Zero-automaticity for groups and monoids

Jean MAIRESSE *

October 17, 2005

Abstract

Consider a group G with finite set of generators Σ. The pair (G, Σ) is 0-automatic if it admits a natural notion of normal form words where only the last letter is modified by multiplication by a generator. We survey the known results on the notion. Then we proceed by giving a complete description of all 0-automatic pairs by ways of an algorithm. In the last part, we initiate a study of 0-automatic monoids.

Keywords: automatic group or monoid, plain group, geodesic, string-rewriting.

1 Introduction

In this paper, we study zero-automatic pairs. They constitute, in a sense, the simplest layer of automatic groups.

Automatic groups stand at the crossroad between group theory and automaton theory. Starting with Cannon and Thurston, the notion was extensively studied in the 80's and 90's [5, 9, 2]. A very good survey, especially intented for theoretical computer scientists, was written by Christian Choffrut [7]. The next informal but crisp description of automatic groups is borrowed verbatim from [7]:

> The basic idea behind automatic groups is to extend the multiplication table from finite to discrete infinite groups. In order to actually manipulate elements of an infinite group, an encoding is needed. This is achieved by ways of normal forms which are specific strings over a suitable set of symbols, the "generators", and whose set is recognized by a finite state machine. Then a computing device is needed to effectively calculate the product of two elements. This second task is done by again using a finite state machine provided with two tapes, one for the input element and the second for the result of the multiplication of that element by some generator.

Automaticity is *intrinsic*: a group which is automatic with respect to a given finite generating set, is automatic with respect to any such set. In this paper, we study the notion of 0-automaticity. It is a non-intrinsic and very specific specialization of the notion of automaticity. The requirement is that only the last letter of a normal form should be modified by multiplication by a generator. It is also possible to give a geometric characterization. Classically, a group is automatic iff it satisfies the k-fellow traveller property for some k. A pair group-generators is 0-automatic iff it satisfies the 0-fellow traveller property. This is our justification for the chosen name.

*LIAFA, CNRS-Université Paris 7, case 7014, 2, place Jussieu, 75251 Paris Cedex 05, France. E-mail: Jean.Mairesse@liafa.jussieu.fr

We now proceed to a more precise definition. A pair (G, Σ) formed by a group (group law $*$, unit element 1_G) and a finite set of generators is called *0-automatic* if the set of words $L(G, \Sigma) = \{u_1 \cdots u_k \mid \forall i, u_i * u_{i+1} \notin \Sigma \cup 1_G\}$ is a cross-section of G. Consider a group $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ which is a free product of a finitely generated free group and a finite family of finite groups, also called *plain group*. Consider the *natural* (but not necessarily minimal) set of generators $\Sigma = S \sqcup S^{-1} \sqcup_i G_i \backslash \{1_{G_i}\}$. Then the pair (G, Σ) is 0-automatic. Now consider an arbitrary 0-automatic pair (G, Σ). Then G is isomorphic to a plain group. On the other hand, Σ may be strictly larger than a natural set of generators of the group seen as a free product. (For examples of this, see the Figures 1 or 2.) Hence it seems appropriate and convenient to coin a specific term as "0-automatic pairs" to highlight the notion.

The purpose and content of the paper is as follows. First, we survey the known results on zero-automaticity. The notion was introduced by Stallings in [15] under another name: *unique factorization groups*, which we did not adopt for several reasons. Following Stallings, the notion was studied by Haring-Smith [10], d'Agostino [8], and Avenhaus, Madlener, & Otto [1]. The results are recalled in Sections 3.2 and 3.3. In Theorem 3.8, we collect five alternative definitions of zero-automatic pairs, four from the above papers, and a new one via the 0-fellow traveller property. Second, and this constitutes a new contribution, we provide a complete description of all 0-automatic pairs by ways of an algorithm in Section 3.4, Theorem 3.12. Third, we initiate a study of 0-automaticity for monoids in Section 4.

Despite their restrictive nature, there may be different motivations for studying 0-automatic pairs. First, they constitute the simplest layer of automatic groups. Second, their definition is natural, and their Cayley graph has remarkable combinatorial properties. Third, and this was our original motivation, they appear in the context of random walks. In [11, 12], it is proved that for any nearest-neighbor random walk on a 0-automatic pair, the harmonic measure has a special Markovian structure. It allows for instance to explicitly compute the drift or the entropy of the walk.

2 Preliminaries

Given a set Σ, the free monoid it generates is denoted by Σ^*. As usual, the elements of Σ and Σ^* are called respectively *letters* and *words*, and the subsets of Σ^* are called *languages*. The empty word is denoted by 1. The concatenation product of two elements u and v of Σ^* is written uv. The *length* (number of letters) of a word u is denoted by $|u|_\Sigma$. The longest prefix of two words u and v is denoted by $u \wedge v$. We denote by $\mathrm{Rat}(\Sigma^*)$, the set of *regular* languages of Σ^*, that is the set of languages *recognized* by a finite automaton.

Consider a discrete group $(G, *)$ with finite set of generators $\Sigma \subset G$. The unit is denoted by 1_G and the inverse of u is denoted by u^{-1}. Throughout the paper, we always assume that $1_G \notin \Sigma$ and that Σ is symmetric, i.e. $u \in \Sigma \implies u^{-1} \in \Sigma$. Denote by $\pi : \Sigma^* \to G$ the monoid homomorphism which associates to a word $a_1 \cdots a_k$ the group element $a_1 * \cdots * a_k$. A word $u \in \pi^{-1}(g)$ is called a *representative* of g. The *length with respect to* Σ of a group element g is:

$$|g|_\Sigma = \min\{k \mid \ g = s_1 * \cdots * s_k, s_i \in \Sigma\}\,. \tag{1}$$

A representative u of g is a *geodesic* if $|u|_\Sigma = |g|_\Sigma$.

The *Cayley graph* $\mathfrak{X}(G, \Sigma)$ of a group G with respect to a set of generators Σ is the directed graph with set of nodes G and with an arc from u to v if $u^{-1} * v \in \Sigma$. As usual, an arc from u to v is represented graphically by $u \longrightarrow v$. It is often convenient to view $\mathfrak{X}(G, \Sigma)$ as a labelled graph with set of labels Σ (with $u \xrightarrow{a} v$ if $u * a = v$). There is a natural undirected graph associated with $\mathfrak{X}(G, \Sigma)$ and obtained by viewing the two directed arcs $u \rightleftarrows u * a$ as an undirected edge $u — u * a$. For examples of Cayley graphs, see Figures 1 or 2. Observe that $|u|_\Sigma$ is the geodesic distance (in number of arcs) from 1_G to u in the Cayley graph.

The symbol $\sqcup$ is used for the disjoint union of sets. Consider a relation $R \subset \Sigma^* \times \Sigma^*$, and let $\sim_R$ be the least congruence on Σ^* such that $u \sim_R v$ if $(u, v) \in R$. Let M be isomorphic to the quotient monoid $(\Sigma^* / \sim_R)$. We say that $\langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ is a *monoid presentation* of M.
Given a set S, denote by $\mathbb{F}(S)$ the free group generated (as a group) by S. A monoid presentation of $\mathbb{F}(S)$ is

$$\langle\, S \sqcup S^{-1} \mid aa^{-1} = 1, a^{-1}a = 1,\ \forall a \in S \,\rangle\,. \tag{2}$$

Consider a relation $R \subset S^* \times S^*$. Let G be isomorphic to the quotient group $(\mathbb{F}(S)/ \sim_R)$. We say that $\langle\, S \mid u = v, (u, v) \in R \,\rangle$ is a *group presentation* of G. A monoid presentation of G is $\langle\, S \sqcup S^{-1} \mid u = v, (u, v) \in R,\ aa^{-1} = 1, a^{-1}a = 1,\ \forall a \in S \,\rangle$.
We write, respectively, $M = \langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ and $G = \langle\, S \mid u = v, (u, v) \in R \,\rangle$, precising when necessary if we are dealing with a monoid or a group presentation.

Given two groups G_1 and G_2, let $G_1 \star G_2$ be the free product of G_1 and G_2. Roughly, the elements of $G_1 \star G_2$ are the finite alternate sequences of elements of $\Sigma_1 = G_1 \backslash \{1_{G_1}\}$ and $\Sigma_2 = G_2 \backslash \{1_{G_2}\}$, and the group law is the concatenation with simplification. More rigorously, set $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and define $R \subset \Sigma^* \times \Sigma^*$ by

$$R = \{(ab, c) \mid a, b \in \Sigma_i, c \in \Sigma_i \cup 1,\ \pi_i(ab) = \pi_i(c),\ i \in \{1, 2\}\}\,,$$

where $\pi_i : \Sigma_i^* \to G_i$ is the canonical monoid homomorphism. Then the monoid presentation $\langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ defines a group called the *free product of* G_1 *and* G_2 and denoted by $G_1 \star G_2$.
Following [10], define a *plain group* as the free product of a finitely generated free group and a finite family of finite groups. Let $G = \mathbb{F}(S) \star G_1 \star \cdots \star G_k$ be a plain group. Then

$$\Sigma = S \sqcup S^{-1} \sqcup_i G_i \backslash \{1_{G_i}\}\,,$$

is a set of generators for G that we call *natural generators*. (Even for a fixed decomposition $(\mathbb{F}(S), G_1, \ldots, G_k)$ of G, the set of natural generators is not unique as soon as $\#S \geq 2$, because of the flexibility in choosing the free generators of the free group.)

3 Zero-Automaticity for groups

3.1 Definition and first properties

Let $(G, *)$ be a group with finite set of generators $\Sigma \subset G$. A language L of Σ^* is a *cross-section* of G if the restriction of π to L defines a bijection, that is if every element of G has a unique representative in L. A word of L is then called a *normal form word*. The map $\phi : G \to L$ which associates to a group element its unique representative in L is called the *normal form map*.

Definition 3.1. *Let G be a group with finite set of generators Σ. Define the language $L(G, \Sigma) \subset \Sigma^*$ of* locally reduced words *by:*

$$L(G, \Sigma) = \{u_1 \cdots u_k \mid \forall i \in \{1, \ldots, k-1\},\ u_i * u_{i+1} \notin \Sigma \cup 1_G\}\,. \tag{3}$$

We say that the pair (G, Σ) is 0-automatic *if $L(G, \Sigma)$ is a cross-section of G.*

A 0-automatic pair (G, Σ) is precisely a *unique factorization group* in Stallings [15], with the additional assumption that Σ be finite.

It is convenient to introduce the sets:

$$\forall a \in \Sigma, \quad \mathrm{Next}(a) = \big\{ b \in \Sigma \mid a * b \notin \Sigma \cup \{1_G\} \big\}, \quad \mathrm{Prev}(a) = \big\{ b \in \Sigma \mid b * a \notin \Sigma \cup \{1_G\} \big\}\,. \quad (4)$$

Then we can reformulate (3) as:

$$\begin{aligned} L(G,\Sigma) &= \{u_1 \cdots u_k \mid \forall i \in \{2,\dots,k\},\ u_i \in \mathrm{Next}(u_{i-1})\} \\ &= \{u_1 \cdots u_k \mid \forall i \in \{1,\dots,k-1\},\ u_i \in \mathrm{Prev}(u_{i+1})\}\,. \end{aligned}$$

There are some direct consequences of Definition 3.1.

Lemma 3.2. *Assume that $L(G, \Sigma)$ is a cross-section, and let $\phi : G \to L(G, \Sigma)$ be the corresponding normal form map. Then, $\forall g \in G$ s.t. $\phi(g) = u_1 \cdots u_k$, $\forall a \in \Sigma$,*

$$\phi(g * a) = \begin{cases} u_1 \cdots u_{k-1} & \text{if } a = u_k^{-1} \\ u_1 \cdots u_{k-1} v & \text{if } u_k * a = v \in \Sigma \\ u_1 \cdots u_{k-1} u_k a & \text{if } u_k * a \notin \Sigma \cup 1_G \end{cases}, \quad (5)$$

and the analog for $\phi(a * g)$ also holds.

The proof is not difficult, see [15]. A direct consequence of Lemma 3.2 is that $L(G, \Sigma)$ belongs to $\mathrm{Rat}(\Sigma^*)$. It is recognized by the following automaton:

Set of states: $\Sigma \cup 1_G$, initial state: 1_G, final states: $\Sigma \cup 1_G$; Transitions: $a \xrightarrow{b} b$ if $a * b \notin \Sigma \cup 1_G$.

The Cayley graph of a 0-automatic pair has a tree-like structure:

Lemma 3.3. *Let (G, Σ) be a 0-automatic pair with G infinite. The Cayley graph $\mathfrak{X}(G, \Sigma)$ has uniform vertex-connectivity 1, that is, the removal of any node disconnects the graph.*

The proof of Lemma 3.3 is easy. The removal of the node g, with $\phi(g) = u$, disconnects the nodes $\{\pi(uv), uv \in L(G, \Sigma)\}$ from the other ones, as a direct consequence of Lemma 3.2.

For comparison, a connected graph with uniform edge-connectivity 1, i.e. such that the removal of any edge disconnects the graph, is a tree. The class of graphs with uniform vertex-connectivity 1, is richer, see Figures 1 or 2.

The converse of Lemma 3.3 does not hold. Consider for instance $\mathbb{Z}^2 \star \mathbb{Z}$ with set of generators $\Sigma = S \sqcup T$, where S and T are generators of $\mathbb{Z}^2$ and $\mathbb{Z}$. Then $\mathfrak{X}(\mathbb{Z}^2 \star \mathbb{Z}, \Sigma)$ has uniform vertex-connectivity 1 but $(\mathbb{Z}^2 \star \mathbb{Z}, \Sigma)$ is not 0-automatic (see Theorem 3.6).

Another way to apprehend the tree-like structure of the Cayley graph is to consider the maps Next and Prev. Define, for $a \in \Sigma$,

$$\Sigma_a = \big\{ b \in \Sigma \mid \exists c \in \Sigma \cup 1_G,\ a * c = b \big\}, \quad \overline{\Sigma}_a = \big\{ b \in \Sigma \mid \exists c \in \Sigma \cup 1_G,\ c * a = b \big\}\,. \quad (6)$$

The following identities are easy to establish:

$$\begin{aligned} &\overline{\Sigma}_a = (\Sigma_{a^{-1}})^{-1}, \quad \Sigma_a = \{x \in \Sigma \mid \mathrm{Prev}(x) = \mathrm{Prev}(a)\}, \quad \overline{\Sigma}_a = \{x \in \Sigma \mid \mathrm{Next}(x) = \mathrm{Next}(a)\} \\ &\mathrm{Next}(a) = \Sigma \backslash \Sigma_{a^{-1}} = \Sigma \backslash \overline{\Sigma}_a^{-1}, \quad \mathrm{Prev}(a) = \Sigma \backslash \overline{\Sigma}_{a^{-1}} = \Sigma \backslash \Sigma_a^{-1}\,. \end{aligned} \quad (7)$$

Denote by $\mathfrak{X}(G, \Sigma)_{|\Sigma}$, the restriction of $\mathfrak{X}(G, \Sigma)$ to the set of nodes Σ. Then the sets of nodes of the maximal strongly connected subgraphs of $\mathfrak{X}(G, \Sigma)_{|\Sigma}$ are precisely the subsets $\Sigma_a, a \in \Sigma$. In $\mathfrak{X}(G, \Sigma)$, if one replaces the subgraph with set of nodes $\{\pi^{-1}(u_1 \dots u_{n-1} x), x \in \Sigma_{u_n}\}, u_1 \dots u_n \in L(G, \Sigma)$, by a single node, then the quotient graph is a tree.

The pair (G, Σ) formed by a group and a finite set of generators has the *unique geodesic property* if any group element has a unique geodesic representative with respect to Σ.

Lemma 3.4. *If (G, Σ) is 0-automatic then (G, Σ) has the unique geodesic property and $L(G, \Sigma)$ is precisely the set of geodesics.*

Proof. Consider a group element g and let $u = u_1 \cdots u_k, u_i \in \Sigma$, be a geodesic representative of g. Obviously, $\forall i, u_i * u_{i+1} \notin \Sigma \cup 1_G$, otherwise we would build a representative of g of length strictly less than k. Hence, u belongs to $L(G, \Sigma)$. Since $L(G, \Sigma)$ is a cross-section, it implies that g has a unique geodesic representative, and also that a non-geodesic representative cannot belong to $L(G, \Sigma)$. $\square$

3.2 Zero-automatic pairs and plain groups

Using the definition, it is easy to prove the following. Let G be a finite group. Then $(G, G \backslash \{1_G\})$ is 0-automatic. Let Σ be a finite set. Recall that $\mathbb{F}(\Sigma)$ is the free group generated (as a group) by Σ. Then $(\mathbb{F}(\Sigma), \Sigma \sqcup \Sigma^{-1})$ is 0-automatic. Let (G_1, Σ_1) and (G_2, Σ_2) be 0-automatic. Then $(G_1 \star G_2, \Sigma_1 \sqcup \Sigma_2)$ is 0-automatic.

Consequently, we get next Lemma.

Lemma 3.5. *The pair $(\ \mathbb{F}(S) \star G_1 \cdots \star G_k\ ,\ S \sqcup S^{-1} \sqcup G_1 \backslash \{1_{G_1}\} \sqcup \cdots \sqcup G_k \backslash \{1_{G_k}\}\ )$, formed by a plain group with natural generators, is 0-automatic.*

Next result is essentially due to Stallings (the result is explicitly stated in [10, Theorem 5]).

Theorem 3.6. *Let G be a group with finite set of generators Σ. If (G, Σ) is 0-automatic, then G is isomorphic to a plain group.*

The proof of Stallings is constructive. We come back to it in §3.4.

Zero-automaticity is not intrinsic, i.e. it depends not on the group but on the pair group - generators. This is illustrated in Example 3.7 below. In particular, Theorem 3.6 is not the converse of Lemma 3.5: if (G, Σ) is 0-automatic then G is isomorphic to a plain group but Σ may be strictly larger than a set of natural generators. This is again illustrated in Example 3.7. We answer the following question in Section 3.4:

Given a plain group G, what are the sets of generators Σ such that (G, Σ) is 0-automatic ?

The following implication was proposed by Shapiro as a conjecture in [13]: [(G, Σ) has the unique geodesic property] $\Longrightarrow$ [G is isomorphic to a plain group]. In view of Theorem 3.6, Shapiro's conjecture, if true, would form a converse to Lemma 3.4.

Example 3.7. Consider the free group $\mathbb{F}(a, b)$ over two generators. Set $S = \{a, a^{-1}, b, b^{-1}\}$. According to Lemma 3.5, the pair $(\mathbb{F}(a, b), S)$ is 0-automatic. The Cayley graph $\mathfrak{X}(\mathbb{F}(a, b), S)$ is an homogeneous tree of degree 4; it is represented on the left of Figure 1.

Now consider the augmented set of generators $\Sigma = \{a, a^{-1}, b, b^{-1}, ab, b^{-1}a^{-1}\}$. Set $c = b^{-1}a^{-1}$. Another possible group presentation for $\mathbb{F}(a, b)$ is $\langle\ a, b, c \mid abc = 1\ \rangle$. The Cayley graph with respect to Σ is represented on the middle of Figure 1. We can check directly that $(\mathbb{F}(a, b), \Sigma)$ is a 0-automatic pair (using for instance the characterization (*iii*) in Theorem 3.8).

Let $\mathfrak{X}(\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, U)$ be the Cayley graph of $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ with respect to the natural generators of the cyclic groups. Clearly, $\mathfrak{X}(G, \Sigma)$ and $\mathfrak{X}(\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, U)$ are isomorphic as unlabelled graphs *but not* as labelled graphs. In particular, the pair $(\mathbb{F}(a, b), \Sigma)$ is not a pair formed by a plain group together with natural generators.

At last, consider the generators $T = S \cup \{uv \mid u, v \in S, v \neq u^{-1}\}$. The corresponding Cayley graph is represented on the right of Figure 1. The pair $(\mathbb{F}(a, b), T)$ is *not* 0-automatic. For instance the group element $a * b * b$ has two representatives in $L(\mathbb{F}(a, b), T)$: $(ab) \cdot b$ and $a \cdot (bb)$.

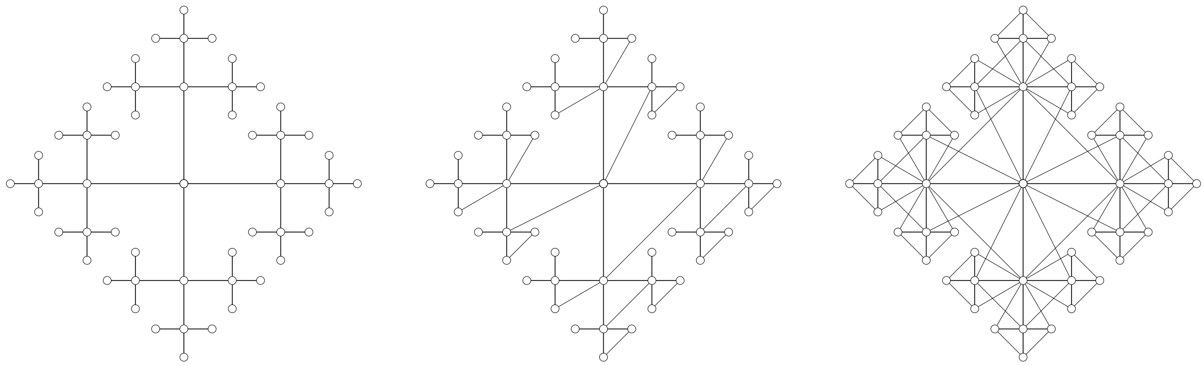

Figure 1: The Cayley graph of $\mathbb{F}(a,b)$ with respect to $S = \{a, a^{-1}, b, b^{-1}\}$ (left), $\{a, a^{-1}, b, b^{-1}, ab, b^{-1}a^{-1}\}$ (middle) and $S \cup \{uv,\ u,v \in \Sigma, v \neq u^{-1}\}$ (right).

3.3 Characterizations of 0-automatic pairs

In this section, we survey various alternative definitions of zero-automatic pairs. We collect five of them in Theorem 3.8, including a new one.

We need to introduce some definitions and notations. First, we review some basics on string-rewriting systems, see for instance [3] for details. A *semi-Thue system* over the set Σ is a subset R of $\Sigma^* \times \Sigma^*$. The elements of R are called the *rules*. For $u, v \in \Sigma^*$, write $u \to v$ if $u = u_1u_2u_3, v = u_1v_2u_3$, and $(u_2, v_2) \in R$. We say that $u \to v$ is a *rewriting*. A word u is *irreducible* if there exists no v such that $u \to v$. View $\to$ as a relation of Σ^* and write $\xrightarrow{*}$ for the reflexive and transitive closure of $\to$. A semi-Thue system is *complete* if it is: (i) *nœtherian*: there exists no infinite sequence of rewritings: $u_1 \to u_2 \to \cdots$; (ii) *confluent*: $\left[u \xrightarrow{*} v, u \xrightarrow{*} w\right] \implies \left[\exists x,\ v \xrightarrow{*} x, w \xrightarrow{*} x\right]$. A semi-Thue system is *finite* if Σ and R are finite. To a finite semi-Thue system R over Σ, one associates naturally the finitely-presented monoid $\langle\, \Sigma \mid u = v, (u,v) \in R\,\rangle$. It is the quotient of Σ^* by the reflexive, symmetric, and transitive closure of $\to$.

For a nœtherian semi-Thue system R, Knuth-Bendix Lemma asserts that R is confluent, hence complete, iff, for each ordered pair $((r,s),(t,u))$ of elements of R, we have:

(i) For all $x, y \in \Sigma^*$ with $t = xry$, there exists $w \in \Sigma^*$ such that $u \xrightarrow{*} w$ and $xsy \xrightarrow{*} w$;

(ii) For all $x, y, z \in \Sigma^*, y \neq 1$, with $r = xy$ and $t = yz$, there exists $w \in \Sigma^*$ such that $sz \xrightarrow{*} w$ and $xu \xrightarrow{*} w$.

The pair (u, xsy) in (i), resp. (sz, xu) in (ii), is classically called a *critical pair*. Using Knuth-Bendix Lemma, it is easily shown that it is algorithmically decidable if a nœtherian finite semi-Thue system is complete.

A *simple circuit* of the graph $\mathfrak{X}(G,\Sigma)$ is a path $u_1 \to u_2 \to \cdots \to u_n$, with $u_n = u_1$ and $u_i \neq u_j$ for all $i,j \in \{1,\ldots,n-1\}, i \neq j$. A *tree* is an acyclic undirected graph. Let $(V,E), E \subset V \times V$, be an undirected graph. A *spanning tree* of (V,E) is an undirected graph $(V,F), F \subset E$, which is a tree. A *spanning tree* T of $\mathfrak{X}(G,\Sigma)$ is a spanning tree of the natural undirected graph associated with $\mathfrak{X}(G,\Sigma)$. We say that $\mathfrak{X}(G,\Sigma)$ is *2-bounded* by T if for all arc $u \to v$ in $\mathfrak{X}(G,\Sigma)$, either $[u \text{ —} v \text{ in } T]$ or $[\exists w, u \text{ —} w \text{ —} v \text{ in } T]$. This last notion is from [8].

There are several equivalent definitions of word automatic groups, see [2, 9] for details. Let us recall the 'geometric' definition via the fellow traveller property. Let G be a group with finite set of generators Σ. Define for $g_1, g_2 \in G$,

$$d_G(g_1, g_2) = \min\{k \mid g_2 = g_1 * u_1 * \cdots * u_k, u_i \in \Sigma\} . \tag{8}$$

Clearly, $d_G(g_1, g_2) = |g_1^{-1} * g_2|_\Sigma$, and d_G is a distance (the *word metric*). Observe that $d_G(g_1, g_2)$ is the geodesic distance (in number of arcs) from g_1 to g_2 in $\mathcal{X}(G, \Sigma)$. For $u \in \Sigma^*$ and $n \in \mathbb{N}$, let $u[n]$ be the prefix of length n of u, with $u[n] = u$ if $n \geq |u|_\Sigma$. A relation $R \subset \Sigma^* \times \Sigma^*$ has the *k-fellow traveller property* if:

$$\forall (u,v) \in R,\ \forall n \leq (|u|_\Sigma \vee |v|_\Sigma) - 1, \qquad d_G(\pi(u[n]), \pi(v[n])) \leq k .$$

(Here we have slightly twisted the usual definition of the k-fellow traveller property for convenience reasons.) Let $L \subset \Sigma^*$ be such that $\pi(L) = G$ and such that $L \in \mathrm{Rat}(\Sigma^*)$. Define the relations

$$R_1 = \big\{(u,v) \mid u,v \in L, \pi(u) = \pi(v)\big\}, \quad \forall a \in \Sigma, \quad R_a = \big\{(u,v) \mid u,v \in L, \pi(ua) = \pi(v)\big\} .$$

The pair (Σ, L) is an *automatic structure* for G if there exists $k \in \mathbb{N}$ such that the relations $R_1, R_a, a \in \Sigma$, have the k-fellow traveller property. A group is *automatic* if it has an automatic structure. This is not obvious on the above definition, but automaticity depends only on the group and not on the finite set of generators.

Given a group G with finite set of generators Σ, define the relation $R(G, \Sigma) \subset \Sigma^* \times \Sigma^*$ by:

$$R(G, \Sigma) = \{(ab, c) \mid a, b \in \Sigma, c \in \Sigma \cup 1,\ \pi(ab) = \pi(c)\} . \tag{9}$$

Define also (FN stands for *Fundamental Neighborhood*, a term coined by Haring-Smith [10]):

$$\mathrm{FN}(G, \Sigma) = \{u \in G \backslash \{1_G\} \mid \exists \text{ simple circuit in } \mathcal{X}(G, \Sigma) \text{ containing } 1_G \text{ and } u\} . \tag{10}$$

Recall that the language $L(G, \Sigma)$ of locally reduced words was defined in (3).

Theorem 3.8. *Consider a group G with finite set of generators Σ. The following properties of the pair (G, Σ) are equivalent and characterize 0-automaticity:*

(i) $L(G, \Sigma)$ is a cross-section of G;

(ii) $\langle \Sigma \mid ab = c, (ab, c) \in R(G, \Sigma) \rangle$ is a monoid presentation of G and the semi-Thue system $R(G, \Sigma)$ is complete;

(iii) $FN(G, \Sigma) = \Sigma$;

(iv) There exists a spanning tree T of $\mathcal{X}(G, \Sigma)$ such that the set of edges of T containing 1_G is $\{1_G — u, u \in \Sigma\}$; and such that $\mathcal{X}(G, \Sigma)$ is 2-bounded by T;

(v) $(\Sigma, L(G, \Sigma))$ is an automatic structure satisfying the 0-fellow traveller property.

Theorem 3.8 collects and completes results from several references. Property (i) is from [15], (ii) from [1], (iii) from [10], (iv) from [8], and (v) seems to be new.

Point (iii) provides a simple combinatorial characterization of 0-automatic pairs. Point (ii) can be used for algorithmic purposes, see Proposition 4.4. Point (v) is our justification for the name '0-automatic' pairs.

Proof. Let us start by showing that $(i) \iff (ii)$. Let Irr be the set of irreducible words of the semi-Thue system $R(G, \Sigma)$. Clearly, $\mathrm{Irr} = L(G, \Sigma)$. Assume first that (ii) holds. It is known (and easy to prove) that in a finite complete semi-Thue system $R(G, \Sigma)$, the set of irreducible words is a cross-section of $\langle \Sigma \mid u = v, (u,v) \in R(G, \Sigma) \rangle$. Hence we obtain that $L(G, \Sigma)$ is a cross-section of G.

Now assume that (i) holds. First, since the rules of $R(G,\Sigma)$ are length-reducing, the system is obviously nœtherian. Second, assume it is not confluent. Then there exist $u, v, w, v \neq w$, such that $u \xrightarrow{*} v, u \xrightarrow{*} w$ and $v, w \in \mathrm{Irr} = L(G,\Sigma)$. But clearly v and w are representatives of the same element of G, which contradicts the fact that $L(G,\Sigma)$ is a cross-section. Therefore $R(G,\Sigma)$ is complete. Now let us prove that $\langle \Sigma \mid ab = c, (ab,c) \in R \rangle$ is a monoid presentation of G. Let u, v be two representatives of the same element of G. Let $\mathrm{Irr}(u)$ and $\mathrm{Irr}(v)$ be the unique (by confluence) irreducible words such that $u \xrightarrow{*} \mathrm{Irr}(u)$ and $v \xrightarrow{*} \mathrm{Irr}(v)$. Since $\mathrm{Irr} = L(G,\Sigma)$ is a cross-section, we must have $\mathrm{Irr}(u) = \mathrm{Irr}(v)$. Therefore $u \xrightarrow{*} \mathrm{Irr}(u) \xleftarrow{*} v$, which means precisely that we can go from u to v using only the relations $ab = c, (ab,c) \in R(G,\Sigma)$. It completes the proof.

In [1], a slightly different result is proved: G is a plain group iff there exists a finite and complete semi-Thue system $T \subset \Sigma^* \times \Sigma^*$ with $\forall (u,v) \in T,\ |u|_\Sigma = 2, |v|_\Sigma = 0$ or 1, and such that $\langle \Sigma \mid u = v, (u,v) \in T \rangle$ is a monoid presentation of G. The proof uses property (iii) instead of (i).

Now let us prove that $(i) \implies (iv)$. For $u \in G$, set $\mathrm{Next}(u) = \{a \in \Sigma \mid \phi(u * a) = \phi(u)a$. Let T be the tree with set of nodes G and set of edges $\{u \text{---} u * a, a \in \mathrm{Next}(u)\}$. Clearly, T is a spanning tree of $\mathcal{X}(G,\Sigma)$. Now using (5), we obtain that $\mathcal{X}(G,\Sigma)$ is 2-bounded by T.

Let us prove that $(iv) \implies (iii)$. In [8], it is proved that (iv) implies that $\mathrm{FN}(G,\Sigma)$ is finite. Observe that $\Sigma \subset \mathrm{FN}(G,\Sigma)$, since $1_G \to a \to 1_G$ is a simple circuit of $\mathcal{X}(G,\Sigma)$ for all $a \in \Sigma$. Now let us prove that $\Sigma = \mathrm{FN}(G,\Sigma)$. Since the set of edges of T containing 1_G is $\{1_G \text{---} u, u \in \Sigma\}$, any simple circuit in $\mathcal{X}(G,\Sigma)$ going through 1_G must contain two successive edges from T. Hence we always are in case (i) of the three cases described in [8, p. 196], with 1_G playing the role of v_1. It is proved in [8] that in this case, all the edges $1_G \text{---} v$, for v belonging to the simple circuit, must belong to T.

The implication $(iii) \implies (i)$ follows immediately from Theorem 3 in [10] (which is exactly Property (b) in Proposition 3.9 below).

Let us prove that $(i) \implies (v)$. Recall that $L(G,\Sigma)$ belongs to $\mathrm{Rat}(\Sigma^*)$. Define

$$R_1 = \{(u,v) \mid u, v \in L(G,\Sigma), \pi(u) = \pi(v)\}, \quad R_a = \{(u,v) \mid u, v \in L(G,\Sigma), \pi(ua) = \pi(v)\},$$

for all $a \in \Sigma$. Let us prove that the relations $R_1, R_a, a \in \Sigma$, satisfy the 0-fellow traveller property. Since $L(G,\Sigma)$ is a cross-section, we have $R_1 = \{(u,u), u \in L(G,\Sigma)\}$. Therefore R_1 satisfies the 0-fellow traveller property. Now consider $a \in \Sigma$ and $(u,v) \in R_a$. If $u = u_1 \cdots u_k$ then, according to Lemma 3.2, we have either

$$v = u_1 \cdots u_{k-1}, \quad v = u_1 \cdots u_{k-1}b, b \neq a, \text{ or } \quad v = u_1 \cdots u_k a .$$

In all three cases, we have: $\forall n \leq (u|_\Sigma \vee |v|_\Sigma) - 1,\ d_G(\pi(u[n]), \pi(v[n])) = 0$. This proves that R_a satisfies the 0-fellow traveller property. We conclude that (v) holds.

At last, let us show that $(v) \implies (i)$, which will complete the proof. Consider $(u_1, u_2) \in R_1$. Assume that $|u_1|_\Sigma < |u_2|_\Sigma$, the case $|u_1|_\Sigma = |u_2|_\Sigma$ is treated similarly. For $|u_1|_\Sigma \leq n < |u_2|_\Sigma$, we have $d_G(\pi(u_1), \pi(u_2[n])) = 0$. Therefore, $u_2 = u_1 a_1 \cdots a_k, a_i \in \Sigma$, and for all $i \leq k$, $\pi(u_1 a_1 \cdots a_i) = \pi(u_1)$. So we get that $\forall i, \pi(a_i) = 1_G$ which is a contradiction. We conclude that $u_1 = u_2$. This shows precisely that $L(G,\Sigma)$ is a cross-section. □

Concerning the points (iii) and (iv) in Theorem 3.8, slightly stronger statements are actually proved in [8] (point a) and in [10] (point b):

Proposition 3.9. *Let G be a group with finite set of generators Σ.*

(a) *If there exists a spanning tree T of $\mathcal{X}(G,\Sigma)$ such that $\mathcal{X}(G,\Sigma)$ is 2-bounded by T, then $FN(G,\Sigma)$ is finite;*

(b) *If $FN(G,\Sigma) = S$ is finite, then $L(G,S)$ is a cross-section of G.*

Let us illustrate Theorem 3.8.

Example 3.10. Define $\mathrm{prod}(a,b;n) = abab\ldots$, with n terms in the product on the right-hand side. Consider the group $G = \langle\, a,b \mid \mathrm{prod}(a,b;2n) = 1 \,\rangle$ for $n \geq 2$ (this is the Artin group of dihedral type $A_{2n} = \langle\, a,b \mid \mathrm{prod}(a,b;2n) = \mathrm{prod}(b,a;2n) \,\rangle$ quotiented by its center). Set $\Sigma = \{\mathrm{prod}(a,b;k), \mathrm{prod}(b,a;k), k = 1,\ldots,2n-1\}$. The Cayley graph $\mathcal{X}(G,\Sigma)$ is shown on Figure 2 for the case $n = 3$. (In this figure, the elements are written using only positive generators, which is helpful in understanding the structure of the Cayley graph.) We check on this Figure that the characterization (iii) in Theorem 3.8 is satisfied. Hence, (G,Σ) is 0-automatic.

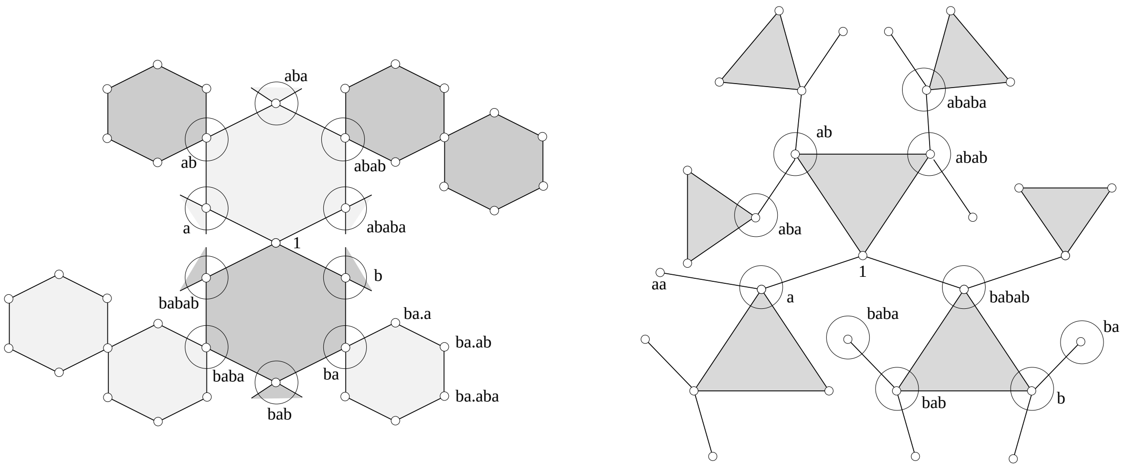

Figure 2: The Cayley graph of $\langle\, a,b \mid \mathrm{prod}(a,b;6) = 1 \,\rangle$ with respect to $\{a, a^{-1}, b, b^{-1}\}$ (left), and $\{a, a^{-1}, ab, abab\}$ (right).

According to Theorem 3.6, G is isomorphic to a plain group. Indeed, we have $G \sim \mathbb{F}(a) \star \{1, \mathrm{prod}(a,b;2k), k = 1,\ldots,n-1\} \sim \mathbb{Z} \star \mathbb{Z}/n\mathbb{Z}$. Set $S = \{a, a^{-1}, \mathrm{prod}(a,b;2k), k = 1,\ldots,n-1\}$ for the corresponding set of natural generators. The elements of Σ are scattered around in $\mathcal{X}(G,S)$: $n+1$ are at distance 1 from 1_G, $2n-2$ at distance 2, and $n-1$ at distance 3, see Figure 2.

3.4 Algorithmic description of 0-automatic pairs

Let (G,Σ) be 0-automatic. Let $(\overline{\Sigma}_i)_{i\in I}$ be the partition of Σ into sets of the type $\overline{\Sigma}_a, a \in \Sigma$, as defined in (6).

Consider the following algorithm which takes in input $(\overline{\Sigma}_i)_{i\in I}$.

ISOMORPHIC-PLAIN-GROUP

Input: $(\overline{\Sigma}_i)_{i\in I}$.

1. Set $H \leftarrow \{1\}$, the group with one element;
2. **While** $I \neq \emptyset$ **do**
3. Pick $i_0 \in I$. Set $K \leftarrow \{a \mid a, a^{-1} \in \overline{\Sigma}_{i_0}\} \cup 1_G$;
4. Set $I_0 \leftarrow \{i \in I \mid \overline{\Sigma}_{i_0} \cap (\overline{\Sigma}_i)^{-1} \neq \emptyset, i \neq i_0\}$ and $L \leftarrow \emptyset$;
5. **For** $i \in I_0$ **do**
6. Pick $a \in \overline{\Sigma}_{i_0} \cap (\overline{\Sigma}_i)^{-1}$, set $L \leftarrow L \cup \{a\}$;
7. **Endfor**
8. Set $H \leftarrow H \star \mathbb{F}(L) \star K$;
9. Set $I \leftarrow I \backslash (I_0 \cup \{i_0\})$;
10. **Endwhile**

Output: H.

In Line 3, $(K, *)$ is a finite group. Hence the output of the algorithm is a plain group. The algorithm ISOMORPHIC-PLAIN-GROUP is a transcription within our notations of a procedure described by Stallings [15], see also [10]. Using [15, 10], we get:

Proposition 3.11. *Let (G, Σ) be a 0-automatic pair and let $(\overline{\Sigma}_i)_{i\in I}$ be the corresponding partition of Σ. Consider the algorithm* ISOMORPHIC-PLAIN-GROUP *with input $(\overline{\Sigma}_i)_{i\in I}$. The output H is a plain group isomorphic to G.*

In each passage in the **while** loop, a pair $(\mathbb{F}(L), K)$ formed by a free group and a finite group is contructed (where either the free or the finite group may be trivial, but not both). Let $\{(\mathbb{F}(L_j), K_j), j \in J\}$ be the set of such pairs constructed in the complete run of the algorithm. In particular, $G \sim \mathbb{F}(\sqcup_{j\in J} L_j) \star (\star_{j\in J} K_j)$. Define

$$\mathcal{T}_j = \{\ (y, u, z^{-1}) \mid y, z \in L_j \cup 1_G, u \in K_j\}, \quad \mathcal{T} = \cup_{j\in J} \mathcal{T}_j\ . \tag{11}$$

According to Stallings [15], we have:

$$\forall (y, u, z^{-1}) \in \mathcal{T},\ \ y * u * z^{-1} \in \Sigma \cup 1_G, \qquad \forall x \in \Sigma,\ \exists!\ (y, u, z^{-1}) \in \mathcal{T},\ \ x = y * u * z^{-1}\ . \tag{12}$$

In particular: $\forall x \in \Sigma,\ |x|_S = 1, 2,$ or 3, where $S = (\sqcup_{j\in J} L_j \sqcup L_j^{-1}) \sqcup_{j\in J} K_j \backslash \{1_{K_j}\}$ is a set of natural generators of $\mathbb{F}(\sqcup_{j\in J} L_j) \star (\star_{j\in J} K_j)$.

The Algorithm ISOMORPHIC-PLAIN-GROUP also enables to give a complete answer to the following natural question:

Let G be a plain group, what are the sets Σ such that (G, Σ) is 0-automatic?

Let $G = \mathbb{F}(X) \star (\star_{i\in I} G_i)$ be a plain group. Define the algorithm:

REVERSE-ISOMORPHIC-PLAIN-GROUP

Input: $X,\ (G_i)_{i\in I}$.

Set $\Sigma \leftarrow \emptyset$;
While $I \neq \emptyset$ **do**
Pick $i_0 \in I, X_0 \subset X$ (X_0 possibly empty). Set $I \leftarrow I \backslash \{i_0\},\ X \leftarrow X \backslash X_0$;
Set $\Sigma \leftarrow \Sigma \sqcup \{x * u * y^{-1} \mid x, y \in X_0 \cup \{1_G\}, u \in G_{i_0}, x * u * y^{-1} \neq 1_G\}$;
Endwhile
While $X \neq \emptyset$ **do**
Pick $X_0 \subset X, X_0 \neq \emptyset$. Set $X \leftarrow X \backslash X_0$;
Set $\Sigma \leftarrow \Sigma \sqcup \{x * y^{-1} \mid x, y \in X_0 \cup \{1_G\}, x \neq y\}$;
Endwhile

Output: Σ.

There are several possible outputs to the algorithm depending on the choices made in lines 3 and 7.

Theorem 3.12. *Consider a plain group $G = \mathbb{F}(X) \star (\star_{i\in I} G_i)$. Let Σ be a possible output of* REVERSE-ISOMORPHIC-PLAIN-GROUP *with input $[X, (G_i)_{i\in I}]$. Then (G, Σ) is 0-automatic. Conversely, if (G, Σ) is 0-automatic, then Σ is a possible output of* REVERSE-ISOMORPHIC-PLAIN-GROUP *for some input $[S, (\widetilde{G}_i)_{i\in I}]$, where $\mathbb{F}(S) \star (\star_{i\in I} \widetilde{G}_i) \simeq \mathbb{F}(X) \star (\star_{i\in I} G_i)$.*

Proof. First of all, let us prove that (G, Σ) is 0-automatic if Σ is a possible output of the algorithm REVERSE-ISOMORPHIC-PLAIN-GROUP with input $[X, (G_i)_{i\in I}]$. One possible way to proceed is to use the characterization (ii) in Theorem 3.8. Define $R(G, \Sigma)$ as in (9). Clearly, we have

$$\begin{aligned} &\forall x \in X, \qquad (xx^{-1}, 1) \in R(G, \Sigma),\ (x^{-1}x, 1) \in R(G, \Sigma) \\ &\forall i, \forall u, v \in G_i \backslash \{1_{G_i}\}, w \in G_i, w = u * v, \qquad (uv, w) \in R(G, \Sigma)\ . \end{aligned}$$

Since the above relations present $\mathbb{F}(X) \star (\star_{i\in I} G_i)$, it follows that $\langle\, \Sigma \mid u = v, (u, v) \in R(G, \Sigma)\,\rangle$ is a monoid presentation of G. Now we need to prove that the semi-Thue system $R(G, \Sigma)$ is complete. First $R(G, \Sigma)$ is nœtherian since it is length-reducing. According to Knuth-Bendix Lemma, the system is confluent iff all the critical pairs are confluent. Now proving that all the critical pairs in $R(G, \Sigma)$ are confluent is not difficult, although slightly tedious. (And the complete proof is left to the reader.)

Now, let Σ be such that (G, Σ) is 0-automatic. Apply the algorithm ISOMORPHIC-PLAIN-GROUP to $(\overline{\Sigma}_i)_{i\in I}$. In each passage in the **while** loop, a pair $(\mathbb{F}(L), K)$ formed by a free group and a finite group is contructed. Let $\{(\mathbb{F}(L_j), K_j), j \in J\}$ be the set of such pairs constructed in the complete run of the algorithm. Set $S = \sqcup_{j\in J} L_j$. Then we have $\mathbb{F}(S) \star (\star_{j\in J} K_j) \simeq \mathbb{F}(X) \star (\star_{i\in I} G_i)$ according to Proposition 3.11. Input $[S, (K_j)_{j\in J}]$ into REVERSE-ISOMORPHIC-PLAIN-GROUP. Now choose the successive pairs (i_0, X_0) in Line 3, and then the succesive sets X_0 in Line 7, in order to recover the decomposition $\{(\mathbb{F}(L_j), K_j), j \in J\}$. According to (12), the corresponding output of the algorithm is then precisely Σ. This completes the proof. ☐

Consider the following slightly different problem:

Given a group G with a finite presentation over the generators Σ, determine if the pair (G, Σ) is 0-automatic.

We treat this problem in the more general context of monoids in Proposition 4.4.

The *graph of successors* $\mathcal{N}\mathrm{ext}(G, \Sigma)$ of a 0-automatic pair (G, Σ) is the directed graph with:

$$\text{nodes: } \Sigma, \quad \text{arcs: } u \to v \text{ if } v \in \mathrm{Next}(u)\ . \tag{13}$$

Proposition 3.11 and Theorem 3.12 enable to get a simple proof of next Lemma.

Lemma 3.13. *Let (G, Σ) be a 0-automatic pair with G infinite and not isomorphic to $\mathbb{Z}$. The graph of successors $\mathcal{N}ext(G, \Sigma)$ is strongly connected.*

Proof. Run the algorithm ISOMORPHIC-PLAIN-GROUP with input $(\overline{\Sigma}_i)_i$. Let $\{(\mathbb{F}(L_j), K_j), j \in J\}$ be the set of pairs free group - finite group constructed by the algorithm. Define the sets $\mathcal{T}_j, j \in J$, as in (11). Consider a, b in Σ. According to (12), $a = x_1 * u_1 * y_1^{-1}$, $(x_1, u_1, y_1^{-1}) \in \mathcal{T}_j$ and $b = x_2 * u_2 * y_2^{-1}$, $(x_2, u_2, y_2^{-1}) \in \mathcal{T}_k$. If $j \neq k$, then $b \in \mathrm{Next}(a)$ and $a \in \mathrm{Next}(b)$. It implies immediately that $\mathcal{N}\mathrm{ext}(G, \Sigma)$ is strongly connected as soon as $|J| \geq 2$.

Assume that $|J| = 1$. Let $(\mathbb{F}(L), K)$ be the pair constructed by the algorithm. Since G is infinite, $\mathbb{F}(L)$ is not trivial, i.e. $|L| > 0$. Assume first that K is not trivial. The set Σ can be partitioned into 5 subsets:

$$\Sigma_1 = K\backslash\{1_G\}, \quad \Sigma_2 = \{x * y^{-1},\ x, y \in L \cup 1_G\}\backslash\{1_G\}, \quad \Sigma_3 = \{x * u,\ x \in L, u \in K\backslash\{1_G\}\}$$
$$\Sigma_4 = \{u * y^{-1},\ y \in L, u \in K\backslash\{1_G\}\}, \quad \Sigma_5 = \{x * u * y^{-1},\ x, y \in L, u \in K\backslash\{1_G\}\}\,.$$

In the graph $\mathcal{N}\mathrm{ext}(G,\Sigma)$, we have, among others, all the following arcs ($\Sigma_i \to \Sigma_j$ stands for $\forall a \in \Sigma_i, \forall b \in \Sigma_j, a \to b$):

$$\Sigma_1 \cup \Sigma_3 \longrightarrow \Sigma_2 \cup \Sigma_3 \cup \Sigma_5, \quad \Sigma_2 \cup \Sigma_4 \cup \Sigma_5 \longrightarrow \Sigma_1 \cup \Sigma_4\,.$$

This shows that $\mathcal{N}\mathrm{ext}(G,\Sigma)$ is strongly connected.

Assume now that K is trivial. Since G is not isomorphic to $\mathbb{Z}$, we must have $|L| \geq 2$. By direct inspection, one shows easily that $\mathcal{N}\mathrm{ext}(G,\Sigma)$ is strongly connected. $\square$

4 Zero-automaticity for monoids

In this section, we initiate a study of 0-automatic monoids. Defining automaticity for monoids is not immediate. Indeed, the equivalent definitions of automatic groups lead to different notions when extended to monoids, see for instance [4, 14]. The same kind of difficulty arises for 0-automaticity.

Let $(M, *)$ be a finitely generated monoid with unit element 1_M. Let $\Sigma \subset M$ be a finite set of generators. We always assume that $1_M \notin \Sigma$. Let $\pi : \Sigma^* \to M$ be the corresponding monoid homomorphism. Mutatis mutandis, the following notions and notations are defined as in §2: a representative word $u \in \pi^{-1}(m)$, a cross-section of M, a normal form word, the normal form map, the length $|\cdot|_\Sigma$, a geodesic, the Cayley graph $\mathcal{X}(M,\Sigma)$.

Define the language $L(M,\Sigma) \subset \Sigma^*$ of *locally reduced words* by:

$$L(M,\Sigma) = \{u_1 \cdots u_k \mid \forall i \in \{1,\dots,k-1\},\ u_i * u_{i+1} \notin \Sigma \cup 1_M\}\,. \tag{14}$$

Define $R(M,\Sigma) \subset \Sigma^* \times \Sigma^*$ by

$$R(M,\Sigma) = \{(ab, c) \mid a, b \in \Sigma, c \in \Sigma \cup 1,\ \pi(ab) = \pi(c)\}\,. \tag{15}$$

The k-fellow traveller property can be extended in several reasonable ways. On the other hand, for the 0-fellow traveller property, the following definition is the natural one.

Let $L \subset \Sigma^*$ be such that $\pi(L) = M$ and such that $L \in \mathrm{Rat}(\Sigma^*)$. Define the relations $R_1 = \{(u,v) \mid u, v \in L, \pi(u) = \pi(v)\}$ and $\forall a \in \Sigma,\ R_a = \{(u,v) \mid u, v \in L, \pi(ua) = \pi(v)\}$. The pair (Σ, L) is an *automatic structure for M with the 0-fellow traveller property* if

$$\forall (u,v) \in R_1 \cup (\cup_{a\in\Sigma} R_a),\ \forall n \leq (|u|_\Sigma \vee |v|_\Sigma) - 1, \qquad \pi(u[n]) = \pi(v[n])\,.$$

It is natural to set the following analog of Definition 3.1.

Definition 4.1. *The pair (M,Σ) is 0-automatic if $L(M,\Sigma)$ is a cross-section of M.*

Here are examples of 0-automatic pairs. If (G,Σ) is 0-automatic in the sense of Definition 3.1 (so G is a group), then it is 0-automatic in the sense of Definition 4.1. Let M be a finite monoid, then $(M, M\backslash\{1_M\})$ is 0-automatic. Let Σ be a finite set and Σ^* the corresponding free monoid.

Then (Σ^*, Σ) is 0-automatic. Define the free product of monoids as in §2. Let (M_1, Σ_1) and (M_2, Σ_2) be 0-automatic, then $(M_1 \star M_2, \Sigma_1 \sqcup \Sigma_2)$ is 0-automatic.

Define a *plain monoid* to be the free product of a finitely generated free group, a finitely generated free monoid, and a finite family of finite monoids. Given a plain monoid $\mathbb{F}(S) \star \Sigma^* \star M_1 \star \cdots \star M_k$, let us call $S \sqcup S^{-1} \sqcup \Sigma \sqcup M_1 \backslash 1_{M_1} \sqcup \cdots \sqcup M_k \backslash 1_{M_k}$ a set of *natural generators*. The pair formed by a plain monoid and natural generators is 0-automatic.

Consider a plain monoid M with natural generators S. As in the group case, it is possible to choose a finite set of generators Σ, strictly larger than S, and such that (M, Σ) is 0-automatic. For instance, consider the monoid M with monoid presentation:

$$M = \langle\, a, b, c \mid ab = c \,\rangle\ .$$

The pair $(M, \{a, b, c\})$ is 0-automatic, and the monoid M is isomorphic to $\{a, b\}^*$. This example is a monoid analog of Example 3.7.

Consider the question:

Given a plain monoid M, what are the finite sets Σ such that (M, Σ) is 0-automatic?

The question is algorithmically decidable as a special case of Proposition 4.4 below. It might be feasible to do better and to characterize all the possible sets Σ along the lines of §3.4. But the above question is not as central as for groups since the family of plain monoids does not exhaust the monoids appearing in 0-automatic pairs, as we now show.

Consider the monoids given by the following monoid presentations:

$$M_1 = \langle\, a, b \mid ab = 1 \,\rangle, \quad M_2 = \langle\, a, b, c \mid ab = 1 \,\rangle\ . \tag{16}$$

The monoid M_1 is known as the *bicyclic monoid*. Set $\Sigma_1 = \{a, b\}$ and $\Sigma_2 = \{a, b, c\}$. It is easy to see that (M_i, Σ_i) is 0-automatic (use Proposition 4.4). However, neither M_1 nor M_2 is isomorphic to a plain monoid. We have represented the Cayley graphs of the two monoids in Figure 3.

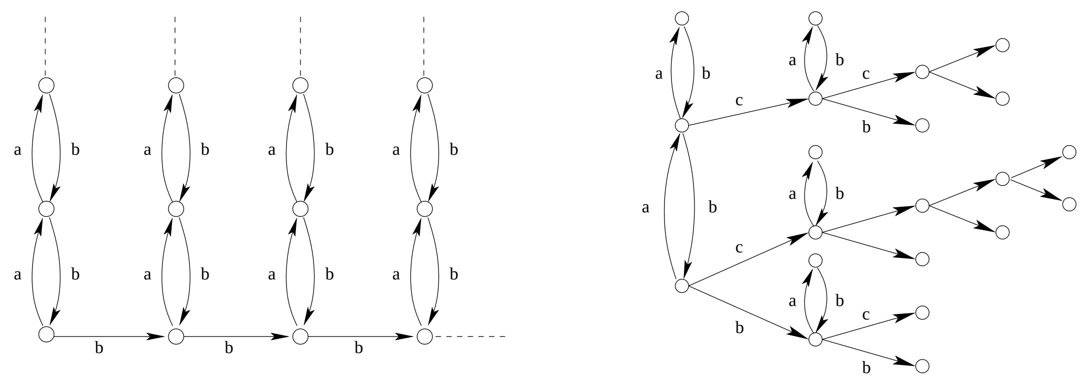

Figure 3: The Cayley graphs of $\langle\, a, b \mid ab = 1 \,\rangle$ (left) and $\langle\, a, b, c \mid ab = 1 \,\rangle$ (right).

Let us investigate some properties of 0-automatic pairs. With the same proof as for Lemma 3.4, we get:

Lemma 4.2. *Let M be a monoid with finite set of generators Σ. If $L(M, \Sigma)$ is a cross-section then (M, Σ) has the unique geodesic property and $L(M, \Sigma)$ is precisely the set of geodesics.*

Let us propose a partial analog of Theorem 3.8.

Proposition 4.3. *Let M be a monoid and let $\Sigma \subset M$ be a finite set of generators. The following properties of the pair (M, Σ) are equivalent:*

(i) $L(M, \Sigma)$ is a cross-section of M;

(ii) $\langle\, \Sigma \mid ab = c, (ab, c) \in R(M, \Sigma) \,\rangle$ is a monoid presentation of M and the semi-Thue system $R(M, \Sigma)$ is complete;

(v) $(\Sigma, L(M, \Sigma))$ is an automatic structure for M with the 0-fellow traveller property.

The proof is exactly the same as the one for groups in Theorem 3.8. For monoids, there does not seem to be a good characterization of 0-automaticity via fundamental neighborhoods (Theorem 3.8, point (iii)). Also, the analog of Lemma 3.2 does not hold. That is, it may not be the case that only the last letter of a normal form word is modified by multiplication by a generator. Consider the finitely-presented monoid:

$$M = \langle\, a, b, c, d, e \mid ad = e, bc = d \,\rangle \,. \tag{17}$$

Set $\Sigma = \{a, b, c, d, e\}$. Using (ii) in Proposition 4.3, we obtain that $L(M, \Sigma)$ is a cross-section. Let ϕ be the corresponding normal form map. We have:

$$\phi(a * b) = ab, \quad \phi(a * b * c) = \phi(a * d) = e \,.$$

Hence, (5) is not satisfied.

In [11], to obtain nice results on random walks, it is necessary to impose the analog of Lemma 3.2 on top of the condition that $L(M, \Sigma)$ be a cross-section.

Zero-automaticity is "decidable":

Proposition 4.4. *Let M be a monoid with a finite monoid presentation $\langle \Sigma \mid x = y, (x, y) \in R \rangle$. Assume that the word problem of M is decidable. Then it is decidable if (M, Σ) is 0-automatic.*

First, let us precise a point which may be ambiguous in the statement of Proposition 4.4. The elements of M are the equivalence classes $[u], u \in \Sigma^*$. In particular, the classes $[u], u \in \Sigma$, may not be all distinct. Now, since the word problem is decidable, starting from a monoid presentation $\langle\, \Sigma_1 \mid R_1 \,\rangle$, we can always compute another equivalent monoid presentation of the type $\langle\, \Sigma_2 \mid R_2 \,\rangle$ with $\Sigma_2 \subset \Sigma_1$ and $\forall u, v \in \Sigma_2, u \neq v \implies [u] \neq [v]$. So we assume, without loss of generality, that the presentation of M is such that: $\forall u, v \in \Sigma, u \neq v \implies [u] \neq [v]$. Then, we can identify the two sets Σ and $\{[u], u \in \Sigma\}$. In particular, the statement "(M, Σ) is 0-automatic" is an abuse of language for "$(M, \{[u], u \in \Sigma\})$ is 0-automatic".

Proof. We need to prove that $L(M, \Sigma)$ is a cross-section. The idea is to use Property (ii) in Proposition 4.3. Define the relation $R(M, \Sigma)$ as in (15). (The set $R(M, \Sigma)$ can be obtained in a constructive way since M has a decidable word problem.) According to Proposition 4.3, (M, Σ) is 0-automatic iff: $(a) : \langle\, \Sigma \mid ab = c, (ab, c) \in R(M, \Sigma) \,\rangle$ is a monoid presentation of M, and (b) : the semi-Thue system $R(M, \Sigma)$ is complete.

Let us show that (a) and (b) can be checked algorithmically. Let us start with (b). Since the rules of the semi-Thue system $R(M, \Sigma)$ are length-reducing, the system is obviously nœtherian. Therefore, the only point is to check if it is confluent. This is algorithmically decidable, see §3.3.

Assume that (b) is satisfied. By confluence, each word $u \in \Sigma^*$ has a unique irreducible descendant with respect to $R(M, \Sigma)$ that we denote by $\mathrm{Irr}(u)$. Given u, computing $\mathrm{Irr}(u)$ is trivial since all the rules in $R(M, \Sigma)$ are length reducing. Now, if (a) is satisfied then for all $(u, v) \in R$, we have $\mathrm{Irr}(u) = \mathrm{Irr}(v)$. Conversely, if for all $(u, v) \in R$, $\mathrm{Irr}(u) = \mathrm{Irr}(v)$, then (a) is satisfied. Hence we can check property (a) by computing the irreducible descendants of all the words appearing in R. □

Acknowledgement. This work has benefited from many exchanges with Frédéric Mathéus. I would also like to thank Géraud Sénizergues and Wolfgang Woess for pointing out several relevant references.

While finishing the study of 0-automatic pairs, I received the preprint [6] from the authors. Their work was based on an earlier version of the present manuscript, and they proved independently an equivalent of Theorem 3.12. Besides, their paper contains a complete proof of the following results (given by Stallings in [15] without full justification): Proposition 3.11 and (12).

References

[1] J. Avenhaus, K. Madlener, and F. Otto. Groups presented by finite two-monadic Church-Rosser Thue systems. *Trans. Amer. Math. Soc.*, 297(2):427–443, 1986.

[2] G. Baumslag, S. Gersten, M. Shapiro, and H. Short. Automatic groups and amalgams. *J. Pure Appl. Algebra*, 76(3):229–316, 1991.

[3] R. Book and F. Otto. *String-rewriting systems.* Texts and Monographs in Computer Science. Springer-Verlag, New York, 1993.

[4] C. Campbell, E. Robertson, N. Ruškuc, and R. Thomas. Automatic semigroups. *Theoret. Comput. Sci.*, 250(1-2):365–391, 2001.

[5] J. Cannon. The combinatorial structure of cocompact discrete hyperbolic groups. *Geom. Dedicata*, 16(2):123–148, 1984.

[6] D. Cartwright and B. Krön. Classification of 0-automatic pairs. Preprint, 2004.

[7] C. Choffrut. A short introduction to automatic group theory. In *Semigroups, algorithms, automata and languages (Coimbra, 2001)*, pages 133–154. World Sci. Publishing, River Edge, NJ, 2002.

[8] G. D'Agostino. Cayley graphs of virtually free groups. *Internat. J. Algebra Comput.*, 3(2):189–199, 1993.

[9] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, and W. Thurston. *Word processing in groups.* Jones and Bartlett, Boston, 1992.

[10] R. Haring-Smith. Groups and simple languages. *Trans. Amer. Math. Soc.*, 279(1):337–356, 1983.

[11] J. Mairesse. Random walks on groups and monoids with a Markovian harmonic measure. LIAFA research report 2004-05, Université Paris 7, 2004.

[12] J. Mairesse and F. Mathéus. Random walks on groups with a tree-like Cayley graph. In *Mathematics and computer science. III. Algorithms, trees, combinatorics and probabilities*, Trends in Mathematics, pages 445–460. Birkhauser Verlag, 2004.

[13] M. Shapiro. Pascal's triangles in abelian and hyperbolic groups. *J. Austral. Math. Soc. Ser. A*, 63(2):281–288, 1997.

[14] P. Silva and B. Steinberg. A geometric characterization of automatic monoids. *Q. J. Math.*, 55(3):333–356, 2004.

[15] J. Stallings. A remark about the description of free products of groups. *Proc. Cambridge Philos. Soc.*, 62:129–134, 1966.