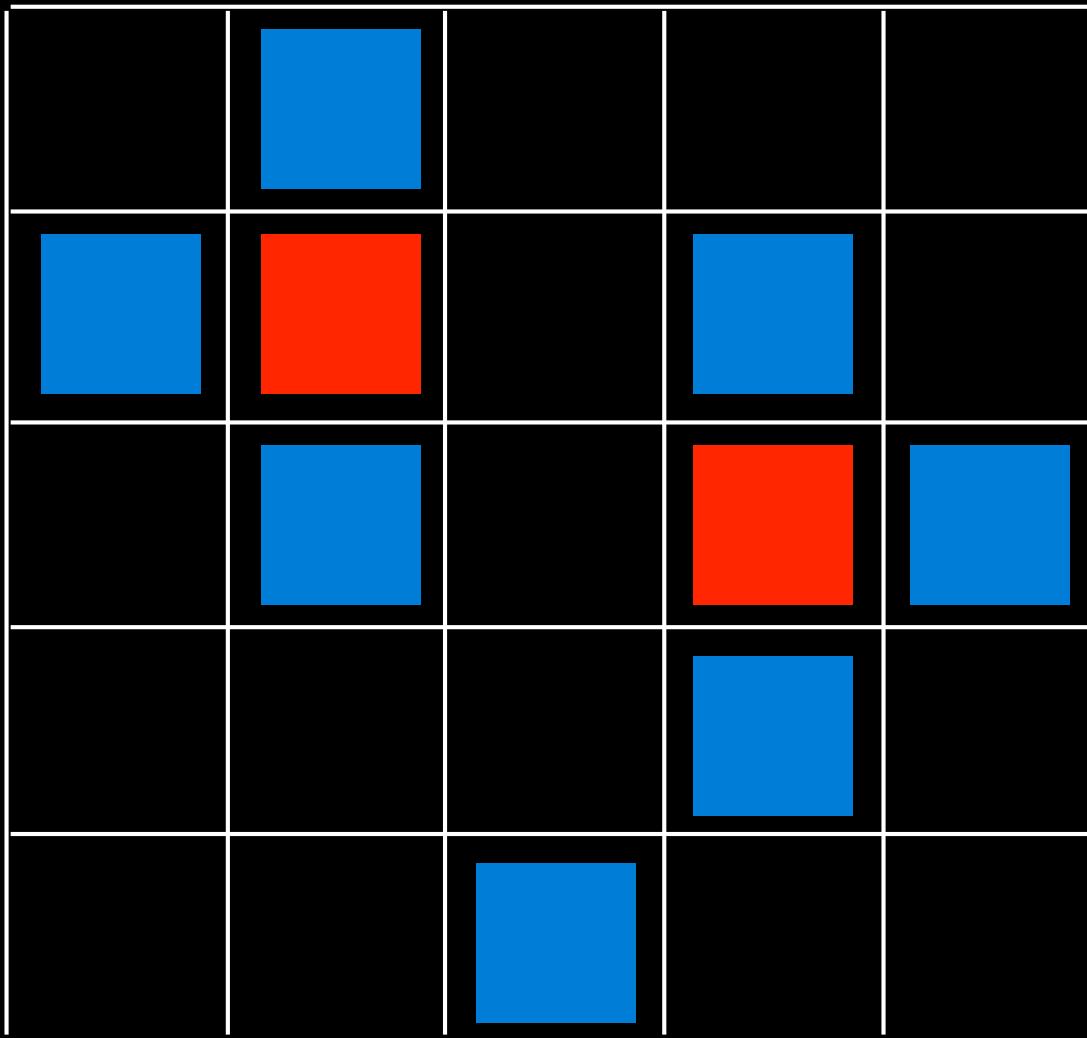
416978352

alternating sign matrices (ASM)
and a quadratic algebra

Def- **ASM** alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: 0, 1, -1
(ii) sum of entries
in each row = 1
(iii) non-zero entries
alternate in
each {row column}



A, A', B, B',

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

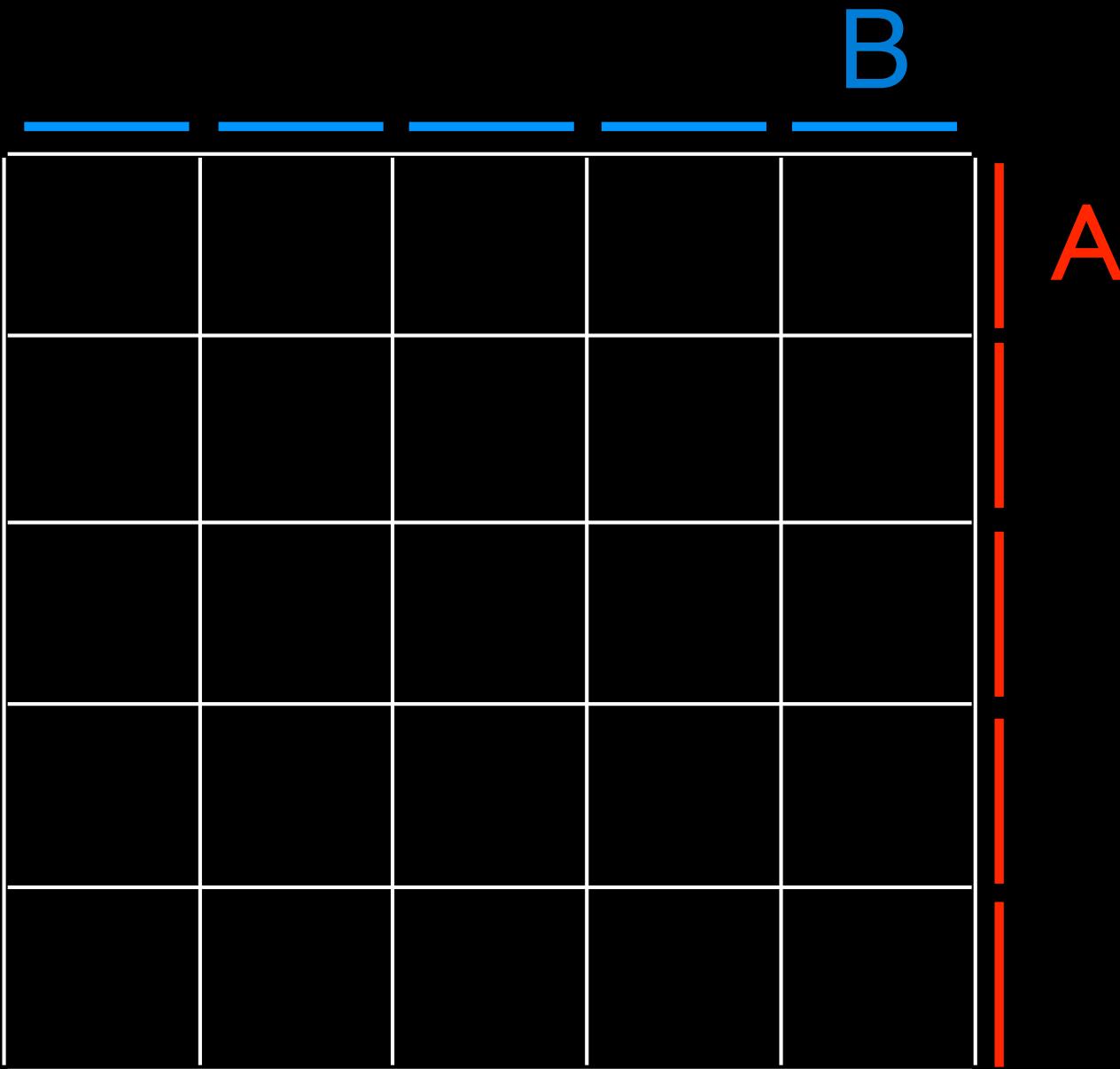
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B' , can be uniquely written

$$\sum C(u, v; w) \underbrace{u(A, A')}_{\substack{\text{word} \\ \text{in } A, A'}} \underbrace{v(B, B')}_{\substack{\text{word} \\ \text{in } B, B'}}$$

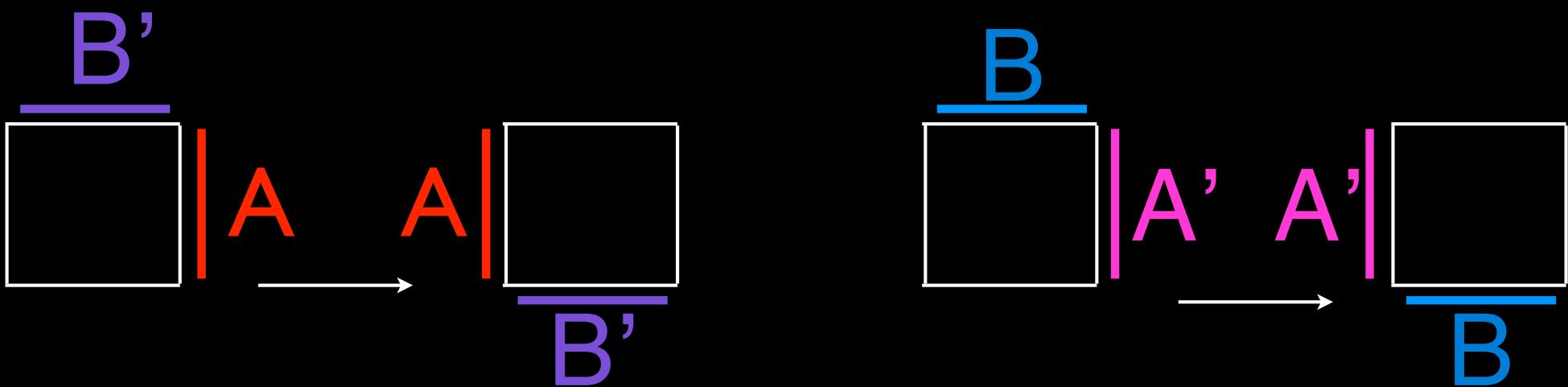
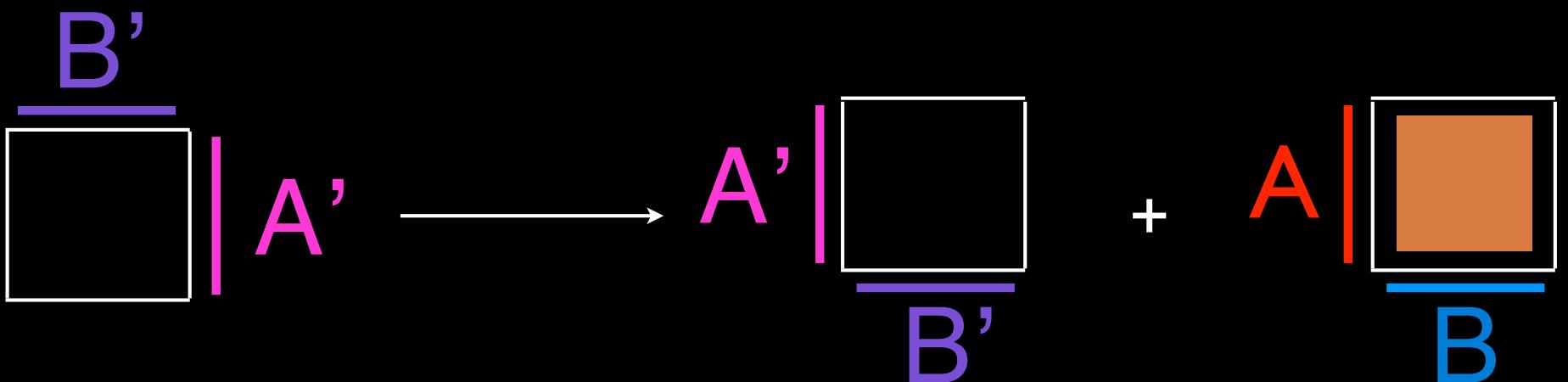
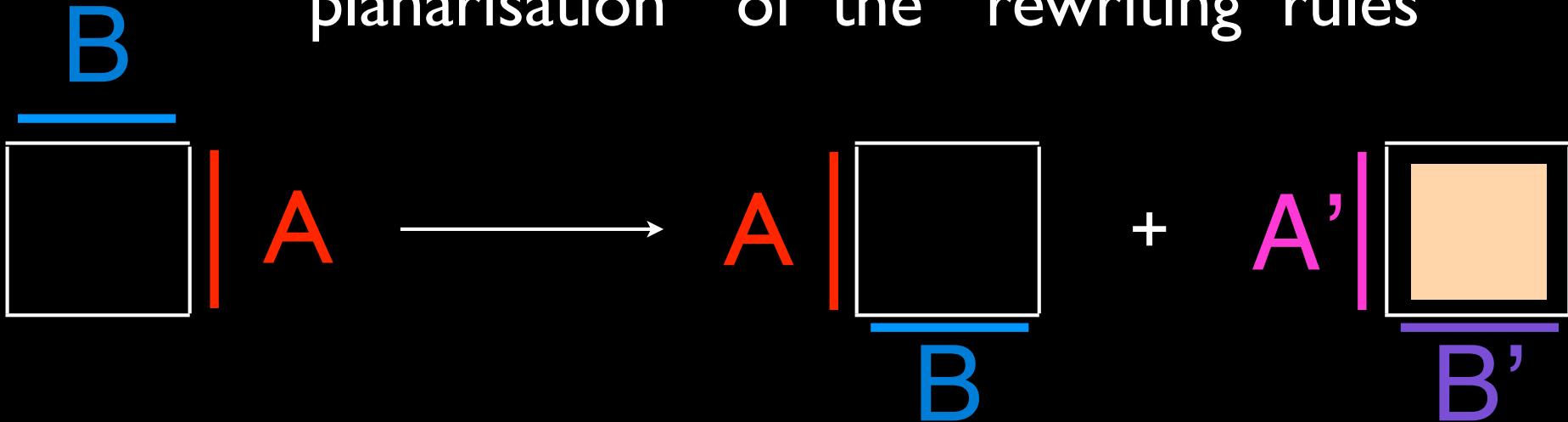
Prop. For $w = B^n A^n$
 $u = A'^n, v = B'^n$

$C(u, v; w)$ = the number of
 $n \times n$ ASM (alternating sign matrices)

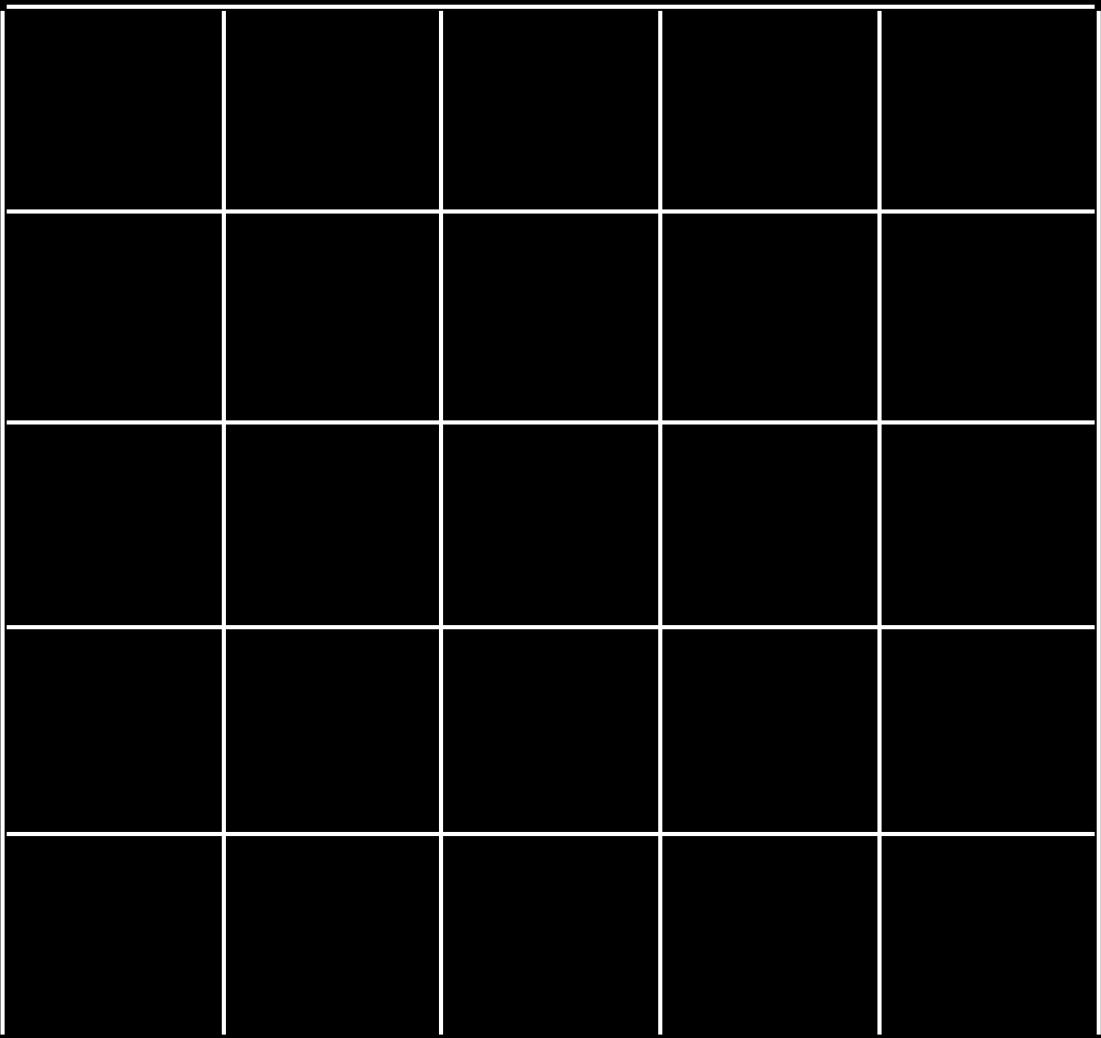
“planar”
proof:



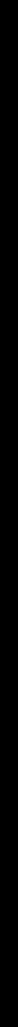
“planarisation” of the “rewriting rules”



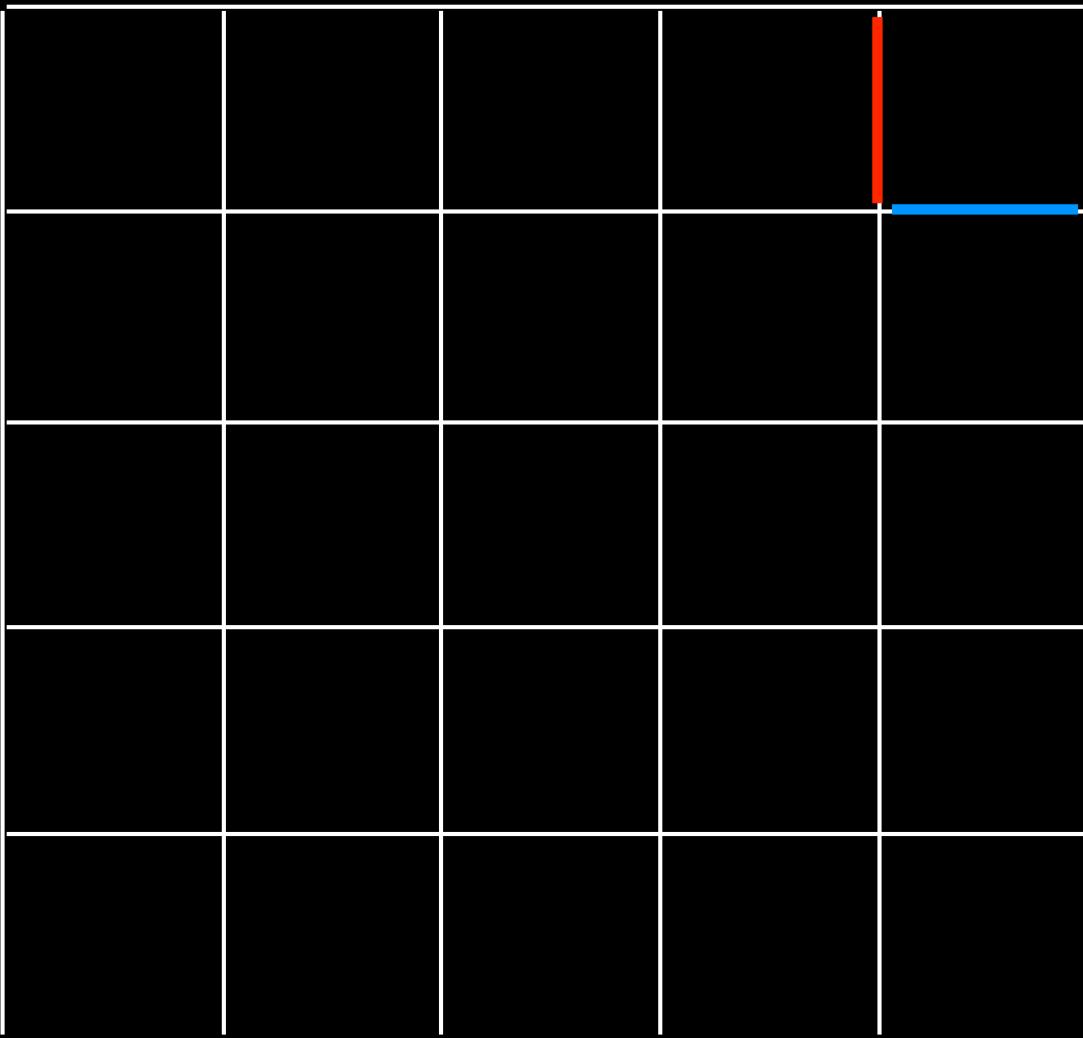
B



A

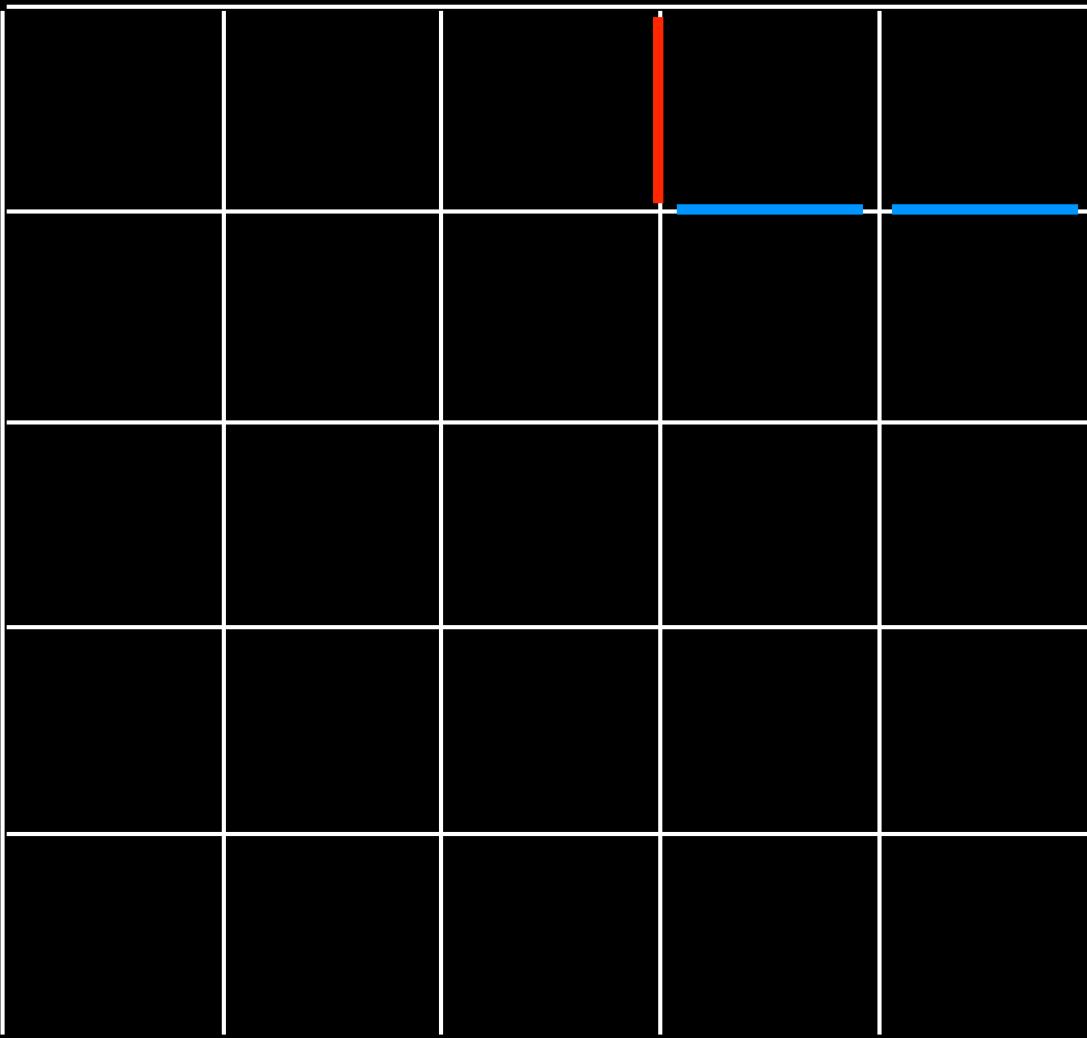


B



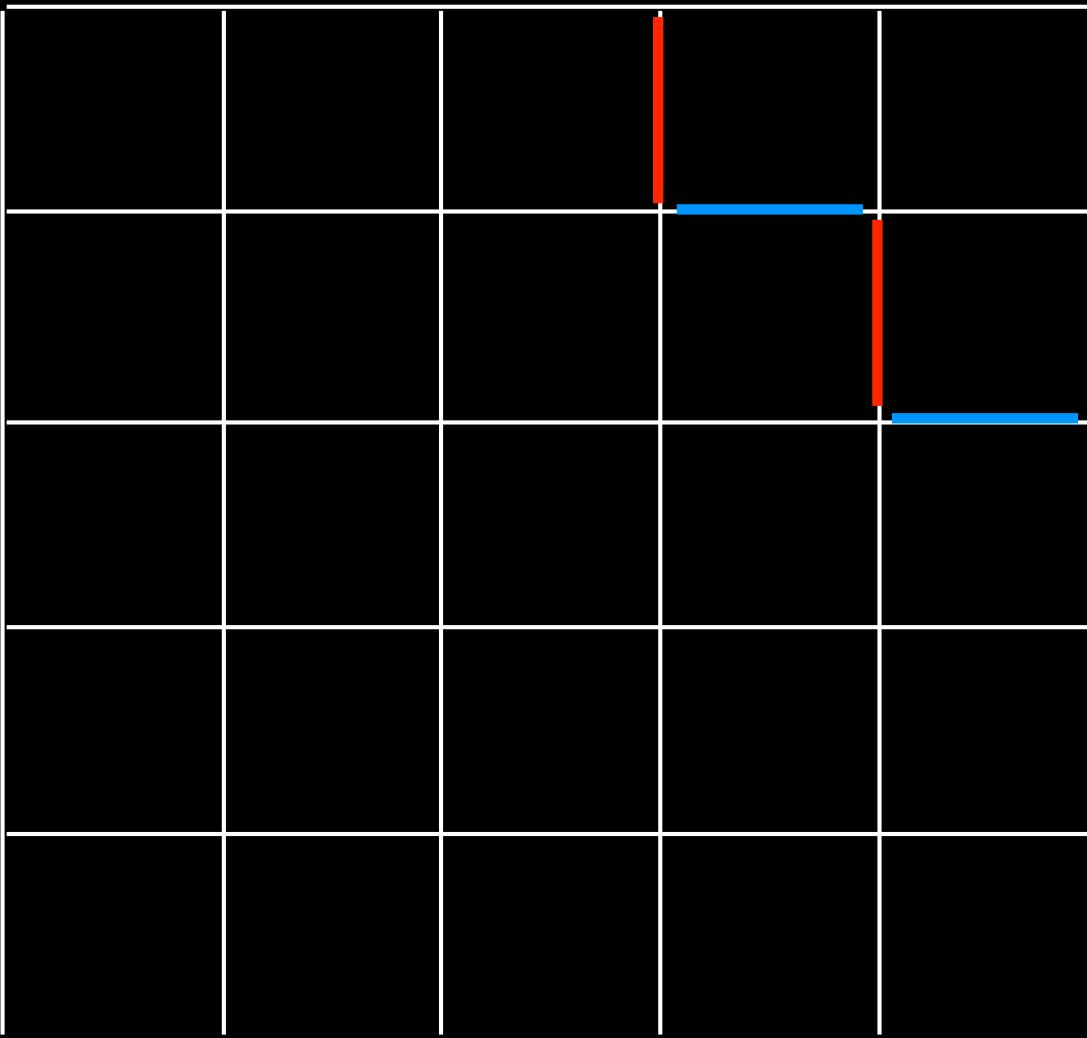
A

B



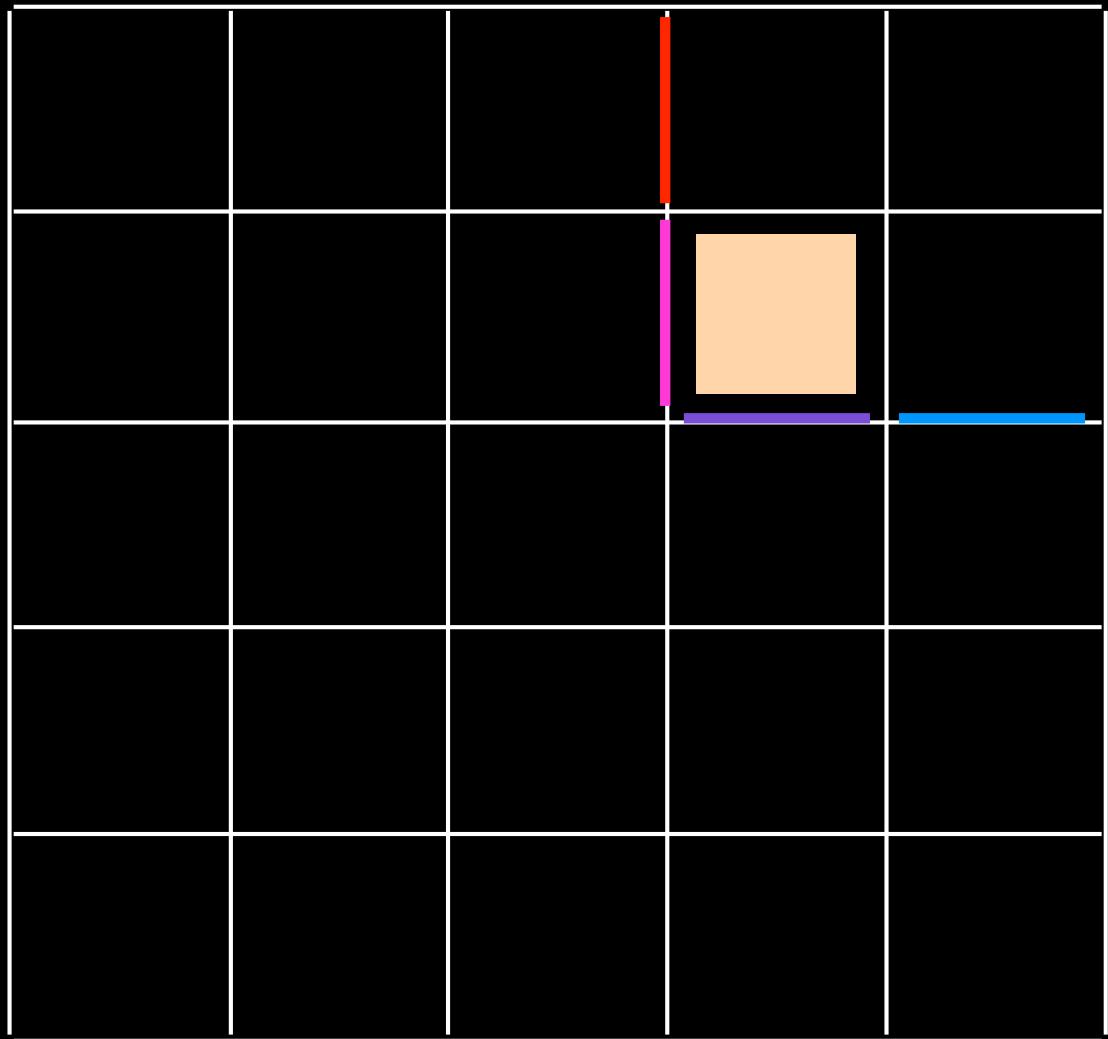
A

B



A

A' |

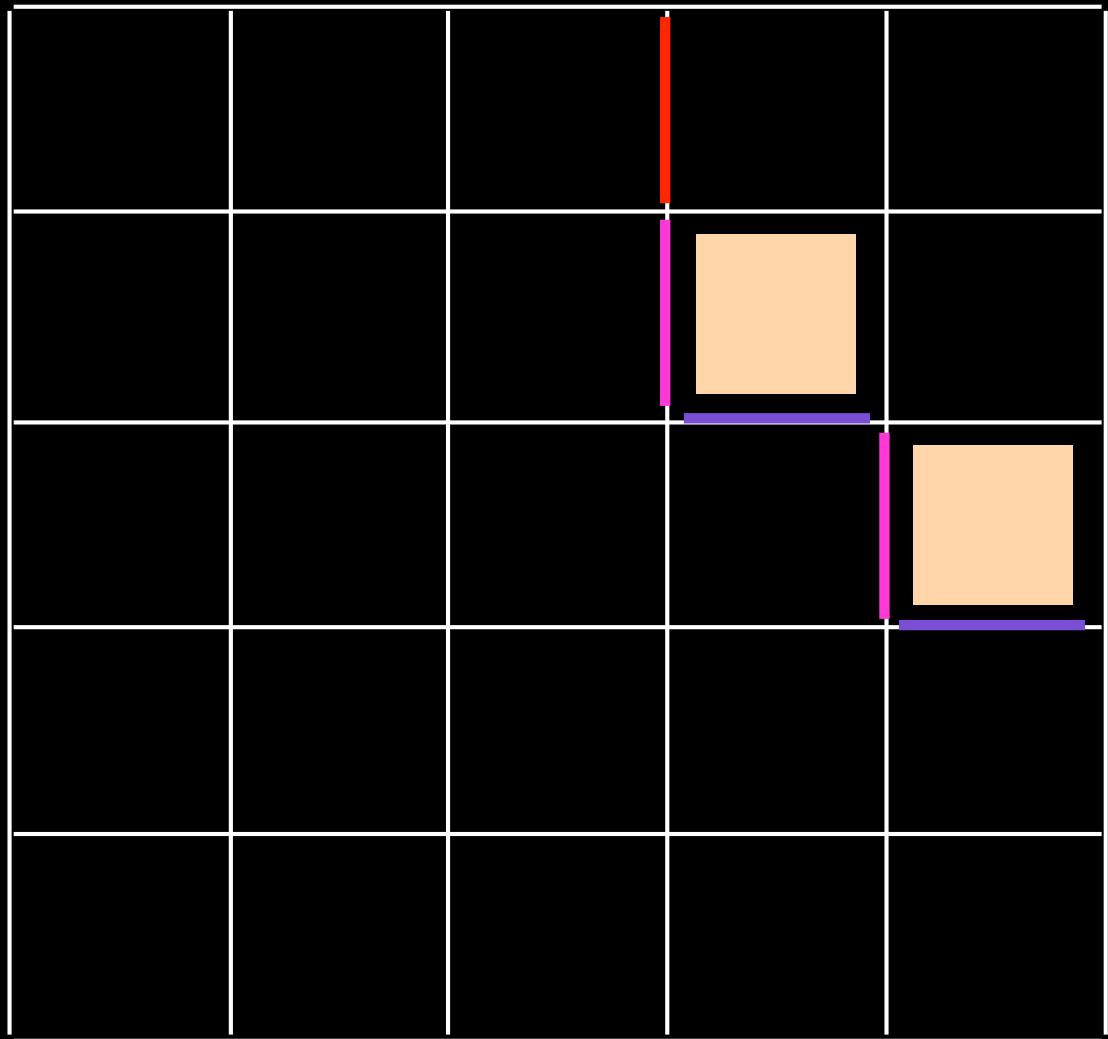


B

A

—
B'

A'

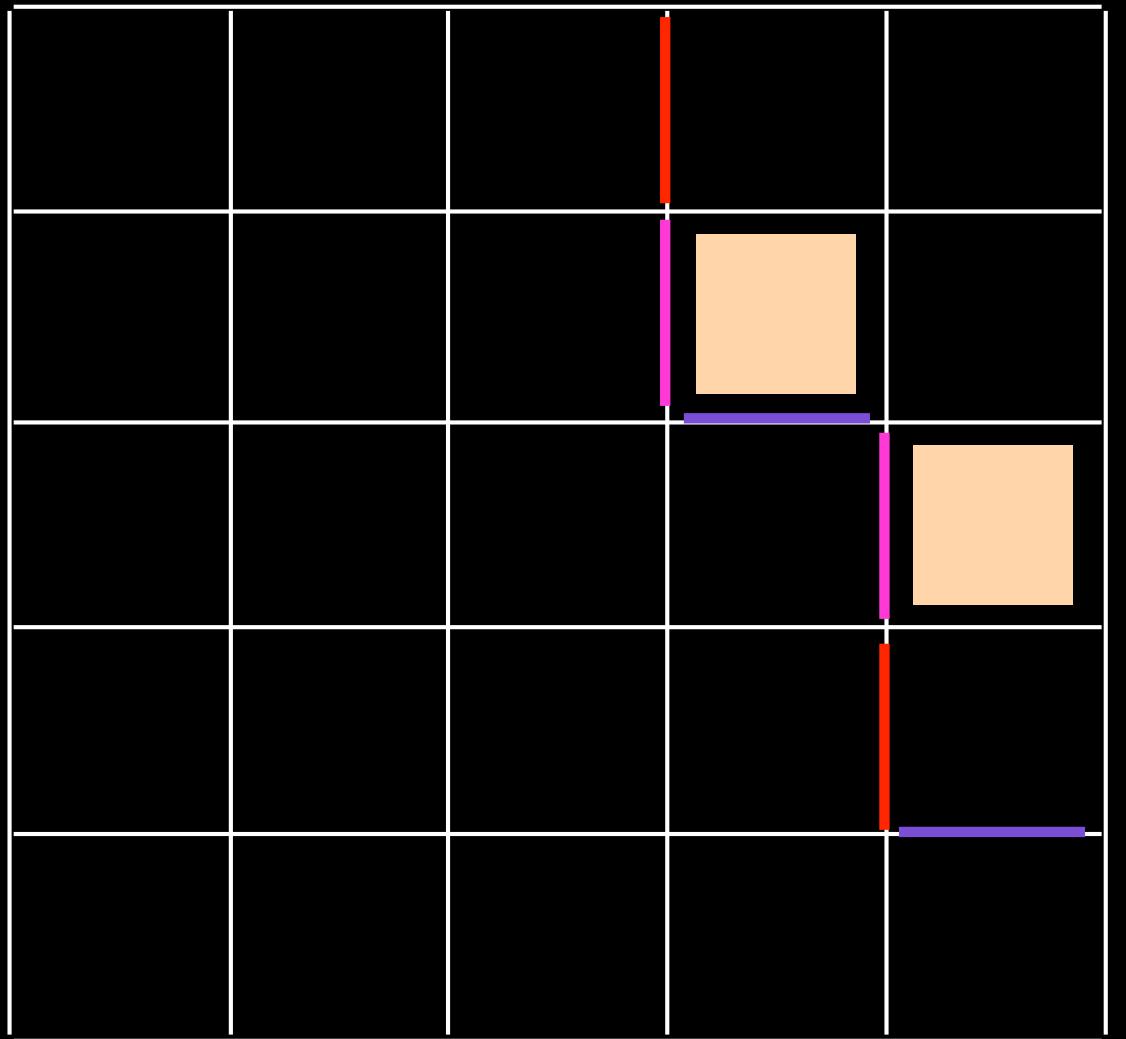


B

A

B'

A'

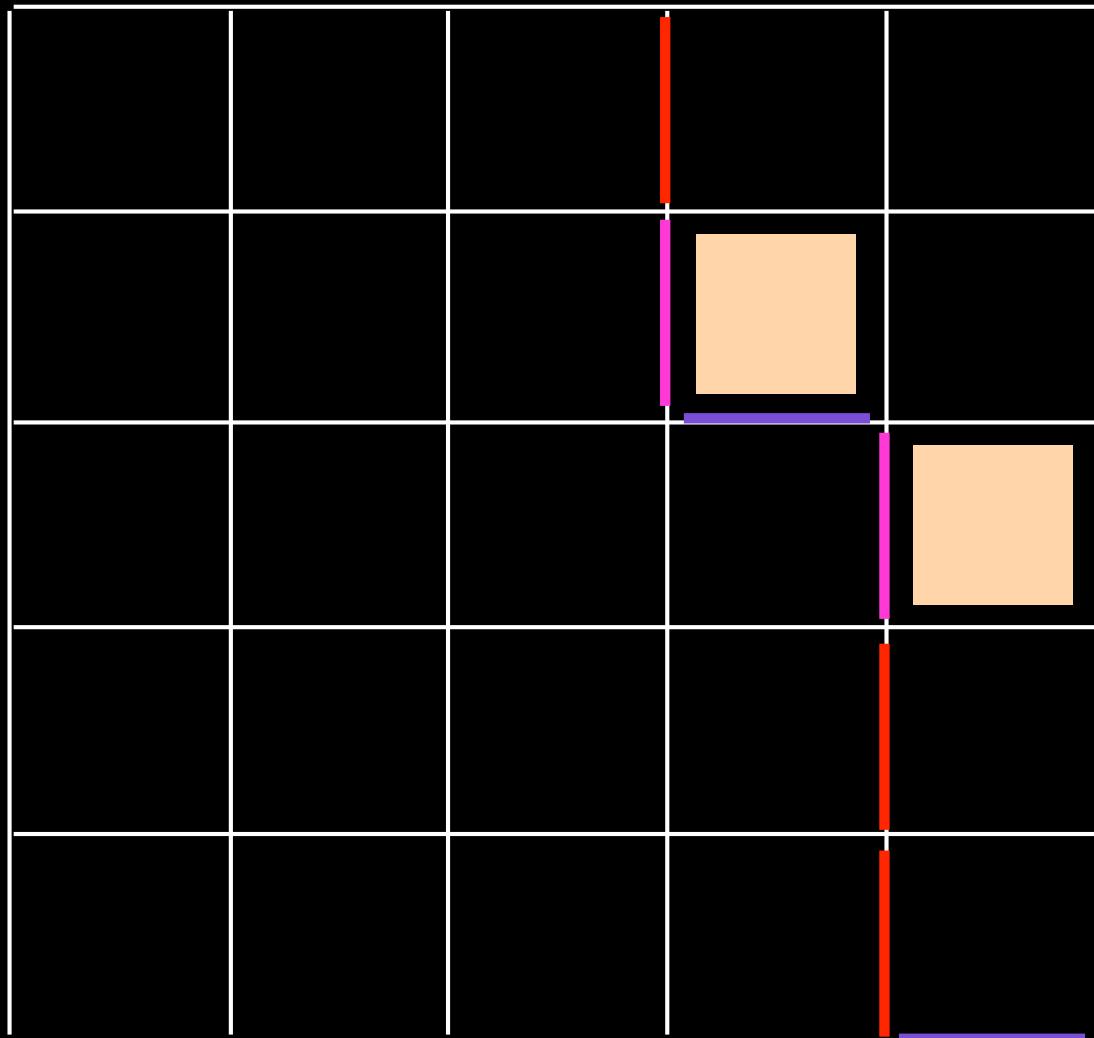


B

A

B'

A'

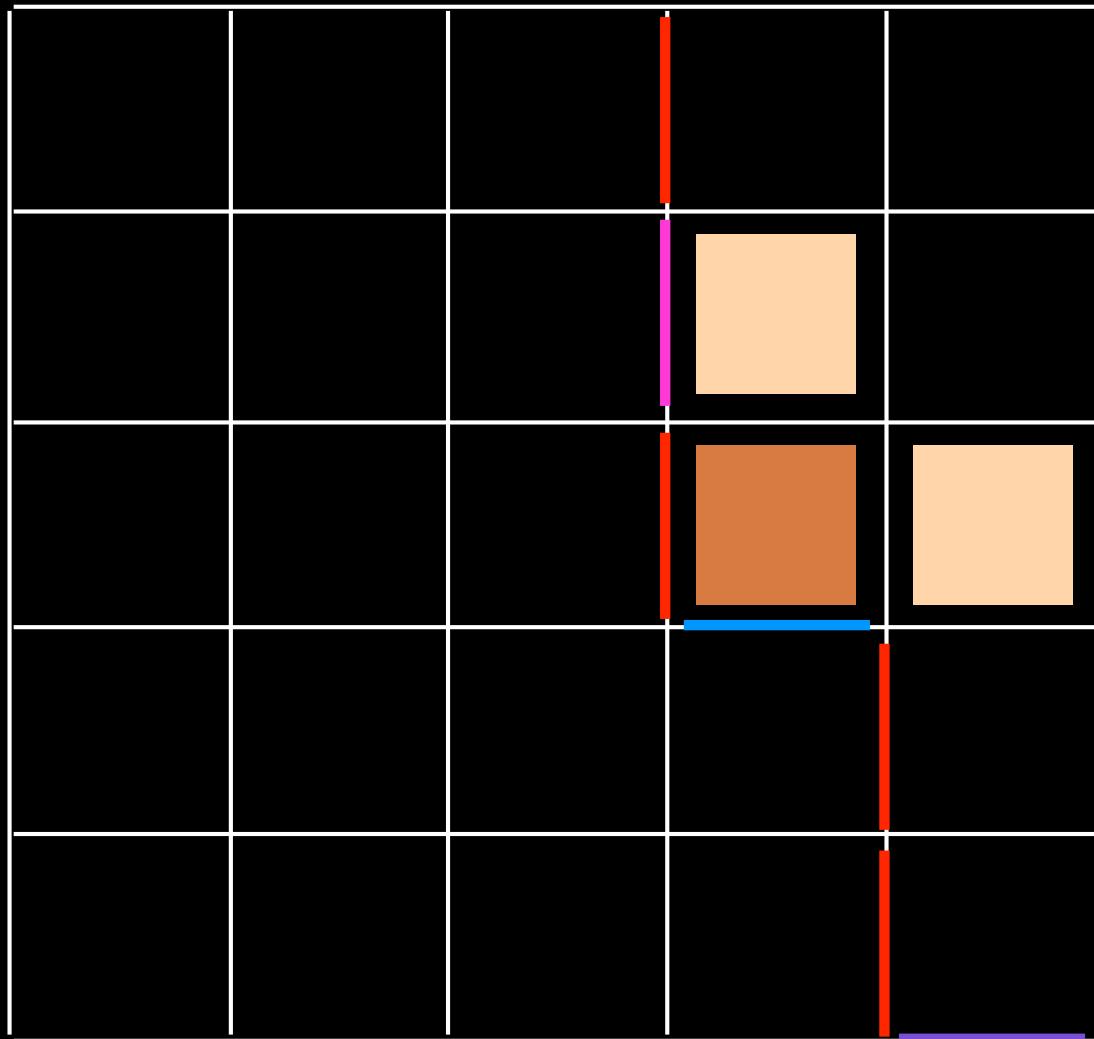


B

A

B'

A'

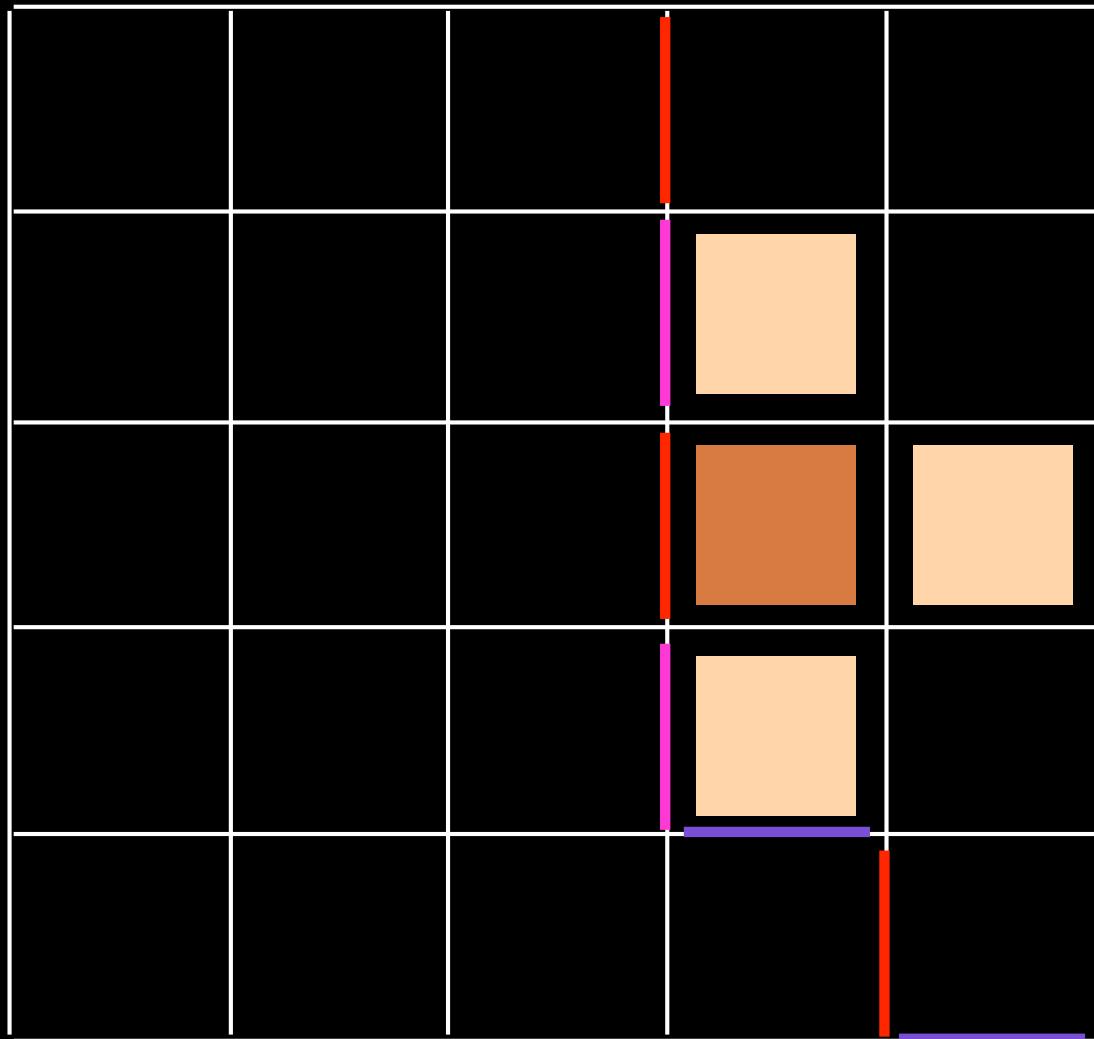


B

A

B'

A' |

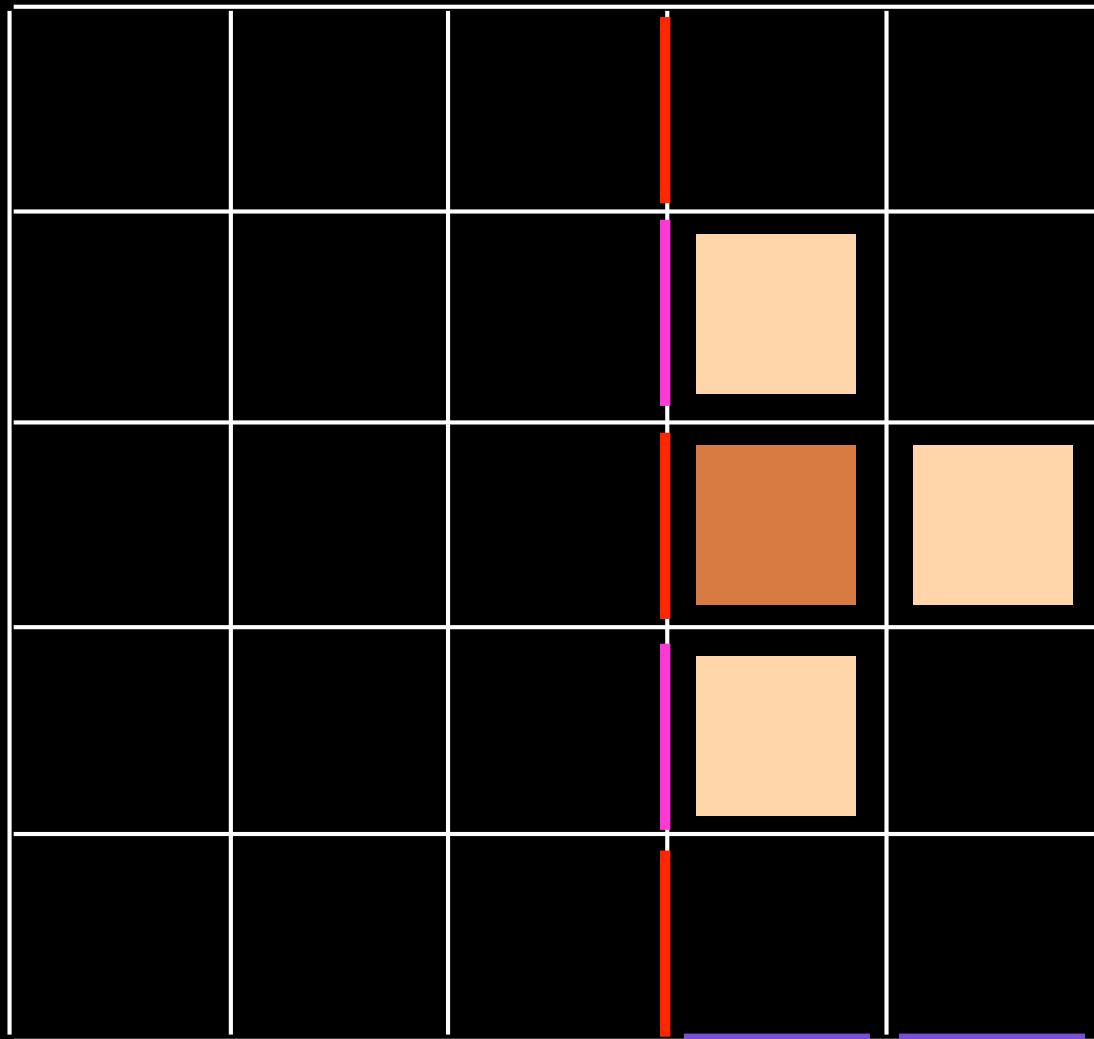


B

A

B'

A' |



B

A

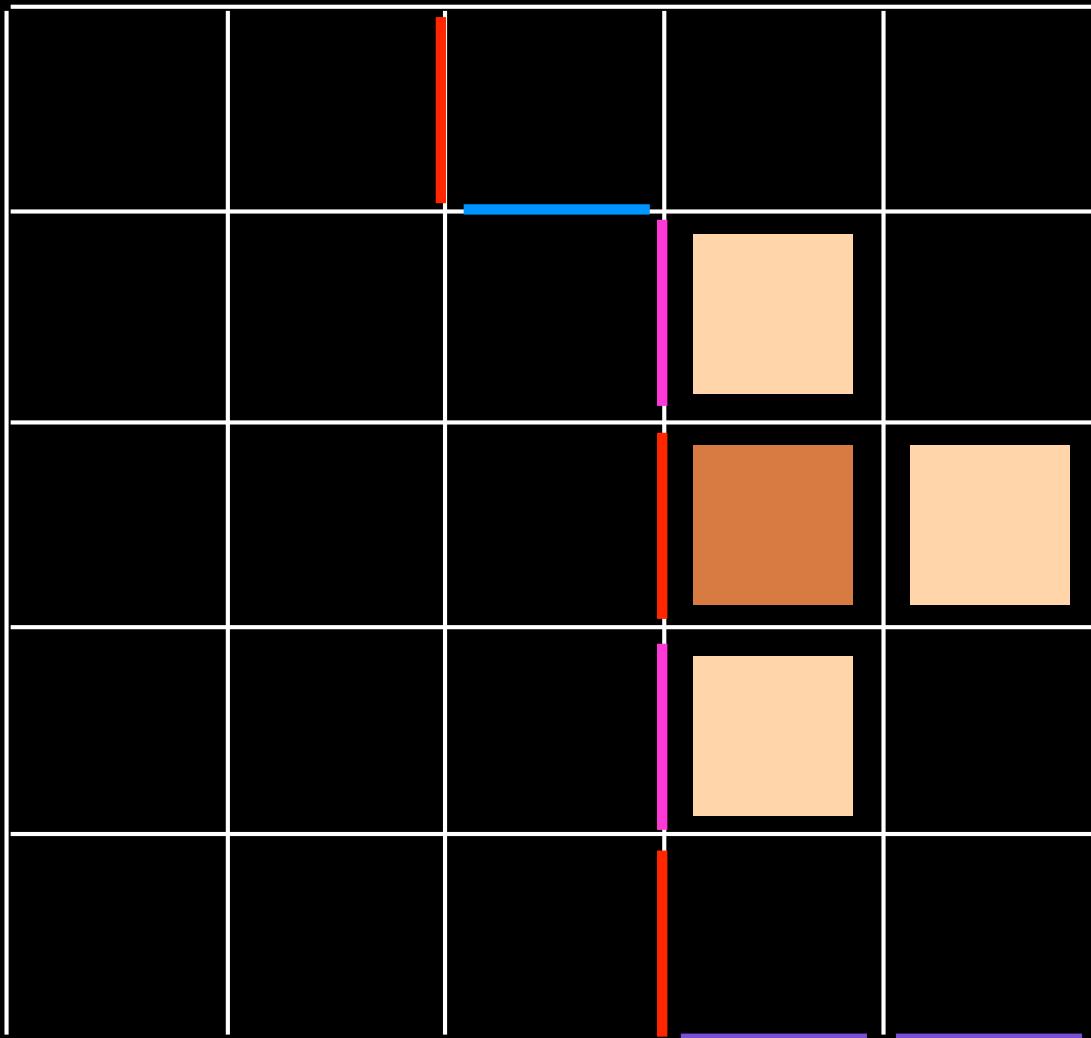
B'

B

A

A'

B'

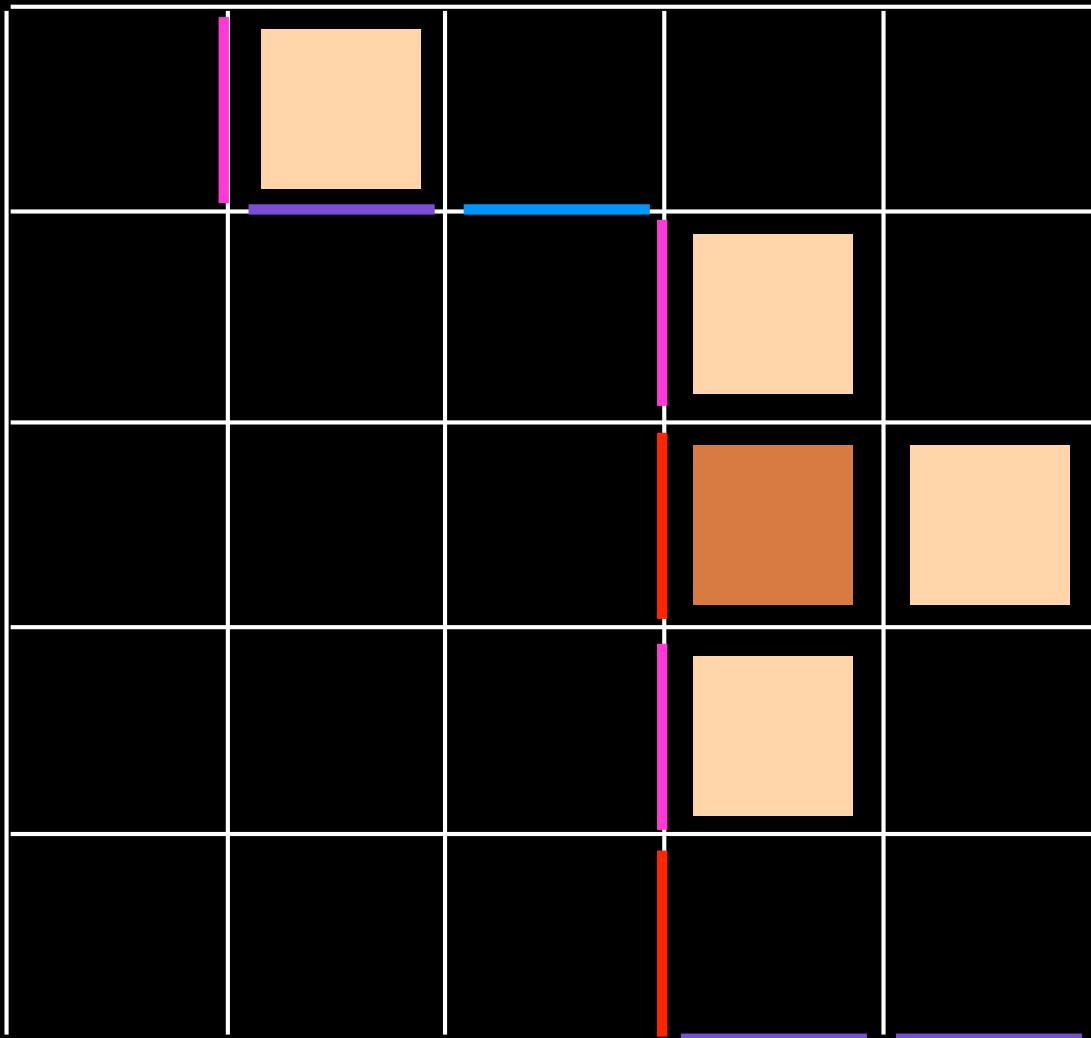


B

A

A'

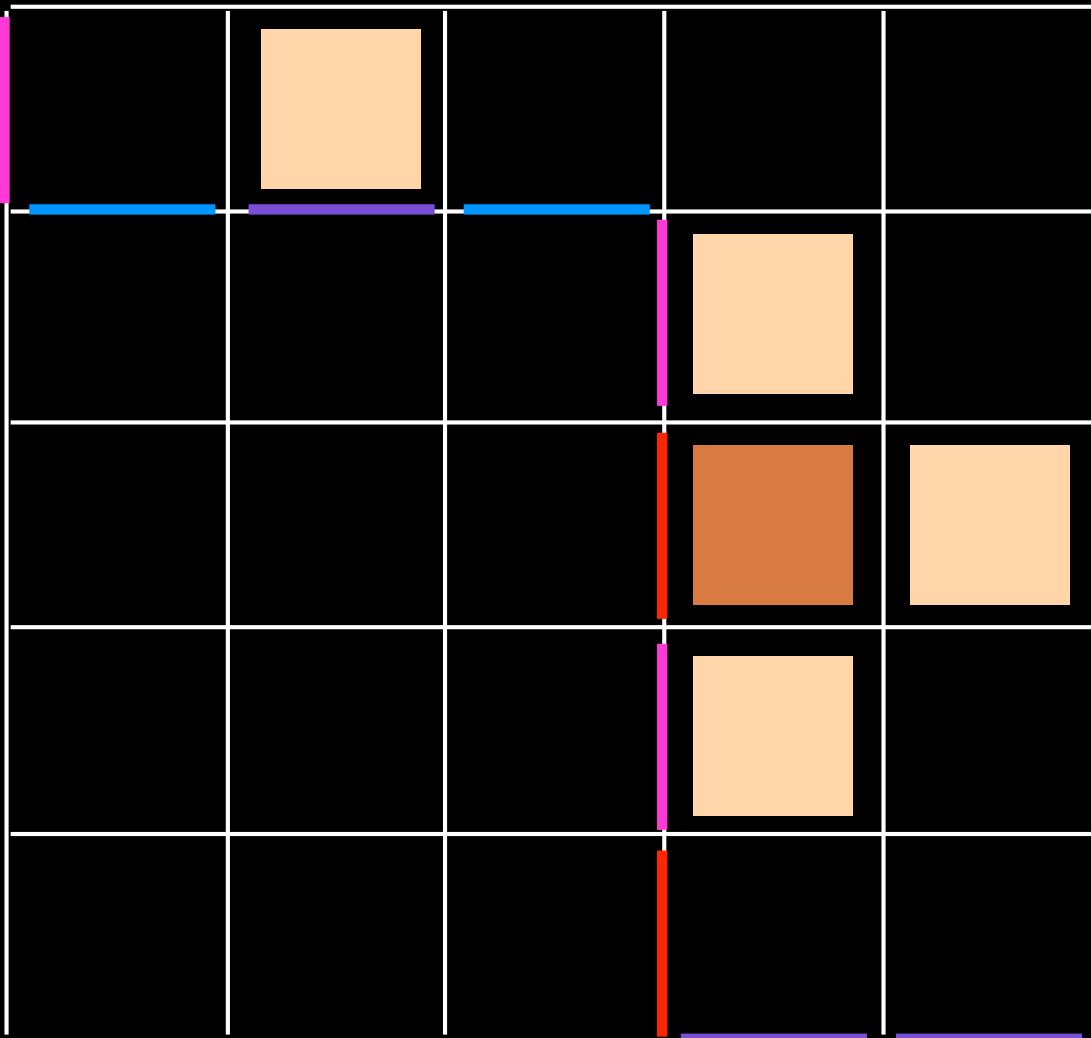
B'



B

A

A'



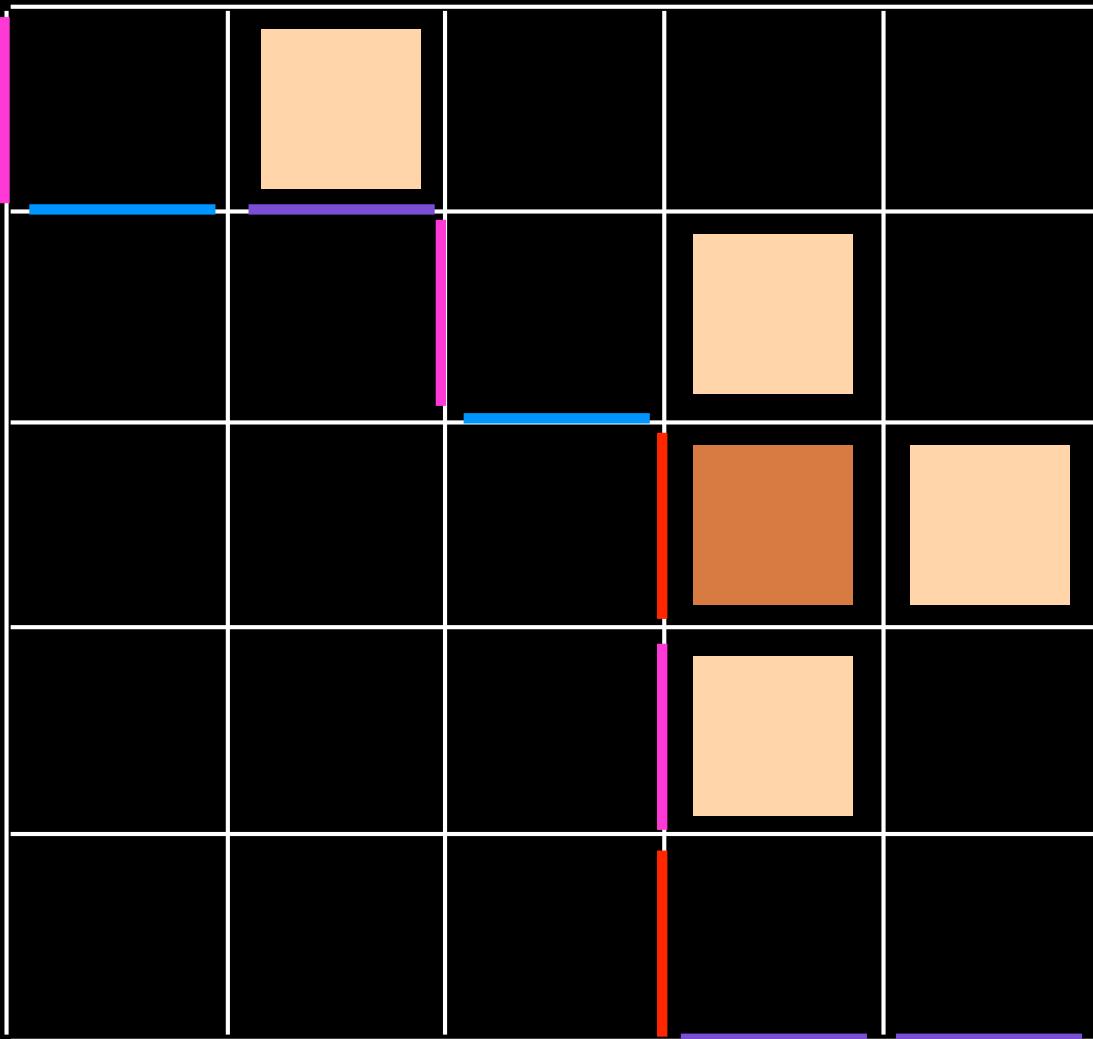
B'

B

A

A'

B'



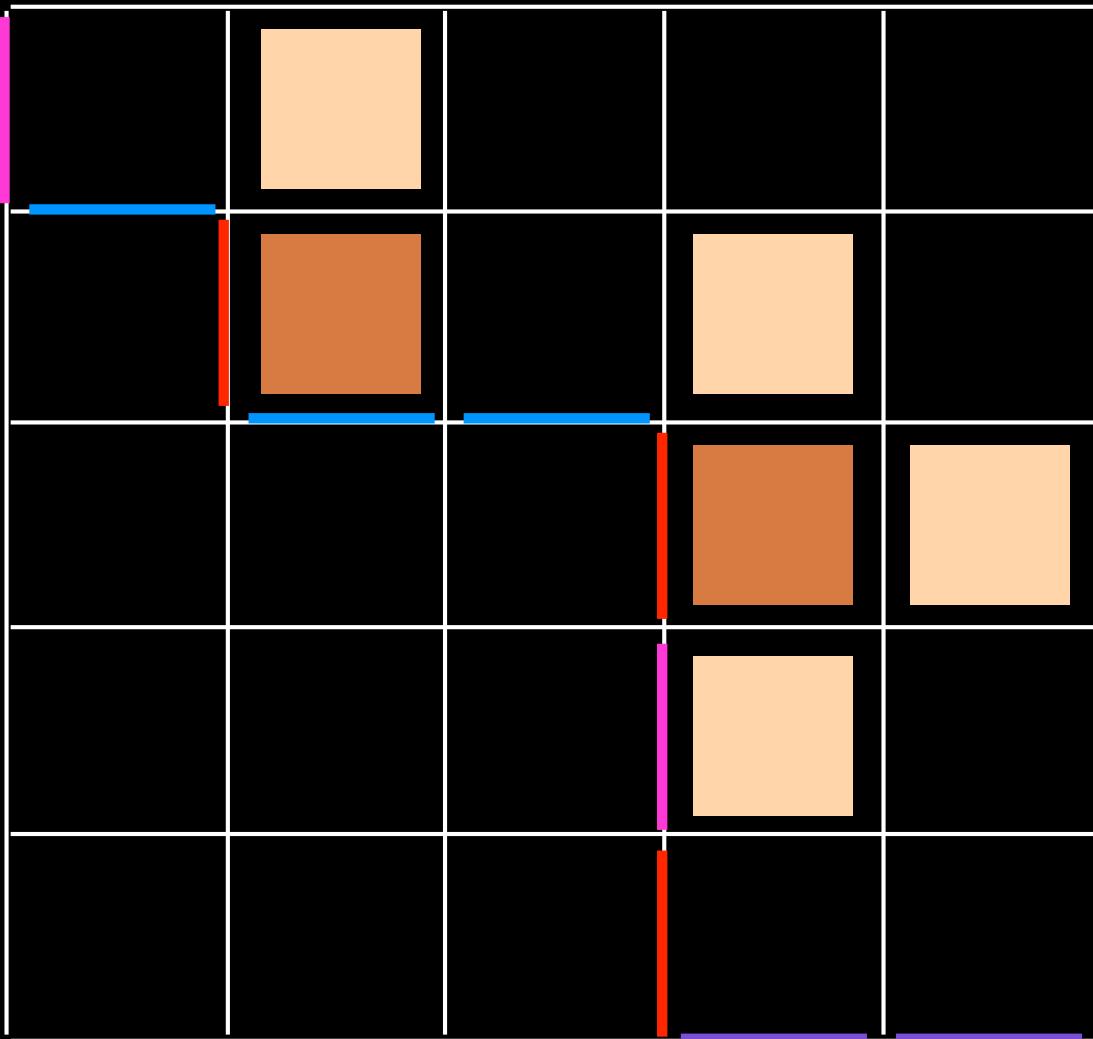
—

B

A

A'

B'



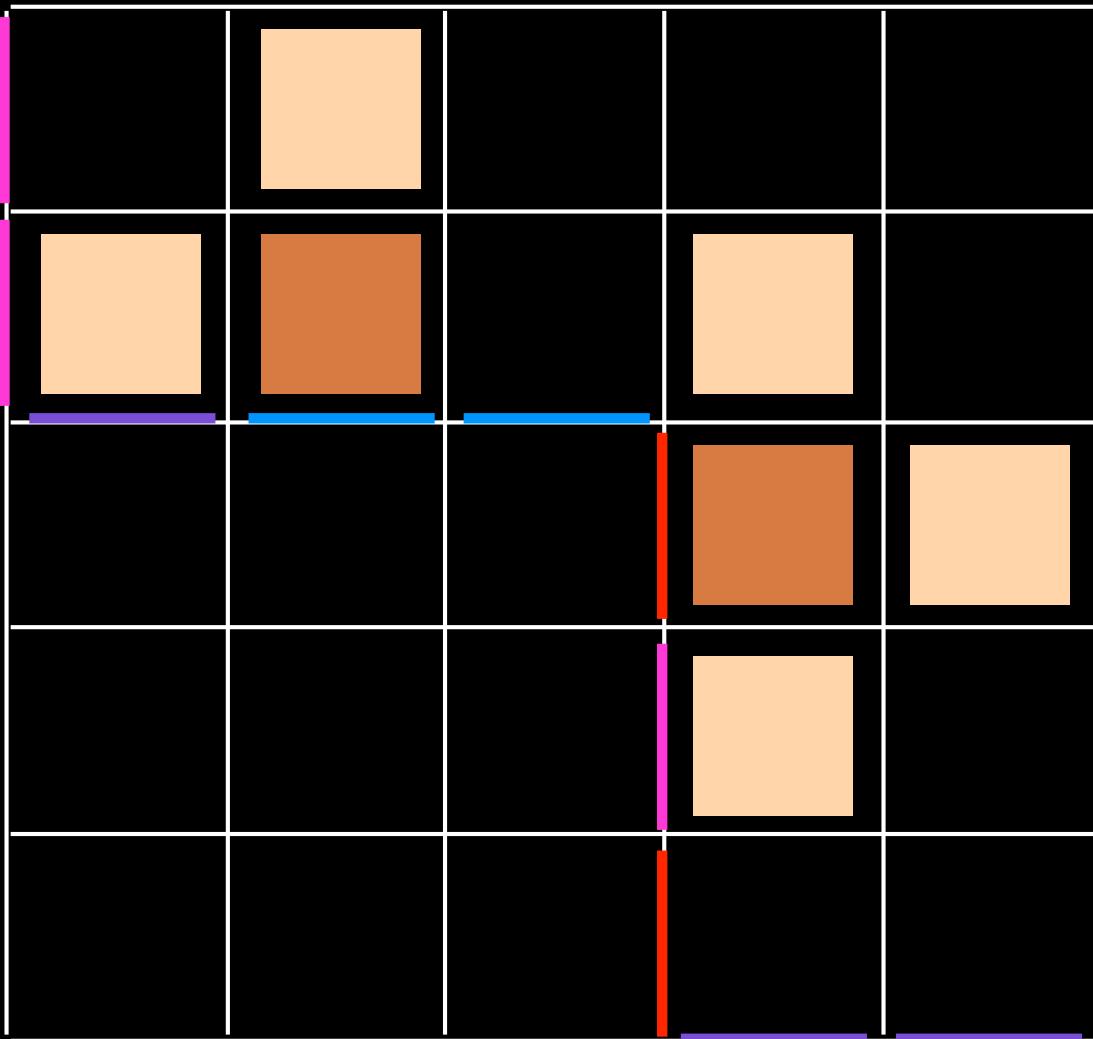
—

B

A

A'

B'



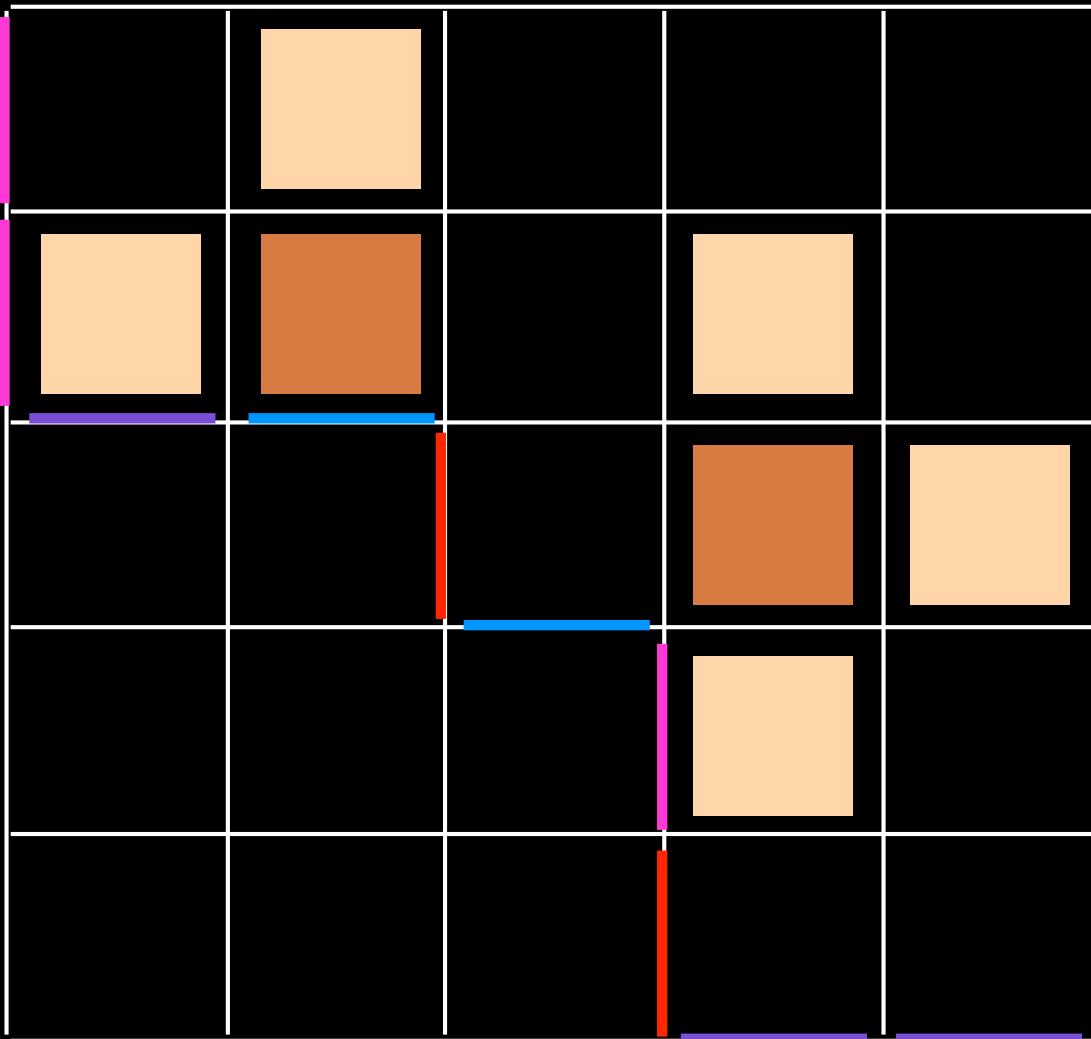
—

B

A

A'

B'



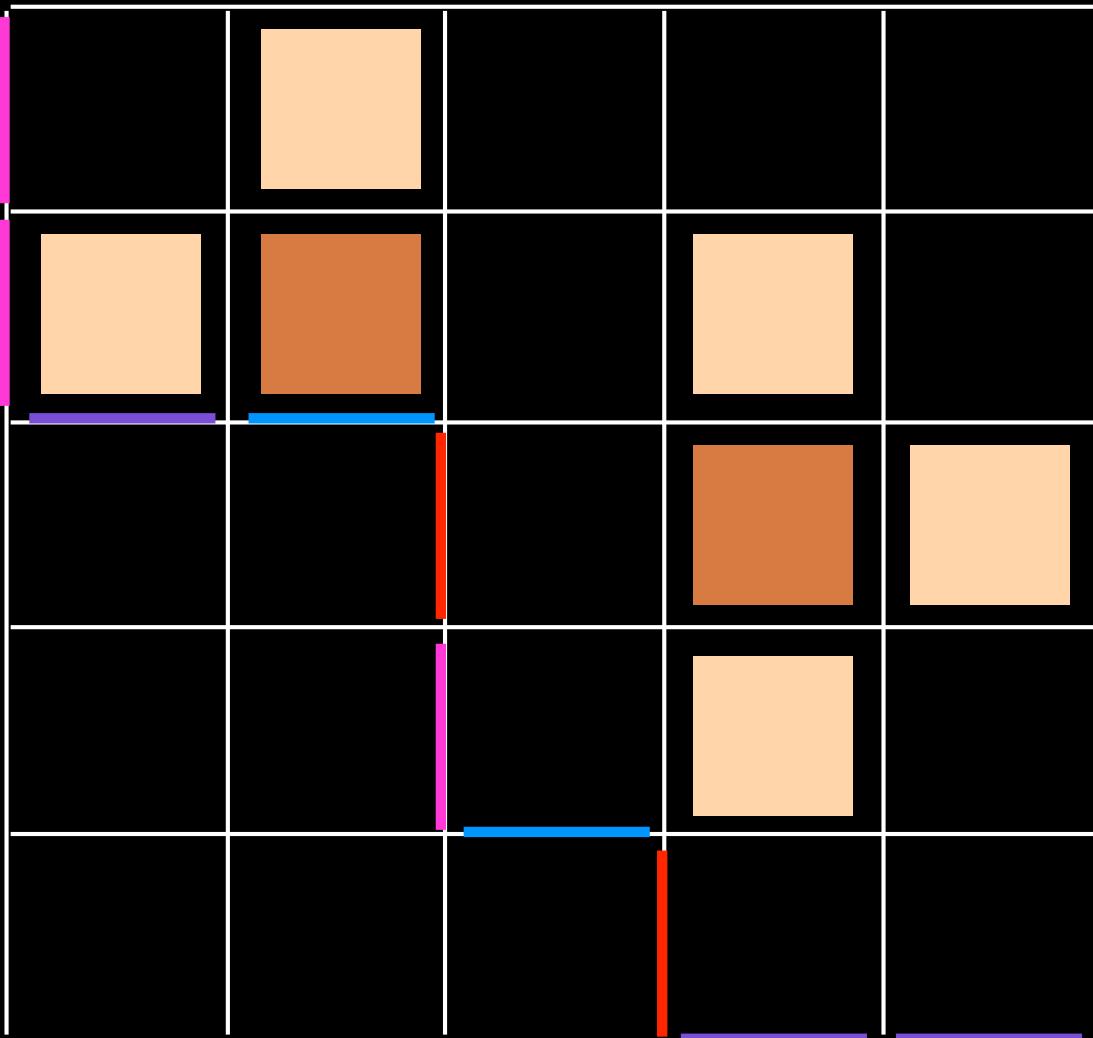
—

B

A

A'

B'

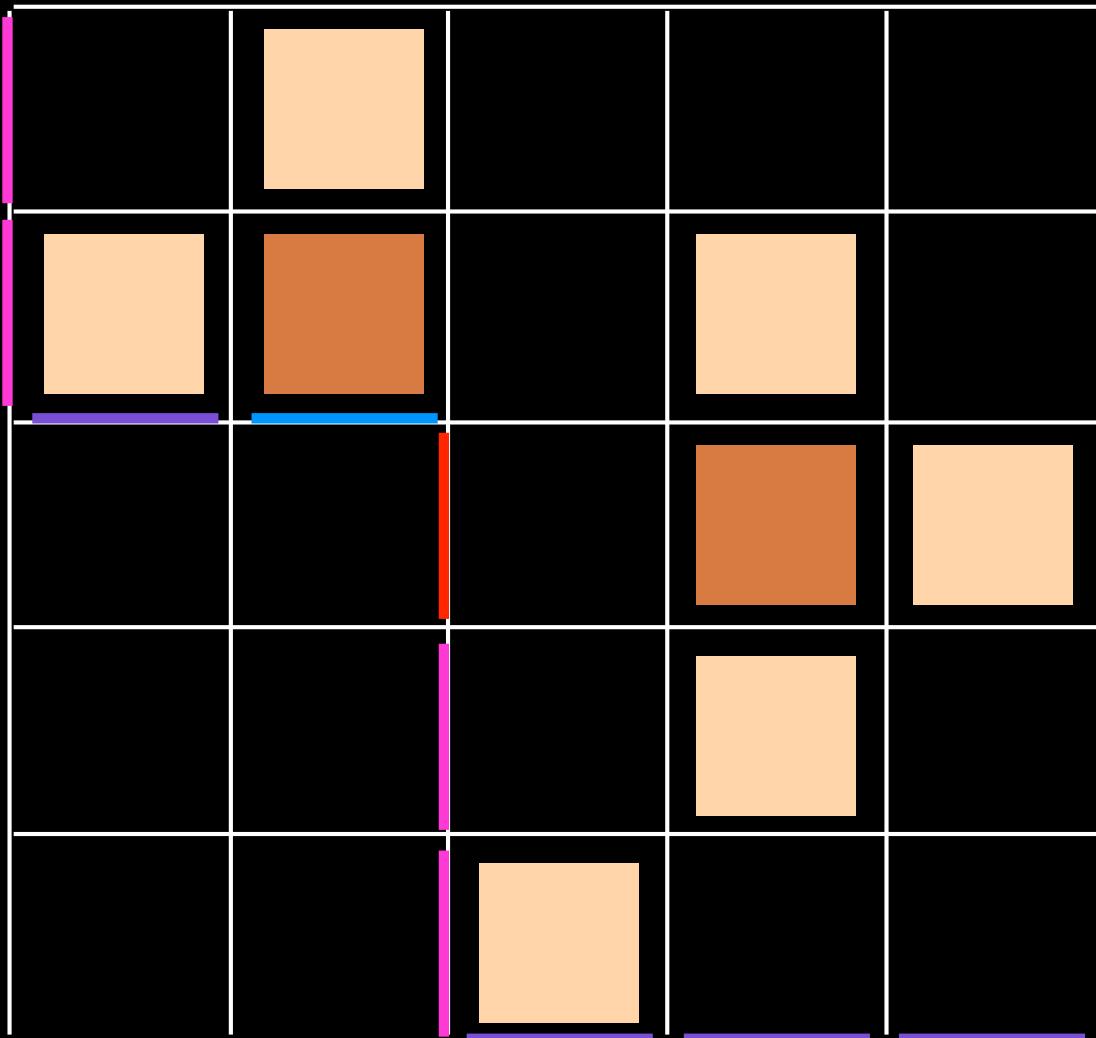


B

A

A'

B'



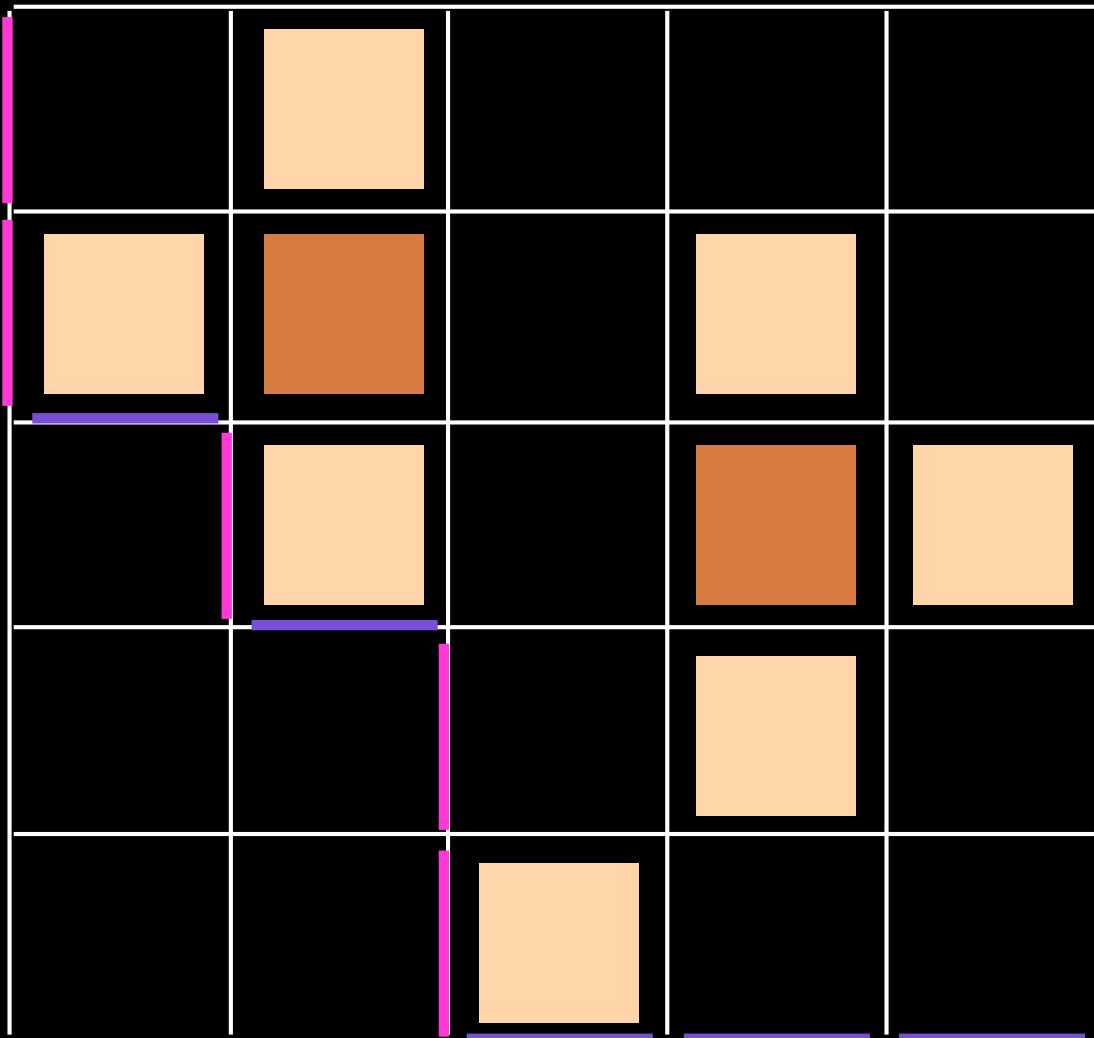
—

B

A

A'

B'



—

B

A

A'

B'



B'

B

A

A'

B'



B'

B

A

A'

B'



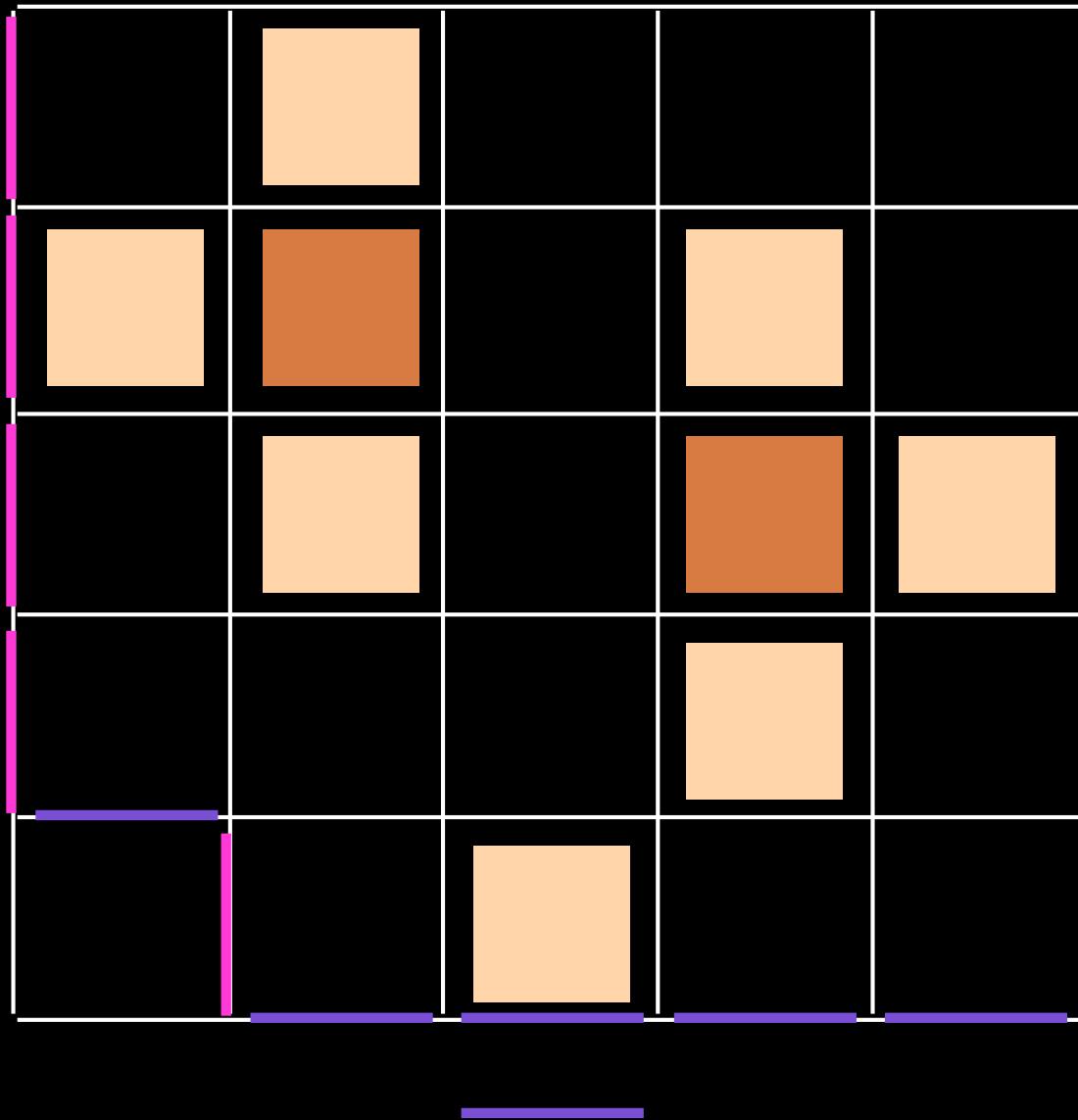
B'

B

A

A'

B'

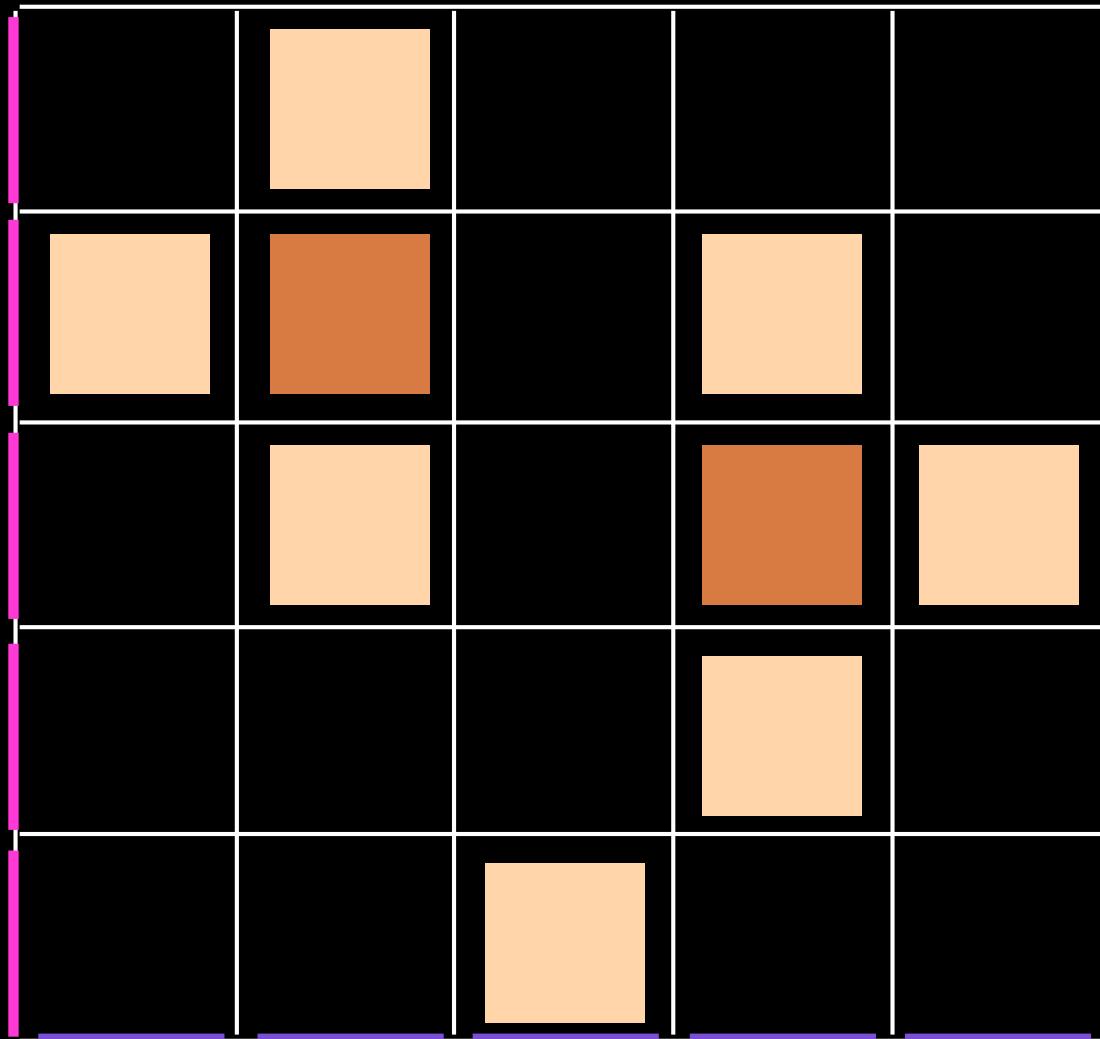


B

A

A'

B'

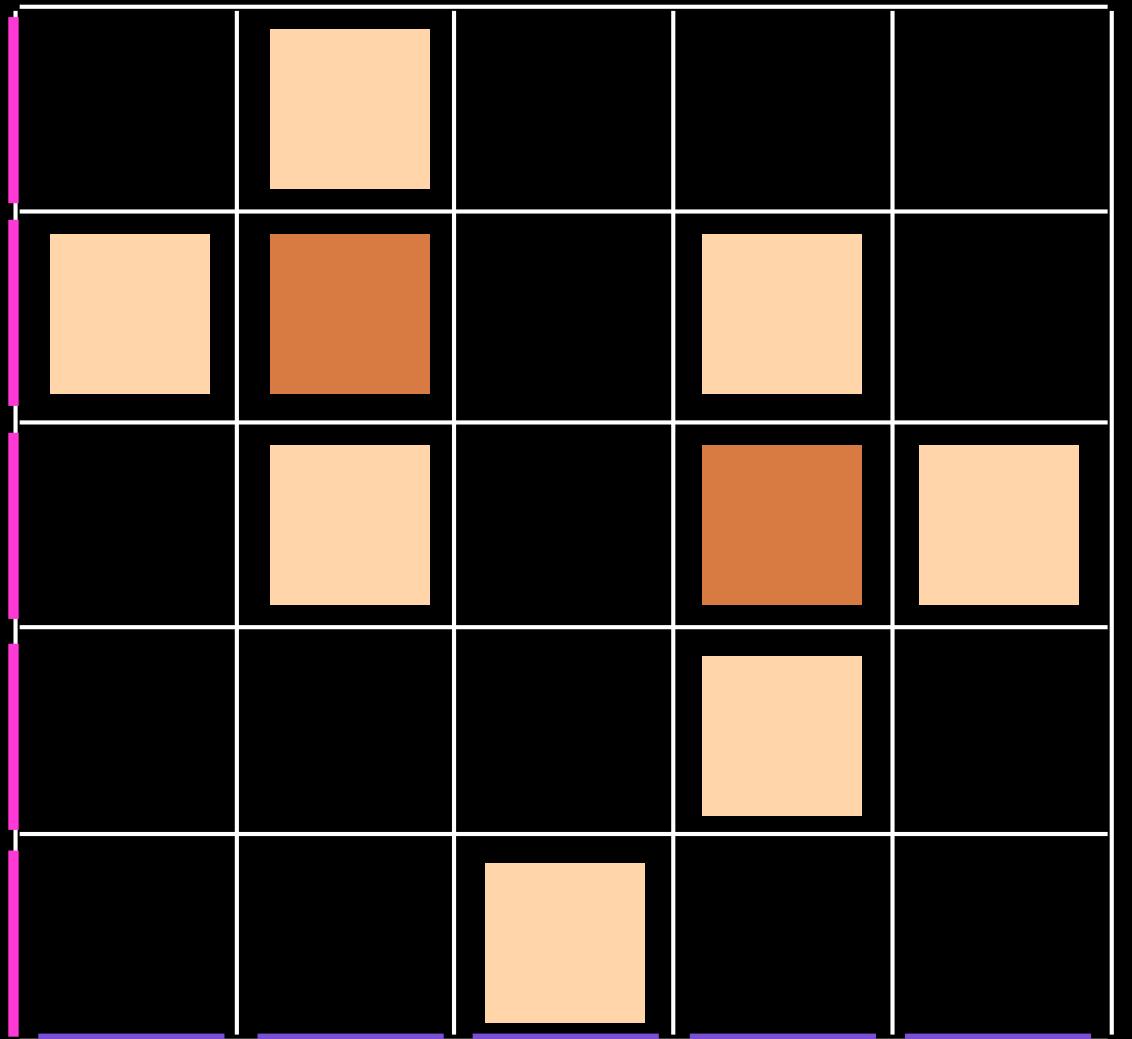


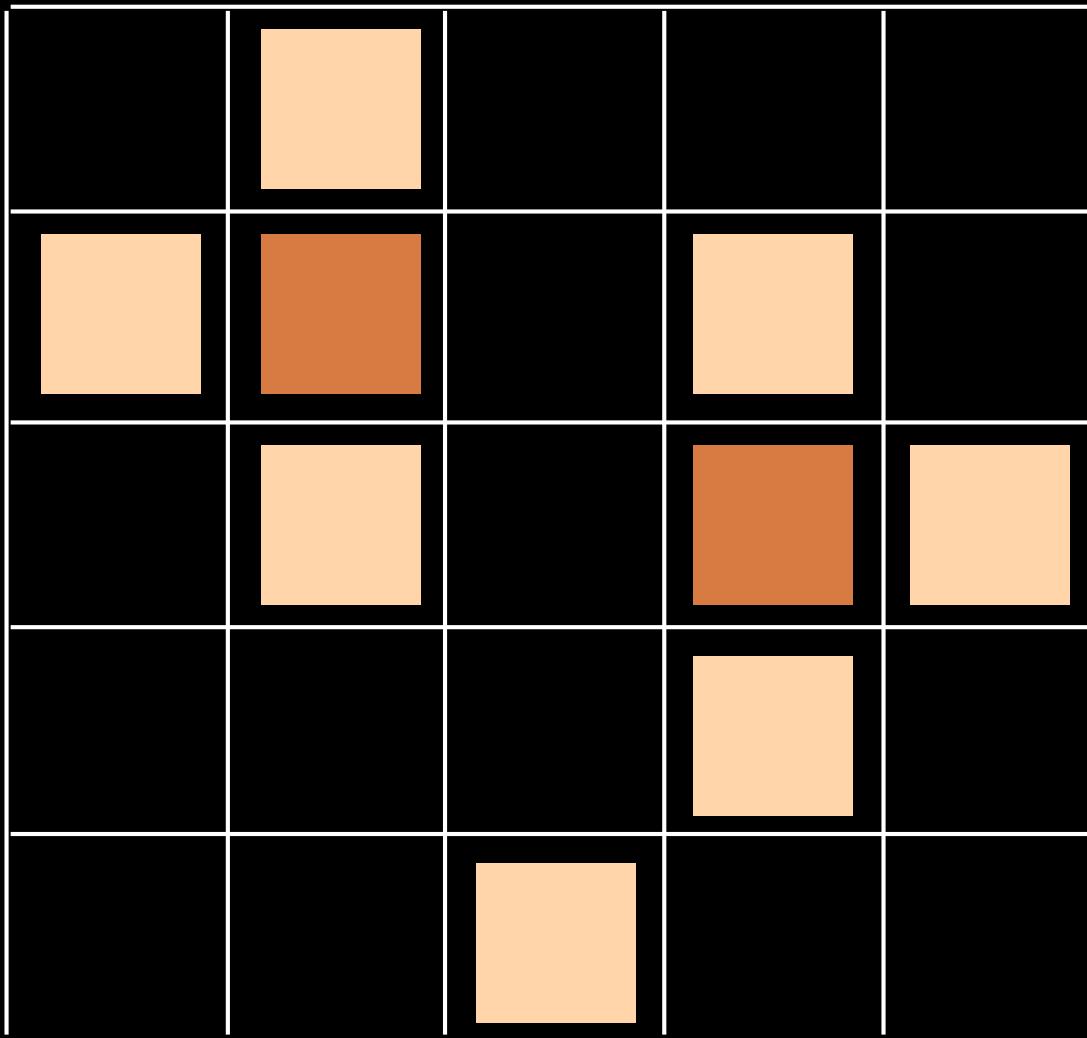
A'

B

A

B'





Questions.

- find a "combinatorial representation" for operators A, A', B, B' .
- analogue of RSK (Robinson-Schensted-Knuth) for ASM ?
- analogue of "local rules" (Fomin)
- direct proof of the formula

$$A_n = \prod_{j=1}^n \frac{(3j-2)!}{(n+j-1)!}$$

(nb of ASM of size n)

= 1, 2, 47, 429, ...

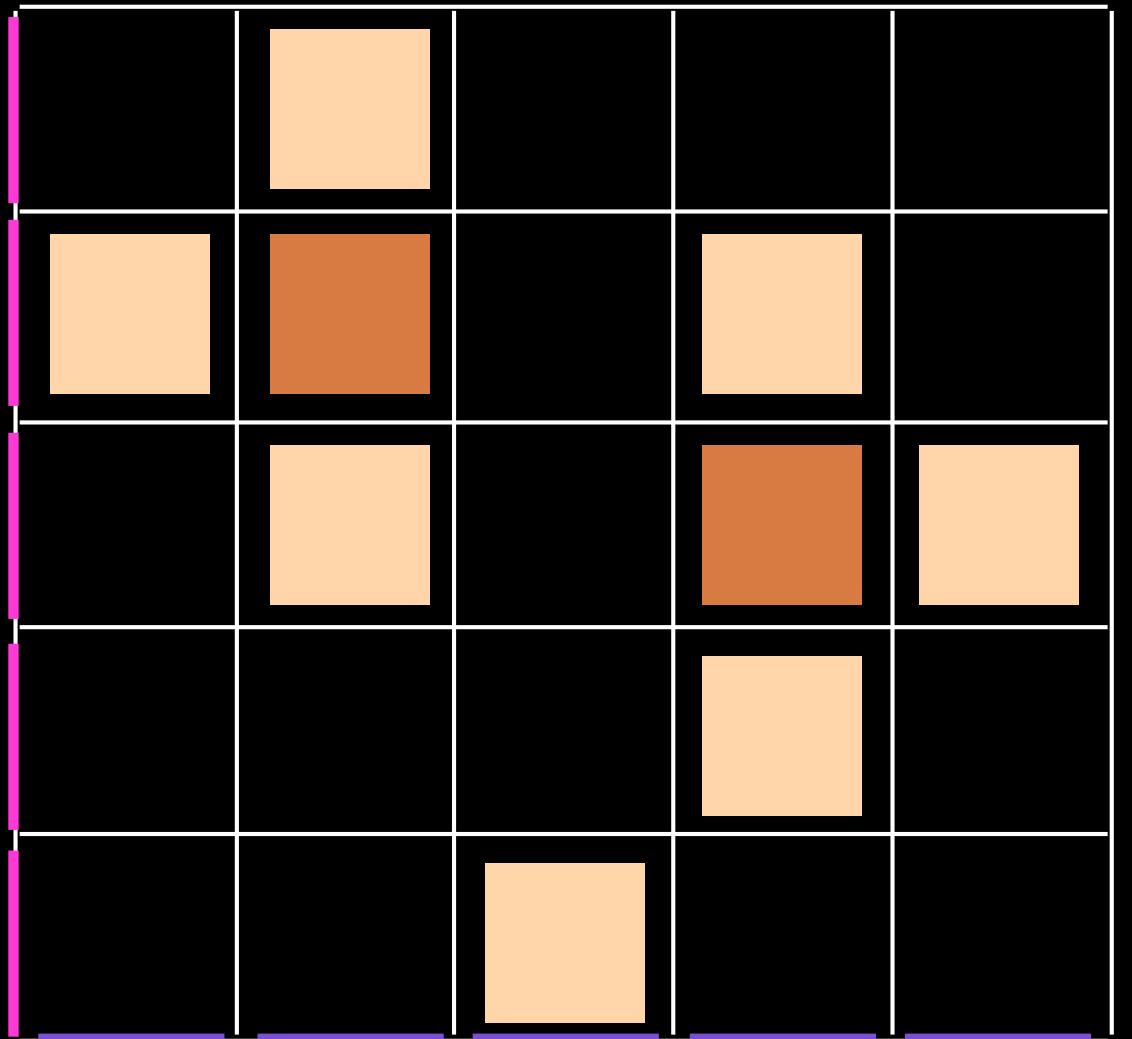
?

A'

B

A

B'



Q-tableaux

Quadratic algebra \mathbb{Q}

generators $B = \{B_j\}_{j \in J}$

$A = \{A_i\}_{i \in I}$

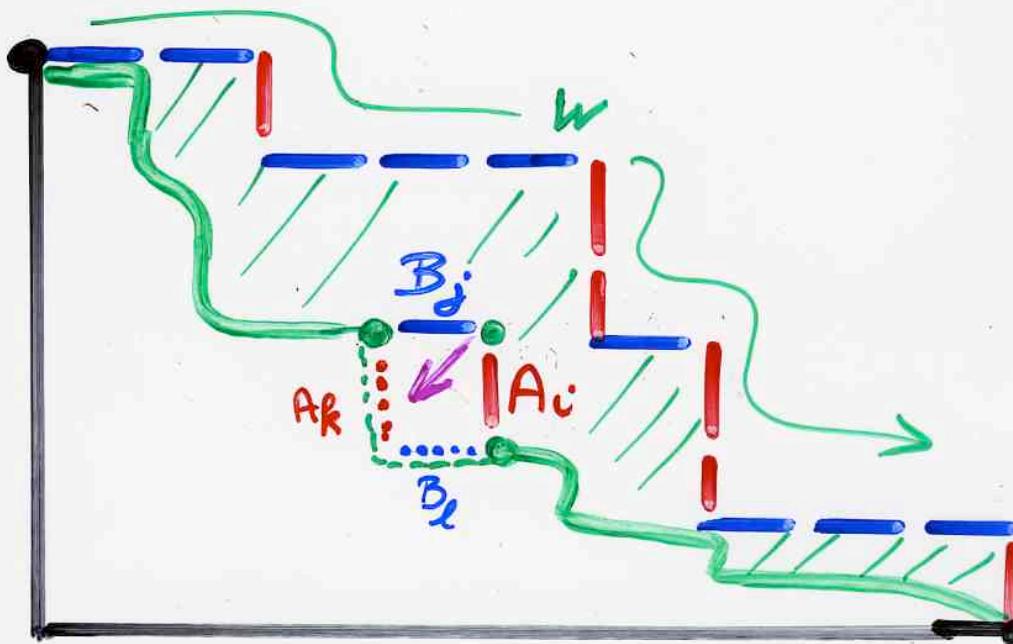
commutation relations

$$B_j A_i = \sum_{k, l} c_{ij}^{kl} A_k B_l \quad \begin{matrix} i \in I \\ j \in J \end{matrix}$$

lemma. In \mathbb{Q} every word $w \in (A \cup B)^*$
can be written in a unique way

$$w = \sum_{\substack{u \in A^* \\ v \in B^*}} c(u, v; w) uv$$

Proof:



S set of labels

$$\varphi : \left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\} = R \longrightarrow S$$

set of
rewriting rules

$$B_j A_i \rightarrow C_{ij}^{kl} A_k B_l$$

Def- **Q-tableau**

"image" by φ of a
"complete Q-tableau"

Planar automaton

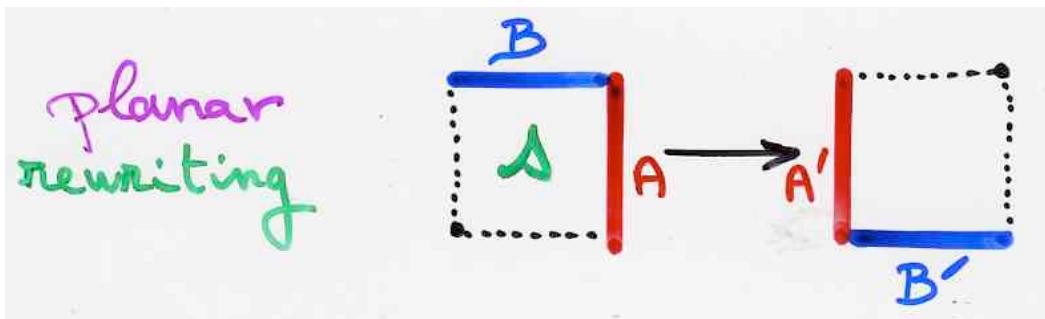
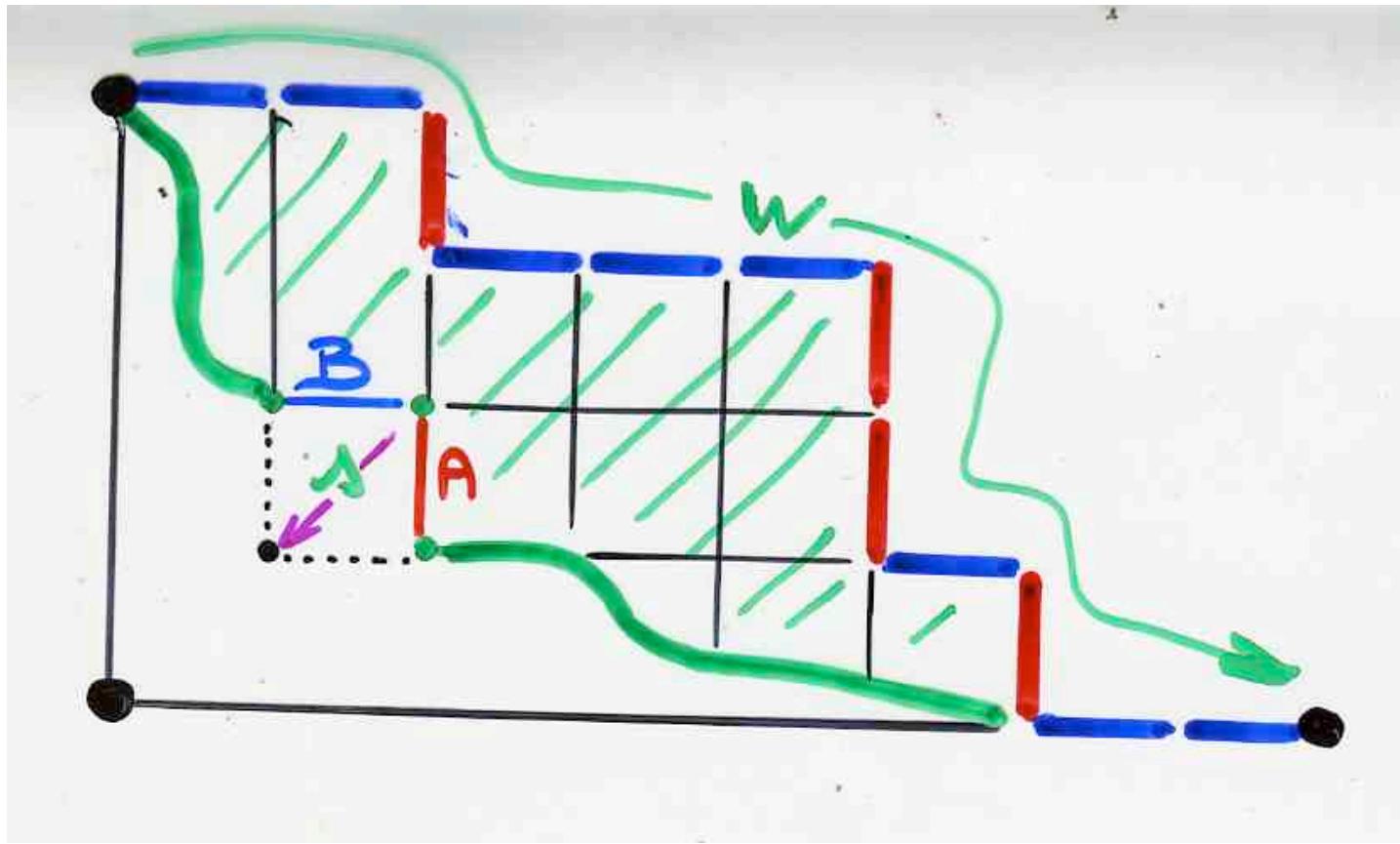
Def. planar automaton P

- 3 finite sets $\{ \cdot \}$
 - : \mathcal{B} horizontal alphabet
 - : α vertical labels
 - : S planar labels (state)

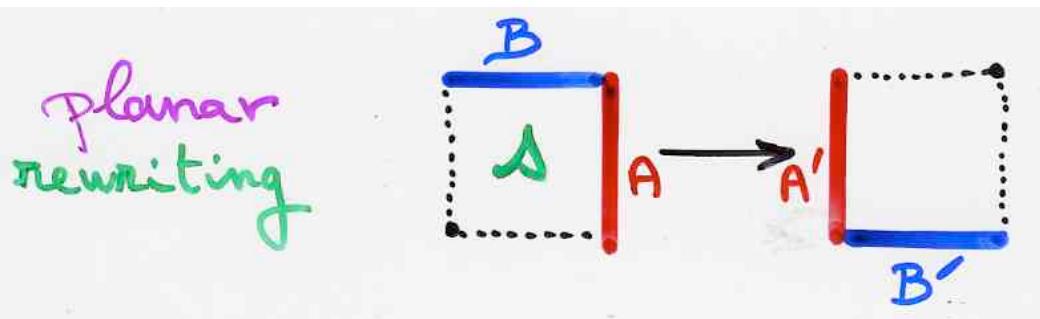
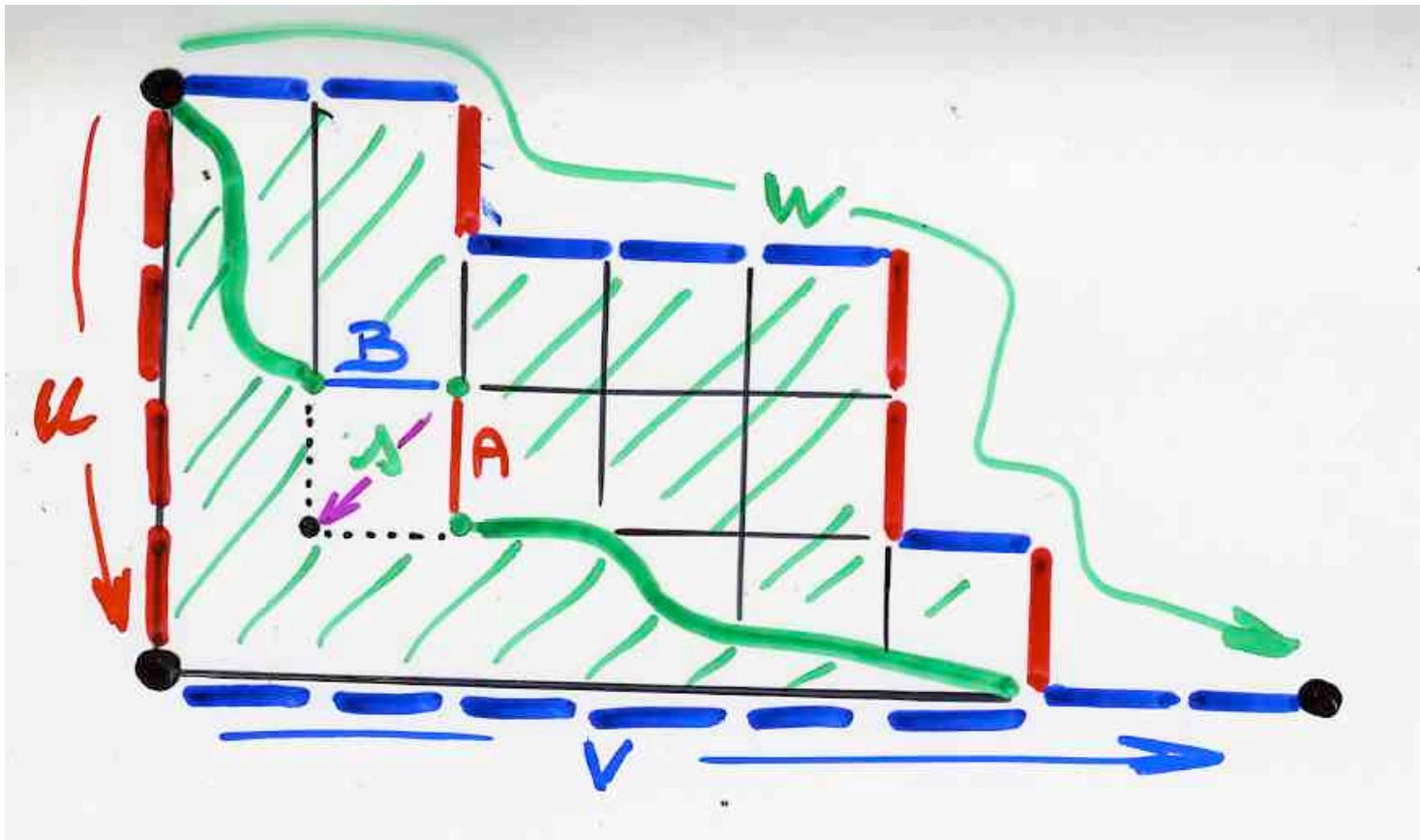
- θ (partial) transition function
 $(s, B, A) \xrightarrow{\theta} (B', A')$ or \emptyset
 $s \in S; B, B' \in \mathcal{B}; A, A' \in \alpha$

- $w \in (\alpha \cup \mathcal{B})^*$ initial word
- $uv, u \in \alpha^*, v \in \mathcal{B}^*$ final

Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \alpha, \theta, w, uv)$



Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \alpha, \theta, w, uv)$



equivalence

Q-tableaux



tableaux

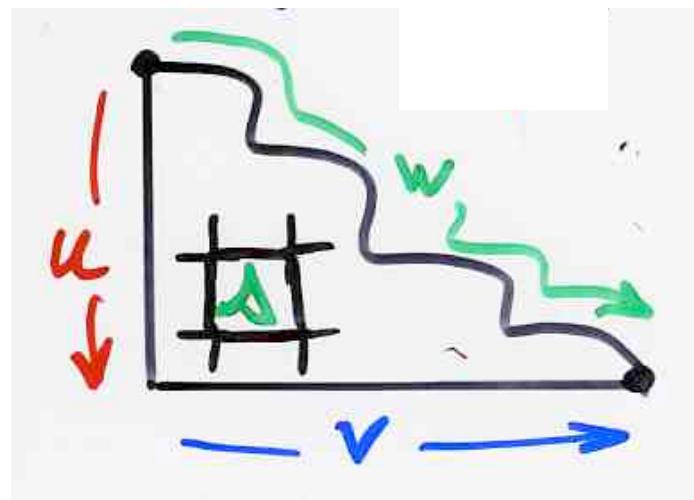
accepted by a

Q quadratic
algebra

φ

planar automaton

$$P = (S, \mathcal{B}, \alpha, \theta, w, uv)$$



$$BA = \sum_{s \in S} A'B'$$
$$(B', A') = \theta(s, B, A)$$

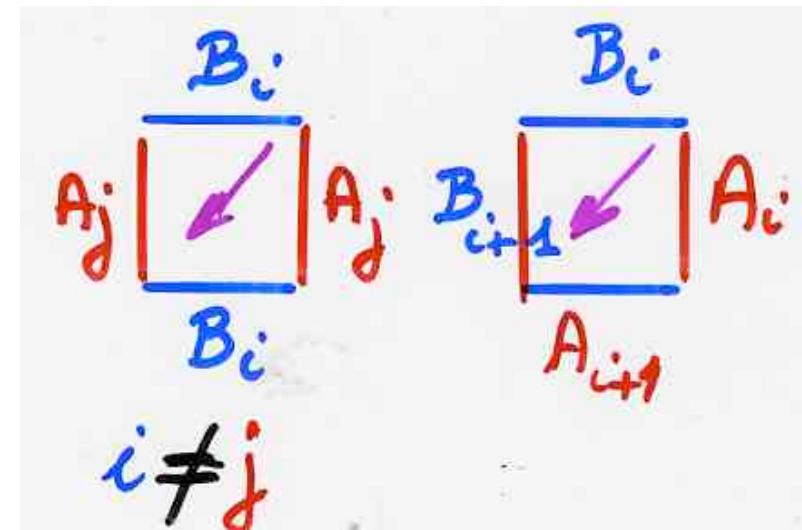
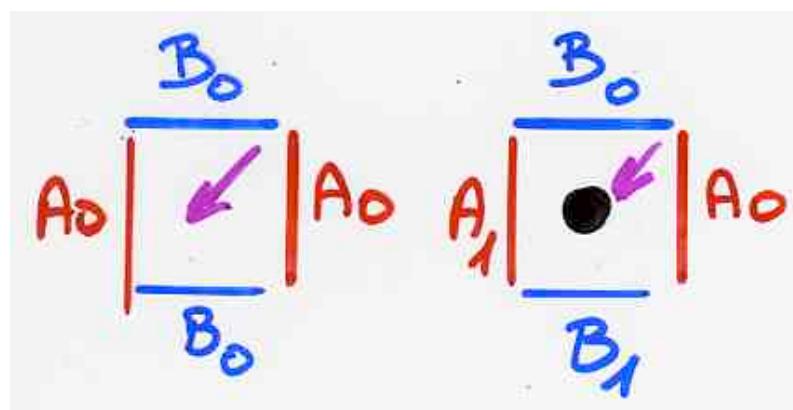
The "RSK planar automaton"

$$\mathcal{B} = \{B_0, B_1, \dots, B_k\}$$

$$\mathcal{A} = \{A_0, A_1, \dots, A_k\}$$

$$w \in \{B_0, A_0\}^*$$

$$S = \{\square, \blacksquare\}$$

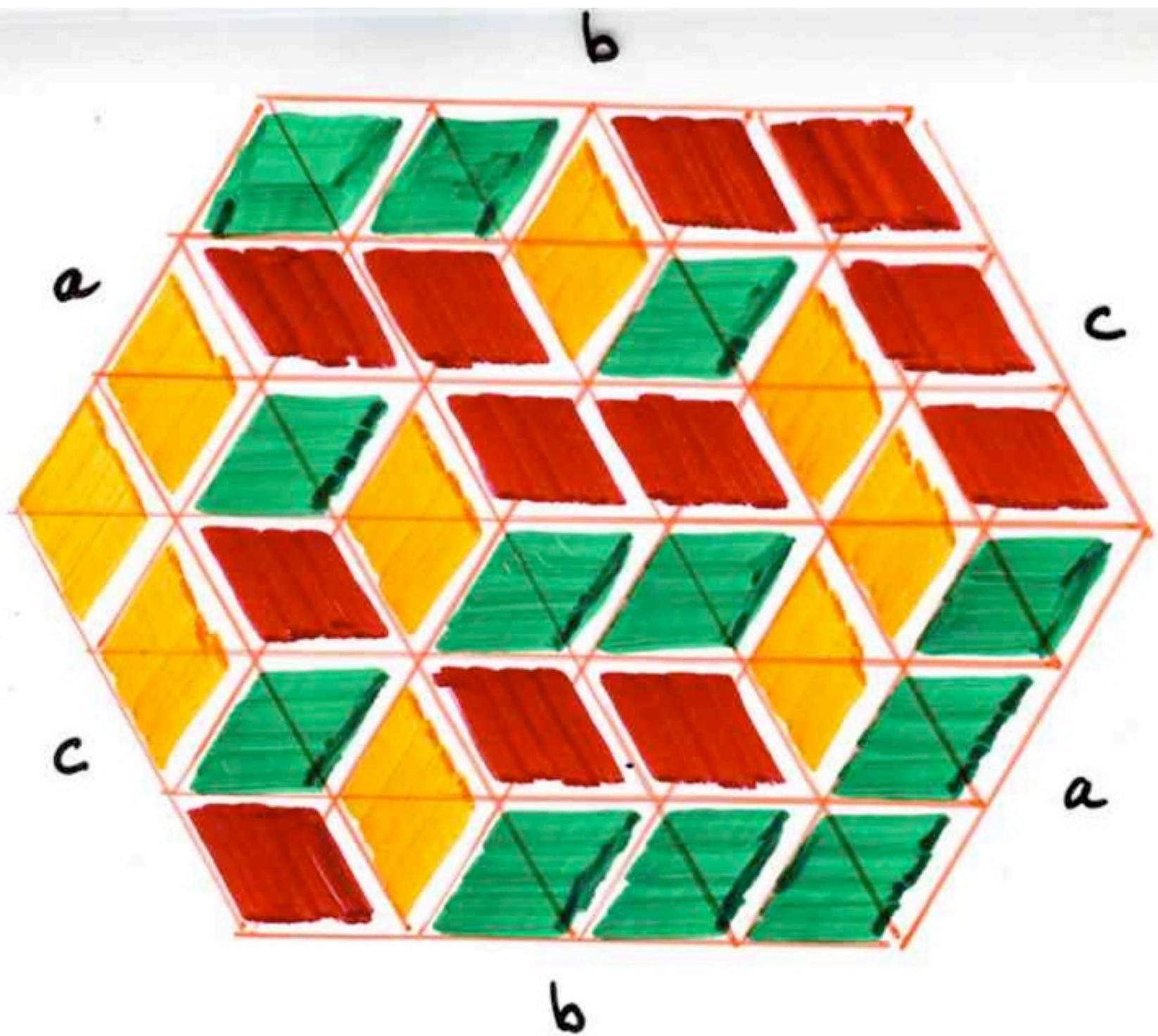


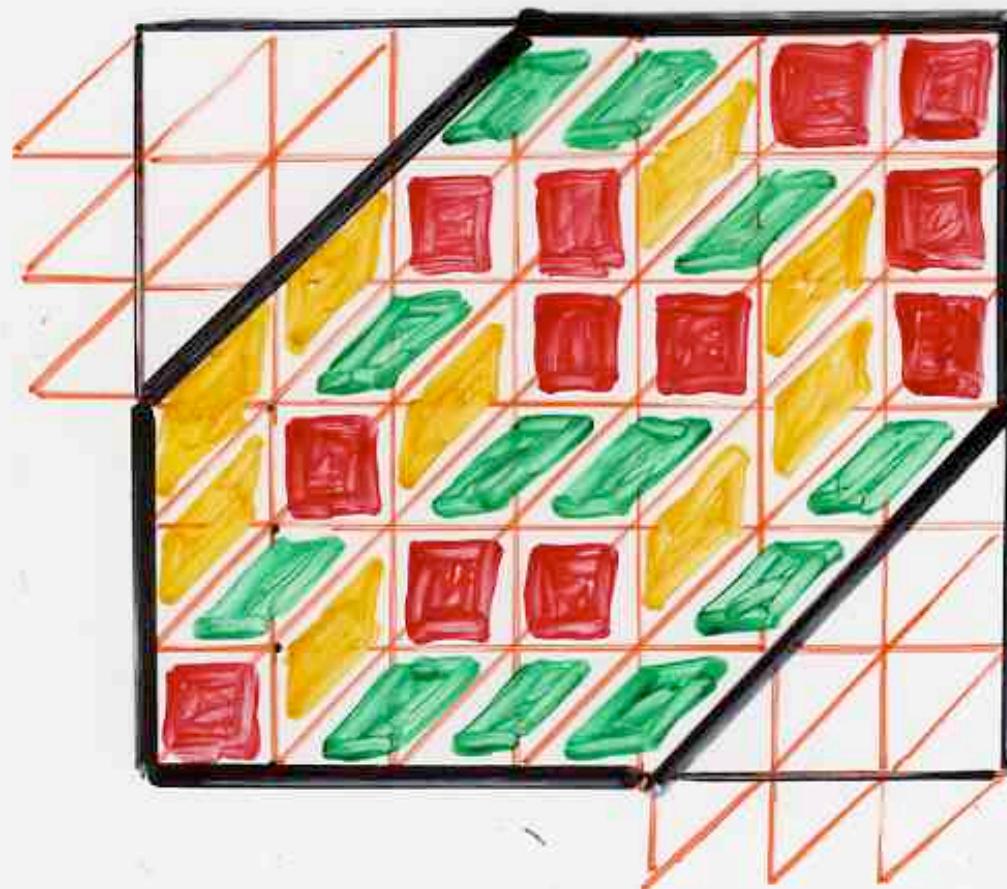
The 8-vertex algebra
(or Z - algebra)

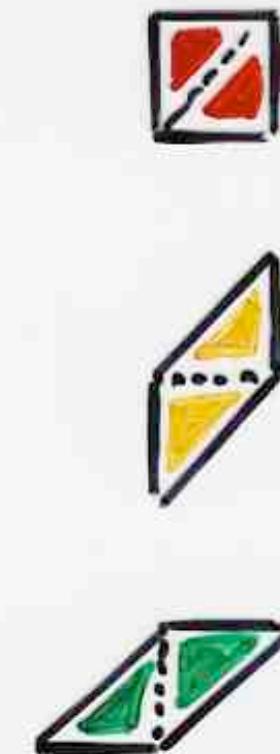
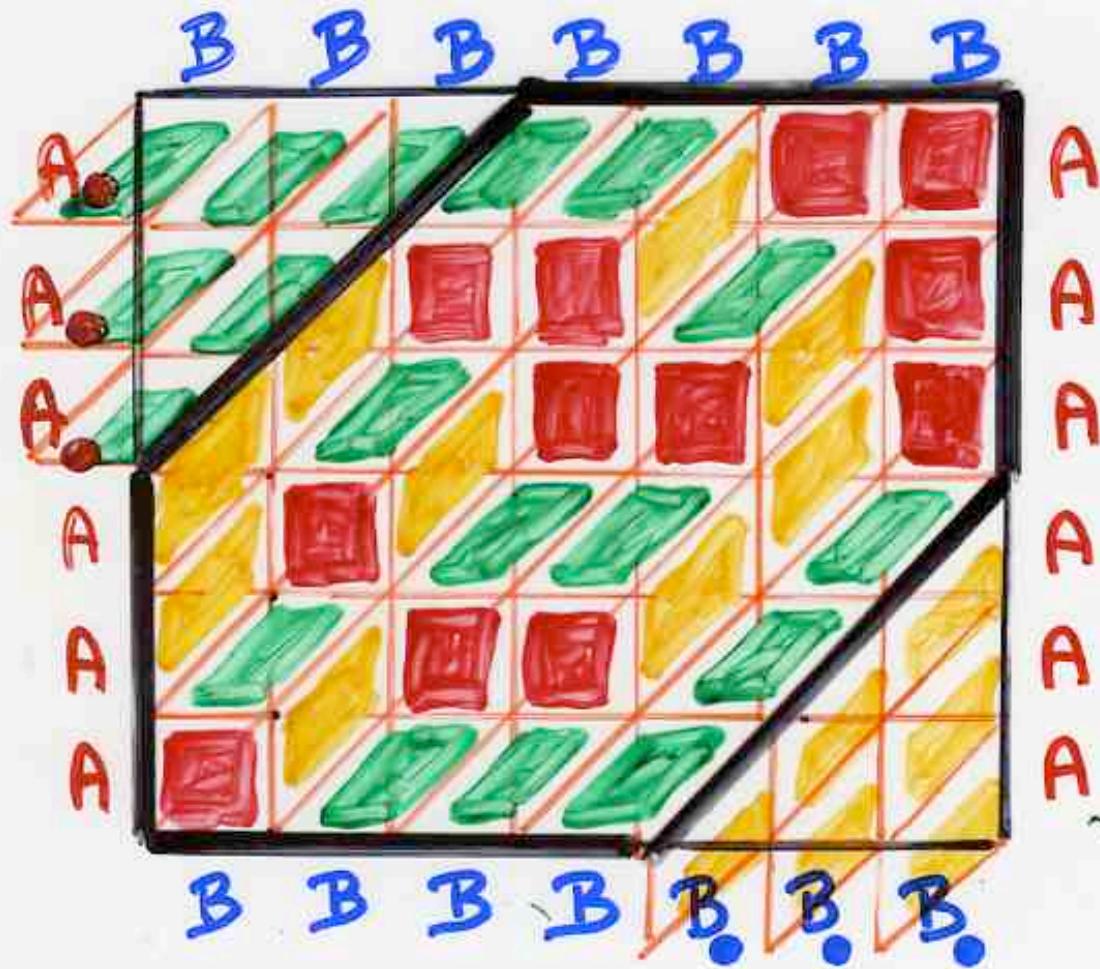
The quadratic algebra \mathbb{Z}

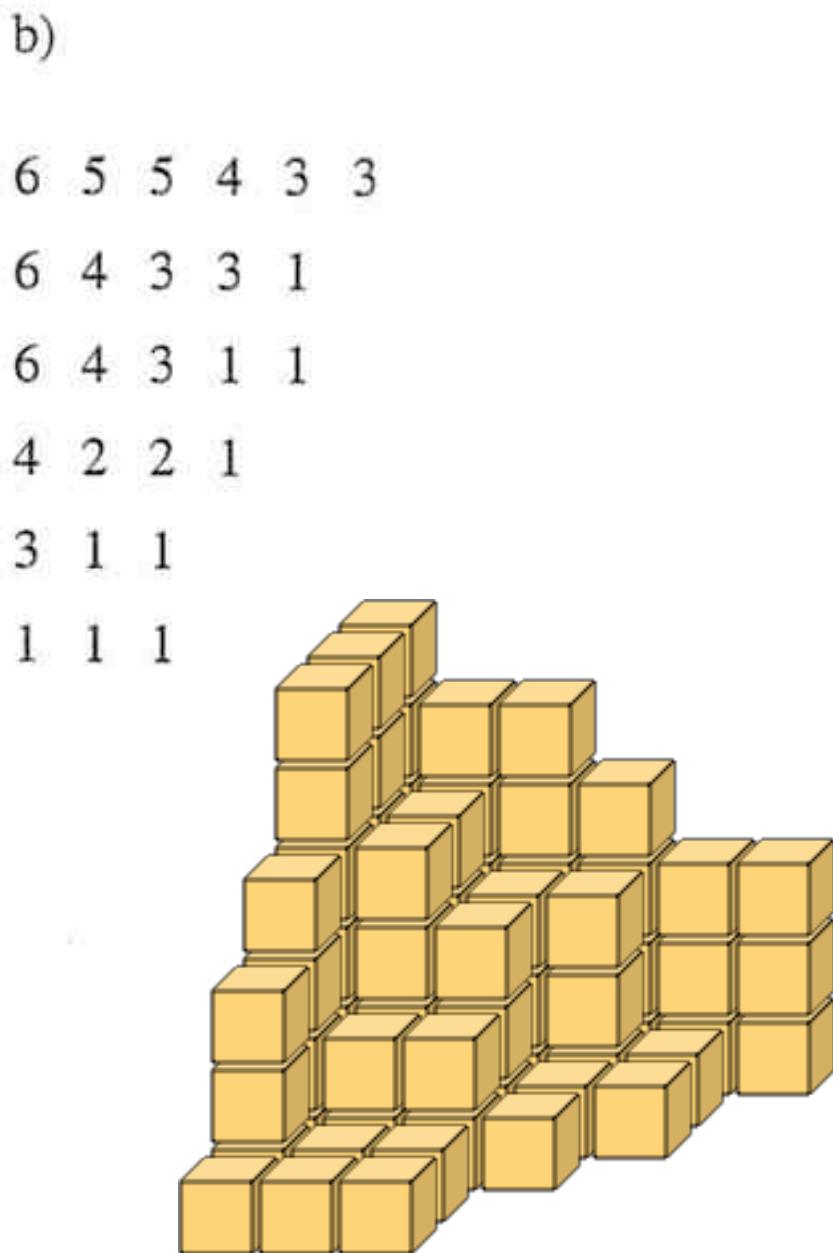
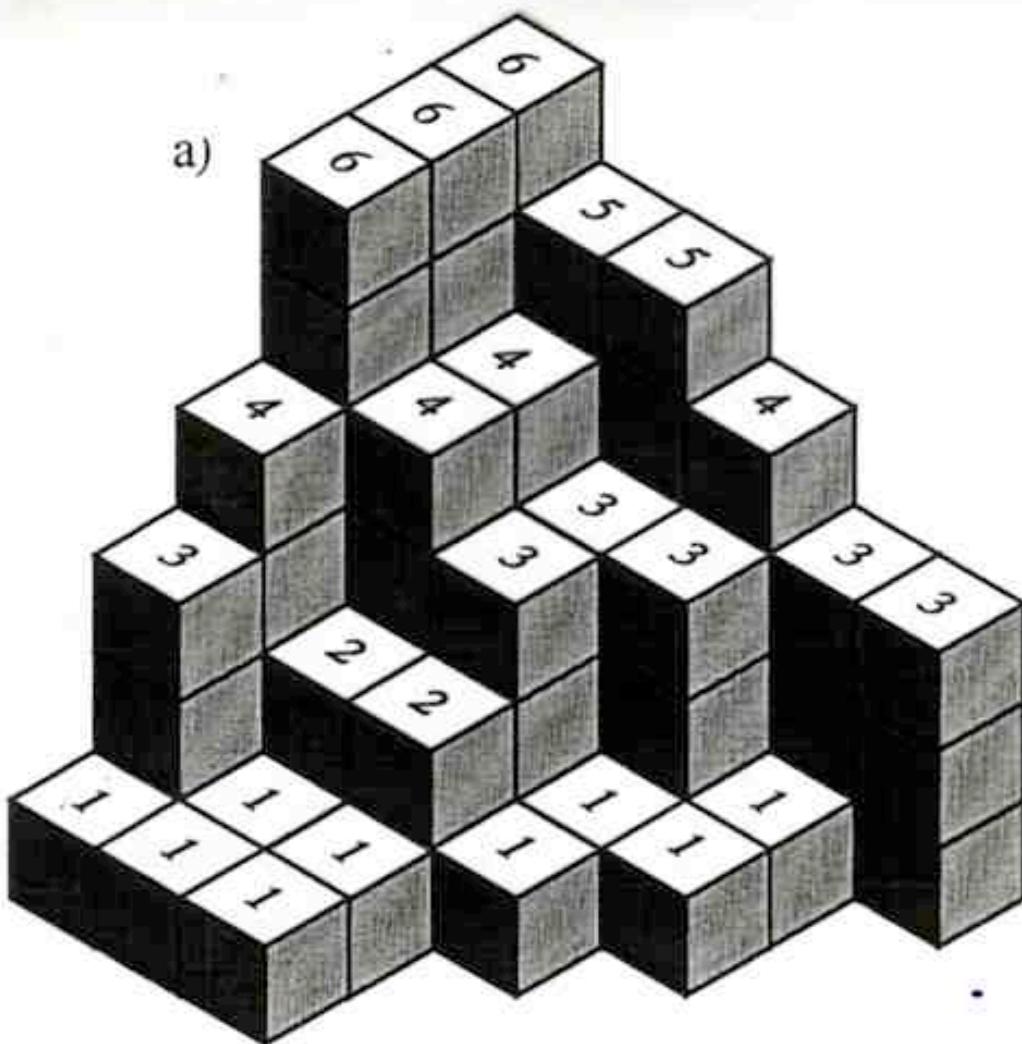
4 generators $B_0 A_0 BA$
8 parameters $q_{...}, t_{...}$

$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A_B \\ B_0 A_0 = q_{00} A_0 B_0 + t_{00} AB \\ B_0 A = q_{00} A B_0 + t_{00} A_B \\ BA_0 = q_{00} A_0 B + t_{00} AB_0 \end{array} \right.$$

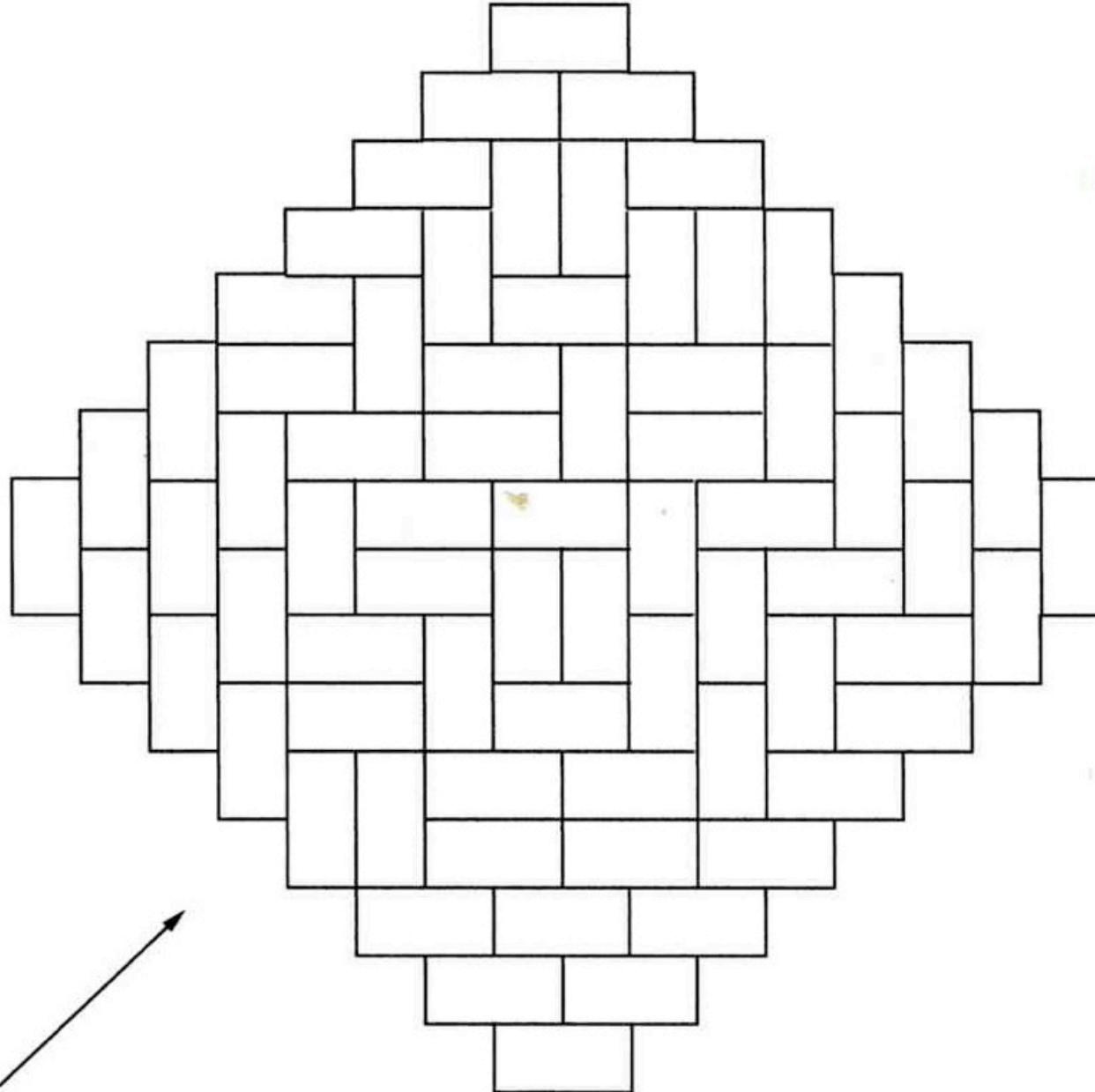






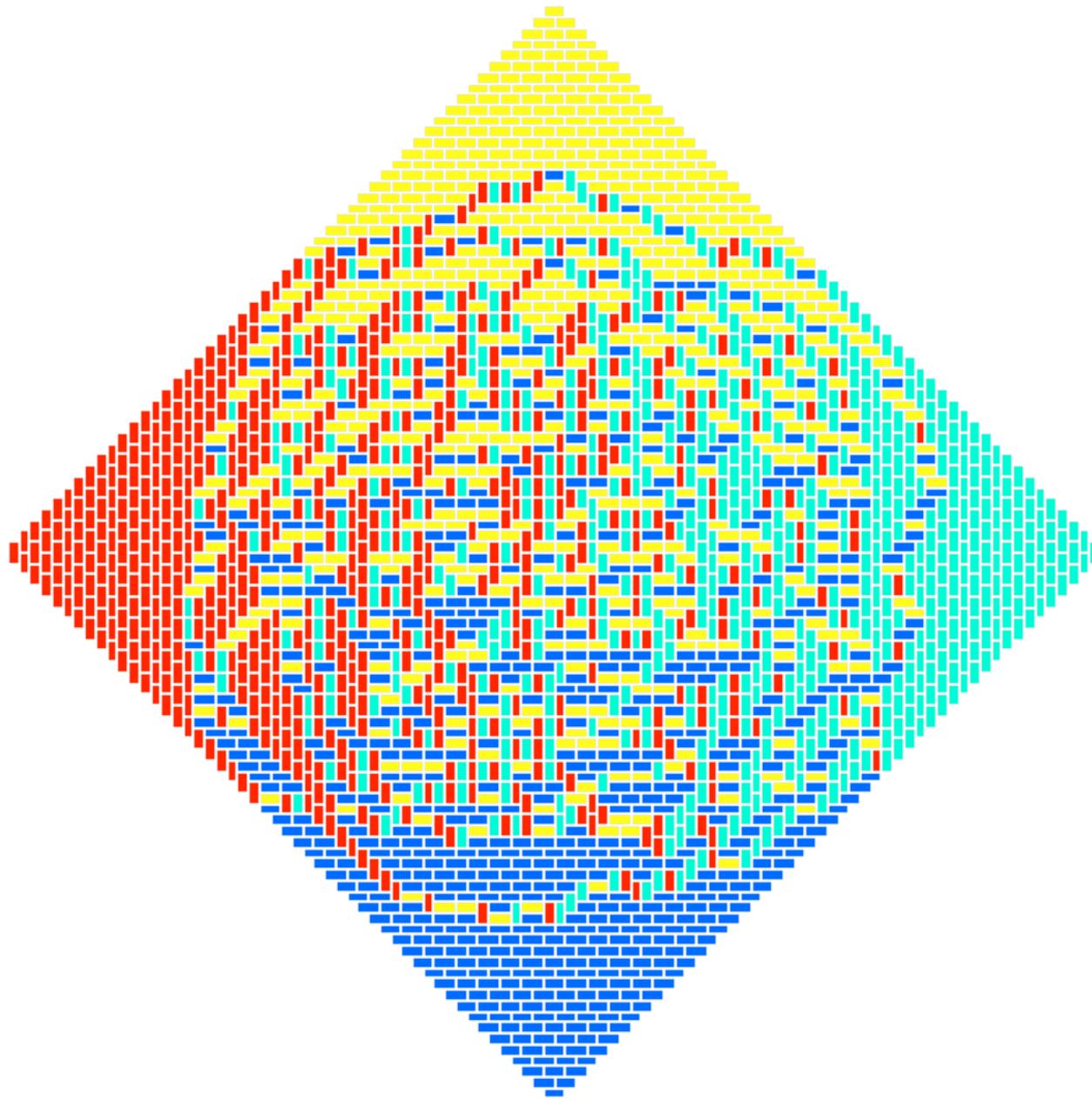


$$2^{n(n-1)/2}$$

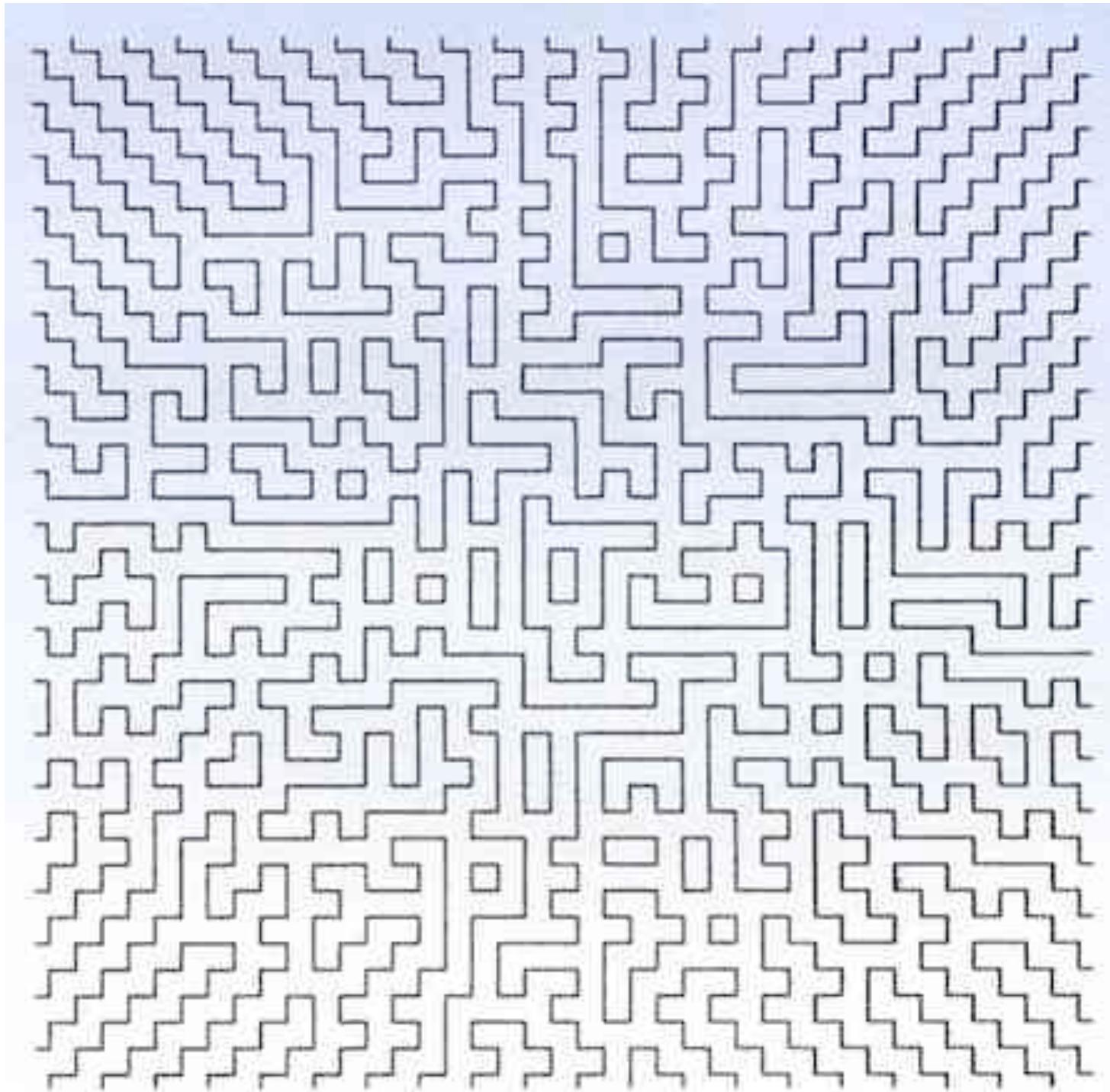


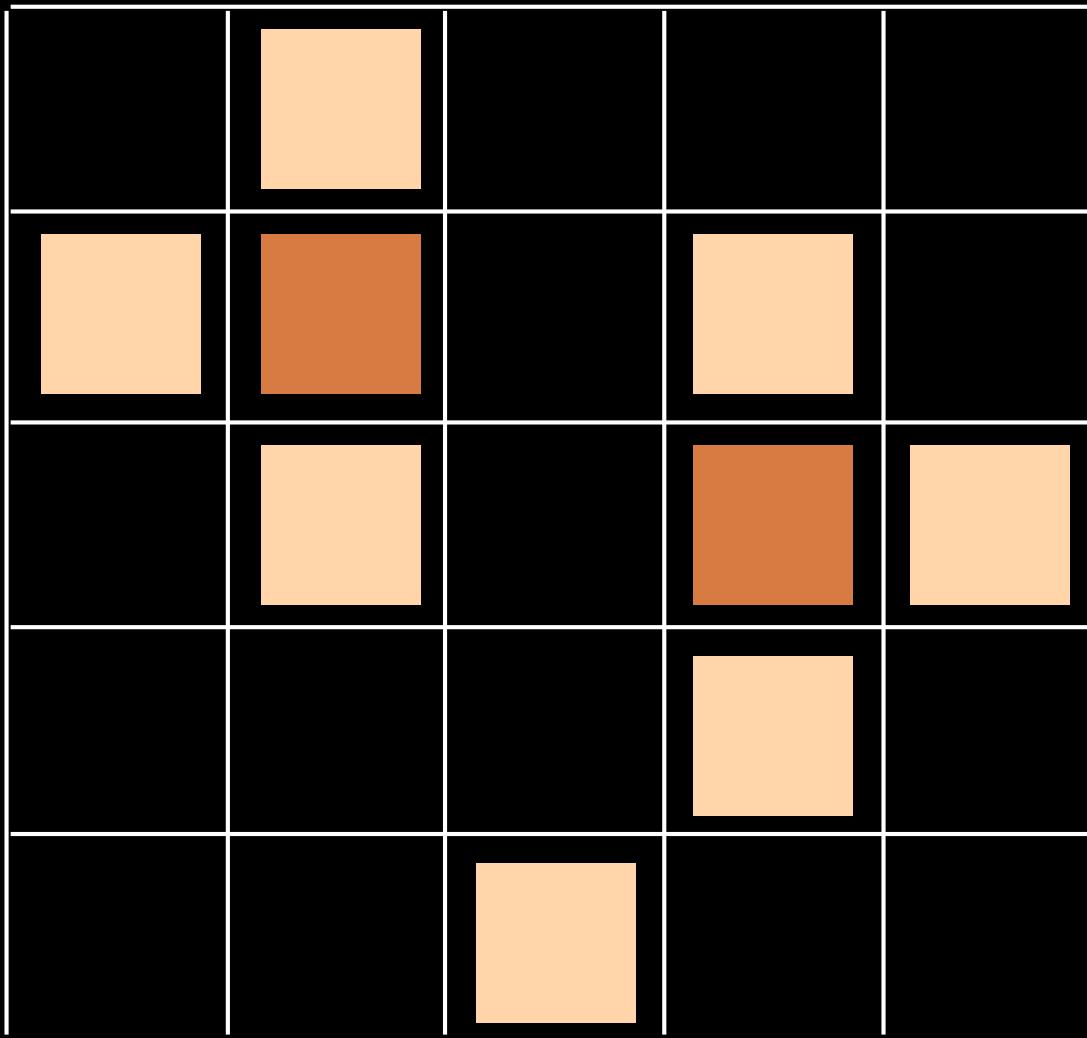
Elkies,
Kuperberg,
Larsen,
Propp
(1992)

random
Aztec
tilings



random
FPL





and beyond

cellular Ansatz 3:
from the quadratic algebra Q
how to guess representations

"L'Ansatz cellulaire"

Physique

"normal ordering"

$$\mathbf{UD} = \mathbf{DU} + \mathbf{Id}$$

Weyl-Heisenberg

$$\mathbf{DE} = q\mathbf{ED} + \mathbf{E} + \mathbf{D}$$

PASEP

algèbre quadratique \mathbf{Q}

commutations
réécritures
planarisation

objets
combinatoires
planarisés

représentation
par opérateurs

histoires
de fichiers
polynômes
orthogonaux

bijections

placements de tours

RSK

permutations

tableaux alternatifs



paires Tableaux Young

permutations

histoires de Laguerre

Q-tableaux
ex: ASM, FPL
pavages, 8-vertex



automates
planaires

Cours I

Cours II

pour plus de détails
voir les diaporamas du cours donné à Talca:

Cours XGV, Universidad de Talca

(December 2010 - January 2011)

Combinatorics and interactions (with physics) (24h)

«The Cellular Ansatz»

accessible sur les sites:

<http://www.labri.fr/perso/viennot/>

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle [ici](#)

Vulgarisation scientifique voir la page de l'association [Cont'Science](#)

http://web.me.com/xgviennot/Xavier_Viennot/

http://web.me.com/xgviennot/Xavier_Viennot2/

Ch 0 Introduction

Ch 1 Ordinary generating function, the Catalan garden

Ch 1a (1/12/2010, 54 p.)

Ch 1b (7/12/2010, 81 p.)

Ch 1c (7/12/2010, 30 p.) algebraic complements in relation with physics

Ch 2 Exponential generating functions, permutations

Ch 2a (22/12/2010, 40 p.)

Ch 2b (4/01/2010, 63 p.)

Ch 2c (4/01/2010, 33 p.) Permutations: Laguerre histories

Ch 3 Permutations and Young tableaux, the Robinson-Schensted correspondence (RSK)

Ch 3a (6/01/2011, 117 p.)

Ch 3b (6, 11/01/2011, 121 p.) RSK and operators

Ch 4 Alternative tableaux and the PASEP (partially asymmetric exclusion process)

Ch 4a (13/01/2011, 98p.)

Ch 4b (13, 18/01/2011, 102 p.) alternative tableaux and the PASEP

Ch 4c (18/01/2011, 81 p.) complements

Ch 5 Combinatorial theory of orthogonal polynomials

(20/01/2011, 110 p.)

Ch 6 "jeu de taquin" for binary trees, Catalan tableaux and the TASEP

Ch 6a (24/01/2011, 98 p.)

Ch 6b (24/01/2011, 111 p.) alternative tableaux and increasing/alternative binary trees

Ch 6c (24/01/2011, 21 p.) Catalan tableaux and the Loday-Ronco algebra

Ch 7 The cellular Ansatz

Ch 7a (25/01/2011, 117 p.)

Ch 7b (25/01/2011, 49 p.) complements

Cours XGV

Universidad de Talca

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24 h

Combinatorics and interactions

(with physics)

«The Cellular Ansatz»

