Bilinear Functions and Trees over the (max, +) Semiring

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Abstract. We consider the iterates of bilinear functions over the semiring $(\max, +)$. Equivalently, our object of study can be viewed as recognizable tree series over the semiring $(\max, +)$. In this semiring, a fundamental result associates the asymptotic behaviour of the iterates of a linear function with the maximal average weight of the circuits in a graph naturally associated with the function. Here we provide an analog for the 'iterates' of bilinear functions. We also give a triple recognizing the formal power series of the worst case behaviour.
Remark. Due to space limitations, the proofs have been omitted. A full version can be obtained from the authors on request.

1 Introduction

Among all the complete binary trees having k internal nodes, what is the largest possible value of the difference between the number of internal nodes having even height and the number of leaves having odd height? As a byproduct of the tools developped in this paper, we will effectively solve this problem, see Example 5 below.

The $(\max, +)$ semiring has been studied in various contexts in the last forty years. It appears in Operations Research for optimization problems (see [9, 14]); it is a useful tool to study some decision problems in formal language theory (see [16, 17, 21, 22]); and it has an important role in the modelling and analysis of Discrete Event Systems (see [1, 6, 13]). In all of these applications, linear functions over the $(\max, +)$ semiring play a preeminent role. To give just one example, the dates of occurrence of events in a Timed Event Graph, a class of Discrete Event Systems, are given by the iterates of a linear function over the $(\max, +)$ semiring, see [1, 6]. It is natural to study the direct generalization of linear functions: bilinear functions over the $(\max, +)$ semiring.

There is another possible way to introduce and motivate our study. Trees are one of the most important structure in computer science. They constitute a basic data structure; they are also the natural way to describe derivations of context free grammars used by compilators. Formal power tree series with coefficients in a semiring were introduced by Berstel and Reutenauer in [2], and further studied for instance in [4,5]. In [2], the authors concentrate on recognizable series with coefficients in a field. They prove, among many other things, that the height of a tree defines a series which is not recognizable over a field (Example 9.2 in [2]). On the other hand, it is straightforward to show that this series is recognizable over the $(\max, +)$ semiring (Example 1 below). In this paper, we study the general class of recognizable tree series in one letter over the $(\max, +)$ semiring.

The $(\max, +)$ semiring $\mathbb{R}_{\max}$ is the set $\mathbb{R} \cup \{-\infty\}$, equipped with the max operation, written additively, i.e., $a \oplus b = \max\{a, b\}$, and the usual sum, written multiplicatively, i.e., $a \otimes b = a + b$. The neutral elements of the two operations are respectively $-\infty$ and 0. The $(\max, +)$ semiring is *idempotent*: $a \oplus a = a$, for all a. When there is no possible confusion, we simplify the notation by writing ab instead of $a \otimes b$. On the other hand, the operations denoted by $+, -, \times$ and $/$ always have to be interpreted in the conventional algebra.

We define accordingly the semimodule $\mathbb{R}_{\max}^n$ (for $u, v \in \mathbb{R}_{\max}^n$, $(u \oplus v)_i = u_i \oplus v_i$, and for $u \in \mathbb{R}_{\max}^n, \lambda \in \mathbb{R}_{\max}, (\lambda u)_i = \lambda u_i$). We denote by $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}_{\max}^n$ (i.e. $(e_i)_i = 0$ and $(e_i)_j = -\infty$ for $j \neq i$).

A *linear form* of dimension n over $\mathbb{R}_{\max}$ is a function $f : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}$ verifying $f(u \oplus v) = f(u) \oplus f(v), \forall u, v \in \mathbb{R}_{\max}^n$, and $f(\lambda u) = \lambda f(u), \forall u \in \mathbb{R}_{\max}^n, \lambda \in \mathbb{R}_{\max}$. A *linear function* of dimension n over $\mathbb{R}_{\max}$ is a function $f : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$ verifying the same two conditions. A *bilinear function* of dimension n over $\mathbb{R}_{\max}$ is a function $f \colon \mathbb{R}_{\max}^n \times \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$, such that for all a in $\mathbb{R}_{\max}^n$ the functions $f_a, g_a \colon \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$ defined by $f_a(x) = f(a, x)$, $g_a(x) = f(x, a)$ are linear.

As usual, we denote by f^k the k-th iterate of a function f. The asymptotic behavior of the iterates of a linear function is well understood. Indeed one of the most famous results in the $(\max, +)$ semiring (see for instance the textbooks [1, 19] and the references therein) is the following one. For all initial vector $\alpha \in \mathbb{R}_{\max}^n$ such that $\forall i, \alpha_i \neq -\infty$, and for all linear form $\beta \colon \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}$ such that $\forall i, \beta(e_i) \neq -\infty$, we have

$$\lim_k \frac{\beta[f^k(\alpha)]}{k} = \rho(f) , \tag{1}$$

where $\rho(f)$ is the maximal average weight of the simple circuits in a graph canonically associated with f.

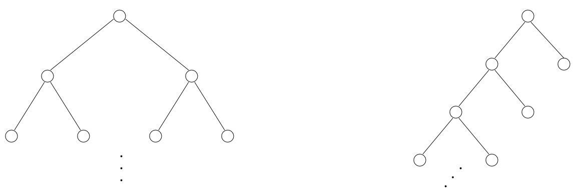
In this paper, we prove analog results for bilinear functions. Let f be a bilinear function of dimension n and let t be a (complete ordered binary) tree. We define recursively the functions $f^t : \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}^n$ by $f^t(u) = u$ when t is the root tree, and $f^t(u) = f(f^{t_1}(u), f^{t_2}(u))$ when t has t_1 and t_2 as left and right subtrees. Let the *size* $|t|$ of a tree t be the number of its internal nodes. Let us consider $\alpha \in \mathbb{R}_{\max}^n$ and a linear form $\beta \colon \mathbb{R}_{\max}^n \to \mathbb{R}_{\max}$ such that $\forall i, \beta(e_i) \neq -\infty$. We prove the following result:

$$\limsup_k \frac{\max_{t, |t|=k} \beta[f^t(\alpha)]}{k} \tag{2}$$

does not depend on β and is equal to the maximal average weight of finitely many "simple" weighted trees. The quantity in (2) is called the *spectral radius* (of f at α). In comparison with the result for linear functions, note that the spectral radius depends on α. A tree attaining the maximum in $\max_{t,|t|=k} \beta[f^t(\alpha)]$ is called a *maximal tree* (of size k).

The computation of the spectral radius for bilinear functions gives rises to situations that are conceptually different from those for linear functions. In order to motivate the reader, and before unfolding our results, we illustrate some of these differences with two simple examples.

As a first example, consider the bilinear function f of dimension 2 defined by $f\colon (u_1, v_1)^T \times (u_2, v_2)^T \longmapsto (u_2 v_2 \oplus u_1 v_2, M u_1 v_1)^T$, where $M \in \mathbb{R}$, and let $\alpha = (-1, 0)^T$ and β be such that $\beta(u) = u_1 \oplus u_2$. The spectral radius is $\max(0, M/3, 2M/3-1)$. Moreover, when $M < 0$, the branch trees (or 'gourmand de la vigne' according to [24], see the right of the figure) are the only maximal trees; when $M = 3$, all trees are maximal; when $M > 0, M \neq 3$, the maximal trees of size k are the ones with respectively no leaf, two leaves or one leaf of odd height if k equals 0,1 or 2 modulo 3 (left of the figure). Thus, although the dependence of the spectral radius on M is continuous, the trees that achieve maximality change drastically.



Now let $f(u, v) = (1 u_2 v_2, u_3 v_3, u_4 v_4, u_1 v_1)^T$ be a bilinear function of dimension 4, and let $\alpha = (0, 0, 0, 0)^T$ and β be such that $\beta(e_i) \neq -\infty$ for all i. As will be easy to deduce from Proposition 1, the spectral radius is then equal to $8/15$, a value that was hard to guess from the coefficients of f and α. We finally illustrate the fact that the spectral radius generally depends on α. For this purpose, consider the same example as above but with $\alpha = (0, c, 0, 0)^T$ for $c \geq 0$. The spectral radius is then $8/15 + c$.

In the last part of the paper, we consider the formal power series over $\mathbb{R}_{\max}$:

$$S(\alpha, f, \beta) = \bigoplus_{k \in \mathbb{N}} \left(\oplus_{t,|t|=k} \beta[f^t(\alpha)] \right) x^k \; .$$

It follows from known results ([2, 8, 18, 12]) that this series is recognizable. Here we provide an alternative proof of this result and we give an explicit construction of a triple recognizing $S(\alpha, f, \beta)$. This construction is a priori different from the known one.

All the results are presented for bilinear functions. This restriction is made only to simplify the presentation. The results are easy to adapt to the case of multilinear functions.

2 Trees

We consider complete ordered binary trees, that is trees where each node has no children or has a left and a right child. The formal definition of a binary tree that we consider is the one given in [20] or [23]. As usual we denote by A^* the free monoid over the set A.

Definition 1. *An* (unlabelled complete ordered binary) tree *is a finite non-empty prefix-closed subset t of $\{0,1\}^*$, such that if $v \in t$, then either both $v \cdot 0 \in t, v \cdot 1 \in t$, or both $v \cdot 0 \notin t, v \cdot 1 \notin t$. Let A be a finite alphabet. A* (complete ordered binary) labelled tree over A *is a partial mapping $\tau\colon \{0,1\}^* \to A$ whose domain* $\mathrm{dom}(\tau)$ *is a tree.*

The definitions below are given for trees. They are easy to extend to labelled trees. Since a tree is a non-empty prefix-closed subset of $\{0,1\}^*$, it always contains the empty word ε that is called the *root* of the tree. The tree $t = \{\varepsilon\}$ is called the *root tree*. The *frontier* of a tree t is the set $\mathrm{fr}(t) = \{v \in t \mid v \cdot 0, v \cdot 1 \notin t\}$. The elements in $t, \mathrm{fr}(t)$, and $t \setminus \mathrm{fr}(t)$ are called respectively *nodes*, *leaves*, and *internal nodes*. The *size* of a tree t, denoted by $|t|$, is the number of its internal nodes. We denote by $\mathcal{T}$ the set of trees, and by $\mathcal{T}^n$ the set of trees of size n.

Given a tree t different from the root tree, we define its *left subtree* $t_1 = \{w \in \{0,1\}^* \mid 0 \cdot w \in t\}$ and its *right subtree* $t_2 = \{w \in \{0,1\}^* \mid 1 \cdot w \in t\}$, and we write $t = \varepsilon(t_1, t_2)$. In the case of a labelled tree τ with left and right subtrees τ_1 and τ_2 and with $\tau(\varepsilon) = a$, we write $\tau = a(\tau_1, \tau_2)$.

3 Linear Functions over the $(\max, +)$ Semiring

The $(\max, +)$ semiring has been defined in the introduction. We use the matrix and vector operations induced by the semiring structure: if A and B are two matrices of appropriate sizes with coefficients in the semiring $\mathbb{R}_{\max}$, we define $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij})$ and $(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj} = \max_k(A_{ik} + B_{kj})$. We still use the simplified notation AB for $A \otimes B$.

Let f be a linear function of dimension n over the $(\max, +)$ semiring, see §1. We associate canonically a matrix A to f by setting $A_{ij} = f(e_i)_j$. Then A^k is the matrix associated with f^k. Below, we have chosen to state the results in terms of powers of matrices instead of iterates of linear functions. All the results are classical; for details see the textbooks [1, 19, 15] and the references therein.

We associate with a square matrix A of dimension n, the valued directed graph $\mathcal{G}(A)$ with nodes $\{1, \ldots, n\}$ and with an arc (i,j) if $A_{ij} \neq -\infty$, this arc being valued by A_{ij}. We use the classical terminology of graph theory. In particular, we use the notation $i \to j$ to denote the existence of a path from node i to node j in the graph.

Let us consider a triple (α, A, β) where $A \in \mathbb{R}_{\max}^{n \times n}$, $\alpha \in \mathbb{R}_{\max}^{1 \times n}$ and $\beta \in \mathbb{R}_{\max}^{n \times 1}$. We say that (α, A, β) is *trim* if for all k, there exist i, j, such that $\alpha_i \neq -\infty$, $\beta_j \neq -\infty$, $i \to k$ and $k \to j$. Given a matrix $A \in \mathbb{R}_{\max}^{n \times n}$, we define

$$\rho(A) = \bigoplus_{l=1,\ldots,n} \bigoplus_{i_1,\ldots,i_l} (A_{i_1 i_2} \otimes A_{i_2 i_3} \otimes \ldots \otimes A_{i_l i_1})^{1/l} .$$

In words, $\rho(A)$ is equal to the maximal average weight of the circuits of $\mathcal{G}(A)$.

Theorem 1. *Let (α, A, β) be a trim triple. We have* $\limsup_k (\alpha A^k \beta)/k = \rho(A)$.

In particular, when A is irreducible, we have $\limsup_k A^k_{ij}/k = \rho(A)$ for all i, j. It is also well known that $\rho(A)$ is equal to the maximal eigenvalue of A (we say that $\lambda \in \mathbb{R}_{\max} \setminus \{-\infty\}$ is an *eigenvalue* of A if there exists $u \in \mathbb{R}^n_{\max}$ such that $A \otimes u = \lambda \otimes u$).

The result in Theorem 1 can be easily extended to a non-trim triple (α, A, β). Indeed, if $(\tilde{\alpha}, \tilde{A}, \tilde{\beta})$ is the trim part of (α, A, β), then we have $\forall k, \alpha A^k \beta = \tilde{\alpha} \tilde{A}^k \tilde{\beta}$.

4 Bilinear Functions over the $(\max, +)$ Semiring

A *bilinear function* of dimension n has the following structure:

$$B\colon \mathbb{R}^n_{\max} \times \mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}, \quad (u, v) \mapsto \begin{pmatrix} \bigoplus_{i,j} B_{1ij} \otimes u_i \otimes v_j \\ \cdots \\ \bigoplus_{i,j} B_{nij} \otimes u_i \otimes v_j \,. \end{pmatrix},$$

where $B_{kij} \in \mathbb{R}_{\max}$ for all i, j, k.

We recall the recursive definition of $B^t : \mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$, for $t \in \mathcal{T}$.

- If $t = \{\varepsilon\}$ then $\forall u,\ B^t(u) = u$;
- if $t = \varepsilon(t_1, t_2)$ then $\forall u,\ B^t(u) = B(B^{t_1}(u), B^{t_2}(u))$.

Let us consider the triple (α, B, β) where $\alpha \in \mathbb{R}^n_{\max}$, $B\colon \mathbb{R}^n_{\max} \times \mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$ is a bilinear function, and $\beta\colon \mathbb{R}^n_{\max} \to \mathbb{R}_{\max}$ is a linear form (see §1). The function *recognized* by the triple is the function $\mu : \mathcal{T} \to \mathbb{R}_{\max}$ defined by $\mu(t) = \beta[B^t(\alpha)]$.

Example 1. The height function $h\colon \mathcal{T} \to \mathbb{N}$ is defined recursively as follows: $h(\{\varepsilon\}) = 0$ and if $t = \varepsilon(t_1, t_2)$, then $h(t) = 1 + \max(h(t_1), h(t_2))$.

Consider the triple (α, B, β) defined as follows:

$$\alpha = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B(u, v) = \begin{pmatrix} u_1 v_1 \\ 1 u_1 v_2 \oplus 1 u_2 v_1 \end{pmatrix}, \quad \beta(u) = u_2 \,.$$

This triple recognizes the height function. Indeed, consider $t = \varepsilon(t_1, t_2)$ and assume that we have $B^{t_1}(\alpha) = (0, h(t_1))$ and $B^{t_2}(\alpha) = (0, h(t_2))$. Then it follows that $B^t(\alpha) = B((0, h(t_1)), (0, h(t_2))) = (0, 1h(t_1) \oplus 1h(t_2)) = (0, h(t))$.

Definition 2. *Given a triple (α, B, β), we define its* spectral radius *as the quantity:*

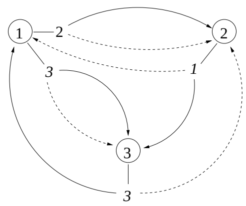
$$\limsup_k \max_{t \in \mathcal{T}^k} \frac{\beta[B^t(\alpha)]}{k}$$

Our goal is to study the spectral radius. To do this, we would like to associate with a bilinear function B a sort of graph describing it (mimicking the situation for linear functions).

5 Tree-graphs, Tree-paths and Tree-circuits

We define a particular type of directed graph, denoted by $\mathcal{G}(B)$, as follows. The set of nodes of $\mathcal{G}(B)$ is $\{1, 2, \ldots, n\}$ (where n is the dimension of B) and for each $B_{kij} \neq -\infty$ there exists in $\mathcal{G}(B)$ a pair of arcs $((k,i),(k,j))$ where (k,i) is the *left arc* and (k,j) is the *right arc*. The pair $((k,i),(k,j))$ is valued by B_{kij}. We say that $\mathcal{G}(B)$ is the *tree-graph associated with* B.

Example 2. Consider the bilinear function $B\colon \mathbb{R}^3_{\max} \times \mathbb{R}^3_{\max} \to \mathbb{R}^3_{\max}$ defined by: $B(u,v) = (\ 2u_2v_2 \oplus 3u_3v_3, 1u_3v_1, 3u_1v_2\ )^T$. The associated graph has three nodes $\{1,2,3\}$. Moreover $B_{122} = 2$, $B_{133} = 3$, $B_{231} = 1$ and $B_{312} = 3$. The corresponding pairs of arcs are shown in the following figure: we draw a continuous line to denote the left arc and a dashed line to denote the right arc.



Remark 1. A triple (α, B, β) can be considered as a bottom-up tree automaton in one letter over $\mathbb{R}_{\max}$. The tree-graph $\mathcal{G}(B)$ can be considered as a visualization of the corresponding top-down tree automaton. An extensive account on tree automata can be found in [7].

We now define the notions of tree-path and tree-circuit in a tree-graph, generalizing the classical notions of path and circuit in a graph.

Definition 3. *Let B be a bilinear function of dimension n. A* tree-path *over $\mathcal{G}(B)$ is a tree τ over the alphabet $\{1, 2, \ldots, n\}$, different from the root tree and such that if $v \in \mathrm{dom}(\tau) \setminus \mathrm{fr}(\tau)$ and $\tau(v) = k$, $\tau(v \cdot 0) = i$ and $\tau(v \cdot 1) = j$, then $B_{kij} \neq -\infty$. A* tree-circuit *over $\mathcal{G}(B)$ is a tree-path where the root and at least one of the leaves have the same label.*

We will denote by $\mathrm{path}(B)$ the set of all tree-paths in $\mathcal{G}(B)$, and by $\mathrm{circ}(B)$ the set of all tree-circuits in $\mathcal{G}(B)$.

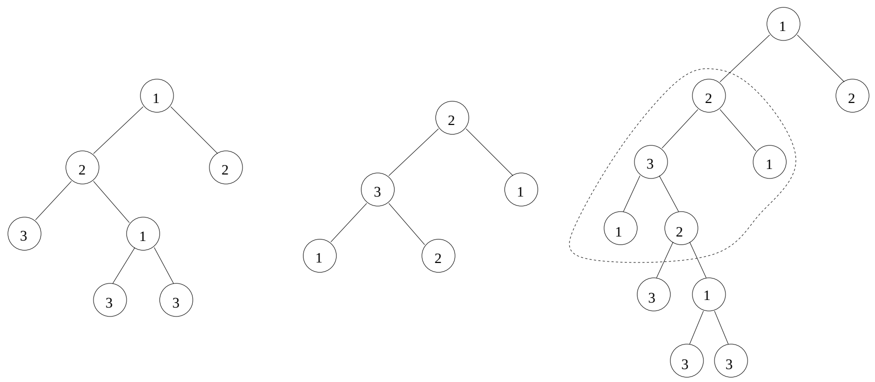
On the free monoid $\{0,1\}^*$, we use the notation $\leq$ for the prefix order (with $0 \leq 1$). Let τ be a tree-path over $\mathcal{G}(B)$, and let $\tau(v) = i$ for some $v \in \mathrm{dom}(\tau)$. Let c be a tree-circuit over $\mathcal{G}(B)$ whose root is labelled by i, and let y be a leaf of c labelled by i. The *composition of the tree-path τ with the tree-circuit c at the nodes v and y* is the tree-path $\tau[v,y]c$ defined as follows:

$$\mathrm{dom}(\tau[v,y]c) = \{u \in \mathrm{dom}(\tau) \mid v \not\leq u\} \cup \{vz \mid z \in \mathrm{dom}(c)\} \cup \{vyw \mid vw \in \mathrm{dom}(\tau)\}$$

$$(\tau[v,y]c)(u) = \begin{cases} \tau(u) & \textit{if} \ \ v \not\leq u \\ c(w) & \textit{if} \ \ u = vz \ \ \textit{and} \ \ z \in \mathrm{dom}(c) \\ \tau(vw) & \textit{if} \ \ u = vyw, \ vw \in \mathrm{dom}(\tau) \end{cases}$$

To avoid ambiguities, it is necessary to use parentheses for iterated compositions. We omit parentheses when the composition is performed from left to right, for instance: $\tau[v,y]c[v',y']c' = (\tau[v,y]c)[v',y']c'$. When we do not need to specify about the exact positions where the composition is performed, we simplify the notation by writing $\tau[\cdot]c$ instead of $\tau[v,y]c$.

Example 3. Below we have represented a tree-path τ over the tree-graph of Example 2 (on the left of the figure) and a tree-circuit c over the same tree-graph (on the middle of the figure). The composition $\tau[0,01]c$ produces the tree on the right of the figure.



Definition 4. *A tree-path over $\mathcal{G}(B)$ is a* simple tree-path *if it can not be obtained by composition of a tree-path with a tree-circuit. A* simple tree-circuit *is a tree-circuit that can not be obtained by composition of two tree-circuits.*

It follows from this definition that every tree-circuit can be written as an iterated composition of simple tree-circuits. Moreover every tree-path can be written as an iterated composition of a simple tree-path with a sequence of simple tree-circuits. We remark that tree-circuits and tree-paths can have several different decompositions.

Lemma 1. *Let τ be a simple tree-path over $\mathcal{G}(B)$. Then we have $h(\tau) \leq n$ and $|\tau| \leq 2^n - 1$ (where h is the height function defined in Example 1). Let c be a simple tree-circuit, then we have $h(c) \leq 2n-1$ and $|c| \leq n + n2^n$.*

6 Asymptotic Behavior

Let us fix a vector α in $\mathbb{R}^n_{\max}$. We define a function $p_\alpha(.)$, depending on B and α, on the set of trees labelled by $\{1,\ldots,n\}$ as follows:

- if $\mathrm{dom}(\tau) = \{\varepsilon\}$, and $\tau(\varepsilon) = k$ then $p_\alpha(\tau) = \alpha_k$;
- if $\tau = k(\tau_1, \tau_2)$, and $\tau_1(\varepsilon) = i, \tau_2(\varepsilon) = j$, then $p_\alpha(\tau) = B_{kij} + p_\alpha(\tau_1) + p_\alpha(\tau_2)$.

The easy but useful relation below provides an interpretation of $B^t(\alpha)$ in terms of maximally weighted tree-paths in $\mathcal{G}(B)$.

$$B^t(\alpha)_i = \max_{\tau\in\mathrm{path}(B),\mathrm{dom}(\tau)=t,\tau(\varepsilon)=i} p_\alpha(\tau) \; . \tag{3}$$

Let c be a tree-circuit and let l be a leaf of c such that $c(l) = c(\varepsilon)$. The *average weight* of c is defined as

$$w_\alpha(c) = \frac{1}{|c|} \Big(\sum_{u\in\mathrm{dom}(c)\setminus\mathrm{fr}(c)} B_{c(u)c(u.0)c(u.1)} + \sum_{u\in\mathrm{fr}(c)\setminus\{l\}} \alpha_{c(u)} \Big) \; .$$

When $\alpha_{c(\varepsilon)} \neq -\infty$, then we have $w_\alpha(c) = (p_\alpha(c) - \alpha_{c(\varepsilon)})/|c|$.

A tree-circuit over $\mathcal{G}(B)$ is *maximal* if its average weight is greater than or equal to the average weight of any other tree-circuit over $\mathcal{G}(B)$. Since there is an infinite number of tree-circuits, the existence of maximal tree-circuits is not a priori guaranteed. We recall that $\mathrm{circ}(B)$ is the set of the tree-circuits of $\mathcal{G}(B)$, and we denote by $\mathrm{simp}(B)$ the set of the simple tree-circuits.

Lemma 2. *There exists a simple tree-circuit with maximal average weight, i.e.:*

$$\sup_{c\in\mathrm{circ}(B)} w_\alpha(c) = \max_{c\in\mathrm{simp}(B)} w_\alpha(c) \; .$$

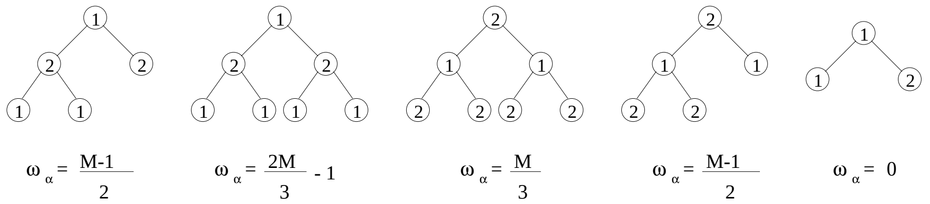
It follows from Lemma 1 and Lemma 2 that there always exist a maximal tree-circuit of height at most $2n-1$.

The result below is to be compared with Theorem 1. We recall that given a triple (α, B, β), we have defined its spectral radius in Definition 2.

Proposition 1. *Let us consider a triple (α, B, β) where $\alpha \in \mathbb{R}^n_{\max}$, $B\colon \mathbb{R}^n_{\max} \times \mathbb{R}^n_{\max} \to \mathbb{R}^n_{\max}$ is a bilinear function, and $\beta\colon \mathbb{R}^n_{\max} \to \mathbb{R}_{\max}$ is a linear form such that $\alpha_i \neq -\infty$ and $\beta(e_i) \neq -\infty$ for all i. The spectral radius depends only on α and B, is denoted $\rho(\alpha, B)$, and is given by $\rho(\alpha, B) = \max_{c\in\mathrm{simp}(B)} w_\alpha(c)$.*

The triple (α, B, β) is said to be *trim* if for all k in $\{1, \dots, n\}$, there exists a tree-path τ over $\mathcal{G}(B)$ such that $k \in \mathrm{dom}(\tau)$, $\beta(e_{\tau(\varepsilon)}) \neq -\infty$, $p_\alpha(\tau) \neq -\infty$. Proposition 1 holds under the more general assumption that the triple is trim.

Example 4. Consider the bilinear function $f(u, v) = (u_2v_2 \oplus u_1v_2, Mu_1v_1)^T$, with $M \in \mathbb{R}$, and let $\alpha = (-1, 0)^T$, and β be such that $\beta(u) = u_1 \oplus u_2$ (same example as in the introduction). There are 9 simple tree-circuits. Only 5 among the 9 can have a maximal weight for some values of M. We have represented below these 5 simple tree-circuits, together with their corresponding average weight w_α.



We deduce the formula for the spectral radius given in the introduction: $\rho(\alpha, f) = \max(0, M/3, 2M/3 - 1)$.

7 Formal Power Series

Let us consider a triple (α, B, β) where $\alpha \in \mathbb{R}^n_{\max}$, B is a bilinear function of dimension n, and β is a linear form of dimension n. We consider the following formal power series in one indeterminate over the $(\max, +)$ semiring:

$$S(\alpha, B, \beta) = \bigoplus_{k \in \mathbb{N}} \left(\oplus_{t,|t|=k} \beta[B^t(\alpha)]\right) x^k \ . \tag{4}$$

For details concerning formal power series over a semiring, see [3, 18]. A series S in one indeterminate over $\mathbb{R}_{\max}$ is *recognizable* if there exists an integer N and a triple (a, A, b) with $a \in \mathbb{R}^{1\times N}_{\max}$, $A \in \mathbb{R}^{N\times N}_{\max}$, $b \in \mathbb{R}^{N\times 1}_{\max}$ and such that

$$S = \bigoplus_{k \in \mathbb{N}} (a \otimes A^k \otimes b) x^k \tag{5}$$

We also say that the triple (a, A, b) *recognizes* S. Using classical results, we obtain that the series $S(\alpha, B, \beta)$ is recognizable. Indeed, according to Theorem 7.1 in [2], the series $S(\alpha, B, \beta)$ is algebraic. Using an adaptation of an original argument by Parikh, see [8, 18], an algebraic series in one indeterminate over a commutative and idempotent semiring is recognizable (the use of Parikh result in the context of (max,+) algebraic series appears in [12]).

Using the notions of simple tree-path and simple tree-circuit defined above, we obtain an alternative proof of this result. We get an explicit construction of a triple having the required property.

Proposition 2. *There exists a triple (a, A, b) of dimension $O(n2^{2n})$ which recognizes $S(\alpha, B, \beta)$.*

Example 5. In order to illustrate the expressiveness of these series, consider the following simple example. We define $f(u, v) = (Ku_2v_2, u_1v_1)^T$, and $\alpha = (0, M)^T$. Then it is easily seen that $f^t(\alpha)$ is equal to (h being the height function):

$$\begin{pmatrix} K \times \#\{u \in t \setminus \mathrm{fr}(t) \mid h(u) \text{ even}\} + M \times \#\{u \in \mathrm{fr}(t) \mid h(u) \text{ odd}\} \\ K \times \#\{u \in t \setminus \mathrm{fr}(t) \mid h(u) \text{ odd}\} + M \times \#\{u \in \mathrm{fr}(t) \mid h(u) \text{ even}\} \end{pmatrix} .$$

For instance, if we choose $K = 1, M = -1$ and $\beta(u) = u_1$, the k-th coefficient in the resulting series $S(\alpha, f, \beta)$ is equal to the largest possible difference between the number of internal nodes of even height and leaves of odd height in trees of size k. The complete version of Proposition 2 (i.e. with proof) gives an explicit construction for computing these quantities for all k.

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