Zero-Automatic Queues and Product Form

Thu-Ha Dao-Thi and Jean Mairesse *

October 10, 2006

Abstract

We introduce and study a new model: *0-automatic queues*. Roughly, 0-automatic queues are characterized by a special buffering mechanism evolving like a random walk on some infinite group or monoid. The salient result is that all stable 0-automatic queues have a product form stationary distribution and a Poisson output process. When considering the two simplest and extremal cases of 0-automatic queues, we recover the simple *M/M/1* queue, and Gelenbe's *G*-queue with positive and negative customers.

Keywords: Queueing theory, M/M/1 queue, G-queue, quasi-reversibility, product form, Quasi-Birth-and-Death process.

AMS classification (2000): Primary 60K25, 68M20.

1 Introduction

Here is an informal description of a special type of 0-automatic queue (corresponding to a free product of three finite monoids). Consider a queue with a single server and an infinite capacity buffer. Customers are colored either in Red, Blue, or Green, with a finite set of possible shades within each color: $\Sigma_R, \Sigma_B, \Sigma_G$. In the buffer, two consecutive customers of the same color either cancel each other or merge to give a new customer of the same color. Customers of different colors do not interact. This is illustrated in Figure 1.

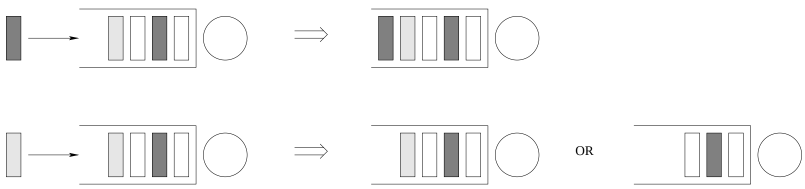

Figure 1: A 0-automatic queue.

The shades get modified in the merging procedure, according to an internal law: $\Sigma_i \times \Sigma_i \to \Sigma_i \cup \{1\}$, with 1 coding for the cancellation. The only but crucial restriction is that each internal law should be associative.

We now give a more detailed account of the model and results. Zero-automatic queues may be viewed as the synthesis of a simple queue and a random walk on a 0-automatic pair. We first recall these last two models.

*LIAFA, CNRS-Université Paris 7, case 7014, 2, place Jussieu, 75251 Paris Cedex 05, France. E-mail: `(daothi,mairesse)@liafa.jussieu.fr`

The $M/M/1/\infty$ FIFO queue, or simply $M/M/1$ queue, is the Markovian queue with arrivals and services occurring at constant rate, say λ and μ, a single server, an infinite capacity buffer, and a First-In-First-Out discipline. This is arguably the simplest and also the most studied model in queueing theory, with at least one book devoted to it [7]. The queue-length process is a continuous time jump Markov process and its infinitesimal generator Q is given by: $\forall n \in \mathbb{N},\ Q(n,n+1)=\lambda,\ Q(n+1,n)=\mu$. Under the stability condition $\lambda < \mu$, the queue-length process is ergodic, and its stationary distribution π is given by:

$$\pi(n) = \big(1-\frac{\lambda}{\mu}\big)\Big(\frac{\lambda}{\mu}\Big)^n . \tag{1}$$

Besides, and this constitutes the celebrated Burke Theorem, the departure process in equilibrium has the same law as the arrival process.

Let us introduce the a priori completely unrelated model of random walk on a plain group studied in [21, 22].

Let X be an infinite group or monoid with a finite set of generators Σ. Let ν be a probability measure on Σ and let $(x_i)_{i\in\mathbb{N}}$ be a sequence of Σ-valued i.i.d. r.v.'s of law ν. Let $(X_n)_n$ be the sequence of X-valued r.v.'s defined by: $X_0 = 1_X,\ X_{n+1} = X_n * x_n = x_0 * x_1 * \cdots * x_n$, where 1_X is the unit element of X and $*$ is the group or monoid law. By definition, $(X_n)_n$ is a realization of the random walk (X,ν).

We now assume that the pair (X,Σ) is formed by a *plain* monoid with *natural* generators. The definition will be given in Section 2. For the moment, it suffices to say that the elements of X can be set in bijection with a regular language $L(X,\Sigma) \subset \Sigma^*$. The random walk $(X_n)_n$ is viewed as evolving on $L(X,\Sigma)$. If $X_n = ua, a \in \Sigma$, and $x_n = b \in \Sigma$, then

$$X_{n+1} = u \ \text{ if } a*b = 1_X, \quad X_{n+1} = uc \ \text{ if } a*b = c \in \Sigma, \quad X_{n+1} = uab \ \text{ otherwise} . \tag{2}$$

Now assume that the random walk is transient. Let $\nu^\infty(u\Sigma^{\mathbb{N}})$ be the probability that the random walk goes to infinity in the "direction" u (i.e. $\nu^\infty(u\Sigma^{\mathbb{N}}) = P\{\exists N, \forall n \geq N, X_n \in u\Sigma^*\}$). The following is the main result in [21]:

$$\forall u = u_1 \cdots u_n \in L(X,\Sigma), \quad \nu^\infty(u\Sigma^{\mathbb{N}}) = \widehat{q}(u_1)\cdots\widehat{q}(u_{n-1})\widehat{r}(u_n) , \tag{3}$$

where $\forall a \in \Sigma,\ \widehat{q}(a) \in (0,1), \widehat{r}(a) \in (0,1)$.

The expressions in (1) and (3) share a common "multiplicative" structure. Guided by this analogy, we want to merge the two models together. To that purpose, we make the following elementary observation: if we block the server in an $M/M/1$ queue, the number of waiting customers after n arrivals is $A_n = n$. And $(A_n)_n$ can be viewed as the (not so random) random walk on the pair $((\mathbb{N},+),\{1\})$ associated with the probability $\nu:\ \nu(1)=1$.

Now, replace the trivial random walk $(A_n)_n$ by another, more complex, random walk $(X_n)_n$ on a plain triple (X,Σ,ν). Hence, the random walk $(X_n)_n$ constitutes the buffering mechanism in a queue with a blocked server. A *0-automatic queue* is the model obtained when unblocking the server. The set Σ is the set of possible *classes* for customers. Customers arrive at constant rate λ. Upon arrival, a new customer (class b) interacts with the customer presently at the back-end of the buffer (class a), according to (2). At the front-end of the buffer, customers are served at constant rate μ.

Let us comment on the name *zero-automatic*. Plain groups, see (7), are *automatic* in the sense of Epstein and al [12]. Automatic groups form an important class of groups extensively studied in geometric group theory, the adjective "automatic" referring to the existence of automata to recognize and multiply elements of the group. Now the pairs (X,Σ) formed by a plain group with natural generators satisfy the *0-fellow traveller property*, see [12]. It was proposed in [21] to call (X, Σ) a *0-automatic pair*. By extension, a queue built upon (X, Σ) is *0-automatic*. The name is also supposed to evoke the local aspect of the interactions between customers in the buffer, see (2).

Let $\widehat{\gamma}$ be the drift or rate of escape to infinity of the random walk $(X_n)_n$. We prove in Section 4 that the stability condition for the 0-automatic queue associated with $(X_n)_n$ is: $\lambda\widehat{\gamma} < \mu$. Under this condition, we prove in Section 5 that the stationary distribution p for the queue-content process has a "multiplicative" structure:

$$\forall u = u_1 \cdots u_n \in L(X, \Sigma), \quad p(u) = (1-\rho)\rho^n q(u_1) \cdots q(u_{n-1}) r(u_n) , \tag{4}$$

for some numbers $\rho \in (0,1)$, $\forall a \in \Sigma$, $q(a) \in (0,1), r(a) \in (0,1)$. (These numbers are in general different from their counterparts in (1) and (3).) Furthermore, the departure process from the queue is a Poisson process of rate $\rho\mu$. Thus we have an analog of Burke Theorem for all 0-automatic queues. Using standard terminology, 0-automatic queues are *quasi-reversible*.

To be more precise, given $(X_n)_n$, several variants of 0-automatic queues can be defined depending on the way customers are incorporated in an empty queue (boundary condition). There is precisely *one* choice for which the result in (4) holds. The numbers $\rho, q(\cdot), r(\cdot)$, as well as the right boundary condition, are obtained implicitly via the unique solution of a set of algebraic equations, see Theorems 5.7 and 5.8 for a precise statement.

Aside from the free monoid, the next simplest example of a plain monoid is the free group over one generator: $(\mathbb{Z}, +)$. The 0-automatic queues associated with $(\mathbb{Z}, \{-1, 1\})$ are variations of Gelenbe's G-queues, or queues with positive and negative customers, which were quite extensively studied in the 90's, see [14, 13] and the bibliography in [15]. General 0-automatic queues can be viewed as a wide generalization of this setting. Indeed, in a 0-automatic queue, different types of tasks (customers) can be modelled. Let us detail four of them which form a representative sample, without exhausting all the types within the realm of 0-automaticity.

- Classical type. Tasks are processed one by one with no simplification occurring in the buffer: $aa = aa$. The corresponding pair is $(\mathbb{N}, \{1\}) \sim (\{a\}^*, \{a\})$.

- Positive/negative type. Tasks are either positive (a) or negative (a^{-1}) and two consecutive tasks of opposite signs cancel each other: $aa^{-1} = a^{-1}a = 1$. The corresponding pair is $(\mathbb{Z}, \{1, -1\}) \sim (\mathbb{F}(a), \{a, a^{-1}\})$. The relevance of this type for applications is discussed in [15].

- "One equals many" type. It takes the same time to process one or several consecutive instances of the same task: $aa = a$. Think for instance of a ticket reservation where the number of requests is only reflected by an integer value in a menu-bar choice. The corresponding pair is $(\mathbb{B}, \{a\})$ where $\mathbb{B}$ is the Boolean monoid $\mathbb{B} = \langle\, a \mid a^2 = a \,\rangle$.

- "Dating agency" type. Two instances of the same task cancel each other: $aa = 1$. Think of a task as being a tennis player looking for a partner (to be provided by the server); when two such tasks are next to each other in the buffer, they leave to play a game instead of waiting in line. The corresponding pair is $(\mathbb{Z}/2\mathbb{Z} = \langle\, a \mid a^2 = 1 \,\rangle, \{a\})$. Instead of tennis players, we may consider music trio players, bridge players, etc, the corresponding group being $\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, etc.

To model a server where several of the above types (and possibly several copies of the same type) can be processed, one just has to perform the free product of the corresponding monoids or groups (see Section 2.1 for the definition).

The $M/M/1$ queue is the basic primitive for building Jackson networks, which have the remarkable property of having a "product-form" stationary distribution. More generally, networks made of quasi-reversible nodes tend to have a product form distribution, see for instance [25]. In a subsequent work [9], we prove that it is indeed the case for Jackson-type and Kelly-type networks of 0-automatic queues.

A preliminary version without proofs of the present paper has appeared in the conference proceedings [8].

2 Preliminaries

Notations. We denote by $\mathbb{N}$ and $\mathbb{R}_+$ the nonnegative integers and reals. We denote by $\mathbb{N}^*$ and $\mathbb{R}_+^*$ the positive integers and reals. The symbol $\sqcup$ is used for the disjoint union of sets. Given a set T and $S \subset T$, define $\mathbb{1}_S : T \to \{0,1\}$ by $\mathbb{1}_S(u) = 1$ if $u \in S$ and $\mathbb{1}_S(u) = 0$ otherwise. Given a set T, a vector $x \in \mathbb{R}^T$, and $S \subset T$, set $x(S) = \sum_{u \in S} x(u)$.

Let us recall the needed material on random walks on plain monoids. The presentation follows [21, 22].

2.1 Monoids and groups

Given a set Σ, the free monoid generated by Σ is denoted by Σ^*. The unit element is denoted by 1 or 1_{Σ^*}. As usual, the elements of Σ and Σ^* are called *letters* and *words*, respectively. The subsets of Σ^* are called *languages*. The *length* (number of letters) of a word u is denoted by $|u|_\Sigma$.

Let $(X, *)$ be a group or monoid with set of generators Σ. The unit element of X is denoted by 1_X. When X is a group, the inverse of $x \in X$ is denoted by x^{-1}. We always assume that: $1_X \notin \Sigma$, and in the group case that: $x \in \Sigma \implies x^{-1} \in \Sigma$. The *length* with respect to Σ of an element x of X is:

$$|x|_\Sigma = \min\{k \mid x = a_1 * \cdots * a_k, a_i \in \Sigma\} \tag{5}$$

The *Cayley graph* $\mathfrak{X}(X, \Sigma)$ of X with respect to Σ is the directed graph with nodes X and arcs $u \to v$ if $\exists a \in \Sigma,\ u * a = v$.

Consider a relation $R \subset \Sigma^* \times \Sigma^*$, and let $\sim_R$ be the least congruence on Σ^* such that $u \sim_R v$ if $(u, v) \in R$. Let X be isomorphic to the quotient monoid $(\Sigma^*/\sim_R)$. We say that $\langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$ is a *monoid presentation* of X and we write $X = \langle\, \Sigma \mid u = v, (u, v) \in R \,\rangle$.

Given a set S, denote by $\mathbb{F}(S)$ the free group generated (as a group) by S. Let S^{-1} be the set of inverses of the generators. A monoid presentation of $\mathbb{F}(S)$ is

$$\mathbb{F}(S) = \langle\, S \sqcup S^{-1} \mid aa^{-1} = 1, a^{-1}a = 1,\ \forall a \in S \,\rangle\,. \tag{6}$$

Given two groups or monoids X_1 and X_2, we denote by $X_1 \star X_2$ the *free product* of X_1 and X_2. Roughly, the elements of $X_1 \star X_2$ are the finite alternate sequences of elements of $X_1 \backslash \{1_{X_1}\}$ and $X_2 \backslash \{1_{X_2}\}$, and the law is the concatenation with simplification. More rigorously, the definition is as follows. Set $S = X_1 \sqcup X_2$. The *free product* $X_1 \star X_2$ is defined by the monoid presentation:

$$\langle\ S \ \mid\ (\forall u, v \in S^*, \forall i \in \{1,2\}),\ u1_{X_i}v = uv, (\forall a, b, c \in X_i,\ \text{s.t. } c = a * b),\ uabv = ucv\ \ \rangle\,.$$

If X_1 and X_2 are groups, then $X_1 \star X_2$ is also a group. The free product of more than two groups or monoids is defined analogously.

The Cayley graph of the plain group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ is represented on Figure 2 (left).

2.2 Plain monoids and groups

A *plain monoid* is a monoid X of the form

$$X = S^* \star \mathbb{F}(T) \star X_1 \star \cdots \star X_k \,, \tag{7}$$

where S and T are finite sets and $X_1, \ldots, X_k$ are finite monoids. A *plain group* is a plain monoid which is also a group. A plain monoid X defined as in (7) is a plain group iff $S = \emptyset$ and $X_1, \cdots, X_k$ are groups.
Define

$$\Sigma = S \sqcup T \sqcup T^{-1} \sqcup X_1 \setminus \{1_{X_1}\} \sqcup \cdots \sqcup X_k \setminus \{1_{X_k}\} \,. \tag{8}$$

The set Σ is a finite set of generators of X, that we call *natural* generators. Define the language $L(X, \Sigma) \subset \Sigma^*$ by:

$$L(X, \Sigma) = \big\{ u_1 \cdots u_k \mid \forall i < k, u_i * u_{i+1} \notin \Sigma \cup \{1_X\} \big\} \,. \tag{9}$$

It is easily seen that the set $L(X, \Sigma)$ is in bijection with the group elements. Below we often identify X and $L(X, \Sigma)$.
It is convenient to introduce the sets: $\forall a \in \Sigma$,

$$\mathrm{Next}(a) = \{ b \in \Sigma \mid b * a \notin \Sigma \cup \{1_X\}\} = \{ b \in \Sigma \mid a * b \notin \Sigma \cup \{1_X\}\} \,.$$

Observe that $L(X, \Sigma) = \big\{ u_1 \cdots u_k \mid \forall i,\ u_{i-1} \in \mathrm{Next}(u_i) \big\} = \big\{ u_1 \cdots u_k \mid \forall i,\ u_{i+1} \in \mathrm{Next}(u_i) \big\}$.

Consider the directed *graph of successors* $(\Sigma, \to)$ where

$$a \to b \quad \text{if} \quad b \in \mathrm{Next}(a) \,. \tag{10}$$

It is easily proved that $(\Sigma, \to)$ is strongly connected.

2.3 Random walks on monoids and groups

Let $(X, *)$ be a group or monoid with finite set of generators Σ. Let ν be a probability distribution over Σ. Consider the Markov chain on the state space X with one-step transition probabilities given by: $\forall x \in X, \forall a \in \Sigma$, $P_{x, x*a} = \nu(a)$. This Markov chain is called the *(right) random walk* (associated with) (X, ν).
Let $(x_n)_n$ be a sequence of i.i.d. r.v's distributed according to ν. Set

$$X_0 = 1_X, \quad X_{n+1} = X_n * x_n = x_0 * \cdots * x_n \,. \tag{11}$$

Then $(X_n)_n$ is a realization of the random walk (X, ν). For all $x, y \in X$, we have $|x * y|_\Sigma \leq |x|_\Sigma + |y|_\Sigma$. Applying Kingman's Subadditive Ergodic Theorem yields the following (first noticed by Guivarc'h [16]): there exists $\gamma \in \mathbb{R}_+$ such that

$$\lim_{n \to \infty} \frac{|X_n|_\Sigma}{n} = \gamma \quad \text{a.s and in } L^p, \tag{12}$$

for all $1 \leq p < \infty$. We call γ the *drift* of the random walk.
To illustrate, consider the plain group $X = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z} = \langle\ a \mid a^2 = 1 \ \rangle \star \langle\ b \mid b^3 = 1\ \rangle$ and the natural generators $\Sigma = \{a, b, b^2 = b^{-1}\}$. Let ν be a probability measure on Σ. On the left of Figure 2, we have represented a finite part of the infinite Cayley graph $\mathfrak{X}(X, \Sigma)$, and the one-step transitions of the random walk (X, ν) starting from the state ba. On the right of the figure, we show the same one-step transitions on the group elements viewed as words of $L(X, \Sigma)$ (written from bottom to top).

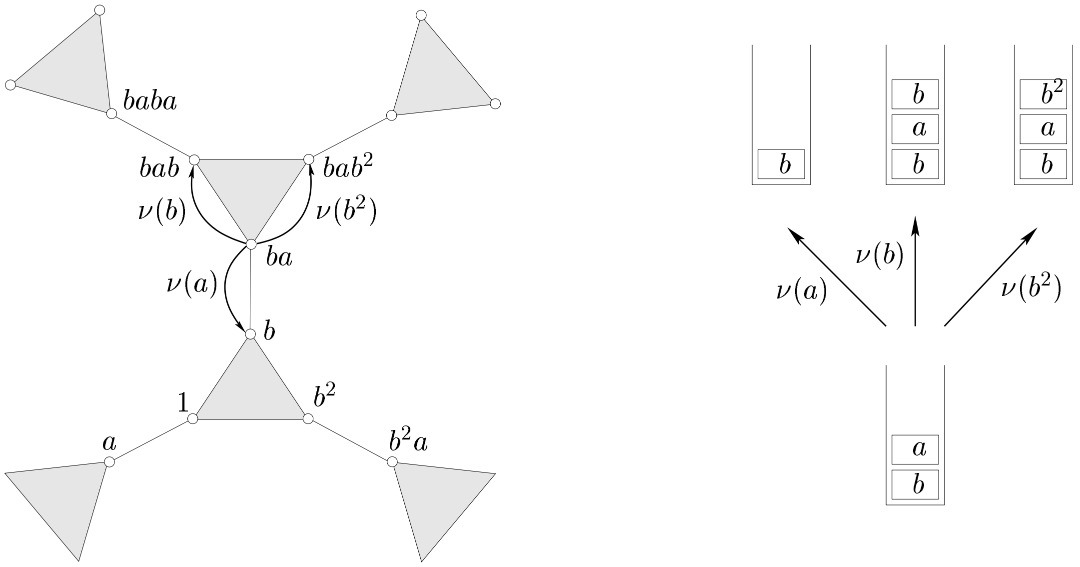

Figure 2: The random walk $(\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, \nu)$.

2.4 Random walks on plain monoids and groups

It is convenient to introduce the notion of a plain triple.

Definition 2.1. *A triple (X, Σ, ν) is* plain *if: (i) X is an infinite plain monoid not isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$; (ii) Σ is a set of natural generators; (iii) ν is a probability measure whose support is included in Σ and generates X.*

Proposition 2.2. *If (X, Σ, ν) is a plain triple, then the random walk (X, ν) is transient.*

If X is an infinite plain monoid with the support of ν generating X, there are only two cases in which (X, ν) is not transient: (1) the triple $(\mathbb{Z}, \{-1, 1\}, \{1/2, 1/2\})$; (2) the triples $(\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}, \{a, b\}, \nu)$, for any ν, where a and b are the respective generators of the two cyclic groups. Since $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ have been excluded from consideration, then the random walk (X, ν) is transient, see [21] for details.

The case $X = \mathbb{Z}$ is specific. Some of the results below remain true but not all of them. For simplicity, we treat this case separately in §6.

Define:

$$\mathcal{B} = \{x \in \mathbb{R}^\Sigma \mid \forall i, x(i) > 0, \sum_i x(i) = 1\}, \qquad \bar{\mathcal{B}} = \{x \in \mathbb{R}^\Sigma \mid \forall i, x(i) \geq 0, \sum_i x(i) = 1\}\,. \tag{13}$$

The Traffic Equations play an essential role in the study of the random walk (X, ν).

Definition 2.3. *The* Traffic Equations (TE) *associated with a plain triple (X, Σ, ν) are the equations of the variables $(x(a))_{a \in \Sigma} \in \mathbb{R}_+^\Sigma$ defined by: $\forall a \in \Sigma$,*

$$x(a) = \nu(a) x(Next(a)) + \sum_{b * d = a} \nu(b) x(d) + \sum_{\substack{d \in Next(a) \\ b * d = 1_X}} \nu(b) \frac{x(d)}{x(Next(d))} x(a)\,. \tag{14}$$

An admissible solution *is a solution belonging to $\mathcal{B}$.*

By multiplying both sides of (14) by $\prod_b x(\text{Next}(b))$, we obtain a new set of Equations without denominators. With some abuse, a solution r in $\bar{\mathcal{B}}$ of this last set of Equations is still called a solution of the TE.

Next result can be easily deduced from the proof of [21, Theorem 4.5].

Proposition 2.4. *Let (X, Σ, ν) be a plain triple. The Traffic Equations have a unique admissible solution.*

The interest of Proposition 2.4 is that the harmonic measure and the drift can be expressed as a function of the solution to the TE. Define the set $L^\infty \subset \Sigma^{\mathbb{N}}$ by

$$L^\infty = \{u_0 u_1 \cdots u_k \cdots \in \Sigma^{\mathbb{N}} \mid \forall i \in \mathbb{N}, u_{i+1} \in \text{Next}(u_i)\} . \tag{15}$$

A word belongs to L^∞ iff all its finite prefixes belong to $L(X, \Sigma)$. The set L^∞ should be viewed as the "boundary" of X.

Let $(X_n)_n$ be a realization of the random walk which is transient by Proposition 2.2. The *harmonic measure* of the random walk is the probability measure ν^∞ on L^∞ with finite-dimensional marginals defined by:

$$\forall u_1 \cdots u_k \in L(X, \Sigma),\ \nu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) = P\{\exists N, \forall n \geq N,\ X_n \in u_1 \cdots u_k \Sigma^*\} .$$

This defines indeed a measure on L^∞ because the random walk is transient, and because X_n and X_{n+1} differ by at most their last symbol. Intuitively, the harmonic measure ν^∞ gives the direction in which $(X_n)_n$ goes to infinity.

For a proof of next result, see [21, Theorem 4.5] and also [22, Theorem 3.3]. In the specific case of the free group, the result appears in [11, 24], see also the survey [20].

Theorem 2.5. *Let (X, Σ, ν) be a plain triple. Let $\widehat{r} = (\widehat{r}(a))_{a \in \Sigma}$ be the unique admissible solution to the Traffic Equations. Set $\widehat{q}(a) = \widehat{r}(a)/\widehat{r}(Next(a))$, for all $a \in \Sigma$. The harmonic measure ν^∞ of the random walk (X, ν) is given by:*

$$\forall u_1 \cdots u_k, \quad \nu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) = \widehat{q}(u_1) \cdots \widehat{q}(u_{k-1}) \widehat{r}(u_k) . \tag{16}$$

The drift of the random walk is given by:

$$\widehat{\gamma} = \sum_{a \in \Sigma} \nu(a) \left[\widehat{r}(Next(a)) - \sum_{b | a * b = 1_X} \widehat{r}(b) \right] . \tag{17}$$

3 The Zero-Automatic Queue

We first define the 0-automatic queue informally, before doing it formally in Definition 3.1. Let X be a plain monoid, Σ be a set of natural generators, and ν a probability measure on Σ. The associated 0-automatic queue is formed by a simple single server queue with FIFO discipline and an infinite capacity buffer in which the buffering occurs according to the random walk (X, ν). It is a multi-class queue (classes Σ) but the class does not influence the way customers get served, only the way they get buffered.

More precisely, the instants of customer arrivals are given by a Poisson process of rate λ, and each customer carries a mark, or *class*, which is an element of Σ. The sequence of marks is i.i.d. of law ν. Upon arrival, a new customer interacts with the customer presently at the back-end of the buffer, and depending on their respective classes, say b and a, one of three possible events occurs: (i) if $b * a = 1_X$, then the two customers leave the queue; (ii) if $b * a = c \in \Sigma$, then the two customers merge to create a customer of type c; (iii) otherwise, customer b takes place at the back-end of the buffer, behind customer a. In the mean time, at the front-end of the buffer, the customers are served one by one and at constant rate μ by the server. To be complete, one needs to specify how customers are incorporated when the buffer is empty. Several variants may be considered, and we view this "boundary condition" as an additional parameter of the model. The resulting flexibility in the definition of a 0-automatic queue will turn out to be a crucial point.

According to the above description, the queue-content (the sequence of classes of customers in the buffer) is a continuous time jump Markov process. The more formal definition of the queue is given via the infinitesimal generator of this process.

Definition 3.1 (Zero-automatic queue). *Consider a plain triple (X, Σ, ν). Let $L(X, \Sigma)$ be the set of words defined in (9). Consider $r \in \tilde{\mathcal{B}}$, see (13), and $\lambda, \mu \in \mathbb{R}_+^*$. The* 0-automatic queue *of type $(X, \Sigma, \nu, r, \lambda, \mu)$ is defined as follows. The* queue-content *$(M(t))_{t \in \mathbb{R}_+}$ is a continuous time jump Markov process on the state space $L(X, \Sigma)$ with infinitesimal generator Q defined by: $\forall u = u_n \cdots u_1 \in L(X, \Sigma) \setminus \cup_{a \in \Sigma} \{a\}^*$,*

$$\begin{cases} Q(u, bu) &= \lambda \nu(b) \ , & \forall b \in \mathit{Next}(u_n) \\ Q(u, cu_{n-1} \cdots u_1) &= \lambda \sum_{b | b * u_n = c} \nu(b) \ , & \forall c \in \Sigma \setminus \{u_n\}, \ \exists b \in \Sigma, \ b * u_n = c \\ Q(u, u_{n-1} \cdots u_1) &= \lambda \sum_{b | b * u_n = 1_X} \nu(b) & \\ Q(u, u_n \cdots u_2) &= \mu & \end{cases} \tag{18}$$

and, for all $a \in \Sigma$ such that $a \in \mathit{Next}(a)$, and for all $n \geq 1$,

$$\begin{cases} Q(a^n, ba^n) &= \lambda \nu(b) \ , & \forall b \in \mathit{Next}(a) \\ Q(a^n, ca^{n-1}) &= \lambda \sum_{b | b * a = c} \nu(b) \ , & \forall c \in \Sigma \setminus \{a\}, \ \exists c \in \Sigma, \ b * a = c \\ Q(a^n, a^{n-1}) &= \mu + \lambda \sum_{b | b * a = 1_X} \nu(b) & \end{cases} \tag{19}$$

and, finally, the boundary condition is,

$$Q(1_{\Sigma^*}, a) \ = \ \lambda \nu(a) r(\mathit{Next}(a)) \ , \quad \forall a \in \Sigma \, . \tag{20}$$

We denote by $M/M/(X, \Sigma)$ any 0-automatic queue of type $(X, \Sigma, \nu, r, \lambda, \mu)$.

Remark 3.2. The intuition behind the form of the boundary condition is as follows: the buffer-content is viewed as the visible part of an iceberg consisting of an infinite word of L^∞, see (15). When the buffer is empty, new customers are incorporated depending on the invisible part of the iceberg, whose first marginal is assumed to be r. This last point will find an a-posteriori justification in Theorem 5.7.

The simplest example of 0-automatic queue is the one associated with the free monoid $(\mathbb{N}, +)$. The triple $(\mathbb{Z}, \{-1, 1\}, \nu)$, where ν is a probability measure on $\{1, -1\}$, is not plain. However, it is simple and interesting to generalize Definition 3.1 in order to define a 0-automatic queue associated with the free group $(\mathbb{Z}, +)$. We now discuss the 0-automatic queues associated with $(\mathbb{N}, +)$ and $(\mathbb{Z}, +)$.

The simple queue. Consider the free monoid $X = \{a\}^* = \{a^k, k \in \mathbb{N}\}$ over the single generator set $\Sigma = \{a\}$. Hence, for any $\lambda, \mu \in \mathbb{R}_+^*$, there is only one possible associated queue: $(X, \Sigma, \nu, r, \lambda, \mu)$, where $\nu(a) = r(a) = 1$. By specializing the infinitesimal generator Q given in Definition 3.1, we get: $\forall n \in \mathbb{N}$,

$$Q(a^n, a^{n+1}) = \lambda, \quad Q(a^{n+1}, a^n) = \mu \, .$$

This is the simple $M/M/1/\infty$ FIFO queue with arrival rate λ and service rate μ.

The G-queue. Consider the free group $X = \mathbb{F}(a) = \{a^k, k \in \mathbb{Z}\}$ and the set of generators $\Sigma = \{a, a^{-1}\}$. Let ν be a probability measure on Σ such that $\nu(a) > 0, \nu(a^{-1}) > 0$. Consider $r \in \mathcal{B}$ and $\lambda, \mu \in \mathbb{R}_+^*$. The 0-automatic queue $(\mathbb{F}(a), \Sigma, \nu, r, \lambda, \mu)$ has an infinitesimal generator Q given by: $\forall n \in \mathbb{N}$,

$$\begin{cases} Q(a^n, a^{n+1}) &= \lambda\nu(a), & Q(a^{n+1}, a^n) &= \mu + \lambda\nu(a^{-1}) \\ Q(a^{-n}, a^{-(n+1)}) &= \lambda\nu(a^{-1}), & Q(a^{-(n+1)}, a^{-n}) &= \mu + \lambda\nu(a) \\ Q(1_{\Sigma^*}, a) &= \lambda\nu(a)r(a), & Q(1_{\Sigma^*}, a^{-1}) &= \lambda\nu(a^{-1})r(a^{-1})\,. \end{cases}$$

This is close to the mechanism of the G-queue, a queue with positive and negative customers introduced by Gelenbe [14, 15]. With respect to the G-queue, one originality of the $M/M/(\mathbb{F}(a), \Sigma)$ queue is that negative and positive customers play symmetrical roles. Another one is the treatment of the boundary condition.

Since the triple $(\mathbb{F}(a), \{a^{-1}, a\}, \nu)$ is not plain according to Def. 2.1, the above queue is not covered by the results in Sections 4 and 5. However, part of the results remain true, and we come back specifically to this model in Section 6.1 and 6.2.4.

Extension. It is possible to generalize Definition 3.1 in order to define a 0-automatic queue of type $GI/GI/(X, \Sigma)$, resp. $G/G/(X, \Sigma)$. Roughly, the description would go as follows. The buffering mechanism is kept unchanged; the sequence of inter-arrival times and classes of customers is i.i.d. (resp. stationary and ergodic); the sequence of service times at the server is i.i.d. (resp. stationary and ergodic) and independent of the arrivals.

3.1 Comparison with other models in the literature

Under stability condition, we will see that the 0-automatic queue has the “Poisson output” property. However, the 0-automatic queue is quite different from the existing models of queues with this property. In particular, it does not fit in the general frameworks of quasi-reversible nodes developped in Kelly [17] or Chao, Miyazawa, and Pinedo [6].

Let us detail the comparison with the models in [6], see also [5]. Their model is a wide generalization of Gelenbe's G-queue with signals, batch arrivals, and batch departures. In a sense, 0-automatic queues can also be viewed as a wide generalization of G-queues. Other common features between the models include: non-linear traffic equations, an output rate different from the input rate, and subtle boundary conditions to get a product form. Despite these similarities, the models are quite orthogonal. The models in [6] have more general state spaces than 0-automatic queues. However, they must satisfy local balance equations for each class of customer, which do not hold for 0-automatic queues.

It is also worth comparing the 0-automatic queue with another model for queues introduced by Yeung and Sengupta [28], see also He [18] (the *YS model* in the following).

A common feature is the structure of the state space : a tree for the YS model (or the cartesian product of a tree and a finite set), and a more general tree-like graph for the 0-automatic queue. In particular, both models correspond to multiclass queues, and the buffer content is coded by a word over the alphabet of classes. Second common feature, the effect of a new arrival is either to add, to modify the class of, or to remove, a customer at the back-end of the buffer (in the YS model, the removal/modification may affect several customers at the back-end of the buffer). Now, and this is the first central difference, departures occur at the front-end of the buffer in the 0-automatic queue, and at the back-end in the YS model. Therefore, the former is a FIFO queue while the latter is a LIFO queue. We have illustrated the FIFO mecanism of the 0-automatic queue in Figure 3. The second important difference concerns the type of results which are

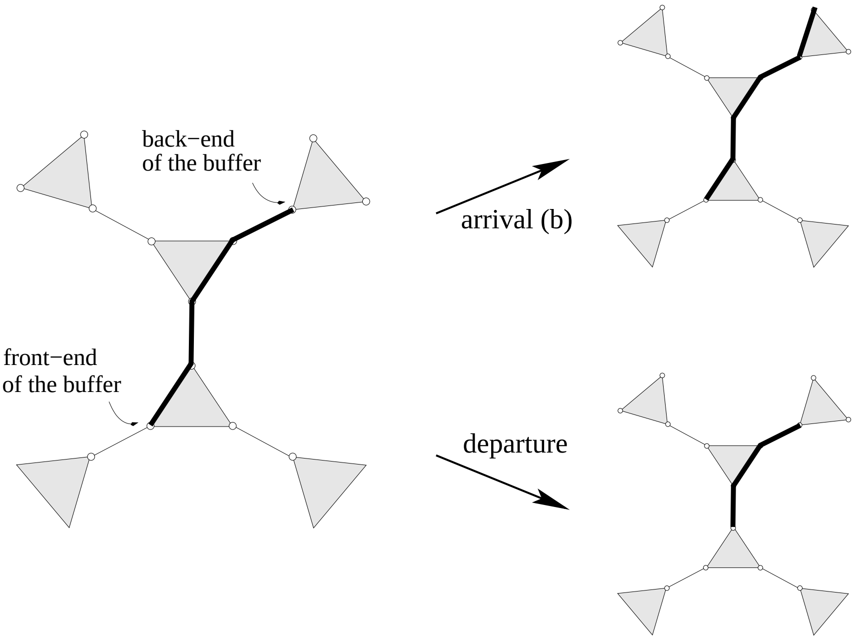

Figure 3: Effect of an arrival and a departure on the content of the buffer in a 0-automatic queue built on the group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$.

proved. In a stable 0-automatic queue, the buffer content has a "product form" stationary distribution, see Theorem 5.7. In the YS model, it has only a "matrix product form", see [28, Section 2] and getting the stronger "product form" requires severe additional assumptions, see [28, Section 6]. To conclude the comparison, the original flavor of the 0-automatic queue comes from the underlying group or monoid structure. It is this algebraic foundation which can be accounted for the ability to get the strong product form results.

4 Stability Condition for a Zero-Automatic Queue

Throughout Sections 4 and 5, the model is as follows. Let (X, Σ, ν) be a plain triple. Fix λ and μ in $\mathbb{R}_+^*$ and r in $\mathcal{B}$. Consider the 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$.

Let $M = (M_t)_t$ be the queue-content process, and Q the infinitesimal generator. Next Lemma is a direct consequence of the strong connectivity of the graph $(\Sigma, \rightarrow)$ defined in (10).

Lemma 4.1. *The process M is irreducible.*

The aim of this Section is to prove Proposition 4.2 which characterizes the stability region of the 0-automatic queue.

Proposition 4.2. *Let $\widehat{\gamma}$ be the drift of the random walk (X, ν). We have:*

$$\begin{aligned}
[\lambda\widehat{\gamma} < \mu] &\iff M \text{ ergodic} \\
[\lambda\widehat{\gamma} = \mu] &\iff M \text{ null recurrent} \\
[\lambda\widehat{\gamma} > \mu] &\iff M \text{ transient}.
\end{aligned}$$

Consider an excursion of M from the instant $t = 0$ at which it is assumed to leave state 1_{Σ^*}, to the instant R_M which corresponds to the first return to state 1_{Σ^*}. Recall that M is transient iff $P\{R_M = \infty\} > 0$, and ergodic iff $E[R_M] < \infty$.

It is convenient to use the following representation for M. Let $A = (A_0 = 0, A_1, A_2, \dots)$ where $(A_1, A_2, \dots)$ are the time points of a time-stationary Poisson process of rate λ on $\mathbb{R}_+$. Let $N_A = (N_A(t))_t$ be the corresponding counting process. Let $N_D = (N_D(t))_t$ be the counting process of a time-stationary Poisson process of rate μ on $\mathbb{R}_+$. Let $\widetilde{X} = (\widetilde{X}_n)_n$ be a realization of the random walk (X, ν) viewed as evolving on $L(X, \Sigma)$, see (11). Assume that $A, \widetilde{X}$, and N_D are mutually independent. Let $(X_t)_t$ be the continuous-time jump Markov process on the state space $L(X, \Sigma)$ defined by:

$$X_t = \widetilde{X}_{n+1} \text{ on } [A_n, A_{n+1}) \,.$$

For all t in the interval $[0, R_M)$, we have:

$$|M_t|_\Sigma = |X_t|_\Sigma - N_D(t) \,. \tag{21}$$

Here X_t is the queue-content at time t *if* no service has been completed. Observe that the first letter of X_t corresponds to the front-end of the buffer (the right-end in Figure 1), and the last letter to the back-end (the left-end in Figure 1).

The counting process of a Poisson process satisfies a Strong Law of Large Numbers. We get, a.s.,

$$\lim_{t\to\infty} \frac{N_A(t)}{t} = \lambda, \qquad \lim_{t\to\infty} \frac{N_D(t)}{t} = \mu \,. \tag{22}$$

We also have, a.s.,

$$\lim_{n\to\infty} \frac{\widetilde{X}_n}{n} = \widehat{\gamma} \,,$$

where $\widehat{\gamma}$ is the drift of the random walk (X, ν). So we have, a.s.,

$$\lim_{n\to\infty} \frac{|X_t|_\Sigma}{t} = \lim_{t\to\infty} \frac{|X_t|_\Sigma}{N_A(t)} \times \frac{N_A(t)}{t} = \lambda\widehat{\gamma} \,. \tag{23}$$

We can now prove the following.

Lemma 4.3. *If $\lambda\widehat{\gamma} < \mu$ then M is recurrent. If $\lambda\widehat{\gamma} > \mu$ then M is transient.*

Proof. We show the first statement by contraposition. If M is transient then $P\{R_M = \infty\} > 0$. Using (21) and (22), we obtain that a.s. on the event $\{R_M = \infty\}$, we have:

$$\lim_t \frac{|M_t|_\Sigma}{t} = \lambda\widehat{\gamma} - \mu \,.$$

To avoid a contradiction, we must have $\lambda\widehat{\gamma} - \mu \geq 0$.

Now assume that $\lambda\widehat{\gamma} > \mu$. Using the independence of $(X_t)_t$, and N, and the regenerative properties of R_M, it is easily shown that $P\{\forall t,\ |M_t|_\Sigma > 0\} > 0$. In particular $P\{R_M = \infty\} > 0$ and M is transient. $\square$

To get the stronger results in Proposition 4.2, the idea is to approximate the 0-automatic queue by a simple queue with a Markov additive arrival process, and then to use standard results from queueing theory.

Since $(X_t)_t$ is transient (Proposition 2.2), there exists an a.s. finite T_0 such that $\forall t \geq T_0, |X_t|_\Sigma \geq 0$. For notational simplicity, assume that $T_0 = 0$. Define the random variables: $\forall k \geq 1$,

$$T_k = \inf\big\{t \mid \forall s \geq t,\ |X_s|_\Sigma \geq k+1\big\}, \qquad \tau_k = T_{k+1} - T_k \,. \tag{24}$$

The r.v.'s T_k are a.s. finite because $(X_t)_t$ is transient, and we have: $(T_0 = 0) < T_1 < T_2 < \dots$ a.s.

By transience, $\lim_t X_t$ is a (random) infinite word on the alphabet Σ, let us write it as $\lim_t X_t = U_0 U_1 U_2 \cdots$. By definition, see Section 2, the law of $U_1 U_2 \cdots$ is the harmonic measure ν^∞ of the random walk (X, ν). Observe that:

$$X_{T_k^-} = U_0 \cdots U_{k-1}, \qquad \forall t \geq T_k,\ X_t = U_0 \cdots U_{k-1} Y_t,\ Y_t \in L(X, \Sigma) \setminus \{1_{\Sigma^*}\}\,.$$

According to Theorem 2.5, we have: $\forall u_0 \cdots u_{k-1} \in L(X, \Sigma)$,

$$\begin{aligned} P\{U_0 \cdots U_{k-1} = u_0 \cdots u_{k-1}\} &= \frac{\widehat{r}(u_0)}{\widehat{r}(\mathrm{Next}(u_0))} \cdots \frac{\widehat{r}(u_{k-2})}{\widehat{r}(\mathrm{Next}(u_{k-2}))} \widehat{r}(u_{k-1}) \\ &= \widehat{r}(u_0) \frac{\widehat{r}(u_1)}{\widehat{r}(\mathrm{Next}(u_0))} \cdots \frac{\widehat{r}(u_{k-1})}{\widehat{r}(\mathrm{Next}(u_{k-2}))}\,. \end{aligned} \tag{25}$$

It follows that $(U_k)_k$ is a Markov chain with initial distribution $\widehat{r}$ and transition matrix P given by:

$$\forall a, b \in \Sigma, \quad P_{a,b} = \begin{cases} \widehat{r}(b)/\widehat{r}(\mathrm{Next}(a)) & \text{if } b \in \mathrm{Next}(a) \\ 0 & \text{otherwise} \end{cases}\,.$$

The matrix P is irreducible as a direct consequence of the strong connectivity of the graph of successors $(\Sigma, \rightarrow)$, see (10). Let π be the stationary distribution of P characterized by $\pi P = \pi$. In general, the Markov chain $(U_k)_k$ is not stationary, i.e. $\widehat{r}$ is different from π. (See [22, Proposition 3.6] for a sufficient condition on (X, ν) ensuring that $\widehat{r} = \pi$.)

Consider now the sequence $(U, \tau) = ((U_k, \tau_k))_k$. A consequence of the above is that (U, τ) is a Markov chain with transition function depending only on the first coordinate. According to the classical terminology, the sequence $(T_k)_k$ is a *Markov additive process (MAP)*.

Consider the simple queue of type MAP$/M/1$ with arrival process $(T_k)_k$, and a service process driven by N. Let $(\sigma_k)_k$ be the corresponding sequence of service times. We deduce from (22), that a.s. and in L^1:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\sigma_i - \tau_i) \;=\; \frac{1}{\mu} - \frac{1}{\lambda \widehat{\gamma}}\,.$$

Let $Z = (Z_t)_t$ be the queue-length process of this queue. Let R_Z be the first instant of return to 0 for the process Z. Applying standard results for MAP$/GI/1$ queues, see for instance [2, Prop. 4.2, Chapter X], we get:

$$[\lambda \widehat{\gamma} < \mu] \implies [E[R_Z] < \infty] \tag{26}$$

$$[\lambda \widehat{\gamma} = \mu] \implies [R_Z \text{ a.s. finite},\ E[R_Z] = \infty]\,. \tag{27}$$

Concentrating on the mechanism of the 0-automatic queue, it is not difficult to see that:

$$\Big[T_k \leq R_Z < T_{k+1}\Big] \implies \Big[T_k \leq R_Z \leq R_M < T_{k+1}\Big]\,. \tag{28}$$

Hence, the queue MAP$/M/1$ is a good approximation of the 0-automatic queue. In particular, the two implications in (26)-(27) also hold for R_M. In view of Lemma 4.3, this completes the proof of Proposition 4.2.

Remark. The above proof does not rely in an essential way on the Markovian assumption. For instance, modulo some care, an analog of Proposition 4.2 can clearly be written for a 0-automatic queue of type $GI/GI/(X, \Sigma)$.

5 Stationary Distribution of a Stable Queue

5.1 The Twisted Traffic Equations

The Traffic Equations, see Definition 2.3, play a central role in studying the random walk. We now introduce equations which play a similar role for the queue.

Definition 5.1 (Twisted Traffic Equations). *The* Twisted Traffic Equations TTE *associated with* $(X,\Sigma,\nu,\lambda,\mu)$ *are the equations of the variables* (η,x), $\eta\in\mathbb{R}_+^*$, $x=(x(a))_{a\in\Sigma}\in\mathbb{R}_+^\Sigma$, *defined by:*

$$\begin{aligned}\eta(\lambda+\mu)x(a) \quad &= \quad \eta^2\mu x(a)+\lambda\nu(a)x(Next(a))+\eta\lambda\sum_{b*d=a}\nu(b)x(d) \\ &\qquad +\eta^2\lambda\sum_{\substack{d\in Next(a)\\ b*d=1_X}}\nu(u)\frac{x(d)}{x(Next(d))}x(a)\ .\end{aligned} \tag{29}$$

To get a hint of the future role of the TTE, let us examine the case $(X,\Sigma)=(\{a\}^*,\{a\})$ considered at the end of Section 3. Recall that there is only one possible variant for the queue $M/M/(\{a\}^*,\{a\})$ which is equivalent to the simple $M/M/1$ queue. By simplifying (29), we get:

$$\rho(\lambda+\mu)=\rho^2\mu+\lambda\ . \tag{30}$$

Compare this with the global balance equations of the $M/M/1$ queue:

$$\pi(n)(\lambda+\mu)=\pi(n-1)\lambda+\pi(n+1)\mu\ . \tag{31}$$

By substituting $\pi(n)=\pi(0)\rho^n$ in (31), we recognize (30).

According to Proposition 2.4, there is a unique admissible solution to the Traffic Equations, that we denote by $\widehat{r}=(\widehat{r}(a))_{a\in\Sigma}$. We denote by $\widehat{\gamma}$ the drift of the random walk (X,ν). Consider $x\in\bar{\mathcal{B}}$. Define

$$A(x)=\sum_{a\in\Sigma}\nu(a)\sum_{b\in\mathrm{Next}(a)}x(b),\quad B(x)=\sum_{a*b\in\Sigma}\nu(a)x(b),\quad C(x)=\sum_{a*b=1_X}\nu(a)x(b)\ . \tag{32}$$

One easily checks that:

$$A(x)+B(x)+C(x)=1\ . \tag{33}$$

This point was the crux of the argument in proving Proposition 2.4 in [21]. Observe that a simple rewriting of (17) gives:

$$\widehat{\gamma}=A(\widehat{r})-C(\widehat{r})\ . \tag{34}$$

Let us investigate some properties of the solutions to the Twisted Traffic Equations.

First, if (ρ,r) is a solution to the TTE with $r\in\bar{\mathcal{B}}$, then r belongs to $\mathcal{B}$. This follows directly from the shape of the TTE and from the strong connectivity of the graph $(\Sigma,\rightarrow)$, see (10).

Second, if we set $\eta=1$ in the Twisted Traffic Equations (29), and perform the obvious simplifications, we obtain the Traffic Equations (14). It implies that $(1,\widehat{r})$ is a solution to the TTE for all λ and μ.

Lemma 5.2. *Let* (ρ,r), $\rho\in\mathbb{R}_+^*, r\in\mathcal{B}$, *be a solution to the TTE. We have either* $(\rho,r)=(1,\widehat{r})$, *or*

$$\rho=\frac{\lambda\sum_{a\in\Sigma}\nu(a)r(Next(a))}{\mu+\lambda\sum_{a*b=1_X}\nu(a)r(b)}=\frac{\lambda A(r)}{\mu+\lambda C(r)}\ . \tag{35}$$

Proof. By summing all the Equations of (29), we get:

$$\begin{aligned}
\rho(\lambda+\mu) &= \rho^2\mu + \lambda\sum_{a\in\Sigma}\nu(a)r(\mathrm{Next}(a)) + \rho\lambda\sum_{a*b\in\Sigma}\nu(a)r(b) \\
&\quad + \rho^2\lambda \sum_{\substack{d\in\mathrm{Next}(a)\\ b*d=1_X}} \nu(b)\frac{r(d)}{r(\mathrm{Next}(d))}r(a) \\
&= \rho^2\mu + \lambda A(r) + \rho\lambda B(r) + \rho^2\lambda C(r)\,.
\end{aligned}$$

Replacing $B(r)$ by $1 - A(r) - C(r)$ in the above, we get:

$$(\rho-1)\big[(\mu+\lambda C(r))\rho - \lambda A(r)\big] = 0\,. \tag{36}$$

If $\rho = 1$, we have seen that the TTE reduces to the TE, which implies by Proposition 2.4 that $(\rho, r) = (1, \widehat{r})$. Otherwise, we must have $(\mu + \lambda C(r))\rho - \lambda A(r) = 0$. This completes the proof. $\square$

The relevant solutions to the TTE will turn out to be the ones satisfying (35). This leads us to the next Definition.

Definition 5.3. *A solution (ρ, r) to the TTE is called an* admissible solution *if $\rho \in \mathbb{R}_+^*$, $r \in \mathcal{B}$, and if (35) is satisfied.*

Lemma 5.4. *If $\lambda\widehat{\gamma} = \mu$, then $(1, \widehat{r})$ is an admissible solution to the TTE. If $(1, r)$ is an admissible solution to the TTE, then $r = \widehat{r}$ and $\lambda\widehat{\gamma} = \mu$.*

Proof. Assume that $\lambda\widehat{\gamma} = \mu$. We know that $(1, \widehat{r})$ is a solution to the TTE. We need to check that it is admissible. By definition, it is admissible if:

$$1 = \frac{\lambda A(\widehat{r})}{\mu + \lambda C(\widehat{r})} \iff \lambda\big(A(\widehat{r}) - C(\widehat{r})\big) = \mu\,.$$

We conclude by recalling that: $\widehat{\gamma} = A(\widehat{r}) - C(\widehat{r})$, see (34).
Assume now that $(1, r)$ is an admissible solution to the TTE. Since $\rho = 1$, the TTE reduce to the TE implying that $r = \widehat{r}$. Now replacing (ρ, r) by $(1, \widehat{r})$ in (35), we get: $\lambda\widehat{\gamma} = \mu$. $\square$

More generally, admissible solutions always exist:

Lemma 5.5. *There exists an admissible solution to the TTE.*

Proof. Consider the Equations (29) and replace η by $\lambda A(x)/(\mu + \lambda C(x))$. The resulting equations in x can be viewed as a fixed point equation of the type $\Psi(x) = x$. The corresponding application $\Psi : (\mathbb{R}_+^*)^\Sigma \longrightarrow (\mathbb{R}_+^*)^\Sigma$ has the following form. For $a \in \Sigma$ and for $x \in (\mathbb{R}_+^*)^\Sigma$,

$$\begin{aligned}
\Psi(x)(a) &= \frac{1}{\lambda+\mu}\Big[\frac{\lambda A(x)}{\mu+\lambda C(x)}\big[\mu x(a) + \lambda\sum_{\substack{d\in\mathrm{Next}(a)\\ b*d=1_X}}\nu(b)\frac{x(d)}{x(\mathrm{Next}(d))}x(a)\big] \\
&\quad + \lambda\sum_{b*d=a}\nu(b)x(d) + \frac{\mu+\lambda C(x)}{A(x)}\nu(a)x(\mathrm{Next}(a))\Big]\,.
\end{aligned} \tag{37}$$

Consider $x \in \mathcal{B}$. By summing the Equations in (37) and using (33), we get:

$$\sum_{a\in\Sigma}\Psi(x)(a) = \frac{1}{\lambda+\mu}\big[\lambda A(x) + \lambda B(x) + \mu + \lambda C(x)\big] = 1\,.$$

We have proved that $\Psi(\mathcal{B}) \subset \mathcal{B}$. The end of the proof follows very closely the proof of Theorem 4.5 in [21]. For the sake of completeness, we recall the argument.

To use a Fixed Point Theorem, we need to define Ψ on a compact and convex set. The set $\bar{\mathcal{B}}$, which is the closure of $\mathcal{B}$, is a compact and convex subset of $\mathbb{R}^{\Sigma}$. But the map Ψ cannot in general be extended continuously on $\bar{\mathcal{B}}$. More precisely, $\Psi(x), x \in \bar{\mathcal{B}} \backslash \mathcal{B}$, can be defined unambiguously iff $x(\text{Next}(u)) \neq 0$ for all u.

For $x \in \bar{\mathcal{B}} \backslash \mathcal{B}$, let $\Psi(x) \subset \bar{\mathcal{B}}$ be the set of possible limits of $\Psi(x_n), x_n \in \mathcal{B}, x_n \to x$. We have extended Ψ to a correspondence $\Psi : \bar{\mathcal{B}} \twoheadrightarrow \bar{\mathcal{B}}$. Clearly this correspondence has a closed graph and nonempty convex values. Therefore, we are in the domain of application of the Kakutani-Fan-Glicksberg Theorem, see [1, Chapter 16]. The correspondence has at least one fixed point: $\exists r \in \bar{\mathcal{B}}$ such that $r \in \Psi(r)$. Now using the shape of the Equations in (37) and the strong connectivity of the graph of succesors $(\Sigma, \to)$, we obtain that $r \in \mathcal{B}$ (see [21, Theorem 4.5] for details).

Set $\rho = \lambda A(r)/(\mu + \lambda C(r))$. The pair (ρ, r) is an admissible solution to the TTE. $\square$

5.2 The main results

Next Lemma begins to establish the link between the Twisted Traffic Equations and the queue $M/M/(X, \Sigma)$.

Lemma 5.6. Let (ρ, r) be an admissible solution to the TTE. Consider the 0-automatic queue of type $(X, \Sigma, \nu, r, \lambda, \mu)$. Let Q_r be the infinitesimal generator of the queue-content process. Consider the measure $p_{\rho,r}$ on $L(X, \Sigma)$ defined by:

$$\forall a_n \cdots a_1 \in L(X, \Sigma), \quad p_{\rho,r}(a_n \cdots a_1) = \rho^n \frac{r(a_n)}{r(\mathit{Next}(a_n))} \cdots \frac{r(a_2)}{r(\mathit{Next}(a_2))} r(a_1) \, . \tag{38}$$

We have $p_{\rho,r} Q_r = 0$. Conversely, assume there exist $\rho \in \mathbb{R}_+^*$ and $r \in \mathcal{B}$ such that the measure $p_{\rho,r}$ defined by (38) satisfies $p_{\rho,r} Q_r = 0$. Then (ρ, r) is an admissible solution to the TTE.

Proof. We have $p_{\rho,r} Q_r = 0$ if and only if: $\forall u \in L(X, \Sigma)$,

$$\sum_{\substack{v \in L(X,\Sigma) \\ v \neq u}} p_{\rho,r}(u) Q_r(u, v) = \sum_{\substack{v \in L(X,\Sigma) \\ v \neq u}} p_{\rho,r}(v) Q_r(v, u) \, . \tag{39}$$

Denote the left and right-hand side of the above equality by L and R, respectively. Define

$$\text{Noact}(a) = \{ b \in \Sigma \mid b * a = a \} \, .$$

The left of (39) is:

$$L = \begin{cases} \sum_{a \in \Sigma} \lambda \nu(a) r(\text{Next}(a)) = \lambda A(r) & \text{if } u = 1_{\Sigma^*} \\ p_{\rho,r}(u)(\lambda(1 - \nu(\text{Noact}(a_n))) + \mu) & \text{otherwise} \end{cases} \, . \tag{40}$$

The right of (39) is given by, for $u = 1_{\Sigma^*}$,

$$R = \sum_{a \in \Sigma} \rho r(a) \big[\mu + \lambda \sum_{b * a = 1_X} \nu(b) \big] = \rho \big(\mu + \lambda C(r) \big) \, , \tag{41}$$

and for $u = a_n \cdots a_1,\ n \geq 1$,

$$
\begin{aligned}
R &= \lambda\nu(a_n)p_{\rho,r}(a_{n-1}\cdots a_1) + \sum_{\substack{d \neq a_n \\ b*d=a_n}} \lambda\nu(b)p_{\rho,r}(da_{n-1}\cdots a_1) \\
&\quad + \sum_{\substack{b\in \mathrm{Next}(a_n) \\ a*b=1_X}} \lambda\nu(a)p_{\rho,r}(ba_n\cdots a_1) + \sum_{b\in\mathrm{Next}(a_1)} p_{\rho,r}(a_n\cdots a_1 b)\mu \\
&= p_{\rho,r}(a_n\cdots a_1)\Big[\frac{1}{\rho}\frac{r(\mathrm{Next}(a_n))}{r(a_n)}\lambda\nu(a_n) + \sum_{\substack{d \neq a_n \\ b*d=a_n}} \frac{r(d)}{r(\mathrm{Next}(d))}\frac{r(\mathrm{Next}(a_n))}{r(a_n)}\lambda\nu(u) \\
&\qquad + \sum_{\substack{b\in \mathrm{Next}(a_n) \\ a*b=1_X}} \rho\lambda\nu(a)\frac{r(b)}{r(\mathrm{Next}(b))} + \sum_{b\in\mathrm{Next}(a_1)} \rho\frac{r(b)}{r(\mathrm{Next}(a_1))}\mu\Big] .
\end{aligned}
$$

Now recall that $b * d \in \Sigma \implies \mathrm{Next}(b*d) = \mathrm{Next}(d)$. We obtain:

$$
\begin{aligned}
R &= p_{\rho,r}(a_n\cdots a_1)\Big[\frac{1}{\rho}\frac{r(\mathrm{Next}(a_n))}{r(a_n)}\lambda\nu(a_n) + \sum_{b*d=a_n} \frac{r(d)}{r(a_n)}\lambda\nu(b) - \lambda\nu(\mathrm{Noact}(a_n)) \\
&\qquad + \sum_{\substack{b\in \mathrm{Next}(a_n) \\ a*b=1_X}} \rho\lambda\nu(a)\frac{r(b)}{r(\mathrm{Next}(b))} + \rho\mu\Big] .
\end{aligned}
$$

We see that for $u \neq 1_{\Sigma^*}$, the equality $L = R$ is precisely equivalent to the fact that (ρ, r) is a solution to the TTE. For $u = 1_{\Sigma^*}$, the equality $L = R$ is precisely equivalent to the fact that ρ and r satisfy (35).

Therefore, the equality $L = R$ is precisely equivalent to the fact that (ρ, r) is an admissible solution to the TTE. This completes the proof. $\square$

We now have all the ingredients to prove the central results of the paper.

Theorem 5.7. Let (X, Σ, ν) be a plain triple. Fix λ and μ in $\mathbb{R}_+^*$. Let (ρ, r) be an admissible solution to the TTE. Consider the 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$. Denote by $M_r = (M_r(t))_t$ the queue-content process and by Q_r its infinitesimal generator. We have:

$$
\begin{array}{ccccc}
[\rho < 1] & \iff & [\lambda\widehat{\gamma} < \mu] & \iff & \big[M_r \text{ ergodic}\big] \\
[\rho = 1] & \iff & [\lambda\widehat{\gamma} = \mu] & \iff & \big[M_r \text{ null recurrent}\big] \\
[\rho > 1] & \iff & [\lambda\widehat{\gamma} > \mu] & \iff & \big[M_r \text{ transient}\big] .
\end{array}
$$

Assume that $\lambda\widehat{\gamma} < \mu$. The stationary distribution $\pi_{\rho,r}$ of the process M_r is given by: $\forall a_n \cdots a_1 \in L(X, \Sigma)$,

$$
\pi_{\rho,r}(a_n\cdots a_1) = (1-\rho)p_{\rho,r}(a_n\cdots a_1) = (1-\rho)\rho^n q(a_n)\cdots q(a_2)r(a_1) , \tag{42}
$$

where $q(a) = r(a)/r(Next(a))$ for all $a \in \Sigma$.

Proof. Let (ρ, r) be an admissible solution to the Twisted Traffic Equations. Let $p_{\rho,r}$ be the measure defined in (38). We have (using that $\sum_{a\in\Sigma} r(a) = 1$):

$$
\sum_{u\in L(X,\Sigma)} p_{\rho,r}(u) = \sum_{n\in\mathbb{N}} \sum_{\substack{u\in L(X,\Sigma) \\ |u|_\Sigma = n}} p_{\rho,r}(u) = \sum_{n\in\mathbb{N}} \rho^n .
$$

Hence, $\sum_u p_{\rho,r}(u) < \infty$ iff $\rho < 1$. Now recall that $p_{\rho,r} Q_r = 0$, Lemma 5.6. It is standard (see for instance [4, Chapter 8]) that the process M_r is ergodic iff $\sum_u p_{\rho,r}(u) < \infty$. Now, according to Proposition 4.2, M_r is ergodic iff $\lambda\widehat{\gamma} < \mu$. By combining the three equivalences, we get:

$$\big[\rho < 1\big] \iff \big[M_r \text{ ergodic }\big] \iff \big[\lambda\widehat{\gamma} < \mu\big] .$$

The result in (42) holds as a direct consequence of Lemma 5.6.

Now let us turn our attention to the null recurrent case. Assume that $\rho = 1$. Using Lemma 5.4 and Proposition 4.2, we have:

$$\big[\rho = 1\big] \quad \Longrightarrow \quad \big[\lambda\widehat{\gamma} = \mu\big] \iff \big[M_r \text{ null recurrent}\big] . \tag{43}$$

Let $(\widetilde{\rho}, \widetilde{r})$ be an admissible solution to the TTE. Using the argumentation in the forthcoming proof of Theorem 5.8, we deduce that we must have $\widetilde{\rho} = \rho = 1$. Hence we have an equivalence on the left of (43). This completes the proof. □

Assume that $\lambda\widehat{\gamma} = \mu$. It follows immediately from Lemma 5.4 and Theorem 5.7 that $(1, \widehat{r})$ is the unique admissible solution to the TTE. We now prove a more interesting result in the same vein.

Theorem 5.8. *Consider the same model as in Theorem 5.7. Assume that $\lambda\widehat{\gamma} < \mu$. Then the TTE have a unique admissible solution. In particular, there is only one variant of the 0-automatic queue $M/M/(X,\Sigma)$ with a product form distribution.*

Proof. Let (ρ, r) and $(\tilde{\rho}, \tilde{r})$ be two admissible solutions to the TTE. According to Theorem 5.7, we have $\rho < 1$ and $\tilde{\rho} < 1$. Let π and $\tilde{\pi}$ be the respective stationary distributions of M_r and $M_{\tilde{r}}$.

We now use a classical result on ergodic Markov processes, cf for instance [4, Chapter 8, Theorem 5.1]: the stationary distribution is proportional to the time spent in each state in an excursion of the process from s to s, for some arbitrary state s.

Assume that the queue-content is 1_{Σ^*} at instant 0. Let T, resp. $\tilde{T}$, be the first instant of jump of M_r, resp. $M_{\tilde{r}}$. Let R, resp. $\tilde{R}$, be the first return instant to 1_{Σ^*}. We have, for all $u \in L(X,\Sigma)$ ($\sim$ stands for 'proportional to'),

$$\pi(u) \sim E\Big[\int_0^R \mathbb{1}_{\{M_r(t)=u\}} dt\Big], \qquad \tilde{\pi}(u) \sim E\Big[\int_0^{\tilde{R}} \mathbb{1}_{\{M_{\tilde{r}}(t)=u\}} dt\Big] .$$

It follows that, for all $u \in L(X,\Sigma)$, $u \neq 1_{\Sigma^*}$,

$$\pi(u) \sim E\Big[\int_T^R \mathbb{1}_{\{M_r(t)=u\}} dt\Big], \qquad \tilde{\pi}(u) \sim E\Big[\int_{\tilde{T}}^{\tilde{R}} \mathbb{1}_{\{M_{\tilde{r}}(t)=u\}} dt\Big] .$$

And, conditioning by the value of $M_r(T)$, resp. $M_{\tilde{r}}(\tilde{T})$, we have, for all $u \in L(X,\Sigma)$, $u \neq 1_{\Sigma^*}$,

$$\begin{aligned} \pi(u) &\sim \sum_{a\in\Sigma} \nu(a) r(\mathrm{Next}(a))\ E\Big[\mathbb{1}_{\{M_r(T)=a\}} \int_T^R \mathbb{1}_{\{M_r(t)=u\}} dt\Big] \\ \tilde{\pi}(u) &\sim \sum_{a\in\Sigma} \nu(a) \tilde{r}(\mathrm{Next}(a))\ E\Big[\mathbb{1}_{\{M_{\tilde{r}}(\tilde{T})=a\}} \int_{\tilde{T}}^{\tilde{R}} \mathbb{1}_{\{M_{\tilde{r}}(t)=u\}} dt\Big] . \end{aligned}$$

Observe that the generators Q_r and $Q_{\tilde{r}}$ differ only in the line indexed by 1_{Σ^*}. In other terms, the conditional law of $(M_r(t))_{t\in[T,R]}$ on the event $\{M_r(T) = a\}$, is equal to the conditional law of $(M_{\tilde{r}}(t))_{t\in[\tilde{T},\tilde{R}]}$ on the event $\{M_{\tilde{r}}(\tilde{T}) = a\}$. Therefore π and $\tilde{\pi}$ are obtained as linear combinations of the same measures p_a defined by:

$$p_a(u) = E\big[\mathbb{1}_{\{M_r(T)=a\}}\int_T^R \mathbb{1}_{\{M_r(t)=u\}}dt\big] = E\big[\mathbb{1}_{\{M_{\tilde{r}}(\tilde{T})=a\}}\int_{\tilde{T}}^{\tilde{R}} \mathbb{1}_{\{M_{\tilde{r}}(t)=u\}}dt\big] .$$

For $n \in \mathbb{N}^*$, set $\pi(n) = \pi\{u \in L(X,\Sigma) \mid |u|_\Sigma = n\}$. Define $\tilde{\pi}(n)$ and $p_a(n)$ accordingly. Using the above, we have, for all $n \in \mathbb{N}^*$,

$$\pi(n) \sim \sum_{a\in\Sigma} \nu(a)r(\mathrm{Next}(a))\ p_a(n), \qquad \tilde{\pi}(n) \sim \sum_{a\in\Sigma} \nu(a)\tilde{r}(\mathrm{Next}(a))\ p_a(n) . \tag{44}$$

Besides, according to (42), we have $\pi(n) \sim \rho^n$ and $\tilde{\pi}(n) \sim \tilde{\rho}^n$. In view of (44), we conclude easily that we must have $\rho = \tilde{\rho}$.

It remains to prove that $r = \tilde{r}$. Let us first show that:

$$\big[\forall a \in \Sigma, q(a) = \tilde{q}(a)\big] \implies \big[\forall a \in \Sigma, r(a) = \tilde{r}(a)\big] . \tag{45}$$

We have: $\forall a,\ q(a) = r(a)/r(\mathrm{Next}(a))$. We can reinterpret this as: $Mr = r$, with r being viewed as a column vector and with M being the matrix of dimension $\Sigma \times \Sigma$ defined by:

$$\forall a, b \in \Sigma, \quad M_{a,b} = \begin{cases} q(a) & \text{if } b \in \mathrm{Next}(a) \\ 0 & \text{otherwise} \end{cases} .$$

Consequently, the matrix M is irreducible. Now invoking the Perron-Frobenius Theorem, since r has all its coordinates positive, it implies that r is necessarily the Perron eigenvector of the matrix, i.e. the unique (up to a multiplicative constant) eigenvector associated with the spectral radius. But we also have: $M\tilde{r} = \tilde{r}$, and $\sum_a r(a) = \sum_a \tilde{r}(a) = 1$. By uniqueness of the Perron eigenvector, we conclude that $r = \tilde{r}$.

Therefore, it remains to prove that: $\forall a \in \Sigma,\ q(a) = \tilde{q}(a)$. Define $C(X,\Sigma) \subset L(X,\Sigma)$ by:

$$C(X,\Sigma) = \{u_1 \cdots u_k \in L(X,\Sigma) \mid u_1 \in \mathrm{Next}(u_k)\} .$$

Observe that: $[u \in C(X,\Sigma)] \implies [\forall n,\ u^n \in C(X,\Sigma)]$.

For $u = u_1 \cdots u_k \in C(X,\Sigma)$, set $q(u) = q(u_1)\cdots q(u_k)$. Define $\tilde{q}(u)$ analogously. Using (42), we have, for all $n \in \mathbb{N}^*$, for some constants C_1, C_2,

$$\pi(u^n) = C_1\rho^{n|u|_\Sigma}q(u)^n r(\mathrm{Next}(u_k)), \qquad \tilde{\pi}(u^n) = C_2\rho^{n|u|_\Sigma}\tilde{q}(u)^n\tilde{r}(\mathrm{Next}(u_k)) . \tag{46}$$

Besides, $\pi(u^n) = C_3\sum_{a\in\Sigma}\nu(a)r(\mathrm{Next}(a))p_a(u^n)$ and $\tilde{\pi}(u^n) = C_4\sum_{a\in\Sigma}\nu(a)\tilde{r}(\mathrm{Next}(a))p_a(u^n)$, for some constants C_3, C_4. Therefore, $\pi(u^n)$ and $\tilde{\pi}(u^n)$ must grow at the same exponential speed as a function of n. In view of (46), we must have:

$$q(u_1)\cdots q(u_k) = \tilde{q}(u_1)\cdots\tilde{q}(u_k) . \tag{47}$$

For the remaining step, the argument depends on the form of X. Set $X = X_1\star\cdots\star X_K\star X_{K+1}\star F$, with $X_{K+1} = \Sigma_{K+1}^*(|\Sigma_{K+1}| \geq 0)$, with $F = \mathbb{F}(\Sigma_{K+2})$ $(|\Sigma_{K+2}| \geq 0)$, with X_i being finite monoids for $1 \leq i \leq K$, and with $\Sigma = \Sigma_{K+2}^{-1} \sqcup_{1\leq i\leq K+2} \Sigma_i$, $\Sigma_i = (X_i \backslash \{1_{X_i}\})$ for $1 \leq i \leq K$.

For $a \in \Sigma_i$, $1 \leq i \leq K$, $\mathrm{Next}(a) = \Sigma \setminus \Sigma_i$. For $b \in \Sigma_{K+1}$, $\mathrm{Next}(b) = \Sigma$. For $c \in \Sigma_{K+2}$, $\mathrm{Next}(c) = \Sigma \setminus \{c^{-1}\}$.

We first treat the case $|\Sigma_{K+1}| + |\Sigma_{K+2}| \geq 1$. Then there exists $i \in \{K+1, K+2\}$ such that $|\Sigma_i| \geq 1$. Set $\mathrm{Gen} = \Sigma_{K+1} \sqcup \Sigma_{K+2} \sqcup \Sigma_{K+2}^{-1}$. For all $a \in \mathrm{Gen}$, $a \in \mathrm{Next}(a)$, so we have

$q(a)^2 = \tilde{q}(a)^2$. It implies that $q(a) = \tilde{q}(a)$. Consider $b \in \Sigma \setminus \text{Gen}$ and $a \in \text{Gen}$, one has $b \in \text{Next}(a)$, so, according to (47), $q(b)q(a) = \tilde{q}(b)\tilde{q}(a)$. Hence, $q(b) = \tilde{q}(b)$. So $q(c) = \tilde{q}(c)$ for all $c \in \Sigma$.

We now consider the case $|\Sigma_{K+1}| + |\Sigma_{K+2}| = 0$ Assume that $K \geq 3$. Consider $a \in \Sigma_1, b \in \Sigma_2, c \in \Sigma_3$. We have: $\text{Next}(a) = \Sigma \backslash \Sigma_1$, $\text{Next}(b) = \Sigma \backslash \Sigma_2$, and $\text{Next}(c) = \Sigma \backslash \Sigma_3$. Therefore, $abc \in C(X, \Sigma)$ and $bc \in C(X, \Sigma)$. Using (47), we deduce that: $q(a)q(b)q(c) = \tilde{q}(a)\tilde{q}(b)\tilde{q}(c)$ and $q(b)q(c) = \tilde{q}(b)\tilde{q}(c)$. We conclude that $q(a) = \tilde{q}(a)$ for all $a \in \Sigma$.

Assume now that $K = 2$. The above argument does not work anymore. First of all, we want to prove that:

$$\forall a \in \Sigma_i, \quad \frac{r(a)}{r(\Sigma_i)} = \frac{\tilde{r}(a)}{\tilde{r}(\Sigma_i)} \,. \tag{48}$$

Consider $a, b \in \Sigma_1$ and $c \in \Sigma_2$. We have: $\text{Next}(a) = \text{Next}(b) = \Sigma_2$ and $\text{Next}(c) = \Sigma_1$. Hence, $ac \in C(X, \Sigma)$ and $bc \in C(X, \Sigma)$. Using (47), we have: $q(a)q(c) = \tilde{q}(a)\tilde{q}(c)$ and $q(b)q(c) = \tilde{q}(b)\tilde{q}(c)$. It implies that: $q(a)/q(b) = \tilde{q}(a)/\tilde{q}(b)$, which is equivalent to (48). Set $R(a) = r(a)/r(\Sigma_i) = \tilde{r}(a)/\tilde{r}(\Sigma_i)$ if $a \in \Sigma_i$.

Now let us sum the TTE corresponding to all the elements of Σ_1, and let us perform the simplifications implied by (48). For instance for (ρ, r), we get:

$$\begin{aligned}
\rho(\lambda+\mu)r(\Sigma_1) &= \rho^2\mu r(\Sigma_1) + \lambda\nu(\Sigma_1)r(\Sigma_2) + \rho\lambda \sum_{a*b\in\Sigma_1} \nu(a)r(b) \\
&\qquad + \rho^2\lambda \sum_{a\in\Sigma_1} \sum_{\substack{b,d\in\Sigma_2 \\ b*d=1_{X_2}}} \nu(b)\frac{r(d)}{r(\Sigma_1)}r(a) \\
&= \rho^2\mu r(\Sigma_1) + \lambda\nu(\Sigma_1)r(\Sigma_2) + \rho\lambda r(\Sigma_1) \sum_{a*b\in\Sigma_1} \nu(a)R(b) \\
&\qquad + \rho^2\lambda r(\Sigma_2) \sum_{\substack{b,d\in\Sigma_2 \\ b*d=1_{X_2}}} \nu(b)R(d)
\end{aligned}$$

Set $B_1 = \sum_{a*b\in\Sigma_1} \nu(a)R(b)$ and $C_2 = \sum_{a,b\in\Sigma_2,\ a*b=1_{X_2}} \nu(a)R(b)$. Using that $r(\Sigma_2) = 1 - r(\Sigma_1)$, we get:

$$r(\Sigma_1) = \frac{\lambda\nu(\Sigma_1) + \rho^2\lambda C_2}{\rho(\lambda+\mu) - \rho^2\mu + \lambda\nu(\Sigma_1) - \rho\lambda B_1 + \rho^2\lambda C_2} \,. \tag{49}$$

But all the terms in the right-hand side of (49) are unchanged when we write the corresponding equation for $\tilde{r}$. So, $r(\Sigma_1) = \tilde{r}(\Sigma_1)$. In view of (48), we deduce that $r(a) = \tilde{r}(a)$ for all $a \in \Sigma$. This completes the proof. □

When the triple is not plain, the TTE may have several admissible solutions. This is for instance the case for the triple $(\mathbb{F}(a), \{a, a^{-1}\}, \{1/2, 1/2\})$ as discussed in Section 6.2.4.

Remark 5.9. In the case $\rho < 1$, if the boundary condition is chosen according to r', where (ρ, r') is not a solution to the TTE, then the stationary distribution of $M_{r'}$ exists (Prop. 4.2). But we do not know how to compute it exactly. See Remark 3.2 for a justification of the form of the boundary condition.

Poisson departure processes.

The celebrated Burke Theorem states that the departure process from a stable $M/M/1$ queue is a Poisson process of the same rate as the arrival process. A nice consequence of Theorem 5.7 is that an analog of Burke Theorem holds for 0-automatic queues.

In a 0-automatic queue, 'departures' occur both at the front-end and at the back-end of the buffer. Here we consider only the front-end departures, i.e. the ones corresponding to service completions and not to buffer cancellations.

Let $M = (M(t))_t$ be the queue-content process of some 0-automatic queue $M/M/(X,\Sigma)$. A *departure* is an instant of jump of M corresponding to a jump of the type: $u_n \cdots u_1 \to u_n \cdots u_2$ for $u = u_n \cdots u_1 \in L(X,\Sigma) \setminus \{1_{\Sigma^*}\}$. When $u = a^n, a \in \Sigma, n \geq 1$ (the case (19) in Definition 3.1), some special care must be taken. The jumps of type $a^n \to a^{n-1}$ which are *departures* occur at rate μ. The *departure process* is the point process of departures.

Theorem 5.10. *The model is the same as in Theorem 5.7. Assume that $\lambda\widehat{\gamma} < \mu$. Let (ρ, r) be an admissible solution of the TTE. Consider the 0-automatic queue $(X, \Sigma, \nu, r, \lambda, \mu)$. The stationary departure process is a Poisson process of rate $\rho\mu$. Furthermore, for all t, the queue-content at time t is independent of the departure process up to time t.*

Proof. The simplest proof of Burke Theorem uses reversibility and is due to Reich, see for instance [17] for details. Here, the argument is similar.

Let M_r be the stationary queue-content process. Its marginal distribution at a given instant is $\pi_{\rho,r}$ given in (42). Let D be the corresponding departure process. By definition, the instantaneous rate of D is $c(t) = \mu$ if $M_r(t) \neq 1_{\Sigma^*}$ and $c(t) = 0$ otherwise.

Now let us consider the time-reversed point process $\widetilde{D}$. The process $\widetilde{D}$ corresponds to the instants of "right-increase" of the time-reversed process $(\widetilde{M_r}(t))_t$. Therefore, the instantaneous rate $\tilde{c}(t)$ of $\widetilde{D}$ is as follows. If $\widetilde{M_r}(t) = a_n \cdots a_1 \in L(X,\Sigma) \setminus \{1_{\Sigma^*}\}$,

$$\tilde{c}(t) = \sum_{a \in \mathrm{Next}(a_1)} \frac{\pi_{\rho,r}(a_n \cdots a_1 a)}{\pi_{\rho,r}(a_n \cdots a_1)} \mu = \sum_{a \in \mathrm{Next}(a_1)} \frac{\rho r(a)}{r(\mathrm{Next}(a_1))} \mu = \rho\mu \,, \tag{50}$$

and if $\widetilde{M_r}(t) = 1_{\Sigma^*}$,

$$\tilde{c}(t) = \sum_{a \in \Sigma} \frac{\pi_{\rho,r}(a)}{\pi_{\rho,r}(1_{\Sigma^*})} \mu = \sum_{a \in \Sigma} \rho r(a) \mu = \rho\mu \,.$$

We conclude that $\widetilde{D}$ is a Poisson process of rate $\rho\mu$. Since Poisson processes are preserved by time-reversal, D is also a Poisson process of rate $\rho\mu$.

Also, using the Markov property of Poisson processes, $\widetilde{M_r}(t)$ is independent of the process $\widetilde{D}$ after time t. Under time-reversal, this translates as: $M_r(t)$ is independent of the departure process D up to time t. $\square$

Here are some additional comments on Theorem 5.10.

1- The infinitesimal generator of the time-reversed process $\widetilde{M_r}$ is certainly not the infinitesimal generator of a 0-automatic queue. This was already the case for the G-queue. But this is in contrast with the situation for the M/M/1 queue.

2- For $a \in \Sigma$, the departure process D_a of customers of class a is not a Poisson process.

3- The saturation principle of Baccelli and Foss [3] holds for many classical queueing systems. Here is a rough description of it.

Consider a queueing system with an infinite capacity buffer. Let μ_0 be the departure rate in the *saturated system* in which an infinite number of customers are stacked in the buffer. Now, if the actual arrival rate in the system is $\lambda < \mu_0$, then the system is stable, and the departure rate is λ. A dual presentation of the same principle is as follows. Let λ_0 be the growth rate of the buffer in the *blocked system* where the server has been shut down. If the actual service rate in the system is $\mu > \lambda_0$, then the system is stable, and the departure rate is λ_0.

Zero-automatic queues do *not* satisfy the saturation principle. This can be viewed on the following inequalities (to be deduced from Theorem 5.10):

$$\lambda\widehat{\gamma} \leq \rho\mu < \mu \ ,$$

where: $\rho\mu$ is the actual departure rate in equilibrium, μ is the departure rate from the saturated system, and $\lambda\widehat{\gamma}$ is the growth rate of the buffer in the blocked system.

5.3 Quasi-Birth-and-Death processes

Quasi-Birth-and-Death (QBD) processes appear naturally in the modelling of several queueing and communication systems. As such, they have been extensively studied, see for instance the monographs [19, 23]. The results in Section 5.2 can be put in perspective by considering the relation between 0-automatic queues and QBD processes.

To that purpose, we define an "approximated" and quite simplified version of the 0-automatic queue. The idea is to keep track of the queue-content only through the number of customers and the class of the back-end customer. Clearly a difficulty arises: if a cancellation occurs at the back-end of the buffer, there is no way to retrieve the class of the new back-end customer. This missing information is compensated as follows: the class is chosen at random according to the relevant conditional law.

Consider a 0-automatic queue. The notations and assumptions are the ones of Theorem 5.7. In particular (ρ, r) is an admissible solution to the TTE. We assume that $\lambda\widehat{\gamma} < \mu$. Recall that Q_r is the infinitesimal generator of the queue-content on the state space $L(X,\Sigma)$, and that $\pi_{\rho,r}$ given by (42) is its stationary distribution.
Consider the application:

$$\begin{array}{rccc} f: & L(X,\Sigma) & \longrightarrow & \{0\} \cup (\mathbb{N}^* \times \Sigma) \\ & u_n \cdots u_1 & \longmapsto & (n, u_n) \\ & 1_{\Sigma^*} & \longmapsto & 0 \end{array}$$

Define the infinitesimal generator $\widetilde{Q}_r$ on the state space $\{0\} \cup (\mathbb{N}^* \times \Sigma)$ by:

$$\widetilde{Q}_r(x,y) = \sum_{u \in f^{-1}(x)} \frac{\pi_{\rho,r}(u)}{\pi_{\rho,r}(f^{-1}(x))} \sum_{v \in f^{-1}(y)} Q_r(u,v) \ . \tag{51}$$

For instance, we obtain by using (42) and simplifying:

$$\widetilde{Q}_r\big((n,a);(n-1,b)\big) = \lambda \sum_{c*a=1_X} \nu(c) \frac{r(b)}{r(\mathrm{Next}(a))} \mathbb{1}_{\{b \in \mathrm{Next}(a)\}} \ .$$

Define a total order on $\{0\} \cup (\mathbb{N}^* \times \Sigma)$ as follows: 0 is the smallest element and $(n,a) \leq (m,b)$ if $n < m$ or $(n = m, a \preceq b)$, where $\preceq$ is some total order on Σ. A couple of lines of computation enable to check the following. If lines and columns are ranked according to the above order, the infinitesimal generator $\widetilde{Q}_r$ is block tridiagonal of the form:

$$\widetilde{Q}_r = \begin{pmatrix} a & A(r) & 0 & 0 & \\ B & A_1 & A_0 & 0 & \ddots \\ 0 & A_2(r) & A_1 & A_0 & \ddots \\ 0 & 0 & A_2(r) & A_1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix} , \tag{52}$$

where a is of dimension 1×1, $A(r)$ is of dimension $1 \times \Sigma$, B is of dimension $\Sigma \times 1$, and $A_0, A_1, A_2(r)$ are of dimension $\Sigma \times \Sigma$. Furthermore the entries of B, A_0, A_1 can be expressed in function of λ, μ, and ν; and the entries of $A(r), A_2(r)$ can be expressed in function of λ, μ, ν, and r.

According to the terminology in Neuts [23], $\widetilde{Q}_r$ is the infinitesimal generator of a QBD process with a complex boundary behavior. For any such ergodic process, the shape of the stationary distribution is known, see for instance [23, Chapter 1.5]. So we assume that $\widetilde{Q}_r$ is ergodic and we apply the general results to get the stationary distribution $\widetilde{\pi}$:

$$\widetilde{\pi}(0) = y, \quad \forall n \geq 1, \forall a \in \Sigma, \ \widetilde{\pi}(n,a) = (xR^{n-1})(a) \ , \tag{53}$$

where

$$\begin{cases} A_0 + RA_1 + R^2 A_2(r) = 0 \\ ay + xB = 0, \ yA(r) + x\big(A_1 + RA_2(r)\big) = 0 \\ y + x(I-R)^{-1}(1,\dots,1)^T = 1 \end{cases} . \tag{54}$$

In (54), R is a matrix of dimension $\Sigma \times \Sigma$ and is the minimal nonnegative solution to the first Equation. The pair (y, x), where y is a scalar and x is a line vector of dimension Σ, is the unique positive solution to the second and third Equations.

The stationary distribution $\widetilde{\pi}$ in (53) has a *matrix product form.* This matrix product form is said to be a *product form* (this is called *Level-Geometric with parameter 0* in [10]) if: $\widetilde{\pi}(n,a) = \eta^{n-1}\widetilde{\pi}(1,a) = \eta^{n-1}x(a)$, for some $\eta \in (0,1)$. Clearly, a necessary and sufficient condition for this to hold is: $xR = \eta x$. Assume that this last equality holds. By multiplying the first Equation in (54) by x, and by simplifying the Equations, we get:

$$\begin{cases} x\big(A_0 + \eta A_1 + \eta^2 A_2(r)\big) = 0 \\ ay + xB = 0, \ yA(r) + x\big(A_1 + \eta A_2(r)\big) = 0 \\ y + (1-\eta)^{-1}x(1,\dots,1)^T = 1 \end{cases} . \tag{55}$$

Now let us replace x by $cr, c \in \mathbb{R}$, and η by ρ in (55). The first Equation yields precisely the Twisted Traffic Equations (29) for the pair (ρ, r). The other two Equations yield exactly: $y = 1 - \rho$, $c = \rho(1-\rho)$.

We conclude that the stationary distribution of $\widetilde{Q}_r$ is given by:

$$\widetilde{\pi}(0) = 1 - \rho, \quad \forall n \geq 1, \forall a \in \Sigma, \ \widetilde{\pi}(n,a) = (1-\rho)\rho^n r(a) \ . \tag{56}$$

This is coherent with the form of $\pi_{\rho,r}$ as given in (42). So the above provides an a-posteriori and partial justification for the shape of $\pi_{\rho,r}$ It also provides another light on the central role of the TTE.

It is classical in QBD theory [19, 10] to modify the boundary condition to get a stationary distribution of product form. Here the situation is more complex since not only the boundary condition, but also the A_2 matrix, are modified in the quest for the product form.

Observe also that the result in Theorem 5.7 is much deeper than the one in (56). In particular, there is a-priori no way to retrieve the result on Q_r from the one on the simplified generator $\widetilde{Q}_r$.

6 Extension and Examples

6.1 Zero-automatic queues built on 0-automatic pairs

All the results in Sections 4 and 5 are derived for queues built on plain monoids not isomorphic to $\mathbb{Z}$ (Definition 2.1). However, the 0-automatic queue built on $\mathbb{Z}$ is interesting in itself (it corresponds to Gelenbe's G-queue, see Section 3) and exhibits new phenomena. It is therefore worthwhile to determine the subset of the above results which remain true for this queue.

In fact, we define a more general framework, and the notion of 0-automatic triples extending the plain triples of Definition 2.1.

Let $(X, *)$ be a group or monoid with set of generators Σ. Denote by $\pi : \Sigma^* \to X$ the monoid homomorphism which associates to a word $a_1 \cdots a_k$ of Σ^* the element $a_1 * \cdots * a_k$ of X. A language L of Σ^* is a *cross-section* of X if the restriction of π to L is a bijection. The inverse map $\Phi : X \to L$ is then called the *normal form* map. Define the language $L(X, \Sigma) \subset \Sigma^*$ as in (9). Define the sets: $\forall a \in \Sigma$,

$$\text{Prev}(a) = \{b \in \Sigma \mid b * a \notin \Sigma \cup \{1_X\}\}, \quad \text{Next}(a) = \{b \in \Sigma \mid a * b \notin \Sigma \cup \{1_X\}\} . \tag{57}$$

In the case of Next(.), this is the same definition as in (10).

Definition 6.1. *Let $(G, *)$ be a group with finite set of generators Σ. We say that the pair (G, Σ) is 0-automatic if $L(G, \Sigma)$ is a cross-section of G.*

Such pairs were first considered by Stallings [26] under another name. It can be proved using the results from Stallings that G is necessarily isomorphic to a plain group. However the set Σ may be larger than a natural (see Section 2.2) set of generators of the plain group.

Now let us extend the notion to monoids. To get good properties it is necessary to choose a more complex definition proposed in [21].

Definition 6.2. *Let $(M, *)$ be a monoid with finite set of generators Σ. Assume that $L(M, \Sigma)$ is a cross-section. Let $\Phi : M \to L(M, \Sigma)$ be the corresponding normal form map. Assume that: $\forall u \in M$ s.t. $\Phi(u) = u_1 \cdots u_k$, $\forall a \in \Sigma$,*

$$\Phi(u*a) = \begin{cases} u_1 \cdots u_{k-1} & \text{if } u_k * a = 1_M \\ u_1 \cdots u_{k-1}v & \text{if } u_k * a = v \in \Sigma \\ u_1 \cdots u_k a & \text{otherwise} \end{cases} , \quad \Phi(a*u) = \begin{cases} u_2 \cdots u_k & \text{if } a * u_1 = 1_M \\ vu_2 \cdots u_k & \text{if } a * u_1 = v \in \Sigma \\ au_1 \cdots u_k & \text{otherwise} \end{cases} \tag{58}$$

Assume furthermore that : $\forall a, b \in \Sigma$ such that $a * b \in \Sigma$,

$$Prev(a * b) = Prev(a), \qquad Next(a * b) = Next(b) . \tag{59}$$

Then we say that the pair (M, Σ) is 0-automatic.

In the group case $M = G$, the conditions (58) and (59) are implied by the fact that the language $L(G, \Sigma)$ is a cross-section.

The pairs formed by a plain monoid and natural generators are 0-automatic. However, in contrast with the group case, plain monoids do not exhaust the family of monoids appearing in 0-automatic pairs. For instance, $(M = \langle\, a, b \mid ab = 1 \,\rangle, \{a, b\})$ is a plain pair, but the monoid M is not isomorphic to a plain monoid. This example is studied in Section 6.2.

Definition 6.3. *A triple (X, Σ, ν) is said to be* 0-automatic *if: (i) (X, Σ) is a 0-automatic pair with X infinite; (ii) ν is a probability measure whose support is included in Σ and generates X.*

Any plain triple, see Def. 2.1, is 0-automatic. Observe that (X, ν) may not be transient in a 0-automatic triple. Also the graph of succesors $(\Sigma, \to)$ defined in (10) may not be strongly connected.

Consider a 0-automatic triple (X, Σ, ν), $\lambda, \mu \in \mathbb{R}_+^*$, and $r \in \bar{\mathcal{B}}$, see (13). The *0-automatic queue* of type $(X, \Sigma, \nu, r, \lambda, \mu)$ is defined exactly as in Definition 3.1.

The corresponding TE and TTE are still defined as in (14) and (29), respectively. We use the convention described after (13) to define a solution to the TTE belonging to $\bar{\mathcal{B}}$.

The TE do not necessarily have a unique solution, in contrast with Prop. 2.4. The TTE as well do not necessarily have a unique solution, and a solution $r \in \bar{\mathcal{B}}$ to the TTE does not necessarily belong to $\mathcal{B}$. This is in contrast with Theorem 5.8.

An analog of Lemma 5.2 holds: if $(\rho, r), r \in \bar{\mathcal{B}}$, is a solution to the TTE then either $(\rho, r) = (1, r)$ and r is a solution to the TE, or

$$\rho = \frac{\lambda A(r)}{\mu + \lambda C(r)} . \tag{60}$$

We can now state the following result. Observe in particuler that there may be several variants (corresponding to different r's) of the 0-automatic queue with a product form.

Proposition 6.4. *Let (X, Σ, ν) be a 0-automatic triple. Let $(\rho, r), r \in \bar{\mathcal{B}}$, be a solution to the TTE satisfying (60). Assume that: $\forall a \in \Sigma$, $[r(a) > 0] \implies [r(Next(a)) > 0]$. Define $\widetilde{\Sigma} = \{a \in \Sigma \mid r(a) > 0\}$ and $\widetilde{L} = L(X, \Sigma) \cap \widetilde{\Sigma}^*$.*

Consider the 0-automatic queue of type $(X, \Sigma, \nu, r, \lambda, \mu)$. Let Q_r be the infinitesimal generator of the queue-content process M_r. Consider the measure $p_{\rho,r}$ on $\widetilde{L}$ defined by:

$$\forall a_1 \cdots a_n \in \widetilde{L}, \quad p_{\rho,r}(a_1 \cdots a_n) = \rho^n \frac{r(a_1)}{r(Next(a_1))} \cdots \frac{r(a_{n-1})}{r(Next(a_{n-1}))} r(a_n) . \tag{61}$$

We have $p_{\rho,r} Q_r = 0$. Besides, we have:

$$\begin{array}{ccccc} [\lambda\widehat{\gamma} < \mu] & \Longrightarrow & [\rho < 1] & \Longrightarrow & [M_r \textit{ ergodic on } \widetilde{L}\,] \\ [\lambda\widehat{\gamma} > \mu] & \Longrightarrow & [\rho > 1] & \Longrightarrow & [M_r \textit{ transient on } \widetilde{L}\,] \end{array} .$$

When $\lambda\widehat{\gamma} < \mu$, the stationary distribution of M_r is $\pi_{\rho,r} = (1-\rho)p_{\rho,r}$. The corresponding stationary departure process is a Poisson process of rate $\rho\mu$.

The proofs of Lemma 5.6, Theorem 5.7, and Theorem 5.10 are easily adapted to get Prop. 6.4.

6.2 Five illustrating examples

We study five particular 0-automatic queues to illustrate the above results. We focus on three aspects: (a) the stability region; (b) the value of the *load* ρ; (c) the existence of several stationary regimes (for the queues built on 0-automatic triples instead of plain triples).

When the model is simple enough, the TTE can be solved explicitly to get closed form formulas as for $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$ below. In all cases and like any set of algebraic equations, the TTE can be solved with any prescribed precision.

Another goal is to convey the idea that 0-automatic queues ought to be pertinent in several modelling contexts, due to the flexibility in their definition. The five examples below should be interpreted having in mind the different "types" of tasks detailed in the Introduction (classical, positive/negative, "one equals many", and "dating agency").

6.2.1 The free product $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$

Consider the plain triple $(\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}, \Sigma = \{a, a^2, b, b^2\}, \nu)$, where $\nu(a) = \nu(b) = p$, $\nu(a^2) = \nu(b^2) = q = 1/2 - p$, $p \in (0, 1/2)$.

In [22, Section 4.2], the drift is computed, it is given by:

$$\widehat{\gamma} = -\frac{1}{4} + \frac{1}{4}\sqrt{16p^2 - 8p + 5}.$$

According to Theorem 5.8, the associated TTE have a unique admissible solution that we denote by (ρ, r). Solving the TTE, we get that:

$$\widehat{r}(a) = \widehat{r}(b) = \frac{4\lambda p^2 - 2\lambda p + 4p\mu + \lambda}{4(4\lambda p^2 - 2\lambda p + \mu + \lambda)} = \frac{-4pq\lambda + \lambda + 4p\mu}{4(-4pq\lambda + \lambda + \mu)}, \quad \widehat{r}(a^2) = \widehat{r}(b^2) = \frac{1}{2} - \widehat{r}(a),$$

$$\rho = 2\frac{4\lambda^2p^2 - 2\lambda^2 p + \lambda\mu + \lambda^2}{4\lambda^2p^2 - 2\lambda^2 p + 4\lambda\mu + \lambda^2 + 4\mu^2}.$$

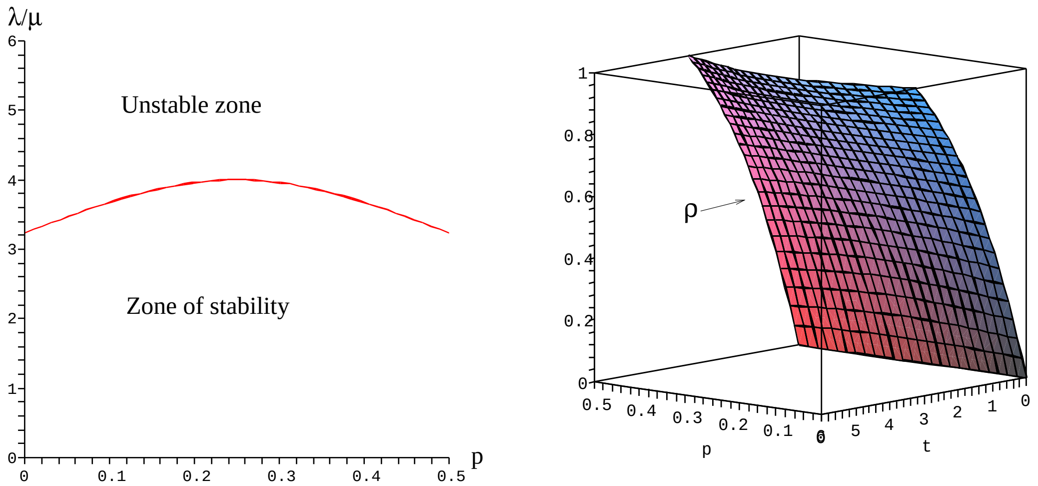

Figure 4: $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$: The stability region (left) and the load ρ (right).

In Figure 4 (left), we show the stability region of the queue. The abscissa is p and the ordinate is λ/μ. In Figure 4 (right), we plot the load ρ as a function of p and $t = \lambda/\mu$, for $p \in (0, 1/2)$ and $\lambda/\mu \in (0, \min(1/\widehat{\gamma}, 6))$. Hence, ρ is always smaller or equal to 1, see Theorem 5.7.

6.2.2 The free product $\mathbb{N} \star \mathbb{B}$

Consider the plain triple $(\{a\}^* \star \langle\, b \mid b^2 = b \,\rangle, \Sigma = \{a, b\}, \nu)$, where $\nu(a) = p$, $\nu(b) = 1 - p$, $p \in (0, 1)$. In Figure 5, we illustrate the corresponding buffering mechanism.

The unique solution $\hat{r}$ of the TE is: $\widehat{r}(c) = p$, $\widehat{r}(b) = 1 - p$. The drift of the random walk is $\widehat{\gamma} = (2 - p)p$.

According to Theorem 5.8, the associated TTE have a unique admissible solution that we denote by (ρ, r). Solving the TTE, we obtain that ρ is a solution of $f(Y) = 0$, where:

$$f(Y) = \mu^2 Y^3 + (\mu^2 + \mu\lambda + \lambda\mu p)Y^2 + (\lambda^2 p + \lambda\mu p)Y - \lambda^2 p^2 + \lambda^2 p\,.$$

The relation between $r(b)$ and ρ is given by: $\rho = [r(b)(1 - p) + p]\lambda/\mu$.

In Figure 6 (left), we show the stability region of the queue. The abscissa is p and the ordinate is λ/μ. In Figure 6 (right), we plot the load ρ as a function of p and $t = \lambda/\mu$, for $p \in (0, 1)$ and $\lambda/\mu \in (0, \min(1/\widehat{\gamma}, 3))$. Hence, ρ is always smaller or equal to 1, see Theorem 5.7.

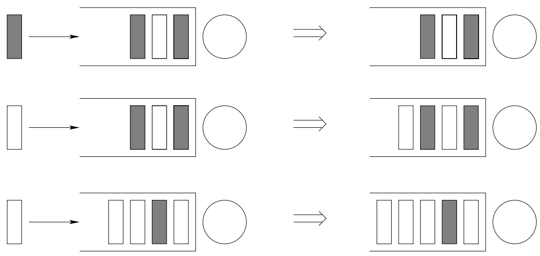

Figure 5: The queue $M/M/(\mathbb{N} \star \mathbb{B}, \Sigma)$ with a in white and b in dark gray.

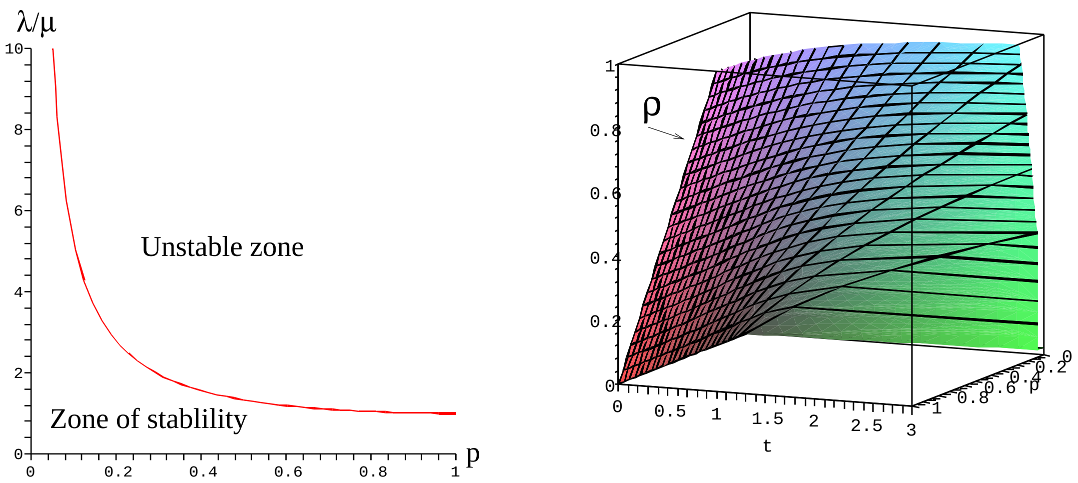

Figure 6: $\mathbb{N} \star \mathbb{B}$: The stability region (left) and the load ρ (right).

6.2.3 The free product $\mathbb{N} \star \mathbb{Z} \star \mathbb{B}$

Consider the queue associated with the plain triple $(\{a\}^* \star \mathbb{F}(b) \star \langle c \mid c^2 = c \rangle, \Sigma = \{a, b, b^{-1}, c\}, \nu)$ where $\nu(a) = p, \nu(b) = \nu(b^{-1}) = q/2$, and $\nu(c) = 1 - p - q$ with $p, q, p + q \in (0, 1)$.
The unique solution $\widehat{r}$ of the associated TE is

$$\widehat{r}(b) = \widehat{r}(b^{-1}) = \frac{1}{2} - \frac{\sqrt{1 - q^2}}{2(1+q)}, \quad \widehat{r}(a) = \frac{p(1 - \widehat{r}(b))}{1 - \widehat{r}(b) - q\widehat{r}(b)}, \quad \widehat{r}(c) = 1 - \widehat{r}(a) - 2\widehat{r}(b).$$

Applying Theorem 2.5, the drift of the random walk is given by: $\widehat{\gamma} = p + (1 - p - q)(1 - \widehat{r}(c)) + q(1 - 2\widehat{r}(b))$. From there, we obtain Figure 7: the stability region is the region below the surface.

6.2.4 The free group $\mathbb{Z}$ and the free product $\mathbb{N} \star \mathbb{Z}$

Consider the 0-automatic queue $(\mathbb{F}(a), \{a, a^{-1}\}, \nu, r, \lambda, \mu)$, where ν is a non-degenerate probability measure on $\Sigma = \{a, a^{-1}\}$. In Figure 8, we illustrate the corresponding buffering mechanism. Such a mechanism is similar to the one of Gelenbe's G-queue.
The underlying triple $(\mathbb{F}(a), \{a, a^{-1}\}, \nu)$ is not plain (Def. 2.1), but it is 0-automatic (Def. 6.3). Here the graph of successors $(\Sigma, \to)$, see (10), is not connected. Also the random walk (X, ν) is not transient but null-recurrent when $\nu(a) = \nu(a^{-1}) = 1/2$.

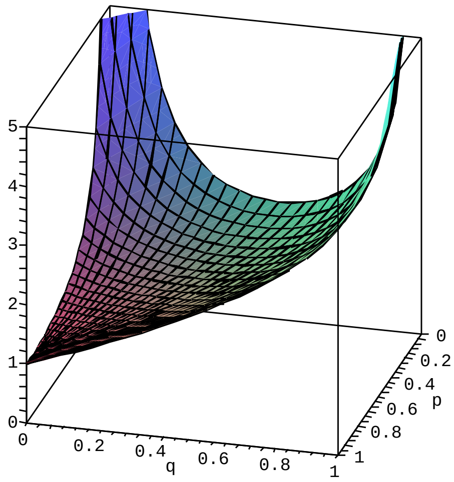

Figure 7: Stability region of the $M/M/1/(\mathbb{N} \star \mathbb{Z} \star \mathbb{B}, \Sigma)$ queue. The axis are p, q, and λ/μ.

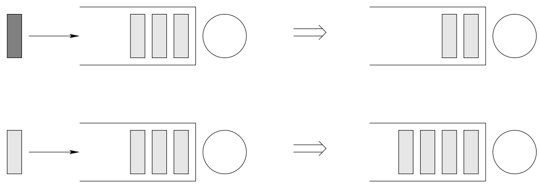

Figure 8: The $M/M/1/(\mathbb{F}(a), \Sigma)$ queue with a in light gray and a^{-1} in dark gray.

The drift of the random walk is easily computed:

$$\widehat{\gamma} = |\nu(a) - \nu(a^{-1})| \,.$$

Assume first that $\nu(a) = \nu(a^{-1})$. Solving the TTE, we get that $(\lambda/(2\mu + \lambda), r)$ is a solution for all $r \in \bar{\mathcal{B}}$. It means that the queue is stable and has a product form distribution under any boundary condition. This interesting behavior can be traced back to the fact that the random walk (X, ν) is not transient.

Assume now that $\nu(a) \neq \nu(a^{-1})$. There are 2 possible solutions for the TTE:

$$(\rho_1, r_1) = \left(\frac{\lambda \nu(a)}{\mu + \lambda \nu(a^{-1})}; (1,0) \right), \qquad (\rho_2, r_2) = \left(\frac{\lambda \nu(a^{-1})}{\mu + \lambda \nu(a)}; (0,1) \right). \tag{62}$$

The two solutions correspond to extremal values for r, it means that in the buffer, there is only one type of customer with probability 1: if $r_1 = (1,0)$, there is only a in the buffer; if $r_2 = (0,1)$, there is only a^{-1} in the buffer. Here we recover a model very close to the classical G-queue. Set $\underline{\rho} = \min\{\rho_1, \rho_2\}$ and $\bar{\rho} = \max\{\rho_1, \rho_2\}$ and define $\underline{r}$ and $\bar{r}$ accordingly. We have:

$$\underline{\rho} < 1, \qquad [\bar{\rho} < 1] \iff [\lambda \widehat{\gamma} < \mu] \,.$$

The stationary distribution of the 0-automatic queue $(\mathbb{F}(a), \Sigma, \nu, \underline{r}, \lambda, \mu)$ is:

$$\pi_{\underline{r}}(1_{\Sigma^*}) = 1 - \underline{\rho}\,, \quad \pi_{\underline{r}}(x^n) = (1 - \underline{\rho})\underline{\rho}^n \,, \forall n \geq 1, \tag{63}$$

where $x = a$ if $\nu(a) < \nu(a^{-1})$, and $x = a^{-1}$ if $\nu(a) > \nu(a^{-1})$. When $\lambda\widehat{\gamma} < \mu$, the 0-automatic queue $(\mathbb{F}(a), \Sigma, \nu, \bar{r}, \lambda, \mu)$ also has a product form stationary distribution of the form (63) with $\bar{\rho}$ instead of $\underline{\rho}$.

Consider now a boundary condition $r \in \mathcal{B}$. In particular, $r \neq r_1$, $r \neq r_2$, so the stationary distribution is not of product form. However, if $\lambda\widehat{\gamma} < \mu$, the stationary distribution π_r can still be determined explicitly by solving the global balance equations. It is given by:

$$\begin{aligned}
\pi_r(1_{\Sigma^*}) &= \Big(1 + r(a)\frac{\rho_1}{1-\rho_1} + r(a^{-1})\frac{\rho_2}{1-\rho_2}\Big)^{-1} \\
\pi_r(a^n) &= \pi_r(1_{\Sigma^*})r(a)\rho_1^n, \qquad \pi_r(a^{-n}) = \pi_r(1_{\Sigma^*})r(a^{-1})\rho_2^n\,,
\end{aligned} \tag{64}$$

where ρ_1 and ρ_2 are defined in (62). The expression in (64) is "almost" of product form. So, why do we prefer an expression like the one in (63) ? The point is that the departure process associated with a stationary distribution of type (64) is not Poisson, as opposed to the one associated with (63). And having a Poisson departure process is crucial to build product form networks, see [9].

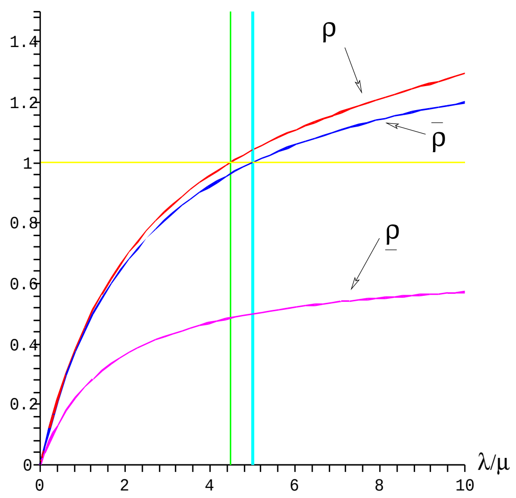

Figure 9: $\mathbb{F}(a)$ and $\mathbb{F}(a) \star \{c\}^*$: the loads as a function of λ/μ.

To summarize, when $\lambda\widehat{\gamma} < \mu$, there are two variants of the 0-automatic queue with a product form. We would like to argue that one of the two makes more "physical" sense.

To that purpose, consider the plain triple $(\mathbb{F}(a) \star \{c\}^*, \{a, a^{-1}, c\}, \nu)$ with $0 < \nu(c) \ll 1$. According to Theorem 5.8, there exists a single variant of the queue with a product form. Let ρ be the corresponding load. The question is to determine which one of the two solutions in (62) is recovered when letting $\nu(c)$ go to 0.

Since the TTE are difficult to solve explicitly, we content ourselves with numerical evidence. In Figure 9, we plot ρ, $\bar{\rho}$, and $\underline{\rho}$ as functions of λ/μ, for $\nu(c) = 0.01$ and $\nu(a) = p = 3/5$. We see that ρ tends to the larger solution $\bar{\rho}$. The two vertical lines correspond to the stability regions. They have an abscissa equal to the inverse of the drift $\widehat{\gamma}^{-1}$ for the random walk on $\mathbb{F}(a) \star \{c\}^*$ and $\mathbb{F}(a)$ respectively.

6.2.5 The monoid $M = \langle\, a, b \mid ab = 1 \,\rangle$

Consider the *bicyclic monoid* $M = \langle\, a, b \mid ab = 1 \,\rangle$. Here we have a new "type" of tasks. It is close to the positive/negative type but with no symmetry between the positive and negative customers.

Consider the triple $(M, \Sigma = \{a, b\}, \nu)$, with $\nu(a) = p \in (0,1)$, $\nu(b) = 1 - p$. It is a 0-automatic triple but not a plain triple. In particular, the graph of successors $(\Sigma, \rightarrow)$ is not strongly connected.

The drift of the random walk is easily computed and given by $\widehat{\gamma} = |1 - 2p|$. Solving the associated TTE, we obtain that there is one solution if $p \leq 1/2$ and two solutions if $p > 1/2$. More precisely, these two solutions are:

$$(\rho_1, r_1) = \left(\frac{\lambda(1-p)}{\mu + \lambda p}; (0,1) \right) \,,\ \forall p$$

$$(\rho_2, r_2) = \left(\frac{\lambda p}{\mu + \lambda(1-p)}; \left(\frac{2p-1}{p}, \frac{1-p}{p}\right) \right) \,,\ \text{if } p > 1/2.$$

We have:

$$\begin{array}{lll} \text{if } p \leq 1/2, & \left[\rho_1 < 1\right] \iff \left[\lambda\widehat{\gamma} < \mu\right] & \\ \text{if } p > 1/2, & \rho_1 < 1, & \left[\rho_2 < 1\right] \iff \left[\lambda\widehat{\gamma} < \mu\right] . \end{array}$$

In Figure 10, we show the solutions to the TTE as a function of p and λ/μ.

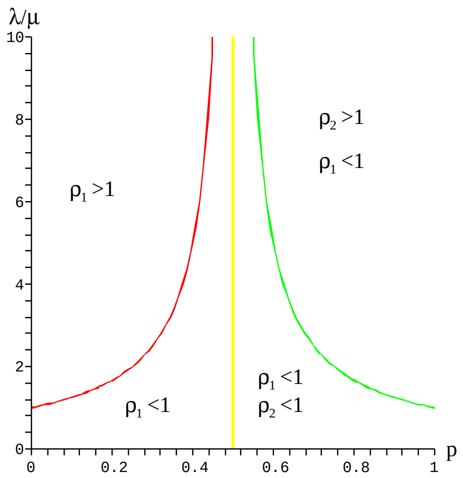

Figure 10: $\mathbb{F}(a)$: The solutions to the TTE.

To discriminate between (ρ_1, r_1) and (ρ_2, r_2), we proceed as for $\mathbb{F}(a)$. Consider $X = \langle\, a, b \mid ab = 1 \,\rangle \star \{c\}^*$. Set $\nu(a) = p$, $\nu(b) = q$, $\nu(c) = 1 - p - q$ with $p, q > 0$, $p + q < 1$. Now, the triple $(X, \{a, b, c\}, \nu)$ is plain.

The TE can be solved explicitly, the unique solution $\widehat{r}$ is determined by:

$$\widehat{r}(a) = \frac{p(1 - \widehat{r}(b))}{1 - p\widehat{r}(b)} \,,\ \widehat{r}(b) = \frac{1 - \sqrt{1 - 4pq}}{2p} \,,\ \widehat{r}(c) = \frac{1 - p - q}{1 - p\widehat{r}(b)}.$$

According to Theorem 2.5, the corresponding drift is $\widehat{\gamma} = \sqrt{1 - 4pq}$. Now, solving the TTE, we find a unique admissible solution (ρ, r), given by:

$$\rho = \frac{\lambda(1 - pr(b))}{\mu + \lambda pr(b)}, \qquad r = \widehat{r}\,.$$

We have $\rho < 1$ iff $\mu > \lambda\widehat{\gamma} = \lambda\sqrt{1-4pq}$.

Let us observe what happens when $\nu(c)$ tends to 0. When $p < 1/2$, there is only one solution (ρ_1, r_1) for the TTE of the first case. And, as expected, (ρ, r) tends to (ρ_1, r_1) when $\nu(c) \to 0$. When $p > 1/2$, there are two possible solutions (ρ_1, r_1) and (ρ_2, r_2) for the TTE of the first case. In this case, (ρ, r) tends to (ρ_2, r_2) when $\nu(c) \to 0$.

In Figure 11 (left), we plot ρ and ρ_1 as functions of p and λ/μ, for $\nu(c) = 0.01$. In Figure 11 (right), we plot ρ and ρ_2.

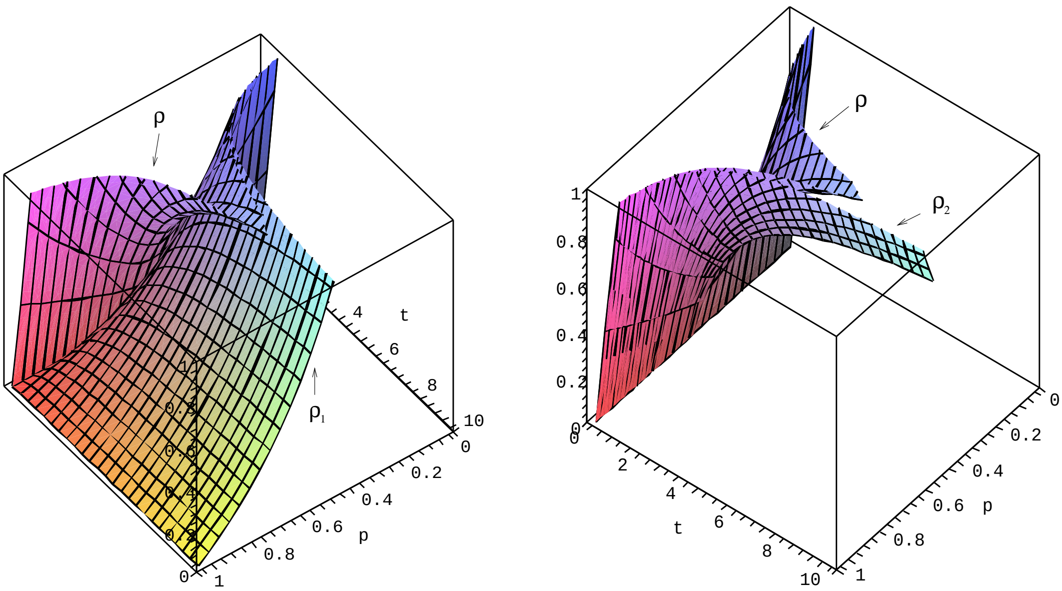

Figure 11: $\langle\, a, b \mid ab = 1 \,\rangle$ and $\langle\, a, b \mid ab = 1 \,\rangle \star \{c\}^*$: the loads in function of $\nu(a)$ and $t = \lambda/\mu$.

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