Zero-Automatic Networks

Thu-Ha Dao-Thi and Jean Mairesse*

Abstract

We continue the study of zero-automatic queues first introduced in [3]. These queues are characterized by a special buffering mechanism evolving like a random walk on some infinite group or monoid. The simple *M/M/1* queue and Gelenbe's G-queue with positive and negative customers are the two simplest 0-automatic queues. All stable 0-automatic queues have an explicit "multiplicative" stationary distribution and a Poisson departure process [3]. In this paper, we introduce and study networks of 0-automatic queues. We consider two types of networks, with either a Jackson-like or a Kelly-like routing mechanism. In both cases, and under the stability condition, we prove that the stationary distribution of the buffer contents has a "product-form" and can be explicitly determined. Furthermore, the departure process out of the network is Poisson.

Keywords: Queueing theory, Jackson network, Kelly network, product form, zero-automatic.

1 Introduction

The study of zero-automatic queues was initiated in [3]. In the present paper, we continue this study by considering networks made of 0-automatic queues. Such networks provide a good compromise between modelling power and mathematical tractability. Let us discuss this in some details. Some parts of the discussion below are borrowed from [3].

Motivating the model

A zero-automatic queue may be viewed as a synthesis of a queue and a random walk on some infinite group or monoid. Alternatively, it may be viewed as a synthesis and wide generalization of the simple queue and the G-queue with positive and negative customers. We detail these two viewpoints.

Let us focus on the buffering mechanism in a queue, that is the mechanism of evolution of the buffer content when services are prevented from occurring.

Consider first a simple queue with an infinite buffer. The buffer content is represented by a non-negative integer n. Upon arrival, a customer simply increments the buffer size by 1, giving: $n \to n+1$. As artificial as it may look, let us view this mechanism as a trivial random walk on the monoid $(\mathbb{N},+)$ with jumps equal to $+1$.

Queues with positive and negative customers were introduced by Gelenbe [7, 8]. Several variants may be considered, but the pertinent one in our context is as follows. A customer is of type $\{1\}$ or $\{-1\}$. The buffer content is an integer n in $\mathbb{Z}$. This is interpreted as a queue of $|n|$ customers which are of type $\{1\}$ if $n > 0$, and of type $\{-1\}$ if $n < 0$. Upon arrival, a $\{1\}$-customer modifies

*LIAFA, CNRS-Université Paris 7, case 7014, 2, place Jussieu, 75251 Paris Cedex 05, France. E-mail: (daothi,mairesse)@liafa.jussieu.fr

the buffer content according to the rule: $n \to n+1$. The interpretation depends on the sign of n. If $n \geqslant 0$, the $\{1\}$-customer simply takes place in the buffer incrementing its size by 1. If $n < 0$, the $\{1\}$-customer cancels with a $\{-1\}$-customer already in the buffer decrementing its size by 1 ($[n < 0, n \to n+1] \implies [|n| \to |n|-1]$). Dually, a $\{-1\}$-customer modifies the buffer content according to the rule $n \to n-1$. Let ν be a probability measure on $\{1,-1\}$. Assume that the types of arriving customers are random, of law ν, and independent of one another. We may view the buffering mechanism as a random walk on the group $(\mathbb{Z},+)$ with jumps made according to ν.

Now let us merge the two models together. The set of possible types for the customers is $\Sigma = \{a, b, b^{-1}\}$. When restricted to customers of type a, the queue is a simple queue. When restricted to customers of type $\{b, b^{-1}\}$, the queue is a G-queue (with b in the role of $\{1\}$ and b^{-1} in the role of $\{-1\}$). Furthermore, customers of type a on the one hand, and customers of types b and b^{-1} on the other, do not interact, and cannot overtake each other. So the buffer content is described by a sequence of symbols in Σ (a word of Σ^*). More accurately, due to the cancellations between customers of types b and b^{-1}, the set of possible buffer contents is the set

$$L = \left\{u_k \cdots u_1 \in \Sigma^* \mid \forall i,\ u_{i+1}u_i \notin \{bb^{-1}, b^{-1}b\}\right\}. \tag{1}$$

(In (1), u_1 is the front-end and first arrived customer, and u_k is the back-end and last arrived customer.) If $u \in L$ is a buffer content and $v \in \Sigma$ is the type of an arriving customer, then we denote the resulting buffer content by $v * u$. The rules are as follows:

$$\begin{aligned} a * u_k \cdots u_1 &= a u_k \cdots u_1 \\ b * u_k \cdots u_1 &= \begin{cases} b u_k \cdots u_1 & \text{if } u_k \neq b^{-1} \\ u_{k-1} \cdots u_1 & \text{if } u_k = b^{-1} \end{cases}, \end{aligned}$$

with the dual rule for $b^{-1} * u_k \cdots u_1$.

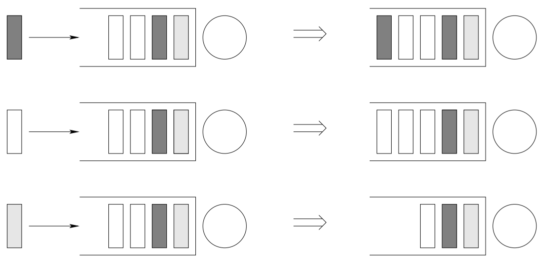

Figure 1: Customers of type a, b, b^{-1} are resp. in dark gray, light gray and white.

Let ν be a probability measure on Σ. Assume that the types of arriving customers are i.i.d. of law ν. Then the buffer content evolves as a Markov chain on L. In more algebraic terms, we view it as the random walk induced by ν on the free product $\{a\}^* \star \mathbb{F}(b)$, where $\{a\}^* \sim \mathbb{N}$ is the free monoid generated by a and $\mathbb{F}(b) \sim \mathbb{Z}$ is the free group generated by b.

In the above description, the originality is to propose an interpretation and synthesis of the simple queue and the G-queue at an algebraic level. The model of 0-automatic queues introduced in [3] exploits this idea.

The starting point consists in defining a whole class of transient random walks on groups or monoids with a particular tree-like structure. This family of groups and monoids includes the free monoids and free groups as its simplest members, and is stable by free product. The associated random walks are well-behaved and have a special and explicit limiting measure which was studied in [12, 13]. Now the random walk is incorporated in the buffer of a queue. At the front-end of the buffer, customers are served at constant rate. At the back-end of the buffer, customers arrive at constant rate and affect the content of the buffer according to the random walk mechanism. In particular, each arriving customer has a type, chosen in an i.i.d. way with a law equal to the law defining the random walk.

In a nutshell, the resulting model has some classical features: it is a Markovian queue with an infinite buffer and a FIFO (First In First Out) service discipline; and some non-classical features: there are several types of customers and the interactions between the types occur according to an underlying associative law. The precise definition of 0-automatic queues will be given in §2.2.

We next argue the interest and pertinence of the model on two grounds. First, 0-automatic queues are flexible and have a good modelling power. Second, they enjoy remarkable mathematical properties, and can be explicitly analyzed.

Modelling flexibility

In a 0-automatic queue, different types of tasks (customers) can be modelled. Let us detail four of them which form a representative sample, without exhausting all the types within the realm of 0-automaticity.

- Classical type. Tasks are processed one by one with no simplification occurring in the buffer: $aa \to aa$.

- Positive/negative type. Tasks are either positive (a) or negative (a^{-1}) and two consecutive tasks of opposite signs cancel each other: $aa^{-1} \to \emptyset, a^{-1}a \to \emptyset$. The relevance of this type for applications is discussed for instance in [8].

- "One equals many" type. It takes the same amount of time to process one or several consecutive instances of the same task: $aa \to a$. Think for instance of a ticket reservation where the number of requests is only reflected by an integer value in a menu-bar choice.

- "Dating agency" type. Two instances of the same task cancel each other: $aa \to \emptyset$. Think of a task as being a tennis player looking for a partner (to be provided by the server); when two such tasks are next to each other in the buffer, they leave to play a game instead of waiting in line. (Instead of tennis players, we may consider music trio players, bridge players, ..., the corresponding simplification rules being $aaa \to \emptyset$, $aaaa \to \emptyset$, ...)

In a 0-automatic queue, several types may coexist as well as several different copies of the same type. The set of types is then of the form $\Sigma = \cup_i \Sigma_i$ with for instance $\Sigma_i = \{a_i\}$ (classical or "one equals many" type) or $\Sigma_i = \{a_i, a_i^{-1}\}$ (positive/negative type) or $\Sigma_i = \{a_i, \dots, a_i^{k-1}\}$ ("dating agency" type with simplification rule $a_i^k \to \emptyset$). In the buffer, two consecutive customers interact (merge or cancel) only if they belong to the same subset Σ_i.

At the algebraic level, each type corresponds to a pair group-generators or monoid-generators. Considering several types in the same queue amounts to considering the free product of the groups or monoids of the different types, together with the union of the generators. The pairs associated with the above types are: $(\mathbb{N}, \{1\}) \sim (\{a\}^*, \{a\})$ for the classical type; $(\mathbb{Z}, \{1, -1\}) \sim (\mathbb{F}(a), \{a, a^{-1}\})$ for the positive/negative type; $(\mathbb{B}, \{a\})$ where $\mathbb{B}$ is the Boolean monoid $\mathbb{B} = \langle a \mid a^2 = a \rangle$ for the "one equals many" type; $(\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = 1 \rangle, \{a\})$ for the tennis "dating agency" type.

Now let us describe informally zero-automatic networks. Consider N zero-automatic queues, numbered from 1 to N, and corresponding to possibly different groups or monoids, hence different sets of types. For a non-empty subset of the queues, there is an external arrival process of customers. After completion of its service, a customer is either routed to another queue or leaves the network. We consider two models corresponding to different routing policies: *Jackson-like* networks and *Kelly-like* networks.

In a Jackson-like network, there is a sub-stochastic matrix $P = (p_{ij})_{ij}$ of size $N \times N$, called the *routing matrix.* A customer completing service at queue i is routed to queue j with probability p_{ij}, and leaves the network with probability $1 - \sum_j p_{ij}$. At each step along its route, a customer chooses its type according to the type-distribution of the queue currently visited.

In a Kelly-like network, there is a finite set of predefined possible routes. A *route* is simply a finite ordered list of queues, to be visited in that order (any given queue may be visited several times along the route). Each customer carries two information: its type and its route. After completion of service, a customer moves to the next queue along its route, or leaves the network if the visited queue was the last one. The type of a customer is reevaluated at each queue as in a Jackson-like network. On the other hand, the route is carried over unchanged. When two customers a and b merge in a buffer to create a new customer c, then the route of c is chosen to be equal to the route of a with probability $1/2$, or to the route of b with probability $1/2$.

Mathematical analysis

In a classical queue or network, the buffer-content is coded by a non-negative integer or a tuple of such integers. In comparison, in a 0-automatic queue or network, the buffer-content is represented by a non-commutative word or a tuple of words over a finite alphabet of types. This is a far more complex situation. Nevertheless, the model remains completely tractable.

We first set up the stage by recalling classical results on simple Markovian queues and networks, see [10, 15] for details.

Consider a simple Markovian queue with arrival rate λ and service rate μ. Under the stability condition $\rho = \lambda/\mu < 1$, the stationary distribution π of the buffer-length is given by:

$$\forall n \in \mathbb{N}, \quad \pi(n) = (1-\rho)\rho^n . \tag{2}$$

Furthermore, the stationary departure process is a Poisson process of rate λ (Burke Theorem).

Consider now a Jackson network of N simple Markovian queues with respective external arrival rate $\lambda_i \in [0, \infty)$, respective service rate $\mu_i \in (0, \infty)$, and routing matrix $P = (p_{ij})_{i,j}$. The *Network Traffic Equations* are the affine equations of the variables $(\tilde{\lambda}_i)_i$ given by

$$\tilde{\lambda}_i = \lambda_i + \sum_{j=1}^{N} \tilde{\lambda}_j p_{ji} . \tag{3}$$

If the Jackson network is connected and without capture, the equations (3) have a unique solution $(\tilde{\lambda}_i)_i$. Under the stability condition $\rho_i = \tilde{\lambda}_i/\mu_i < 1$ for all i, the stationary distribution π of the buffer contents can be explicited. It is given by, $\forall n = (n_i)_i \in \mathbb{N}^N$,

$$\begin{aligned} \pi(n) &= \prod_{i=1}^{N} \pi_i(n_i) && (4) \\ &= \prod_{i=1}^{N} (1-\rho_i)\rho_i^{n_i} . && (5) \end{aligned}$$

The distribution π is said to have a "product form" since it is the product of the stationary distributions $(\pi_i)_i$ of the different queues considered in isolation.

Similarly, Kelly networks with fixed routes have an explicit "product form" stationary distribution π. The capacity of determining explicitly π for such networks is one of the cornerstones of Markovian queueing theory.

For 0-automatic queues and networks, we obtain a description which is more complex, but still remarkably close to the above one.

First let us consider a 0-automatic queue with arrival rate λ and service rate μ. Let Σ be the set of types and ν the probability distribution of types. As in (1), the set of possible buffer contents is of the form $L = \{u_k \cdots u_1 \in \Sigma^* \mid u_i \in \text{Next}(u_{i+1})\}$, where $\text{Next}(a) \subset \Sigma$ is the subset of types which do not interact (merge or cancel) with type a. Let $(X_n)_n$ be the L-valued r.v.'s corresponding to the successive buffer contents if we block the services. Alternatively, $(X_n)_n$ is the transient random walk on the infinite group or monoid associated with the queue. Let $|X_n|$ be the length of X_n (number of customers in the buffer). Set $\widehat{\gamma} = \lim_{n\to\infty} |X_n|/n$. This is the growth rate of the buffer length in the absence of services. The stability condition for the queue is $\lambda\widehat{\gamma} < \mu$. Let us assume that this condition holds. A set of so-called *Twisted Traffic Equations* is associated with the queue, see (20). These equations are polynomial and explicitly defined in function of Σ, ν, λ and μ. They have a unique solution of the form $(\rho, r) \in (0,1) \times \{x \in \mathbb{R}^\Sigma \mid \forall i \in \Sigma, x(i) > 0,\ \sum_i x(i) = 1\}$. Set $q(a) = r(a)/r(\text{Next}(a))$ for all $a \in \Sigma$. The stationary distribution of the buffer content satisfies (cf. [3]), $\forall u = u_k \cdots u_1 \in L$,

$$\pi(u) = (1-\rho)\rho^k q(u_k) \cdots q(u_2) r(u_1) . \tag{6}$$

There are two parts in π: the term $(1-\rho)\rho^k$ which is the same as in (2) and which accounts for the buffer length, and the term $q(u_k) \cdots q(u_2)r(u_1)$ which accounts for the precise buffer content. In view of the shape of (6), we are tempted to say that π has a "product form" with respect to its content.

Now let us move on to the network level, of Jackson or Kelly type. Here we have two interacting layers of Traffic-like Equations: the *Twisted Traffic Equations* of the individual queues, and the *Network Traffic Equations*, which are similar but more complicated than (3), see Definitions 3.2 and 4.4. The stationary distribution has a "double product form": with respect to the buffer content for each individual queue (as in (6)) and with respect to the different queues (as in (4)). The precise statements appear in Theorems 3.6 and 4.5.

The paper is organized as follows: in Section 2, we recall the needed results on 0-automatic queues from [3]; in Sections 3 and 4 we study the networks of Jackson type and Kelly type respectively. A short version without proofs of the present paper appeared in the Proceedings of the Valuetools'06 Conference [4].

2 Preliminaries

Notations. We denote by $\mathbb{N}$ and $\mathbb{R}_+$ the nonnegative integers and reals. We denote by $\mathbb{N}^*$ and $\mathbb{R}_+^*$ the positive integers and reals. The symbol $\sqcup$ is used for the disjoint union of sets. Given a set T and $S \subset T$, define $\mathbb{1}_S : T \to \{0,1\}$ by $\mathbb{1}_S(u) = 1$ if $u \in S$ and $\mathbb{1}_S(u) = 0$ otherwise. Given a set T, a vector $x \in \mathbb{R}^T$, and $S \subset T$, set $x(S) = \sum_{u\in S} x(u)$.

2.1 Random walks on groups and monoids

We recall the needed material on random walks on plain groups and monoids. The presentation closely follows [12, 13].

Monoids and groups. Given a set Σ, the free monoid generated by Σ is denoted by Σ^*. The unit element is denoted by 1 or 1_{Σ^*}. As usual, the elements of Σ and Σ^* are called *letters* and *words*, respectively. The subsets of Σ^* are called *languages*. The *length* (number of letters) of a word u is denoted by $|u|_\Sigma$.

Let $(X,*)$ be a group or monoid with set of generators Σ. The unit element of X is denoted by 1_X. When X is a group, the inverse of $x \in X$ is denoted by x^{-1}. We always assume that: $1_X \notin \Sigma$, and in the group case that: $x \in \Sigma \implies x^{-1} \in \Sigma$. The *length* with respect to Σ of an element x of X is :

$$|x|_\Sigma = \min\{k \mid x = a_1 * \cdots * a_k, a_i \in \Sigma\}\,. \tag{7}$$

The *Cayley graph* $\mathfrak{X}(X,\Sigma)$ of X with respect to Σ is the directed graph with nodes X and arcs $u \to v$ if $\exists a \in \Sigma,\ u * a = v$.

Consider a relation $R \subset \Sigma^* \times \Sigma^*$, and let $\sim_R$ be the least congruence on Σ^* such that $u \sim_R v$ if $(u,v) \in R$. Let X be isomorphic to the quotient monoid $(\Sigma^*/\sim_R)$. We say that $\langle\, \Sigma \mid u = v, (u,v) \in R\,\rangle$ is a *monoid presentation* of X and we write $X = \langle\, \Sigma \mid u = v, (u,v) \in R\,\rangle$.

Given a set S, denote by $\mathbb{F}(S)$ the free group generated (as a group) by S. Let S^{-1} be the set of inverses of the generators. A monoid presentation of $\mathbb{F}(S)$ is

$$\mathbb{F}(S) = \langle\, S \sqcup S^{-1} \mid aa^{-1} = 1, a^{-1}a = 1,\ \forall a \in S\,\rangle\,. \tag{8}$$

Given two groups or monoids X_1 and X_2, we denote by $X_1 \star X_2$ the *free product* of X_1 and X_2. Roughly, the elements of $X_1 \star X_2$ are the finite alternate sequences of elements of $X_1 \backslash \{1_{X_1}\}$ and $X_2 \backslash \{1_{X_2}\}$, and the law is the concatenation with simplification. More rigorously, the definition is as follows. Set $S = X_1 \sqcup X_2$. The *free product* $X_1 \star X_2$ is defined by the monoid presentation:

$$\langle\ S \ \mid\ (\forall u,v \in S^*, \forall i \in \{1,2\}),\ u1_{X_i}v = uv, (\forall a,b,c \in X_i,\ \text{s.t. } c = a*b),\ uabv = ucv\ \rangle\,.$$

If X_1 and X_2 are groups, then $X_1 \star X_2$ is also a group. The free product of more than two groups or monoids is defined analogously.

The Cayley graph of the group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$ is represented on Figure 2 (left). For simplicity, bidirected arcs $\leftrightarrow$ have been represented by undirected edges. On the figure, the simple edges correspond to copies of $\mathbb{Z}/2\mathbb{Z}$ and the squares to copies of $\mathbb{Z}/4\mathbb{Z}$.

Plain monoids and groups. A *plain monoid* is a monoid X of the form

$$X = S^* \star \mathbb{F}(T) \star X_1 \star \cdots \star X_k\,, \tag{9}$$

where S and T are finite sets and $X_1,\ldots,X_k$ are finite monoids. A *plain group* is a plain monoid which is also a group. A plain monoid X defined as in (9) is a plain group iff $S = \emptyset$ and $X_1,\cdots,X_k$ are groups.

Define

$$\Sigma = S \sqcup T \sqcup T^{-1} \sqcup X_1 \setminus \{1_{X_1}\} \sqcup \cdots \sqcup X_k \setminus \{1_{X_k}\}\,. \tag{10}$$

The set Σ is a finite set of generators of X, that we call *natural* generators. Define the sets: $\forall a \in \Sigma$,

$$\mathrm{Next}(a) = \{b \in \Sigma \mid b * a \notin \Sigma \cup \{1_X\}\} = \{b \in \Sigma \mid a * b \notin \Sigma \cup \{1_X\}\}\,.$$

The last equality in (11) follows form the definition of a plain monoid, and can be checked case by case. Indeed, we have the following: let $\Sigma_i = X_i \setminus \{1_{X_i}\}$, then

$$\mathrm{Next}(a) = \begin{cases} \Sigma & \text{if } a \in S \\ \Sigma \setminus \{a^{-1}\} & \text{if } a \in T \sqcup T^{-1} \\ \Sigma \setminus \Sigma_i & \text{if } a \in \Sigma_i \end{cases}\,.$$

Define the language $L(X, \Sigma) \subset \Sigma^*$ of *locally reduced words* by:

$$L(X, \Sigma) = \{u_1 \cdots u_k \mid \forall i < k, u_i * u_{i+1} \notin \Sigma \cup \{1_X\}\} . \tag{11}$$

Observe that $L(X, \Sigma) = \{u_1 \cdots u_k \mid \forall i,\ u_{i-1} \in \text{Next}(u_i)\} = \{u_1 \cdots u_k \mid \forall i,\ u_{i+1} \in \text{Next}(u_i)\}$. It is easily seen that the set $L(X, \Sigma)$ is in bijection with the monoid elements. Below we often identify X and $L(X, \Sigma)$.

Next property, to be used later on, is a direct consequence of the definition of a plain monoid :

$$a * b \in \Sigma \implies \text{Next}(a) = \text{Next}(b) = \text{Next}(a * b) . \tag{12}$$

Random walks on monoids and groups. Let $(X, *)$ be a group or monoid with finite set of generators Σ. Let ν be a probability distribution over Σ. Consider the Markov chain on the state space X with one-step transition matrix P defined by: $\forall x \in X, \forall a \in \Sigma$, $P_{x,x*a} = \nu(a)$. This Markov chain is called the *(right) random walk* (associated with) (X, ν).
Let $(x_n)_n$ be a sequence of i.i.d. r.v's distributed according to ν. Set

$$X_0 = 1_X, \quad X_{n+1} = X_n * x_n = x_0 * \cdots * x_n . \tag{13}$$

Then $(X_n)_n$ is a realization of the random walk (X, ν). For all $x, y \in X$, we have $|x * y|_\Sigma \leq |x|_\Sigma + |y|_\Sigma$. Applying Kingman's Subadditive Ergodic Theorem yields the following (first noticed by Guivarc'h [9]): there exists $\gamma \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \frac{|X_n|_\Sigma}{n} = \gamma \quad \text{a.s and in } L^p, \tag{14}$$

for all $1 \leq p < \infty$. We call γ the *drift* of the random walk.

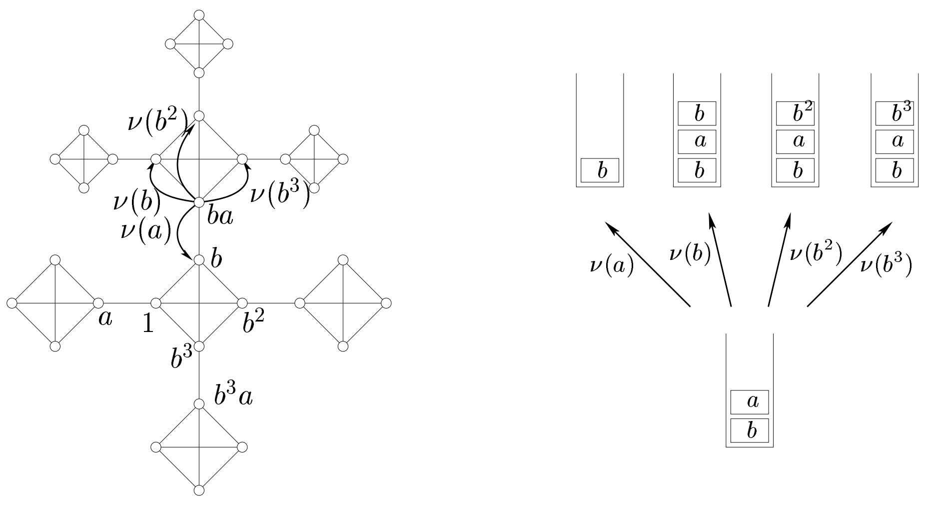

Figure 2: The random walk $(\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}, \nu)$.

To illustrate, consider the plain group $X = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z} = \langle\ a \mid a^2 = 1\ \rangle \star \langle\ b \mid b^4 = 1\ \rangle$ and the natural generators $\Sigma = \{a, b, b^2, b^3 = b^{-1}\}$. Let ν be a probability measure on Σ. On the left of Figure 2, we have represented a finite part of the infinite Cayley graph $\mathfrak{X}(X, \Sigma)$, and the one-step transitions of the random walk (X, ν) starting from the state ba. On the right of the figure, we show the same one-step transitions on the group elements viewed as words of $L(X, \Sigma)$ (written from bottom to top).

Random walks on plain monoids and groups. It is convenient to introduce the notion of a plain triple.

Definition 2.1. *A triple* (X, Σ, ν) *is* plain *if: (i)* X *is an infinite plain monoid not isomorphic to* $\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$*; (ii)* Σ *is a set of natural generators; (iii)* ν *is a probability measure whose support is included in* Σ *and generates* X*.*

Proposition 2.2. *If* (X, Σ, ν) *is a plain triple, then the random walk* (X, ν) *is transient.*

If X is an infinite plain monoid with the support of ν generating X, there are only two cases in which (X, ν) is not transient: (1) the triple $(\mathbb{Z}, \{-1, 1\}, \{1/2, 1/2\})$; (2) the triples $(\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}, \{a, b\}, \nu)$, for any ν, where a and b are the respective generators of the two cyclic groups. Since $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ have been excluded from consideration, then the random walk (X, ν) is transient, see [12] for details.

The case $X = \mathbb{Z}$ is specific. Some of the results below remain true but not all of them. For simplicity, we treat this case separately in §2.3.

Define:

$$\mathcal{B} \quad = \quad \{x \in \mathbb{R}^{\Sigma} \mid \forall i, x(i) > 0, \ \sum_i x(i) = 1\} \, . \tag{15}$$

The Traffic Equations play an essential role in the study of the random walk (X, ν).

Definition 2.3. *The* Traffic Equations (TE) *associated with a plain triple* (X, Σ, ν) *are the equations of the variables* $(x(a))_{a \in \Sigma} \in \mathbb{R}_+^{\Sigma}$ *defined by:* $\forall a \in \Sigma$,

$$x(a) = \nu(a) x(Next(a)) + \sum_{b * d = a} \nu(b) x(d) + \sum_{\substack{d \in Next(a) \\ b * d = 1_X}} \nu(b) \frac{x(d)}{x(Next(d))} x(a) \, .$$

An admissible solution *is a solution belonging to* $\mathcal{B}$*.*

Next result can be easily deduced from the proof of [12, Theorem 4.5].

Proposition 2.4. *Let* (X, Σ, ν) *be a plain triple. The Traffic Equations have a unique admissible solution.*

The interest of Proposition 2.4 is that the drift, which is the speed of escape to infinity of the random walker, can be expressed as a linear function of the solution to the TE.

Theorem 2.5. *Let* (X, Σ, ν) *be a plain triple. Let* $\widehat{r} = (\widehat{r}(a))_{a \in \Sigma}$ *be the unique admissible solution to the Traffic Equations. The drift of the random walk, defined in (14), is given by:*

$$\widehat{\gamma} = \sum_{a \in \Sigma} \nu(a) \left[\widehat{r}(Next(a)) - \sum_{b \mid a * b = 1_X} \widehat{r}(b) \right] \, . \tag{16}$$

Having such a simple formula for the drift is a remarkable feature. It depends strongly on the fact that the random walker evolves on a plain monoid, and it would not be true in a more general context. The above theorem is in fact obtained as a corollary of a stronger result on the explicit characterization of the harmonic measure of the random walk. Actually, $\widehat{r}(a)$ is exactly the probability that the random walker goes to infinity in the "direction" "$a\Sigma^{\mathbb{N}}$". This provides a justification for the equation (16). For the complete statement and for a proof, see [12, Theorem 4.5] and also [13, Theorem 3.3]. In the specific case of the free group, the result appears in [5, 14], see also the survey [11].

2.2 The zero-automatic queue

This section is taken verbatim from [3].

Definition of a zero-automatic queue. Consider a plain triple (X, Σ, ν). The associated 0-automatic queue is formed by a Markovian single server queue with FIFO discipline and an infinite capacity buffer in which the buffering occurs according to the random walk (X, ν). It is a multi-type queue (types Σ) but the type does not influence the way customers get served, only the way they get buffered.

More precisely, the instants of customer arrivals are given by a Poisson process of rate λ, and each customer carries a mark, or *type*, which is an element of Σ. The sequence of types is i.i.d. of law ν. Upon arrival, a new customer interacts with the customer presently at the back-end of the buffer, and depending on their respective classes, say b and a, one of three possible events occurs: (i) if $b * a = 1_X$, then the two customers leave the queue; (ii) if $b * a = c \in \Sigma$, then the two customers merge to create a customer of type c; (iii) otherwise, customer b takes place at the back-end of the buffer, behind customer a. In the mean time, at the front-end of the buffer, the customers are served one by one and at constant rate μ by the server. To be complete, one needs to specify how customers are incorporated when the buffer is empty. Several variants may be considered, and we view this "boundary condition" as an additional parameter of the model. The resulting flexibility in the definition of a 0-automatic queue turns out to be a crucial point.

According to the above description, the queue-content (the sequence of types of customers in the buffer) is a continuous time jump Markov process. The more formal definition of the queue is given via the infinitesimal generator of this process.

Definition 2.6 (Zero-automatic queue). *Let (X, Σ, ν) be a plain triple. Let $L(X, \Sigma)$ be the set defined in (11). Consider $r \in \{x \in \mathbb{R}^{\Sigma} \mid \forall i, x(i) \geq 0, \ \sum_i x(i) = 1\}$, and $\lambda, \mu \in \mathbb{R}_+^*$. The* 0-automatic queue *of type $(X, \Sigma, \nu, r, \lambda, \mu)$ is defined as follows. The* queue-content $(M(t))_{t \in \mathbb{R}_+}$ *is a continuous time jump Markov process on the state space $L(X, \Sigma)$ with infinitesimal generator Q defined by: $\forall u = u_n \cdots u_1 \in L(X, \Sigma) \setminus \cup_{a \in \Sigma} \{a\}^*$,*

$$\left\{ \begin{array}{lcll} Q(u, bu) & = & \lambda \nu(b) \ , & \forall b \in \mathit{Next}(u_n) \\ Q(u, cu_{n-1} \cdots u_1) & = & \lambda \sum_{b | b * u_n = c} \nu(b) \ , & \forall c \in \Sigma \setminus \{u_n\}, \ \exists b \in \Sigma, \ b * u_n = c \\ Q(u, u_{n-1} \cdots u_1) & = & \lambda \sum_{b | b * u_n = 1_X} \nu(b) & \\ Q(u, u_n \cdots u_2) & = & \mu & \end{array} \right. \tag{17}$$

and, for all $a \in \Sigma$ such that $a \in \mathit{Next}(a)$, and for all $n \geq 1$,

$$\left\{ \begin{array}{lcll} Q(a^n, ba^n) & = & \lambda \nu(b) \ , & \forall b \in \mathit{Next}(a) \\ Q(a^n, ca^{n-1}) & = & \lambda \sum_{b | b * a = c} \nu(b) \ , & \forall c \in \Sigma \setminus \{a\}, \ \exists c \in \Sigma, \ b * a = c \\ Q(a^n, a^{n-1}) & = & \mu + \lambda \sum_{b | b * a = 1_X} \nu(b) & \end{array} \right. \tag{18}$$

and, finally, the boundary condition is,

$$Q(1_{\Sigma^*}, a) \ = \ \lambda \nu(a) r(\mathit{Next}(a)) \ , \quad \forall a \in \Sigma \, . \tag{19}$$

The intuition behind the form of the boundary condition is as follows: the buffer-content is viewed as the visible part of an iceberg consisting in an infinite word. When the buffer is empty, new customers are incorporated depending on the invisible part of the iceberg, whose first marginal is assumed to be r. This last point will find an a-posteriori justification in Theorem 2.8.

Stationary distribution of a stable queue. The Traffic Equations, see Definition 2.3, play a central role in studying the random walk. We now introduce equations playing a similar role for the 0-automatic queue.

Definition 2.7 (Twisted Traffic Equations)**.** *Let (X, Σ, ν) be a plain triple. Consider $\lambda \in \mathbb{R}_+^*$ and $\mu \in \mathbb{R}_+^*$. The* Twisted Traffic Equations TTE *associated with $(X, \Sigma, \nu, \lambda, \mu)$ are the equations of the variables (η, x), $\eta \in \mathbb{R}_+^*$, $x = (x(a))_{a \in \Sigma} \in \mathbb{R}_+^{\Sigma}$, defined by:*

$$\begin{aligned} \eta(\lambda+\mu)x(a) &= \eta^2 \mu x(a) + \lambda \nu(a) x(\mathit{Next}(a)) + \eta\lambda \sum_{b*d=a} \nu(b) x(d) \\ &\quad + \eta^2 \lambda \sum_{\substack{d \in \mathit{Next}(a) \\ b*d=1_X}} \nu(b) \frac{x(d)}{x(\mathit{Next}(d))} x(a) \, . \end{aligned} \tag{20}$$

Next Theorem is the central result in [3].

Theorem 2.8. *Let (X, Σ, ν) be a plain triple. Fix λ and μ in $\mathbb{R}_+^*$. Let (ρ, r) be a solution in $\mathbb{R}_+^* \times \mathcal{B}$ of the Twisted Traffic Equations. Consider the 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$. Denote by $M_r = (M_r(t))_t$ the queue-content process. We have:*

$$\begin{array}{ccccc} [\rho < 1] & \iff & [\lambda\widehat{\gamma} < \mu] & \iff & [M_r \textit{ ergodic}] \\ [\rho = 1] & \iff & [\lambda\widehat{\gamma} = \mu] & \iff & [M_r \textit{ null recurrent}] \\ [\rho > 1] & \iff & [\lambda\widehat{\gamma} > \mu] & \iff & [M_r \textit{ transient}] \, . \end{array}$$

Assume that $\lambda\widehat{\gamma} < \mu$. Then the Twisted Traffic Equations have a unique solution (ρ, r) in $(0,1) \times \mathcal{B}$. Consider the queue $(X, \Sigma, \nu, r, \lambda, \mu)$. The stationary distribution $\pi_{\rho,r}$ of the process M_r is given by: $\forall a_n \cdots a_1 \in L(X, \Sigma)$,

$$\pi_{\rho,r}(a_n \cdots a_1) = (1-\rho)\rho^n q(a_n) \cdots q(a_2) r(a_1) \, , \tag{21}$$

where $q(a) = r(a)/r(\mathit{Next}(a))$ for all $a \in \Sigma$.

It is not elementary to come up with an a-priori intuitive explanation for the TTE in (20). But, going the other way round, if one assumes that (21) holds, writes the balance equations and simplifies them, then the resulting equations are (20), see [3] for details.

The celebrated Burke Theorem states that the departure process from a stable $M/M/1$ queue is a Poisson process of the same rate as the arrival process. A nice consequence of Theorem 2.8 is that an analog of Burke Theorem holds for 0-automatic queues.

In a 0-automatic queue, 'departures' occur both at the front-end and at the back-end of the buffer. Here we consider only the front-end departures, i.e. the ones corresponding to service completions and not to buffer cancellations.

Let $M = (M(t))_t$ be the queue-content process of some 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$. A *departure* is an instant of jump of M corresponding to a jump of the type: $u_n \cdots u_1 \to u_n \cdots u_2$ for $u = u_n \cdots u_1 \in L(X, \Sigma) \setminus \{1_{\Sigma^*}\}$. When $u = a^n, a \in \Sigma, n \geq 1$ (the case (18) in Definition 2.6), some special care must be taken. The jumps of type $a^n \to a^{n-1}$ which are *departures* occur at rate μ. The *departure process* is the point process of departures.

Theorem 2.9. *The model is the same as in Theorem 2.8. Assume that $\lambda\widehat{\gamma} < \mu$. Let (ρ, r) in $(0,1) \times \mathcal{B}$ be the unique solution of the TTE. Consider the 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$. The stationary departure process is a Poisson process of rate $\rho\mu$. Furthermore, for all t, the queue-content at time t is independent of the departure process up to time t.*

The networks obtained by connecting together queues with Poisson departure processes, generally have good properties reflected in the existence of a product-form stationary distribution, see [2, 10, 15]. Below, we carry out the study of networks of 0-automatic queues.

Remark. In [3, §6], an extension of the definition of 0-automatic queues is proposed: the pairs (X, Σ) are still formed by an infinite plain monoid X but the set of generators Σ may be strictly larger than a set of natural generators of X as defined in (10). Most of the results in §3 and §4 below would remain true for this extended definition of 0-automatic queues.

2.3 The G-queue

The above results are derived under the assumption that X is not isomorphic to $\mathbb{Z}$. Let us now assume that $X \sim \mathbb{Z}$. The results below are detailed in [3, §6.2.4].

Set

$$\bar{\mathcal{B}} = \{x \in \mathbb{R}^{\Sigma} \mid \forall i, x(i) \geq 0, \sum_i x(i) = 1\} . \tag{22}$$

Consider a triple $(\mathbb{F}(a), \Sigma = \{a, a^{-1}\}, \nu)$ and assume that $\nu(a) \in (0,1)$, $\nu(a) \neq \nu(a^{-1})$, so that the random walk $(\mathbb{F}(a), \nu)$ is non-degenerate and transient. Here the drift of $(\mathbb{F}(a), \nu)$ is easily computed: $\widehat{\gamma} = |\nu(a) - \nu(a^{-1})|$. Consider $\lambda, \mu \in \mathbb{R}_+^*$. For any boundary condition $r \in \bar{\mathcal{B}}$, we define the 0-automatic queue $(\mathbb{F}(a), \Sigma, \nu, r, \lambda, \mu)$ exactly as in Definition 2.6. The TTE associated with $(\mathbb{F}(a), \Sigma, \nu, \lambda, \mu)$ are still defined as in (20). By multiplying both sides of (20) by $\prod_b x(\text{Next}(b))$, we obtain a new set of equations without denominators. With some abuse, a solution r in $\mathbb{R}_+^* \times \bar{\mathcal{B}}$ of this last set of equations is still called a solution of the TTE.

By solving explicitly the TTE, we obtain that there exist two possible solutions in $\mathbb{R}_+^* \times \bar{\mathcal{B}}$ (which do not belong to $\mathbb{R}_+^* \times \mathcal{B}$):

$$(\rho^1, R^1) = \left(\frac{\lambda\nu(a)}{\mu + \lambda\nu(a^{-1})}; (1,0)\right), \qquad (\rho^2, R^2) = \left(\frac{\lambda\nu(a^{-1})}{\mu + \lambda\nu(a)}; (0,1)\right) . \tag{23}$$

Let us consider the variants of the 0-automatic queue with boundary conditions R^1 and R^2. In the buffer, there is only one type of customer with probability 1: if $r = R^1 = (1,0)$, there is only a in the buffer; if $r = R^2 = (0,1)$, there is only a^{-1} in the buffer.
Observe that at least one of ρ^1 and ρ^2 is strictly less than 1. If $\rho^1 < 1$, then the queue $(\mathbb{F}(a), \{a, a^{-1}\}, \nu, R^1, \lambda, \mu)$ is stable and the stationary distribution π of the buffer content satisfies:

$$\forall n \in \mathbb{N}, \quad \pi(a^n) = (1 - \rho^1)(\rho^1)^n .$$

An analog result holds when $\rho^2 < 1$. For some values of λ, μ, we have both $\rho^1 < 1$ and $\rho^2 < 1$ implying that the two variants of the queue are stable.
The above model is very close to the classical G-queue.

3 Jackson-like network of 0-automatic queues

3.1 Description of the model

Consider a network consisting of N zero-automatic queues $(X_i, \Sigma_i, \nu_i, r_i, \lambda_i, \mu_i)_{1 \leq i \leq N}$. We recall the meaning of the queue $(X_i, \Sigma_i, \nu_i, r_i, \lambda_i, \mu_i)$. Customers arrive at rate λ_i, choose their type in Σ_i according to ν_i, line up according to the random walk (X_i, ν_i), and are served in a FIFO way at rate μ_i. The vector r_i determines the way customers are incorporated in an empty queue.

After completing its service at queue i, a customer goes to queue j with probability p_{ij} or leaves definitely the network with probability $d_i = 1 - \sum_j p_{ij}$. The matrix $P = (p_{ij})_{ij}$ of dimension $N \times N$ is called the *routing matrix*. The graph of P is assumed to be connected. We also assume that the spectral radius of P is strictly less than 1, which is equivalent to the assumption that the network is without capture: a customer entering the network will eventually leave it.

The types of customers are not preserved from queue to queue (in fact, different queues need not have the same sets of types). More precisely, when a customer arrives at queue i coming from another queue, then it chooses its type according to ν_i, as if it was arriving from the outside.

We call this model a *Jackson-like network of 0-automatic queues*.

In Figure 3, a generic network of three queues is represented.

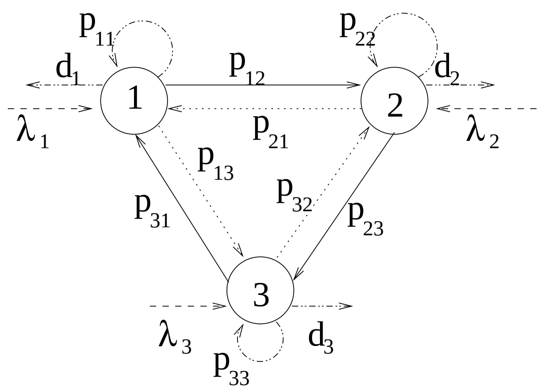

Figure 3: Jackson-like network of three queues.

Once again the boundary conditions r_i will play a central role, with one specific choice leading to good properties for the network.

The more formal definition of the network is done via the infinitesimal generator of the queue-content process.

Definition 3.1. *Let $\mathcal{X}_i = (X_i, \Sigma_i, \nu_i, r_i, \lambda_i, \mu_i)$, $1 \leq i \leq N$, be N zero-automatic queues. Let $P = (p_{ij})_{ij} \in \mathbb{R}_+^{N \times N}$ be a sub-stochastic matrix, whose graph is assumed to be connected, and whose spectral radius is strictly less than 1. The* Jackson-like network *associated with $((\mathcal{X}_i)_i, P)$ is defined as follows. The* queue-content *process $(M(t))_t$ of the network is a continuous time jump Markov process on the state space $\prod_i L(X_i, \Sigma_i)$, see (11). Denote by Q its infinitesimal generator. Denote by $Q^i_{\lambda,\mu}$ the infinitesimal generator of the 0-automatic queue $(X_i, \Sigma_i, \nu_i, r_i, \lambda, \mu)$. For all $u^i = a^i_{n_i} \cdots a^i_1$, $n_i \geq 1$, in $L(X_i, \Sigma_i)$, set $Dep(u^i) = a^i_{n_i} \cdots a^i_2$ (Dep for "departure").*

Assume first that for all a in $\cup_i \Sigma_i$, we have $a * a \neq a$. The infinitesimal generator Q is then defined by: $\forall u = (u^i)_i$, $u^i \in L(X_i, \Sigma_i)$,

$$\begin{cases} Q(u, u^1 \cdots u^{i-1} v^i u^{i+1} \cdots u^N) & = & Q^i_{\lambda_i,0}(u^i, v^i) + Q^i_{\mu_i p_{ii},0}(Dep(u^i), v^i), \\ & & \qquad \forall v^i,\ v^i \neq u^i,\ v^i \neq Dep(u^i) \\ Q(u, \cdots Dep(u^i) \cdots) & = & \mu_i d_i, \\ & & \qquad \forall u^i \notin \cup_{a \in \Sigma_i} \{a\}^* \\ Q(u, \cdots a^{n_i - 1} \cdots) & = & \mu_i d_i + Q^i_{\lambda_i,0}(a^{n_i}, a^{n_i-1}), \\ & & \qquad \forall u^i = a^{n_i} \\ Q(u, \cdots Dep(u^i) \cdots v^j \cdots) & = & Q^j_{\mu_i p_{ij}}(u^j, v^j), \\ & & \qquad \forall v^j,\ v^j \neq u^j, v^j \neq Dep(u^j) \\ Q(u, \cdots a^{n_i-1} \cdots b^{n_j-1} \cdots) & = & Q^i_{\mu_j p_{ji},0}(a^{n_i}, a^{n_i-1}) + Q^j_{\mu_i p_{ij},0}(b^{n_i}, b^{n_i-1}), \\ & & \qquad \forall u^i = a^{n_i}, u^j = b^{n_j} \end{cases} \tag{24}$$

In general, i.e. when for some generators a, we have $a * a = a$, then Q has a more complicated form. The lines 1, 4, and 5 of (24) are unchanged, but the lines 2 and 3 are modified as follows:

$$
\left\{
\begin{array}{lcl}
Q(u, \cdots Dep(u^i) \cdots) & = & \mu_i - \sum_{j \neq i, v^j \neq u^j} Q(u, \cdots Dep(u^i) \cdots v^j \cdots) \\
 & & - \sum_{v^i \neq u^i} Q^i_{p_{ii}\mu_i,0}(Dep(u^i), v^i) \ , \qquad \forall u^i \notin \cup_{a \in \Sigma_i} \{a\}^* \\
Q(u, \cdots a^{n_i - 1} \cdots) & = & \mu_i - \sum_{j \neq i, v^j \neq u^j} Q(u, \cdots a^{n_i-1} \cdots v^j \cdots) \\
 & & - \sum_{v^i \neq u^i} Q^i_{p_{ii}\mu_i,0}(a^{n_i-1}, v^i) \\
 & & + \sum_{k \neq i, u^k = (a_1^k)^{n_k}} Q^i_{p_{ki}\mu_k,0}(a^{n_i}, a^{n_i-1}) \\
 & & + Q^i_{\lambda_i,0}(a^{n_i}, a^{n_i-1}), \qquad \forall u^i = a^{n_i}
\end{array}
\right. \tag{25}
$$

All the other non-diagonal elements of Q are equal to 0.

Let us comment on the various cases in (24)-(25). In the first three lines, the evolution of the buffer-content affects only queue i. The first line corresponds to an arrival at queue i, which may come either from the outside or from queue i itself (feedback), which accounts for the two terms on the right-hand side. The second line corresponds to a departure from queue i. The third line corresponds to an evolution of the type $a^{n_i} \to a^{n_i-1}$, which may derive either from a departure or an arrival with cancellation. In the lines 4 and 5, the evolution of the buffer content affects two different queues, i and j. In line 5, the evolution may result either from a transfer from i to j, or from j to i, the two cases being undistinguishable due to the specific form of the buffer-contents.

3.2 Traffic Equations

The Network Traffic Equations (3) play a central role in the study of the classical Jackson networks. We now introduce their counterpart : the Jackson-like Network Traffic Equations.

Definition 3.2 (J-NTE). *The* Jackson-like Network Traffic Equations *(J-NTE) associated with a Jackson-like network of N queues $((X_i, \Sigma_i, \nu_i,\ \lambda_i, \mu_i)_i, P)$ are the equations in the variables $(\tilde{\lambda}_i)_i$ defined by:*

$$\tilde{\lambda}_i = \lambda_i + \sum_j p_{ji}\mu_j\tilde{\rho}_j \ , \tag{26}$$

where $(\tilde{\rho}_i, \tilde{r}_i)$ is a solution of the TTE associated with $(X_i, \Sigma_i, \nu_i, \tilde{\lambda}_i, \mu_i)$.

The interpretation of the J-NTE (26) is that $\tilde{\lambda}_i$ can be viewed as the "total" arrival rate in queue i, which is the sum of the external arrival rate λ_i and all the transfer rates coming from other queues.
The subtlety in (26) lies in the fact that the equations are defined in a completely implicit way : $(\tilde{\lambda}_i)_i$ is a function of $(\tilde{\rho}_i)_i$ (via the J-NTE) and $\tilde{\rho}_i$ is itself a function of $\tilde{\lambda}_i$ (via the TTE). On the other hand, the J-NTE do not depend on the boundary conditions $(r_i)_i$ defining the 0-automatic queues.

To carry on with the analysis, we need to state a conjecture which has stood up to our efforts.

Conjecture 3.3. *Consider the same model as in Theorem 2.8, except that $\lambda \in \mathbb{R}_+^*$ is no longer fixed. For all λ, there is a unique solution (ρ, r) in $\mathbb{R}_+^* \times \mathcal{B}$ of the TTE corresponding to λ. Let us set $\rho_{(\lambda)} = \rho$. Then $\rho_{(\lambda)}$ is a continuous and increasing function of λ.*

This conjecture is reasonable. In particular, it holds in all the specific examples that we have been able to treat completely.

For a line vector $\Lambda = (\Lambda_i)_i$ in $\mathbb{R}_+^N$, denote by $Diag(\Lambda)$ the diagonal matrix of size $N \times N$ defined by $Diag(\Lambda)_{ii} = \Lambda_i$. Assume that Conjecture 3.3 holds. Denote by $\rho(\Lambda)$ the line vector in $\mathbb{R}_+^N$ whose i-th coordinate is $\rho_{i(\Lambda_i)}$ where $(\rho_{i(\Lambda_i)}, R_{i(\Lambda_i)})$ is the unique solution of the TTE of $(X_i, \Sigma_i, \nu_i, \Lambda_i, \mu_i)$. By convention, $\rho_{i(0)} = 0$.

Fix the line vectors $\lambda = (\lambda_i)_i$ in $\mathbb{R}_+^N$ and $\mu = (\mu_i)_i$ in $(\mathbb{R}_+^*)^N$. Define the function:

$$\begin{array}{rcl} \Phi : \mathbb{R}_+^N & \to & \mathbb{R}_+^N \\ \Lambda & \mapsto & \Phi(\Lambda) = \lambda + \mu Diag(\rho(\Lambda))P \ , \end{array} \tag{27}$$

where $P = (p_{ij})_{ij}$ is the routing matrix. Conjecture 3.3 implies that Φ is an increasing function. We can rewrite the J-NTE (26) as follows:

$$\tilde{\lambda} = \lambda + \mu Diag(\rho(\tilde{\lambda}))P = \Phi(\tilde{\lambda}) \ , \tag{28}$$

where $\tilde{\lambda} = (\tilde{\lambda}_i)_i$, $\lambda = (\lambda_i)_i$, $\mu = (\mu_i)_i$ are line vectors of dimension N.

3.3 Stationary distribution

Lemma 3.4. *Assume that Conjecture 3.3 holds. Then there exists a solution for the J-NTE (26).*

Proof. Let $(\rho_{i(\Lambda_i)}, R_{i(\Lambda_i)})$ be the unique solution to the TTE for the queue i with an arrival rate $\Lambda_i > 0$. Using the TTE, we get, $\forall a \in \Sigma_i$,

$$\rho_{i(\Lambda_i)}(\Lambda_i + \mu_i)R_{i(\Lambda_i)}(a) > \rho^2_{i(\Lambda_i)}\mu_i R_{i(\Lambda_i)}(a) \ . \tag{29}$$

Simplifying (29), we have:

$$\rho_{i(\Lambda_i)}\mu_i < (\Lambda_i + \mu_i) \ . \tag{30}$$

Consider the function:

$$\begin{array}{rcl} \varphi : \mathbb{R}_+^N & \to & \mathbb{R}_+^N \\ \Lambda & \mapsto & \lambda + (\mu + \Lambda)P \ . \end{array} \tag{31}$$

Using (30), we get:

$$\varphi(\Lambda) = \lambda + (\mu + \Lambda)P > \lambda + \mu Diag(\rho(\Lambda))P = \Phi(\Lambda) \ . \tag{32}$$

Consider the sequence $(\Lambda^n)_n$ in $\mathbb{R}_+^N$ defined by:

$$\begin{array}{rcl} \Lambda^0 & = & 0 \ , \\ \Lambda^1 & = & \varphi(\Lambda^0) = \lambda + \mu P \ , \\ \Lambda^2 & = & \varphi(\Lambda^1) = \lambda + \lambda P + \mu P + \mu P^2 \ , \\ & \cdots & \\ \Lambda^n & = & \varphi(\Lambda^{n-1}) = \lambda + \lambda P + \cdots + \lambda P^{n-1} \\ & & \qquad + \mu P + \cdots + \mu P^n \ , \\ & \cdots & \end{array} \tag{33}$$

The matrix P has a spectral radius strictly less than 1, so the sequence $(\Lambda^n)_n$ converges. Now, consider another sequence $(\tilde{\lambda}^n)_n$ in $\mathbb{R}_+^N$ defined by:

$$
\begin{array}{rcl}
\tilde{\lambda}^0 & = & 0 \ , \\
\tilde{\lambda}^1 & = & \Phi(\tilde{\lambda}^0) \ , \\
 & \cdots & \\
\tilde{\lambda}^n & = & \Phi(\tilde{\lambda}^{n-1}) \ , \\
 & \cdots &
\end{array}
\tag{34}
$$

Since Φ is increasing, the sequence $\tilde{\lambda}^n$ is also increasing. The inequality (32) implies that $\tilde{\lambda}^n < \Lambda^n$ for all n. So there exists a limit $\tilde{\lambda}$ of $\tilde{\lambda}^n$ when n tends to infinity. Since ρ is a continuous function of λ (Conjecture 3.3), this is equivalent to the existence of a fixed point for the function Φ, or to the existence of a solution for the J-NTE (26). This completes the proof. $\square$

Lemma 3.5. *Let $(\tilde{\lambda}_i)_i$ be a solution to the J-NTE (26) and let $(\tilde{\rho}_i, \tilde{r}_i)_i$ be the corresponding solutions of the TTE. Consider the Jackson-type network of N queues $\big[(X_i, \Sigma_i, \nu_i, \tilde{r}_i, \lambda_i, \mu_i)_i, P\big]$. Let Q be the infinitesimal generator of the queue-content process. Consider the measure p on $L(X_1, \Sigma_1) \times \cdots \times L(X_i, \Sigma_i) \times \cdots \times L(X_N, \Sigma_N)$ defined by:*

$$
\forall u = (u^i)_i \ , \ u^i \in L(X_i, \Sigma_i), \quad p(u) = \prod_i p^i(u^i) \ ,
$$

where $p^i(u^i)$ has the form given by:

$$
\forall u^i = a^i_{n_i} \cdots a^i_1 \in L(X_i, \Sigma_i), \quad p^i(u^i) = (\tilde{\rho}_i)^n \tilde{q}_i(a^i_{n_i}) \cdots \tilde{q}_i(a^i_2) \tilde{r}_i(a^i_1) \ , \tag{35}
$$

where $\tilde{q}_i(a) = \tilde{r}_i(a)/\tilde{r}_i(Next(a))$ for all $a \in \Sigma_i$. We have $pQ = 0$.

Proof. The equation $pQ = 0$ is equivalent to the balance equation:

$$
\sum_{v \neq u} p(u) Q(u, v) = \sum_{v \neq u} p(v) Q(v, u) \ . \tag{36}
$$

Denote the left and right-hand side of the above equality by L and R, respectively. Define

$$
A^i(\tilde{r}_i) = \sum_{a \in \Sigma_i} \nu_i(a) \tilde{r}_i(\text{Next}(a)) \ , \quad \nu_i(\text{Noact}(a)) = \sum_{b \in \Sigma_i | b * a = a} \nu_i(b) \ . \tag{37}
$$

Using the shape of Q, see Def. 3.1, the left of (36) can be written as:

$$
\begin{aligned}
L \ = \ & p(u) \sum_i \mathbb{1}_{\{n_i = 0\}} \lambda_i A^i(\tilde{r}_i) + p(u) \sum_i \mathbb{1}_{\{n_i > 0, \ u^i \neq (a^i_{n_i})^{n_i}\}} (\lambda_i - \lambda_i \nu_i(\text{Noact}(a^i_{n_i})) + \mu_i) \\
& + p(u) \sum_i \mathbb{1}_{\{n_i > 0, \ u^i = (a^i_{n_i})^{n_i}\}} (\lambda_i - \lambda_i \nu_i(\text{Noact}(a^i_{n_i})) + \mu_i - p_{ii} \mu_i \nu_i(a^i_{n_i})) \ .
\end{aligned}
\tag{38}
$$

Using Theorem 2.8, one obtains N balance equations associated with the queues $(X_i, \Sigma_i, \nu_i, \tilde{r}_i, \tilde{\lambda}_i, \mu_i)$ for $1 \leq i \leq N$:

$$
\begin{aligned}
\sum_{v^i \neq u^i} p^i(v^i) Q^i_{\tilde{\lambda}_i, \mu_i}(v^i, u^i) \ & = \ \sum_{v^i \neq u^i} p^i(u^i) Q^i_{\tilde{\lambda}_i, \mu_i}(u^i, v^i) \\
& = \ \mathbb{1}_{\{n_i > 0\}} p^i(u^i) \big[\tilde{\lambda}_i \big(1 - \nu_i(\text{Noact}(a^i_{n_i}))\big) + \mu_i\big] \\
& \quad + \mathbb{1}_{\{n_i = 0\}} p^i(u^i) \tilde{\lambda}_i A^i(\tilde{r}_i) \ .
\end{aligned}
\tag{39}
$$

The right of (36) can be written as:

$$
\begin{aligned}
R &= \sum_i \frac{p(u)}{p^i(u^i)} \sum_{v^i \neq u^i} p^i(v^i) Q(u^1 \cdots v^i \cdots u^N, u) \\
&\quad + \sum_{i,j|i\neq j} \frac{p(u)}{p^i(u^i)p^j(u^j)} \sum_{v^i\neq u^i, v^j \neq u^j} p^i(v^i)p^j(v^j)Q(u^1\cdots v^i \cdots v^j \cdots u^N, u) \qquad (40)\\
&= \sum_i \frac{p(u)}{p^i(u^i)} B_1 + \sum_{i,j|i\neq j} \frac{p(u)}{p^i(u^i)p^j(u^j)} B_2 \,.
\end{aligned}
$$

Let us consider the term B_1. Using the equation (39) and the form of the infinitesimal generator Q given in (25), it implies:

$$
\begin{aligned}
B_1 &= \sum_{v^i} p^i(v^i) Q^i_{\lambda_i,0}(v^i,u^i) + \sum_{v^i a} p^i(v^i a) Q^i_{p_{ii}\mu_i,0}(v^i,u^i) + \sum_{v^i|\mathrm{Dep}(v^i)=u^i} p^i(v^i) \\
&\quad \times \Big\{ \mu_i d_i + \sum_j p_{ij}\mu_i\nu_j(\mathrm{Noact}(a^j_{n_j})) + \sum_j \mathbb{1}_{n_j=0}\mu_i p_{ij}\big(1 - A^j(\tilde{r}_j)\big)\Big\} \\
&= \frac{\lambda_i}{\tilde{\lambda}_i} p^i(u^i) \Big\{ \mathbb{1}_{n_i>0}\big[\tilde{\lambda}_i\big(1-\nu_i(\mathrm{Noact}(a^i_{n_i}))\big) + \mu_i\big] + \mathbb{1}_{n_i=0}\tilde{\lambda}_i A^i(\tilde{r}_i) - \mu_i\tilde{\rho}_i \Big\} \\
&\quad + \tilde{\rho}_i \sum_{v_i} p^i(v^i) Q^i_{p_{ii}\mu_i,0}(v^i,u^i) - \mathbb{1}_{\{u^i=(a^i_1)^{n_i}\}} p^i(u^i) p_{ii}\mu_i\nu_i(a^i_1) + p^i(u^i)\tilde{\rho}_i\mu_i(1-\sum p_{ij}) \\
&\quad + p^i(u^i)\tilde{\rho}_i \Big\{ \sum_j p_{ij}\mu_i\nu_j(\mathrm{Noact}(a^j_{n_j})) + \sum_j \mathbb{1}_{n_j=0}\mu_i p_{ij}\big(1-A^j(\tilde{r}_j)\big)\Big\} \\
&= \frac{\lambda_i}{\tilde{\lambda}_i} p^i(u^i) \Big\{ \mathbb{1}_{n_i>0}\big[\tilde{\lambda}_i\big(1-\nu_i(\mathrm{Noact}(a^i_{n_i}))\big) + \mu_i\big] + \mathbb{1}_{n_i=0}\tilde{\lambda}_i A^i(\tilde{r}_i) - \mu_i\tilde{\rho}_i \Big\} \\
&\quad + \tilde{\rho}_i \frac{p_{ii}\mu_i}{\tilde{\lambda}_i} p^i(u^i) \Big\{ \mathbb{1}_{n_i>0}\big[\tilde{\lambda}_i\big(1-\nu_i(\mathrm{Noact}(a^i_{n_i}))\big) + \mu_i\big] + \mathbb{1}_{n_i=0}\tilde{\lambda}_i A^i(\tilde{r}_i) - \mu_i\tilde{\rho}_i \Big\} \\
&\quad - \mathbb{1}_{\{u^i=(a^i_1)^{n_i}\}} p^i(u^i) p_{ii}\mu_i\nu_i(a^i_1) + p^i(u^i)\tilde{\rho}_i\mu_i(1-\sum p_{ij}) \\
&\quad + p^i(u^i)\tilde{\rho}_i \Big\{ \sum_j p_{ij}\mu_i\nu_j(\mathrm{Noact}(a^j_{n_j})) + \sum_j \mathbb{1}_{n_j=0}\mu_i p_{ij}\big(1-A^j(\tilde{r}_j)\big)\Big\} \,. \qquad (41)
\end{aligned}
$$

Similarly, the term B_2 is:

$$
\begin{aligned}
B_2 &= \sum_{v^i|\mathrm{Dep}(v^i)=u^i} p^i(v^i) \sum_{j\neq i,\ v^j} p^j(v^j) Q^j_{p_{ij}\mu_i,0}(v^j,u^j) \qquad (42)\\
&= p^i(u^i)\tilde{\rho}_i \frac{\mu_i p_{ij}}{\tilde{\lambda}_j} p^j(u^j) \Big\{ \mathbb{1}_{n_j>0}\big[\tilde{\lambda}_j\big(1-\nu_j(\mathrm{Noact}(a^j_{n_j}))\big) + \mu_j\big] + \mathbb{1}_{n_j=0}\tilde{\lambda}_j A^j(\tilde{r}_j) - \tilde{\rho}_j\mu_j \Big\} \,.
\end{aligned}
$$

Substitute B_1 and B_2 in (40) and simplify by using the TE (26), one gets:

$$\begin{aligned}
\frac{R}{p(u)} &= \sum_i \frac{\lambda_i + \sum_j \mu_j p_{ji}\tilde{\rho}_j}{\tilde{\lambda}_i}\Big\{\mathbb{1}_{n_i>0}\big[\tilde{\lambda}_i\big(1-\nu_i(\mathrm{Noact}(a^i_{n_i}))\big)+\mu_i\big] + \mathbb{1}_{n_i=0}\tilde{\lambda}_i A^i(\tilde{r}_i) - \mu_i\tilde{\rho}_i\Big\} \\
&\quad -\sum_i \mathbb{1}_{\{u^i=(a^i_1)^{n_i}\}} p_{ii}\mu_i\nu_i(a^i_1) + \sum_i \tilde{\rho}_i\mu_i - \sum_i \tilde{\rho}_i\mu_i \sum p_{ij} \\
&\quad +\sum_{i,j}\Big\{\tilde{\rho}_i p_{ij}\mu_i\nu_j(\mathrm{Noact}(a^j_{n_j})) + \mathbb{1}_{n_j=0}\tilde{\rho}_i\mu_i p_{ij}\big(1-A^j(\tilde{r}_j)\big)\Big\} \\
&= \sum_i \Big\{\mathbb{1}_{n_i>0}\big[\tilde{\lambda}_i\big(1-\nu_i(\mathrm{Noact}(a^i_{n_i}))\big)+\mu_i\big] + \mathbb{1}_{n_i=0}\tilde{\lambda}_i A^i(\tilde{r}_i)\Big\} - \sum_i(\tilde{\lambda}_i - \lambda_i) \\
&\quad -\sum_i \mathbb{1}_{\{u^i=(a^i_1)^{n_i}\}} p_{ii}\mu_i\nu_i(a^i_1) + \sum_j (\tilde{\lambda}_i-\lambda_i)\Big[\nu_j(\mathrm{Noact}(a^j_{n_j})) + \mathbb{1}_{n_j=0}\big(1-A^j(\tilde{r}_j)\big)\Big] \\
&= \sum_i \mathbb{1}_{\{n_i=0\}}\lambda_i A^i(\tilde{r}_i) + \sum_i \mathbb{1}_{\{n_i>0,\ u^i\neq(a^i_{n_i})^{n_i}\}}(\lambda_i - \lambda_i\nu_i(\mathrm{Noact}(a^i_{n_i})) + \mu_i) \\
&\quad +\sum_i \mathbb{1}_{\{n_i>0,\ u^i=(a^i_{n_i})^{n_i}\}}(\lambda_i - \lambda_i\nu_i(\mathrm{Noact}(a^i_{n_i})) + \mu_i - p_{ii}\mu_i\nu_i(a^i_{n_i}))\ . \qquad (43)
\end{aligned}$$

Therefore, the balance equation $L = R$ is proved, or $pQ = 0$. $\square$

Remark. It is possible to bypass the above proof. Indeed, a zero-automatic queue is "quasi-reversible" in the sense of Chao, Miyazawa & Pinedo [2, Definition 3.4]. Then by invoking the general results in [2] for Jackson-like networks of quasi-reversible queues, we get the product form result of Lemma 3.5.

Next theorem establishes the following result: the stationary distribution of a stable Jackson-like network of 0-automatic queues has a product form, which means that it is obtained as the product of the stationary distributions of the individual queues in isolation.

Theorem 3.6. *Consider a Jackson-like network of N zero-automatic queues $((X_i,\Sigma_i,\nu_i,\lambda_i,\mu_i)_i,P)$. Let $\widehat{\gamma}_i$ be the drift of the random walk (X_i,ν_i). Set $\widehat{\gamma} = (\widehat{\gamma}_i)_{1\leq i\leq N}$.*
Consider the J-NTE (26) associated with the network. Let $(\tilde{\lambda}_i)_i$ be a solution to the J-NTE. Let $(\tilde{\rho}_i,\tilde{r}_i)$ be a solution of the TTE associated with $(X_i,\Sigma_i,\nu_i,\tilde{\lambda}_i,\mu_i)$.
Consider now the Jackson-like network of N queues $((X_i,\ \Sigma_i,\nu_i,\tilde{r}_i,\lambda_i,\mu_i)_i,P)$ where we have specified the boundary conditions $(\tilde{r}_i)_i$. Denote by $M = (M_1(t),\dots,M_N(t))_t$ the jump Markov process associated with this network (where $(M_i(t))_t$ is the queue-content process in queue i). Then we have the following equivalences:

$$\big[\ M \text{ is ergodic }\ \big] \iff \big[\ \forall i,\ \tilde{\rho}_i < 1\ \big] \iff \big[\ \forall i,\ \tilde{\lambda}_i\widehat{\gamma}_i < \mu_i\big]\ .$$

Assume that $\tilde{\lambda}_i\widehat{\gamma}_i < \mu_i$, for all i. The stationary distribution of the process M exists and has a product form:

$$\forall u = (u^i)_i,\ u^i \in L(X_i,\Sigma_i), \quad \pi(u) = \prod_i \pi^i_{\tilde{\rho}_i,\tilde{r}_i}(u^i)\ , \qquad (44)$$

where $\pi^i_{\tilde{\rho}_i,\tilde{r}_i}(u^i)$ has the form given in (21):

$$\forall u^i = a^i_{n_i}\cdots a^i_1 \in L(X_i,\Sigma_i), \quad \pi^i_{\tilde{\rho}_i,\tilde{r}_i}(u^i) \ = \ (1-\tilde{\rho}_i)(\tilde{\rho}_i)^{n_i}\tilde{q}_i(a^i_{n_i})\cdots\tilde{q}_i(a^i_2)\tilde{r}_i(a^i_1)\ ,$$

where $\tilde{q}_i(a) = \tilde{r}_i(a)/\tilde{r}_i(Next(a))$ for all $a \in \Sigma_i$.

Proof. Let p be the measure defined in (35). We have:

$$\sum_u p(u) = \prod_{i=1}^N \sum_{u^i \in L(X_i,\Sigma_i)} p^i(u^i) = \prod_{i=1}^N \sum_{n\in\mathbb{N}} \tilde{\rho}_i^n \ .$$

Therefore, $\sum_u p(u) < \infty$ iff $\tilde{\rho}_i < 1$ for all i. Lemma 3.5 states that $pQ = 0$. It is standard that the process M is ergodic iff $\sum_u p(u) < \infty$, see for instance [1, Chapter 8]. So we have established that M is ergodic iff $\tilde{\rho}_i < 1$ for all i.

If $\tilde{\rho}_i < 1$, then $\sum_{u^i\in L(X_i,\Sigma_i)} p^i(u^i) = 1/(1-\tilde{\rho}_i)$. It yields that under the stability condition, the stationary distribution is given by (44). Hence, this completes the proof. □

As in §3.2, let us consider front-end departures, that is departures corresponding to service completions and not to buffer cancellations. Furthermore, we consider the departures out of the network, that is the front-end departures for which the customer is routed to the outside.

Corollary 3.7. *The model is the same as in Theorem 3.6. Assume that $\tilde{\lambda}_i\widehat{\gamma}_i < \mu_i$ for all i. Then the departure process out of the network is a Poisson process of rate $\sum_i \mu_i\tilde{\rho}_i d_i$, where $d_i = 1 - \sum_j p_{ij}$. Furthermore, for all t, the queue-content at time t is independent of the departure process up to time t.*

Proof. Let $M = (M^i)_i$ be the jump Markov process of the buffer-content. Let $Q = (q(u,v))_{u,v}$ be the infinitesimal generator of M. Let D be the departure process out of the network. Define

$$\mathcal{D} = \left\{(u,u') \mid u = (u^i)_i \in L(X,\Sigma),\ u' = u^1 \cdots \mathrm{Dep}(u^i)\cdots u^N\right\} ,$$

where $\mathrm{Dep}(\cdot)$ is defined in Definition 3.1. The set $\mathcal{D}$ is the set of couples of buffer-contents corresponding to a departure out of the network.

Consider the time-reversed process $\widetilde{D}$ of D. The process $\widetilde{D}$ corresponds to the instants of "right-increase" in one of the N time-reversed processes $(\widetilde{M^i}(t))_t$. Consequently, the instantaneous rate $\tilde{c}(t)$ of $\widetilde{D}$ is as follows. If $\widetilde{M}(t) = u = u^1\cdots u^N$, $u^i \in L(X_i,\Sigma_i)$, $u^i = a^i_{n_i}\cdots a^i_1$, $n_i \geq 1$, then, using Theorem 3.6, we get

$$\begin{aligned}
\tilde{c}(t) &= \sum_{u'} \pi(u')q(u',u)\mathbb{1}_{(u',u)\in\mathcal{D}}/\pi(u) \\
&= \sum_i \sum_{a\in\mathrm{Next}(a^i_1)} \pi(u^1\cdots[a^i_{n_i}\cdots a^i_1 a]\cdots u^N)\mu_i d_i/\pi(u) \\
&= \sum_i \sum_{a\in\mathrm{Next}(a^i_1)} \tilde{\rho}_i r(a)/r(\mathrm{Next}(a^i_1))\mu_i d_i \\
&= \sum_i \tilde{\rho}_i\mu_i d_i \ .
\end{aligned}$$

When u^i is equal to the empty word, the summation over $a \in \mathrm{Next}(a^i_1)$ is replaced by a summation over $a \in \Sigma$. After simplification, we still get $\tilde{c}(t) = \sum_i \tilde{\rho}_i\mu_i d_i$.

Therefore, $\widetilde{D}$ is a Poisson process of rate $\sum_i \tilde{\rho}_i\mu_i d_i$. Since Poisson processes are preserved by time-reversal, D is also a Poisson process of rate $\sum_i \tilde{\rho}_i\mu_i d_i$. This completes the proof.

Also, using the Markov property of Poisson processes, $\widetilde{M}(t)$ is independent of the process $\widetilde{D}$ after time t. Under time-reversal, this translates as: $M(t)$ is independent of the departure process D up to time t. □

Remark. Using exactly the same proof as above, we can also show that the departure process out of the network from queue i is a Poisson process of rate $\tilde{\rho}_i \mu_i d_i$.

In the case of one 0-automatic queue, the stability condition ($\lambda < \mu$) depends only on the input variables which define the model. The same holds for a classical Jackson network where the stability condition can be rewritten as : $\lambda(I-P)^{-1} < \mu$, where λ and μ are line vectors, and where I is the identity matrix (see (3)). Here, on the other hand, the stability condition of the Jackson-like network depends on $\tilde{\lambda}$, which is an implicit solution of the J-NTE. It is hard to give an explicit stability condition for the network. Nevertheless, we found the following explicit necessary and sufficient conditions.

Define $Diag(\cdot)$ and Φ as in §3.2. Let $\widehat{\Lambda}$ be the line vector defined by:

$$\widehat{\Lambda} Diag(\widehat{\gamma}) = \mu \ . \tag{45}$$

Theorem 2.8 implies that $Diag(\rho(\widehat{\Lambda})) = I$. Hence, $\Phi(\widehat{\Lambda}) = \lambda + \mu P$.

The next two lemmas provide respectively a sufficient and a necessary condition of stability for a Jackson-like network of 0-automatic queues. Observe that the condition $(\lambda + \mu P) Diag(\widehat{\gamma}) < \mu$ is equivalent to

$$\lambda + \mu P < \widehat{\Lambda} \ . \tag{46}$$

Lemma 3.8. *Assume that Conjecture 3.3 is true. If we have $(\lambda + \mu P)Diag(\widehat{\gamma}) < \mu$, then the J-NTE (26) have a solution $\tilde{\lambda}$ such that $\rho_i(\tilde{\lambda}) < 1$ for all i.*

Proof. Consider the sequence $\tilde{\lambda}^i$ defined by (34). The following inequality will be proved by induction:

$$\forall i, \qquad \tilde{\lambda}^i < \lambda + \mu P \ . \tag{47}$$

For $i = 1$, we have: $\tilde{\lambda}^1 = \lambda < \lambda + \mu P$.

Suppose that $\tilde{\lambda}^i < \lambda + \mu P$. Then, since Φ is increasing, we get:

$$\tilde{\lambda}^i < \lambda + \mu P < \widehat{\Lambda} \implies \tilde{\lambda}^{i+1} = \Phi(\tilde{\lambda}^i) < \Phi(\widehat{\Lambda}) = \lambda + \mu P \ . \tag{48}$$

Hence, (47) is proved. Therefore, the sequence $\tilde{\lambda}^i$ is increasing and bounded by $\lambda + \mu P$. It implies that the limit $\tilde{\lambda}$ of the sequence $\tilde{\lambda}^i$ is strictly less than $\widehat{\Lambda}$. According to Conjecture 3.3, the function ρ is increasing which yields that: $\rho_i(\tilde{\lambda}) < \rho_i(\widehat{\Lambda}) = 1$ for all i. This completes the proof. ☐

Lemma 3.9. *Assume that Conjecture 3.3 is valid. If the solution $\tilde{\lambda}$ of the J-NTE (26) is such that $\rho_i(\tilde{\lambda}) < 1$ for all i, then $\lambda(I - Diag(\widehat{\gamma})P)^{-1} < \widehat{\Lambda}$.*

Proof. Suppose that the network is stable, or equivalently that the solution $\tilde{\lambda}$ of the J-NTE (26) satisfies $\rho_i(\tilde{\lambda}) < 1$ for all i.

In a stable 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$, where (ρ, r) is an admissible solution to the associated TTE, one has $\lambda \widehat{\gamma} \leq \rho\mu$, where $\widehat{\gamma}$ is the drift of the random walk (X, ν), see [3]. It implies that:

$$\tilde{\lambda} = \lambda + \mu Diag(\rho(\tilde{\lambda}))P \geq \lambda + \tilde{\lambda} Diag(\widehat{\gamma})P \ . \tag{49}$$

This is equivalent to:

$$\tilde{\lambda}(I - Diag(\widehat{\gamma})P) \geq \lambda \ . \tag{50}$$

The matrix P has a spectral radius strictly less than 1 by hypothesis. So this is a fortiori also the case for the matrix $Diag(\widehat{\gamma})P$. So the inverse matrix $(I - Diag(\widehat{\gamma})P)^{-1}$ is well-defined. From (50), we get:

$$\tilde{\lambda} \geq \lambda(I - Diag(\widehat{\gamma})P)^{-1} . \tag{51}$$

We conclude by observing that the network is stable if and only if $\tilde{\lambda} < \widehat{\Lambda}$. This completes the proof. □

3.4 Examples

The simple $M/M/1$ queue and the G-queue are the two simplest examples of 0-automatic queues. A classical Jackson network is a 0-automatic Jackson-like network of simple queues. A G-network, as studied for instance in [7, 6, 8], is close to a 0-automatic Jackson-like network of G-queues. In a classical G-network, the routing is slightly more complex, and the customer types are slightly less complex.

3.4.1 The classical Jackson network

Consider a network of 0-automatic queues $(X_i, \Sigma_i, \nu_i, r_i, \lambda_i, \mu_i)_{1 \leq i \leq N}$, where $X_i = \{a_i^k, k \in \mathbb{N}\}$, $\Sigma_i = \{a_i\}$, and a routing matrix $P = (p_{ij})_{ij}$. Here, one has $\nu_i(a_i) = r_i(a_i) = 1$, so there is a single possible variant for the 0-automatic network. This variant is the classical Jackson network.

In queue i, the solution to the TTE is $\rho_i(\Lambda_i) = \Lambda_i/\mu_i$. If we simplify the J-NTE (26) accordingly, we obtain precisely the NTE (3).

The necessary condition in Lemma 3.9 is also the sufficient condition of stability for the network.

3.4.2 0-automatic Jackson-like network of G-queues

Consider N zero-automatic queues $(\mathbb{F}(a_i), \{a_i, a_i^{-1}\}, \nu_i, r_i, \lambda_i, \mu_i)_i$ where $\nu_i(a_i) \in (0,1) \setminus \{1/2\}$, for all i. Let $P = (p_{ij})_{ij}$ be the routing matrix (connected, with a spectral radius strictly less than 1).

We recall that each TTE has 2 possible solutions, see (23). If the arrival rate to queue i is Λ_i, we denote these two solutions by $(\rho_i^1(\Lambda_i), R^1)$ and $(\rho_i^2(\Lambda_i), R^2)$.

Corresponding to these solutions, we may define 2^N different variants of the Jackson-like network of G-queues. In particular, there are 2^N possible J-NTE:

$$\tilde{\lambda}_i = \lambda_i + \sum_j p_{ji}\mu_j\tilde{\rho}_i \ , \tag{52}$$

where $\tilde{\rho}_i = \rho_i^1(\tilde{\lambda}_i)$ or $\tilde{\rho}_i = \rho_i^2(\tilde{\lambda}_i)$. If $\tilde{\rho}_i = \rho_i^1(\tilde{\lambda}_i)$, then there are only customers of type a_i in queue i; and if $\tilde{\rho}_i = \rho_i^2(\tilde{\lambda}_i)$, then there are only customers of type a_i^{-1} in queue i.

For each i, (23) implies immediately that at least one of $\rho_i^1(\tilde{\lambda}_i)$ and $\rho_i^2(\tilde{\lambda}_i)$ is strictly less than 1. Hence, among the 2^N variants of the network, there is at least one variant in which the network is stable.

Consider the case $N = 2$ where $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 2$, $\nu_1(a_1) = \nu_2(a_2) = 4/9$, $\nu_1(a_1^{-1}) = \nu_2(a_2^{-1}) = 5/9$. Let us consider the four variants.

1. $\tilde{\rho}_1 = \rho_1^1(\tilde{\lambda}_1)$, $\tilde{\rho}_2 = \rho_2^1(\tilde{\lambda}_2)$. The J-NTE have a unique solution s.t. $\tilde{\rho}_1 = \tilde{\rho}_2 = 1/5$.

2. $\tilde{\rho}_1 = \rho_1^1(\tilde{\lambda}_1)$, $\tilde{\rho}_2 = \rho_2^2(\tilde{\lambda}_2)$. The J-NTE have a unique solution s.t. $\tilde{\rho}_1 = 0.208$, $\tilde{\rho}_2 = 0.265$.

3. $\tilde{\rho}_1 = \rho_1^2(\tilde{\lambda}_1)$, $\tilde{\rho}_2 = \rho_2^1(\tilde{\lambda}_2)$. The J-NTE have a unique solution s.t. $\tilde{\rho}_1 = 0.265$, $\tilde{\rho}_2 = 0.208$.

4. $\tilde{\rho}_1 = \rho_1^2(\tilde{\lambda}_1)$, $\tilde{\rho}_2 = \rho_2^2(\tilde{\lambda}_2)$. The J-NTE have a unique solution s.t. $\tilde{\rho}_1 = \tilde{\rho}_2 = 0.276$.

Hence, in the numerical example above, each possible J-NTE has a unique solution such that the corresponding network is stable and has a product form.

In comparison with the above model, the G-networks defined in [7] are slightly more flexible in the way the routing mechanism is defined. On the other hand, and because of this flexibility, it is not possible to define several variants of the network as above.

3.4.3 Zero-automatic Jackson-like network of a $\mathbb{N}$-queue and a $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$-queue

Consider a network of two queues: the simple queue $(\{c\}^*, \{c\}, \nu_1(c) = 1, r_1(c) = 1, \lambda_1, \mu_1)$ and a $(\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, \{a, a^2, b, b^2\}, \nu_2, r_2, \lambda_2, \mu_2)$ - queue, where $\nu_2(a) = \nu_2(b) = p$ and $\nu_2(a^2) = \nu_2(b^2) = q = 1/2 - p$. The network is represented in Figure 4. For simplicity, we take $p_{11} = p_{22} = 0$.

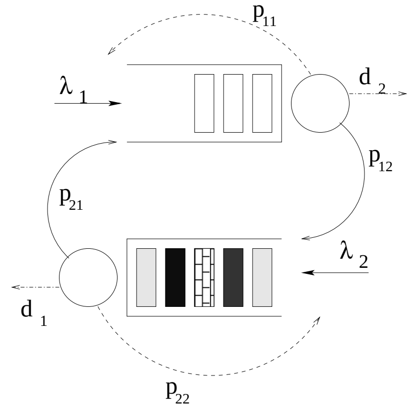

Figure 4: Network of two queues.

In [3], the solution $\rho_2(\lambda, \mu)$ to the TTE of the zero-automatic queue $(\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, \{a, b, a^2, b^2\}, \nu_2, \lambda, \mu)$ is given:

$$\rho_2(\lambda, \mu) = 2\frac{4\lambda^2p^2 - 2\lambda^2p + \lambda\mu + \lambda^2}{4\lambda^2p^2 - 2\lambda^2p + 4\lambda\mu + \lambda^2 + 4\mu^2} \ . \tag{53}$$

Therefore, the J-NTE (26) of the network are:

$$\begin{cases} \tilde{\lambda}_1 = \lambda_1 + \mu_2 p_{21} \rho_2(\tilde{\lambda}_2, \mu_2) \\ \tilde{\lambda}_2 = \lambda_2 + \tilde{\lambda}_1 p_{12} \end{cases} \ . \tag{54}$$

The J-NTE (54) have a unique positive solution which can be expressed as a root of a polynomial of degree 3.

For simplicity, let us carry on with the computations for the model obtained by setting $p = 1/3, p_{12} = 1/4, p_{21} = 1/2$, $\mu_1 = 2, \mu_2 = 3$. The arrival rates λ_1 and λ_2 may vary.

The vector $\widehat{\Lambda}$ defined in (45) is given by: $\widehat{\Lambda} = (2, (\sqrt{37} - 3)/4)$. We obtain that $\tilde{\lambda}_2$ is the positive root of the polynomial:

$$f(x) = 28x^3 + (411 - 28\lambda_2 - 7\lambda_1)x^2 + (-432\lambda_2 + 1215 - 108\lambda_1)x - 324\lambda_1 - 1296\lambda_2 \ .$$

Then $\tilde{\lambda}_1$ can be determined via $\tilde{\lambda}_2$:

$$\tilde{\lambda}_1 = 4(\tilde{\lambda}_2 + \lambda_2) \ .$$

In particular, we can explicitly determine the stability region which, we recall, is characterized by $\tilde{\lambda}_1 \widehat{\gamma}_1 < \mu_1, \ \tilde{\lambda}_2 \widehat{\gamma}_2 < \mu_2$ (Theorem 3.6). It is instructive to compare this stability region with the regions provided by the sufficient condition of Lemma 3.8 and the necessary condition of Lemma 3.9.

The necessary condition in Lemma 3.9 is equivalent to:

$$0 < \lambda_1 < 2, \ 0 < \lambda_2 < \widehat{\Lambda}_2, \ \lambda_1 + \lambda_2 \widehat{\Lambda}_2/6 < 2, \ \lambda_2 + 1/4\lambda_1 < \widehat{\Lambda}_2 \ . \tag{55}$$

Consider now the sufficient condition in Lemma 3.8. In the present situation, we obtain:

$$0 < \lambda_1 < 1/2, \quad 0 < \lambda_2 < \frac{\sqrt{37} - 5}{4} \ . \tag{56}$$

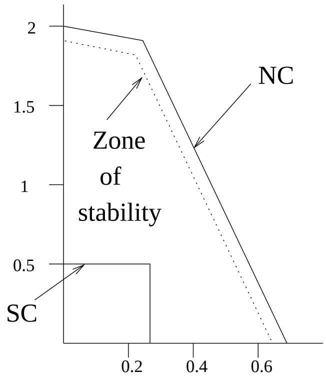

Figure 5: Stability region, the sufficient condition (SC) and the necessary condition (NC).

Figure 5 shows the stability region, together with the regions associated with the sufficient condition (56) and the necessary condition (55). The stability region is very close to the necessary condition, but very far from the sufficient condition.

4 Kelly-like network of 0-automatic queues

4.1 Zero-automatic queues with classes

Consider a 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$ and denote by $Q_{\lambda,\mu}$ the infinitesimal generator of the buffer-content. Let $\mathcal{C}$ be a non-empty finite set of *classes*. We refine the model by assuming that each customer carries a *class* (in $\mathcal{C}$) on top of its type (in Σ). The customers of class c enter the network according to a Poisson process of rate λ_c, so that $\lambda = \sum_{c \in \mathcal{C}} \lambda_c$.

The classes of the customers do not influence the buffering mechanism. When two customers of *types* a and b merge in the buffer to create a new customer of *type* $c = a * b$, then the *class* of the customer c is chosen to be equal to the class of the customer a with probability 1/2, and to the class of the customer b with probability 1/2.

The *0-automatic queue with classes* $(X, \Sigma, \nu, \ r, \mathcal{C}, (\lambda_c)_c, \mu)$ is defined formally as follows. The *buffer-content* $(M(t))_{t \in \mathbb{R}_+}$ is a continuous time jump Markov process on the state space

$$\mathcal{S} = \big\{ (u, \alpha) \in L(X, \Sigma) \times \mathcal{C}^* \mid |u|_\Sigma = |\alpha|_\mathcal{C} \big\} \tag{57}$$

with infinitesimal generator Q defined by:
$\forall (u,\alpha)$, $u = u_n \cdots u_1 \in L(X,\Sigma)$, $\alpha = \alpha_n \cdots \alpha_1 \in \mathcal{C}^*$,

$$\left\{ \begin{array}{lcll} Q((u,\alpha),(bu,c\alpha)) & = & Q_{\lambda_c,0}(u,bu), & \forall b \in \text{Next}(u_n) \\ Q((u,\alpha),(bu_{n-1}\cdots u_1, c\alpha_{n-1}\cdots\alpha_1)) & = & \mathbb{1}_{c=\alpha_n}(1/2)Q_{\lambda+\lambda_c,0}(u,bu_{n-1}\cdots u_1) & \\ & & +\mathbb{1}_{c\neq\alpha_n}(1/2)Q_{\lambda_c,0}(u,bu_{n-1}\cdots u_1), & \text{for } b\in\Sigma \\ Q((u,\alpha),(u,c\alpha_{n-1}\cdots\alpha_1)) & = & \mathbb{1}_{c\neq\alpha_n}(1/2)\lambda_c\nu(\text{Noact}(u_n)), & \end{array} \right. \tag{58}$$

where $\text{Noact}(a) = \{b \in \Sigma \mid b * a = a\}$.
The others values of Q are given by: if $(u,\alpha) \notin \cup_{a\in\Sigma, c\in\mathcal{C}}\{a\}^* \times \{c\}^*$,

$$\left\{ \begin{array}{lcl} Q((u,\alpha),(u_{n-1}\cdots u_1,\alpha_{n-1}\cdots\alpha_1)) & = & Q_{\lambda,0}(u,u_{n-1}\cdots u_1) \\ Q((u,\alpha),(u_n\cdots u_2,\alpha_n\cdots\alpha_2)) & = & Q_{0,\mu}(u,u_n\cdots u_2), \end{array} \right. \tag{59}$$

and, if $(u,\alpha) = (a^n, c^n)$ for some $a \in \Sigma$ and $c \in C$,

$$Q((u,\alpha),(a^{n-1},c^{n-1})) = Q_{\lambda,\mu}(a^n,a^{n-1}) \ . \tag{60}$$

Theorem 4.1. *Consider a 0-automatic queue with classes $(X,\Sigma,\nu,r,\mathcal{C},(\lambda_c)_c,\mu)$, where (ρ,r) is an admissible solution to the TTE associated with $(X,\Sigma,\nu,\lambda,\mu)$, where $\lambda = \sum_{c\in\mathcal{C}}\lambda_c$. Denote by $M = (M(t))_t$ the queue-content process. Assume that $\lambda\widehat{\gamma} < \mu$, where $\widehat{\gamma}$ is the drift of the random walk (X,Σ,ν). The stationary distribution $\pi_{\rho,r}$ of the process M is given by: $\forall (u,\alpha) = (u_n\cdots u_1, \alpha_n\cdots\alpha_1) \in L(X,\Sigma)\times\mathcal{C}^*$,*

$$\pi_{\rho,r}(u,\alpha) = (1-\rho)\rho^n q(u_n)\cdots q(u_2) r(a_1) p_{\alpha_n}\cdots p_{\alpha_1} \ , \tag{61}$$

where $q(a) = r(a)/r(Next(a))$ for all $a\in\Sigma$, $p_c = \lambda_c/\lambda$, for all $c \in \mathcal{C}$.

There are two parts in (61): the same term as in (21), i.e. $(1-\rho)\rho^n q(u_n)\cdots q(u_2)r(a_1)$ which accounts for the "types" of the customers, and the term $p_{\alpha_n}\cdots p_{\alpha_1}$ which accounts for the "classes" of the customers.

Proof. Under the stability condition $\lambda\widehat{\gamma} < \mu$, the admissible solution (ρ,r) satisfies $\rho < 1$, see Theorem 2.8. Hence, proving Theorem 4.1 is equivalent to proving the balance equations:

$$\sum_{(v,\beta)\neq(u,\alpha)} \pi_{\rho,r}(u,\alpha)Q((u,\alpha),(v,\beta)) = \sum_{(v,\beta)\neq(u,\alpha)} \pi_{\rho,r}(v,\beta)Q((v,\beta),(u,\alpha)) \ . \tag{62}$$

Denote the left and the right-hand side of the above equality by L and R, respectively. The left of (62) is:

$$L = \left\{ \begin{array}{ll} \sum_{a\in\Sigma,c\in\mathcal{C}}(1-\rho)\lambda_c\nu(a)r(\text{Next}(a)) & \text{if } u = 1_{\Sigma^*} \\ \pi_{\rho,r}(u,\alpha)\{\lambda[1-1/2(1+p_{\alpha_n})\nu(\text{Noact}(a_n))]+\mu\} & \text{otherwise} \end{array} \right. . \tag{63}$$

The right of (62) is given by, for $u = 1_{\Sigma^*}$,

$$R = \sum_{\substack{a\in\Sigma \\ c\in\mathcal{C}}} (1-\rho)\rho r(a)p_c[\mu + \lambda\sum_{b*a=1_X}\nu(b)] = (1-\rho)\rho[\mu+\lambda\sum_{b*a=1_X}\nu(b)r(a)] \ . \tag{64}$$

Now the formula
$$\rho = \frac{\lambda\sum_{a\in\Sigma}\nu(a)r(\text{Next}(a))}{\mu+\lambda\sum_{a*b=1_X}\nu(a)r(b)}$$
is proved in [3, Lemma 5.2]. This concludes the proof of $L = R$ for $u = 1_{\Sigma^*}$.

For $(u,\alpha) = (u_n \cdots u_1, \alpha_n \cdots \alpha_1)$, $n \geq 1$, using the fact that if $a * b \in \Sigma$, then $\text{Next}(a*b) = \text{Next}(b)$, one has:

$$\begin{aligned}
R &= \lambda_{\alpha_n}\nu(u_n)\pi_{\rho,r}(u_{n-1}\cdots u_1, \alpha_{n-1}\cdots\alpha_1) + \sum_{\substack{b\in\text{Next}(u_n)\\ a*b=1_X\\ c_1,c_2\in\mathcal{C}}} \lambda_{c_2}\nu(a)\pi_{\rho,r}(bu, c_1\alpha) \\
&\quad + \sum_{\substack{b\in\text{Next}(u_1)\\ c\in\mathcal{C}}} \mu\pi_{\rho,r}(ub,\alpha c) + \sum_{\substack{a*u_n=u_n\\ c\in\mathcal{C}, c\neq c_n}} \frac{1}{2}\lambda_c\nu(a)\pi_{\rho,r}(u, c\alpha_{n-1}\cdots\alpha_1) \\
&\quad + \sum_{\substack{a*b=u_n, b\neq u_n\\ c,c_1\in\mathcal{C}}} (\mathbb{1}_{c_1=c=\alpha_n} + \frac{1}{2}(\mathbb{1}_{c\neq c_n, c_1=\alpha_n} + \mathbb{1}_{c=\alpha_n, c_1\neq\alpha_n}))\lambda_{c_1}\nu(a) \\
&\quad \pi_{\rho,r}(bu_{n-1}\cdots u_1, c\alpha_{n-1}\cdots\alpha_1) \\
&= \pi_{\rho,r}(u,\alpha)\Big[\frac{1}{\rho}\frac{r(\text{Next}(u_n))}{r(u_n)}\lambda\nu(u_n) + \sum_{\substack{b\in\text{Next}(a_n)\\ a*b=1_X}} \rho\lambda\nu(a)\frac{r(b)}{r(\text{Next}(b))} + \rho\mu \\
&\quad + \sum_{a*u_n=u_n} \lambda\nu(a)\frac{1}{2}(1-p_{\alpha_n}) + \sum_{\substack{b\neq u_n\\ a*b=u_n}} \lambda\nu(a)\frac{r(b)}{r(u_n)}\big(\frac{1}{2}(1+p_{\alpha_n}) + \frac{1}{2}(1-p_{\alpha_n})\big)\Big] \\
&= \pi_{\rho,r}(u,\alpha)\Big[\frac{1}{\rho}\frac{r(\text{Next}(u_n))}{r(u_n)}\lambda\nu(u_n) + \sum_{\substack{b\in\text{Next}(a_n)\\ a*b=1_X}} \rho\lambda\nu(a)\frac{r(b)}{r(\text{Next}(b))} + \rho\mu \\
&\quad + \sum_{a*b=u_n} \lambda\nu(a)\frac{r(b)}{r(u_n)} - \sum_{a*u_n=u_n} \lambda\nu(a)\frac{1}{2}(1+p_{\alpha_n})\Big] .
\end{aligned}$$

For $u \neq 1_{\Sigma^*}$, the equality $L = R$ is precisely equivalent to the fact that (ρ, r) is a solution to the TTE associated with $(X, \Sigma, \nu, \lambda, \mu)$. This completes the proof. □

Corollary 4.2. *The model is the same as in Theorem 4.1. Assume that $\lambda\widehat{\gamma} < \mu$. The stationary departure process of customers of class c, $c \in \mathcal{C}$, is a Poisson process of rate $p_c\rho\mu$.*

Proof. Let M be the stationary buffer-content process whose marginal distribution at a given instant is $\pi_{\rho,r}$ given in (61). Let D_c be the departure process of customers of class c (departures at the front-end of the buffer).

Consider the time-reversed process $\widetilde{D_c}$ of D_c. The process $\widetilde{D_c}$ corresponds to the instants of "right-increase" of the time-reversed process $(\widetilde{M}(t))_t$, where the new "right-increase" class is c. Therefore, the instantaneous rate $\tilde{d}_c(t)$ of $\widetilde{D_c}$ is as follows. If $\widetilde{M}(t) = (u,\alpha) = (u_n \cdots u_1, \alpha_n \cdots \alpha_1)$, $n \geq 1$,

$$\tilde{d}_c(t) = \sum_{a\in\text{Next}(u_1)} \frac{\pi_{\rho,r}(ua,\alpha c)}{\pi_{\rho,r}(u,\alpha)}\mu = \sum_{a\in\text{Next}(u_1)} \frac{\rho r(a)p_c}{r(\text{Next}(u_1))}\mu = p_c\rho\mu ,$$

and if $\widetilde{M}(t) = (1_{\Sigma^*}, 1_{\mathcal{C}^*})$,

$$\tilde{d}_c(t) = \sum_{a\in\Sigma} \frac{\pi_{\rho,r}(a,c)}{\pi_{\rho,r}(1_{\Sigma^*}, 1_{\mathcal{C}^*})}\mu = \sum_{a\in\Sigma} p_c\rho r(a)\mu = p_c\rho\mu .$$

We conclude that $\widetilde{D_c}$ is a Poisson process of rate $p_c\rho\mu$. Since Poisson processes are preserved by time-reversal, D_c is also a Poisson process of rate $p_c\rho\mu$.

□

4.2 Description of the network model

Consider a network consisting of N zero-automatic queues with classes $(X_i, \Sigma_i, \nu_i, r_i, \mathcal{C}, (\lambda_c^i), \mu_i)$. Observe that the set of classes $\mathcal{C}$ is the same for all the queues. Let $\mathcal{C}_I$, the set of *initial* classes, be a non-empty subset of $\mathcal{C}$. Let:

$$\Phi : \mathcal{C} \to \mathcal{C} \times \{0\}, \quad \xi : \mathcal{C} \to \{1, \cdots, N\}$$

be two maps, with the following interpretation: a customer of class c queues up at queue $\xi(c)$ and, after completing its service there, he transmutes into a customer of class $\Phi(c)$ or leaves the network if $\Phi(c) = 0$. The function Φ satisfies the following two conditions:

$$\forall c \in \mathcal{C}_I, \quad \Phi^{-1}(c) = \varnothing \tag{65}$$

$$\forall c \in \mathcal{C} \setminus \mathcal{C}_I, \ \exists! c' \in \mathcal{C}, \quad \Phi(c') = c\,. \tag{66}$$

Here is an interpretation of the above two conditions: there is a finite number of possible customer routes and a customer's class determines both its global route and its current position within the route. A class in $\mathcal{C}_I$ corresponds to a route with the current position being the first queue along the route. The equality $\Phi(c') = c$ in (66) means that c' and c are two classes corresponding to the same global route with the current position being one step ahead in c with respect to c'. More precisely, assume that $c_1 \in \mathcal{C}_I$ and that $c_2, \ldots, c_k$ are defined by $c_{i+1} = \Phi(c_i)$ and $\Phi(c_k) = 0$. Then the classes $c_1, \ldots, c_k$ correspond to the same route, and this route is of length k and formed by the sequence of queues $\xi(c_1), \ldots, \xi(c_k)$. Observe that nothing prevents the same queue to appear several times along a given route.

Customers enter the network from the outside according to independent Poisson streams. Let λ_c^i be the rate of arrival of customers of class c in queue i. In accordance with the above interpretation, we have:

$$\forall c \in \mathcal{C}_I, \qquad \lambda_c^{\xi(c)} > 0, \quad \forall i \neq \xi(c), \quad \lambda_c^i = 0 \tag{67}$$

$$\forall c \in \mathcal{C} \setminus \mathcal{C}_I, \forall i, \quad \lambda_c^i = 0\,. \tag{68}$$

Define $E \subset \prod_i L(X_i, \Sigma_i) \times \mathcal{C}^*$ as follows:

$$E = \left\{ (u^1 \cdots u^N, \alpha^1 \cdots \alpha^N) \mid \forall i, \ |u^i|_{\Sigma_i} = |\alpha^i|_{\mathcal{C}}, \ u^i \in L(X_i, \Sigma_i), \ \alpha^i = \alpha^i_{n_i} \cdots \alpha^i_1, \forall j, \xi(\alpha^i_j) = i \right\}. \tag{69}$$

The set E corresponds to the set of all possible buffer-contents in the network.

The resulting network is a *Kelly-like network of zero-automatic queues.* We now propose a more formal definition.

Definition 4.3. *Let $\Phi : \mathcal{C} \to \mathcal{C} \times \{0\}$ and $\xi : \mathcal{C} \to \{1, \cdots, N\}$ be two maps with Φ satisfying (65)-(66). Let $\mathcal{X}_i = (X_i, \Sigma_i, \nu_i, r_i, \mathcal{C}, (\lambda_c^i), \mu_i)$, $1 \leq i \leq N$, be N zero-automatic queues with classes, where (λ_c^i) satisfy the condition (67)-(68). The* Kelly-like network *associated with $((\mathcal{X}_i)_i, \Phi, \xi)$ is defined as follows. The buffer-content process of the network is a continuous time jump Markov process on the state space E defined in (69). Denote by Q its infinitesimal generator. Denote by $Q^i_{(\lambda_c),\mu}$ the infinitesimal generator of the 0-automatic queue with classes $(X_i, \Sigma_i, \nu_i, r_i, \mathcal{C}, (\lambda_c)_c, \mu)$. For $u = u_n \cdots u_1$, $n \geq 1$, set $Dep(u) = u_n \cdots u_2$.*
The generator Q is given by:

$\forall (u,\alpha) \in E,\ (u,\alpha) = (u^i,\alpha^i)_i,\ u^i = a^i_{n_i}\cdots a^i_1,\ \alpha^i = \alpha^i_{n_i}\cdots\alpha^i_1,$

$$\left\{\begin{array}{l}
Q((u,\alpha),(u^1\cdots v^i\cdots u^N,\alpha^1\cdots\beta^i\cdots\alpha^N)) \\
\quad = Q^i_{(\lambda^i_c),0}((u^i,\alpha^i),(v^i,\beta^i))\ ,\quad \forall (v^i,\beta^i)\neq(u^i,\alpha^i)\ , \\
Q((u,\alpha),(u^1\cdots Dep(u^i)\cdots u^N,\alpha^1\cdots Dep(\alpha^i)\cdots\alpha^N)) \\
\quad = \mu_i\ ,\quad \textit{if}\ \Phi(\alpha^i_1)=0\ , \\
Q((u,\alpha),(u^1\cdots Dep(u^i)\cdots v^{\xi\circ\Phi(\alpha^i_1)}\cdots u^N,\alpha^1\cdots Dep(\alpha^i)\cdots\beta^{\xi\circ\Phi(\alpha^i_1)}\cdots\alpha^N)) \\
\quad = Q^{\xi\circ\Phi(\alpha^i_1)}_{(\kappa_c),0}((u^{\xi\circ\Phi(\alpha^i_1)},\alpha^{\xi\circ\Phi(\alpha^i_1)}),(v^{\xi\circ\Phi(\alpha^i_1)},\beta^{\xi\circ\Phi(\alpha^i_1)}))\ , \\
\qquad \textit{if}\ \Phi(\alpha^i_1)\neq 0, \xi\circ\Phi(\alpha^i_1)\neq i, \kappa_c=\mu_i\mathbb{1}_{c=\Phi(\alpha^i_1)},\ \forall (v^{\xi\circ\Phi(\alpha^i_1)},\beta^{\xi\circ\Phi(\alpha^i_1)})\neq(u^{\xi\circ\Phi(\alpha^i_1)},\alpha^{\xi\circ\Phi(\alpha^i_1)}), \\
Q((u,\alpha),(u^1\cdots v^i\cdots u^N,\alpha^1\cdots\beta^i\cdots\alpha^N)) \\
\quad = Q^i_{(\kappa_c),0}((Dep(u^i),Dep(\alpha^i)),(v^i,\beta^i))\ , \\
\qquad \textit{if}\ \xi\circ\Phi(\alpha^i_1)=i, \kappa_c=\mu_i\mathbb{1}_{c=\Phi(\alpha^i_1)},\ \forall (v^i,\beta^i)\neq(u^i,\alpha^i)\ , \\
Q((u,\alpha),(u^1\cdots Dep(u^i)\cdots u^N,\alpha^1\cdots Dep(\alpha^i)\cdots\alpha^N)) \\
\quad = \mu_i - \sum_{(v^{\xi\circ\Phi(\alpha^i_1)},\beta^{\xi\circ\Phi(\alpha^i_1)})\neq(u^i,\alpha^i)} Q^{\xi\circ\Phi(\alpha^i_1)}_{(\kappa_c),0}((u^{\xi\circ\Phi(\alpha^i_1)},\alpha^{\xi\circ\Phi(\alpha^i_1)}),(v^{\xi\circ\Phi(\alpha^i_1)},\beta^{\xi\circ\Phi(\alpha^i_1)}))\ , \\
\qquad \textit{if}\ \kappa_c=\mu_i\mathbb{1}_{c=\Phi(\alpha^i_1)}\ , \\
Q((u,\alpha),(u^1\cdots Dep(u^i)\cdots u^N,\alpha^1\cdots Dep(\alpha^i)\cdots\alpha^N)) \\
\quad = \mu_i - \mathbb{1}_{\xi\circ\Phi(\alpha^i_1)=i}\sum_{(v^i,\beta^i)\neq(u^i,\alpha^i)} Q^i_{(\kappa_c),0}((Dep(u^i),Dep(\alpha^i)),(v^i,\beta^i))\ , \\
\qquad \textit{if}\ \kappa_c=\mu_i\mathbb{1}_{c=\Phi(\alpha^i_1)}\ .
\end{array}\right. \tag{70}$$

Let us comment on the different subcases in (70).

- First case: an external arrival to queue i of a customer of class c.
- Second case: a customer of class c leaves the network.
- Third case: a customer of class c leaves queue $\xi(c)$ and transmutes into a customer of class $\Phi(c)$, where $\xi\circ\Phi(c)\neq\xi(c)$.
- Fourth case: a customer of class c leaves queue $\xi(c)$ and transmutes into a customer of class $\Phi(c)$, where $\xi\circ\Phi(c)=\xi(c)$.
- Fifth case: a customer of class c leaves queue $\xi(c)$, transmutes into a customer of class $\Phi(c)$, then arrives to queue $\xi\circ\Phi(c)\neq\xi(c)$ and has no effect.
- Sixth case: a customer of class c leaves queue $\xi(c)$, transmutes into a customer of class $\Phi(c)$, then arrives to queue $\xi\circ\Phi(c)=\xi(c)$ and has no effect.

4.3 Traffic Equations

Definition 4.4 (K-NTE). *The* Kelly-like Network Traffic Equations *(K-NTE) associated with a Kelly-like network* $((X_i,\Sigma_i,\nu_i,\mathcal{C},(\lambda^i_c),\mu_i)_i,\Phi,\xi)$ *are defined by:*

$$\begin{array}{rcl}
\tilde\lambda^{\xi(c)}_c & = & \left\{\begin{array}{ll} \lambda^{\xi(c)}_c, & \textit{if}\ c\in\mathcal{C}_i \\ \mu_{\xi\circ\Phi^{-1}(c)}\tilde\rho_{\xi\circ\Phi^{-1}(c)}p^{\xi\circ\Phi^{-1}(c)}_{\Phi^{-1}(c)}, & \textit{otherwise}\end{array}\right. \\
\tilde\lambda^i_c & = & 0\ ,\quad \textit{if}\ \ \xi(c)\neq i\ ,
\end{array} \tag{71}$$

where $\tilde\lambda_i=\sum_c\tilde\lambda^i_c$*, where* $p^i_c=\tilde\lambda^i_c/\tilde\lambda_i$*, where* $\tilde\rho_i=\tilde\rho_i(\tilde\lambda_i)$*, where* $(\tilde\rho_i(\tilde\lambda_i),\tilde r_i)$ *is a solution of the TTE associated with* $(X_i,\Sigma_i,\nu_i,\tilde\lambda_i,\mu_i)$*.*

As for the J-NTE, the K-NTE are defined in an intricate and implicit way. Observe also that the K-NTE do not depend on the boundary conditions $(r_i)_i$.

We do not know if the K-NTE always have a solution. In particular, we do not have a counterpart to Lemma 3.4. However, we do have a counterpart to Theorem 3.6.

Theorem 4.5. *Consider a Kelly-like network of N zero-automatic queues $((X_i, \Sigma_i, \nu_i, \mathcal{C}, (\lambda_c^i), \mu_i)_i, \Phi, \xi)$. Assume there exists a solution $(\tilde{\lambda}_c^i)_{i,c}$ to the K-NTE (71). Set $\tilde{\lambda}_i = \sum_c \tilde{\lambda}_c^i$. Let $(\tilde{\rho}_i, \tilde{r}_i)$ be a solution to the TTE associated with $(X_i, \Sigma_i, \nu_i, \tilde{\lambda}_i, \mu_i)$.*

Consider the Kelly-like network $((X_i, \Sigma_i, \nu_i,\ \tilde{r}_i, \mathcal{C}, (\lambda_c^i), \mu_i)_i, \Phi, \xi)$, where we have specified the boundary conditions $(\tilde{r}_i)_i$. Denote by M the jump Markov process of the buffer-contents. We have the equivalences:

$$\big[M \text{ is ergodic}\big] \iff \big[\forall i, \tilde{\rho}_i < 1\big] \iff \big[\forall i, \tilde{\lambda}_i \widehat{\gamma}_i < \mu_i\big] \ ,$$

where $\widehat{\gamma}_i$ is the drift of the random walk associated with (X_i, Σ_i, ν_i).

Assume that $\tilde{\lambda}_i \widehat{\gamma}_i < \mu_i$, for all i. The stationary distribution of the process M has the product form given by: $\forall (u, \alpha) \in E$, $(u, \alpha) = (u^i, \alpha^i)_i$

$$\pi(u, \alpha) = \prod_i \pi^i(u^i, \alpha^i) \ , \tag{72}$$

where $\pi^i(u^i, \alpha^i)$ has the form given by: for $u^i = u^i_{n_i} \cdots u^i_1$, $\alpha^i = \alpha^i_{n_i} \cdots \alpha^i_1$,

$$\pi^i(u^i, \alpha^i) = (1 - \tilde{\rho}_i)(\tilde{\rho}_i)^{n_i} \tilde{q}_i(u^i_{n_i}) \cdots \tilde{q}_i(u^i_2) \tilde{r}_i(u^i_1) p^i_{\alpha_{n_i}} \cdots p^i_{\alpha_1} \ , \tag{73}$$

where $p^i_c = \tilde{\lambda}^c_i / \tilde{\lambda}_i$, $\tilde{q}_i(a) = \tilde{r}_i(a) / \tilde{r}_i(Next(a))$ for $a \in \Sigma_i$.

Proof. We have to prove the balance equations:

$$\sum_{(v,\beta) \neq (u,\alpha)} \pi(u, \alpha) Q((u, \alpha), (v, \beta)) = \sum_{(v,\beta) \neq (u,\alpha)} \pi(v, \beta) Q((v, \beta), (u, \alpha)) \tag{74}$$

Define

$$A^i(\tilde{r}_i) = \sum_{a \in \Sigma_i} \nu_i(a) \tilde{r}_i(\mathrm{Next}(a)) \ , \quad \lambda_i = \sum_c \lambda^i_c \ . \tag{75}$$

The left of (74) is:

$$\begin{aligned} L \quad &= \quad \pi(u, \alpha) \sum_i \mathbb{1}_{n_i > 0} \Big(\lambda_i - \nu_i(\mathrm{Noact}(a^i_{n_i})) \big(\frac{1}{2} \lambda_i + \frac{1}{2} \lambda^i_{\alpha^i_{n_i}} \big) + \mu_i \Big) \\ &\quad + \pi(u, \alpha) \sum_i \mathbb{1}_{n_i = 0} \lambda_i A^i(\tilde{r}_i) \ . \end{aligned} \tag{76}$$

For all i, Theorem 4.1 yields the following equations:

$$\begin{aligned} \sum_{(v^i,\beta^i)} \pi^i(v^i, \beta^i) Q^i_{(\tilde{\lambda}^i_c),\mu_i}((v^i, \beta^i), (u^i, \alpha^i)) \quad &= \quad \sum_{(v^i,\beta^i)} \pi^i(u^i, \alpha^i) Q^i_{(\tilde{\lambda}^i_c),\mu_i}((u^i, \alpha^i), (v^i, \beta^i)) \\ &= \quad \mathbb{1}_{n_i=0} \pi^i(u^i, \alpha^i) \tilde{\lambda}_i A^i(\tilde{r}_i) \ + \mathbb{1}_{n_i>0} \pi^i(u^i, \alpha^i) \\ &\qquad \big[\mu_i + \tilde{\lambda}_i \big(1 - \frac{1}{2}(1 + p^i_{\alpha^i_{n_i}}) \nu_i(\mathrm{Noact}(a^i_{n_i}))\big)\big] . \end{aligned} \tag{77}$$

The right of (74) is:

$$
\begin{aligned}
R &= \sum_i \frac{\pi(u,\alpha)}{\pi^i(u^i,\alpha^i)} \sum_{(v^i,\beta^i)} \pi^i(v^i,\beta^i) Q^i_{(\lambda^i_c),0}((v^i,\beta^i),(u^i,\alpha^i)) \\
&\quad + \sum_i \frac{\pi(u,\alpha)}{\pi^i(u^i,\alpha^i)} \sum_{\substack{v^i|\mathrm{Dep}(v^i)=u^i \\ \beta^i|\mathrm{Dep}(\beta^i)=\alpha^i}} \pi^i(v^i,\beta^i) \mathbb{1}_{\xi(\beta^i_1)=i} \mathbb{1}_{\Phi(\beta^i_1)=0} \mu_i \\
&\quad + \sum_{j\neq i} \frac{\pi(u,\alpha)}{\pi^j(u^j,\alpha^j)} \sum_{\substack{v^j|\mathrm{Dep}(v^j)=u^j \\ \beta^j|\mathrm{Dep}(\beta^j)=\alpha^j}} \pi^j(v^j,\beta^j) \mathbb{1}_{\xi(\beta^j_1)=j} \mathbb{1}_{(\Phi,\xi)(\beta^j_1)=(c,i)} \\
&\quad \Big\{ \mathbb{1}_{n_i=0}\mu_j(1-A^i(\tilde r_i)) + \mathbb{1}_{n_i>0}\mu_j \nu_i(\mathrm{Noact}(a^i_{n_i}))\big(\mathbb{1}_{c=\alpha^i_{n_i}} + \frac{1}{2}\mathbb{1}_{c\neq\alpha^i_{n_i}}\big) \\
&\quad + \sum_{v^i,\beta^i} \frac{\pi^i(v^i,\beta^i)}{\pi^i(u^i,\alpha^i)} Q^i_{(\lambda_{c'}=\mu_j\mathbb{1}_{c'=c}),0}((v^i,\beta^i),(u^i,\alpha^i)) \Big\} \\
&\quad + \sum_i \mathbb{1}_{\xi(\beta^i_1)=i} \mathbb{1}_{(\Phi,\xi)(\beta^i_1)=(c,i)} \Big\{ \sum_{\substack{v^i|\mathrm{Dep}(v^i)=u^i \\ \beta^i|\mathrm{Dep}(\beta^i)=\alpha^i}} \pi^i(v^i,\beta^i) \\
&\quad \big[\mathbb{1}_{n_i=0}\mu_i(1-A^i(\tilde r_i)) + \mathbb{1}_{n_i>0}\mu_i\nu_i(\mathrm{Noact}(a^i_{n_i}))\big(\mathbb{1}_{c=\alpha^i_{n_i}} + \frac{1}{2}\mathbb{1}_{c\neq\alpha^i_{n_i}}\big)\big] \\
&\quad + \sum_{v^i,\beta^i} \frac{\pi^i(v^i,\beta^i)}{\pi^i(u^i,\alpha^i)} Q^i_{(\lambda_{c'}=\mu_j\mathbb{1}_{c'=c}),0}((\mathrm{Dep}(v^i),\mathrm{Dep}(\beta^i)),(u^i,\alpha^i)) \Big\} \\
&= B_1 + B_2 + B_3 + B_4 \ . \qquad (78)
\end{aligned}
$$

Let us consider the term $B_3 + B_4$. Using the K-NTE (71) and the form of the infinitesimal generator Q given in (70), it implies:

$$
\begin{aligned}
\frac{B_3+B_4}{\pi(u,\alpha)} &= \sum_j \tilde\rho_j \sum_{\substack{\beta^j|\xi(\beta^j_1)=j \\ \mathrm{Dep}(\beta^j)=\alpha^j}} \mathbb{1}_{(\Phi,\xi)(\beta^j_1)=(c,i)} p^j_{\beta^j_1} \Big\{ \mathbb{1}_{n_i>0}\mu_j \big[\nu_i(\mathrm{Noact}(a^i_{n_i}))\big(\mathbb{1}_{c=\alpha^i_{n_i}} + \frac{1}{2}\mathbb{1}_{c\neq\alpha^i_{n_i}}\big) \\
&\quad -1\big] - \mathbb{1}_{n_i=0}\mu_j A^i(\tilde r_i) + \frac{1}{\pi^i(u^i,\alpha^i)} \sum_{v^i,\beta^i} \pi^i(v^i,\beta^i) Q^i_{(\lambda_{c'}=\mu_j\mathbb{1}_{c'=c}),0}((v^i,\beta^i),(u^i,\alpha^i)) \Big\} \\
&= \sum_i \sum_{c|\exists\Phi^{-1}(c)} \Big\{ \mathbb{1}_{n_i>0}\tilde\lambda^i_c \big[\nu_i(\mathrm{Noact}(a^i_{n_i}))\big(\mathbb{1}_{c=\alpha^i_{n_i}} + \frac{1}{2}\mathbb{1}_{c\neq\alpha^i_{n_i}}\big) - 1\big] \\
&\quad - \mathbb{1}_{n_i=0}\tilde\lambda^i_c A^i(\tilde r_i) + \frac{1}{\pi^i(u^i,\alpha^i)} \sum_{v^i,\beta^i} \pi^i(v^i,\beta^i) Q^i_{(\lambda_{c'}=\tilde\lambda^i_c\mathbb{1}_{c'=c}),0}((v^i,\beta^i),(u^i,\alpha^i)) \Big\}
\end{aligned}
$$

Substitute $B_3 + B_4$ in (78) and using (77), one gets:

$$
\begin{aligned}
\frac{R}{\pi(u,\alpha)} &= \sum_i \Big\{ \mathbb{1}_{n_i=0}\tilde\lambda_i A^i(\tilde r_i) \ + \mathbb{1}_{n_i>0}\pi^i(u^i,\alpha^i)\big[\mu_i + \tilde\lambda_i\big(1 - \frac{1}{2}(1+p^i_{\alpha^i_{n_i}})\nu_i(\mathrm{Noact}(a^i_{n_i}))\big)\big] \Big\} \\
&\quad + \sum_i \Big\{ \mathbb{1}_{n_i>0}\big[-(\tilde\lambda_i - \lambda_i) + \nu_i(\mathrm{Noact}(a^i_{n_i})) \sum_{c|\exists\Phi^{-1}(c)} \tilde\lambda^i_c\big(\mathbb{1}_{c=\alpha^i_{n_i}} + \frac{1}{2}\mathbb{1}_{c\neq\alpha^i_{n_i}}\big)\big] \\
&\quad - \mathbb{1}_{n_i=0}(\tilde\lambda_i - \lambda_i)A^i(\tilde r_i) \Big\} \qquad (79) \\
&= \sum_i \mathbb{1}_{n_i=0}\lambda_i A^i(\tilde r_i) + \sum_i \mathbb{1}_{n_i>0} \Big\{ \lambda_i + \mu_i - \nu(\mathrm{Noact}(a^i_{n_i}))\big(\frac{1}{2}\lambda_i + \frac{1}{2}\lambda^i_{\alpha^i_{n_i}}\big) \Big\}
\end{aligned}
$$

Therefore, the balance equation $L = R$ is proved. This completes the proof.

$\square$

4.4 Examples

4.4.1 The FIFO Kelly network

Consider a Kelly-like network of N queues built on the pairs $(X_i = \{a_i^k, k \in \mathbb{N}\}, \Sigma_i = \{a_i\})$. Obviously, $\nu_i(a_i) = r_i(a_i) = 1$. There is no simplification in the buffers, and the service discipline is FIFO.

Let$\mathcal{C}$ be the set of possible classes. Let Φ and ξ be the two maps defining the Kelly-like network, see Section 4.2. Let:

$$\Phi^- : \mathcal{C} \to \mathcal{C}$$

be the map defined by: if $\Phi^{-n}(c) = \varnothing$ and $\Phi^{-(n-1)}(c) \neq \varnothing$ then $\Phi^-(c) = \Phi^{-(n-1)}(c)$. The interpretation is that $\Phi^-(c)$ is the class in $\mathcal{C}_I$ in which a customer of class c "enters" the network.

The K-NTE associated with the Kelly-like network $((X_i, \Sigma_i, \nu_i, \mathcal{C}, (\lambda_c^i), \mu_i)_i, \Phi, \xi)$ are given by:

$$\tilde{\lambda}_c^{\xi(c)} = \lambda_{\Phi^-(c)}^{\xi\circ\Phi^-(c)} , \qquad \tilde{\lambda}_c^i = 0, \quad \text{if } \xi(c) \neq i \ . \tag{80}$$

Then $\tilde{\rho}_i$ and $\tilde{\lambda}_i$ are given by:

$$\tilde{\lambda}_i = \sum_{c|\xi(c)=i} \lambda_{\Phi^-(c)}^{\xi\circ\Phi^-(c)} , \qquad \tilde{\rho}_i = \frac{\tilde{\lambda}_i}{\mu_i} \ .$$

Set $\tilde{\rho}_c^i = p_c^i \tilde{\rho}_i = \tilde{\lambda}_c^i/\mu_i$. Under the stability condition $\tilde{\rho}_i < 1$ for all i, the stationary distribution of the network is given by: $\forall (u, \alpha) \in E, u = (n_i)_i, \alpha = (\alpha^i)_i$,

$$\pi(u, \alpha) = \prod_i \pi^i(n_i, \alpha^i) \ ,$$

where $\pi^i(n_i, \alpha^i)$ has the form given by: for $\alpha^i = \alpha_{n_i} \cdots \alpha_1$,

$$\pi^i(n_i, \alpha^i) = \tilde{\rho}_{\alpha_{n_i}}^i \cdots \tilde{\rho}_{\alpha_1}^i \ .$$

Here, we recover exactly the Kelly network with FIFO discipline, see [10].

4.4.2 A job-shop model

Consider the network represented in Figure 6. There are three possible routes: $(1) = 1 \to 3 \to 4$, $(2) = 2 \to 3 \to 4$, $(3) = 2 \to 3 \to 5$. This model is picked from [10]. The set of classes is $\mathcal{C} = \{11, 13, 14, 22, 23, 24, 32, 33, 35\}$. The class $ki \in \mathcal{C}$ corresponds to the route (k) and to the current queue i.

Consider five plain triples (X_i, Σ_i, ν_i). The external arrival rates are λ_{11}^1, λ_{22}^2, λ_{32}^2. The service rate in queue i is μ_i. Denote by $(\rho_i(\Lambda_i), r_i(\Lambda_i))$ an admissible solution to the TTE associated

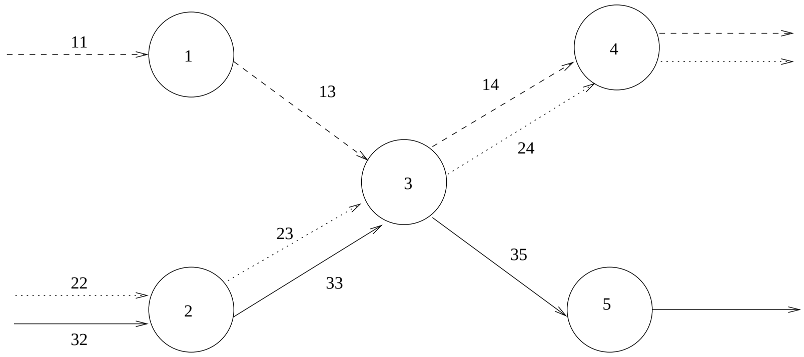

Figure 6: A job-shop model.

with $(X_i, \Sigma_i, \nu_i, \Lambda_i, \mu_i)$. Hence, the K-NTE associated with the network are given by:

$$\left\{ \begin{array}{rcl} \tilde{\lambda}_{11}^{1} & = & \lambda_{11}^{1}, \quad \tilde{\lambda}_{22}^{2} = \lambda_{22}^{2}, \quad \tilde{\lambda}_{32}^{3} = \lambda_{32}^{3} \ , \\ \tilde{\lambda}_{13}^{1} & = & \rho_1(\lambda_{11}^{1})\mu_1 \ , \\ \tilde{\lambda}_{23}^{2} & = & \rho_2(\lambda_{22}^{2} + \lambda_{32}^{3})\mu_2 \dfrac{\lambda_{22}^{2}}{\lambda_{22}^{2} + \lambda_{32}^{3}} \ , \\ \tilde{\lambda}_{33}^{3} & = & \rho_2(\lambda_{22}^{2} + \lambda_{32}^{3})\mu_2 \dfrac{\lambda_{32}^{3}}{\lambda_{22}^{2} + \lambda_{32}^{3}} \ , \\ \tilde{\lambda}_{14}^{1} & = & \rho_2(\tilde{\lambda}_{13}^{1} + \tilde{\lambda}_{23}^{2} + \tilde{\lambda}_{33}^{3})\mu_3 \dfrac{\tilde{\lambda}_{13}^{1}}{\tilde{\lambda}_{13}^{1} + \tilde{\lambda}_{23}^{2} + \tilde{\lambda}_{33}^{3}} \ , \\ \tilde{\lambda}_{24}^{2} & = & \rho_2(\tilde{\lambda}_{13}^{1} + \tilde{\lambda}_{23}^{2} + \tilde{\lambda}_{33}^{3})\mu_3 \dfrac{\tilde{\lambda}_{23}^{2}}{\tilde{\lambda}_{13}^{1} + \tilde{\lambda}_{23}^{2} + \tilde{\lambda}_{33}^{3}} \ , \\ \tilde{\lambda}_{35}^{3} & = & \rho_2(\tilde{\lambda}_{13}^{1} + \tilde{\lambda}_{23}^{2} + \tilde{\lambda}_{33}^{3})\mu_3 \dfrac{\tilde{\lambda}_{33}^{3}}{\tilde{\lambda}_{13}^{1} + \tilde{\lambda}_{23}^{2} + \tilde{\lambda}_{33}^{3}} \ . \end{array} \right. \tag{81}$$

Since the routing is simple enough, the K-NTE (81) can be solved explicitly as soon as the TTE associated with $(X_i, \Sigma_i, \nu_i, \Lambda_i, \mu_i)$ are solvable.

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