Services within a Busy Period of an M/M/1 Queue and Dyck Paths

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Abstract

We analyze the service times of customers in a stable M/M/1 queue in equilibrium depending on their position in a busy period. We give the law of the service of a customer at the beginning, at the end, or in the middle of the busy period. It enables as a by-product to prove that the process of instants of beginning of services is not Poisson. We then proceed to a more precise analysis. We consider a family of polynomial generating series associated with Dyck paths of length $2n$ and we show that they provide the correlation function of the successive services in a busy period with $n+1$ customers.

Résumé

On s'intéresse à l'analyse des temps de service des clients d'une file M/M/1 stable et en équilibre selon leur position dans une période d'activité. On donne la loi d'un service sachant que le client se trouve au début, à la fin ou au milieu de la période d'activité. Ceci permet, au passage, de prouver que le processus des instants de début de service n'est pas un processus de Poisson. On mène ensuite une étude plus fine. On exhibe une famille de séries génératrices polynômiales associées aux chemins de Dyck de longueur $2n$ et on montre qu'il s'agit de la fonction de corrélation des différents services dans une période d'activité comportant $n+1$ clients.

Keywords: M/M/1 queue, busy period, Dyck paths.

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1 Introduction

The M/M/1/∞/FIFO queue (or M/M/1 queue) is the queue with a Poissonian arrival stream, exponential services, a single server, an unlimited buffer capacity, and a First-In-First-Out service discipline. It can be argued that the M/M/1 queue is the most elementary and the most studied system in queueing theory, see for instance [3, 11, 8, 9]. Quoting [9], "most likely, any book with *queueing* in the title has something to say on the subject".

Let λ be the intensity of the Poisson arrival process and let μ be the parameter of the exponential service times. Assume that the stability condition $\lambda < \mu$ holds and consider the queue in equilibrium. Our objective is to get precise information on the distribution of the service of a customer based on its position in the busy period.

First of all, recall that the distribution of the first, respectively last, service is an exponential of parameter μ, respectively $(\lambda + \mu)$. We are then able to compute the distribution of a service in

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the "middle" of a busy period (i.e. neither at the beginning nor at the end). As a by-product, we also get the distribution of the duration between two successive beginning of services. Since it is not an exponential, we conclude that the point process of the instants of beginning of services is not Poisson (as opposed to the point process of completion of services).

Then we study the service time of the k-th customer in a busy period of length $n+1$ (i.e. containing $n+1$ customers). Consider a busy period conditionned to be of length $n+1$, and let $(q_i)_{0,\dots,2n+2}$ be the corresponding embedded queue-length excursion. Its trajectories are equiprobable and it is easy to see that they are in bijection with Dyck paths of length $2n$. If we condition $(q_i)_{0,\dots,2n+2}$ to be associated with a given Dyck path π of length $2n$ then we observe that the law of the service time of the k-th customer is equal to the convolution product of l_k+1 exponentials of parameter $\lambda+\mu$ where l_k is the length of the intersection of π with the line $y = x - 2k$. By summing over Dyck paths of length $2n$, we get an expression for the joint law of the services in a busy period of length $n+1$. Then using elementary properties of Dyck paths, we obtain results on services within a busy period somewhat difficult to obtain by direct probabilistic arguments (Section 4). The correlation function of the services is a natural generating polynomial of Dyck paths following a simple integral recursion (Section 5).

Using the combinatorial properties of lattice paths to study the busy period of simple queues is classical, see [4, 6, 12] and references therein. In these articles, quantities such that the area swept by the queue-length process during a busy period are studied, with a much more involved combinatorial analysis than what is presented below for the sequence of services within a busy period. This should come as no surprise. The area and related quantities, are derived by counting in a Dyck path the number of ascents and descents of a given vertical coordinate (Dyck paths are lattice paths in $\mathbb{N}\times\mathbb{N}$, see Section 3). On the contrary, the sequence of services is derived by counting in a Dyck path the number of ascents of a given horizontal coordinate (roughly speaking). This is in essence like working with generating polynomials of Dyck paths in *non-commuting* variables. It is therefore hopeless to get as precise information.

2 In the Middle of the Busy Period

Given a positive real random variable X with law μ, denote its Laplace transform by $\Psi_X(s) = \Psi_\mu(s) = \int \exp(-sx)d\mu(x), s \in \mathbb{C}, \mathrm{Re}(s) \geq 0$. We write $\mathcal{L}[X \mid \mathcal{A}] = \mathbb{P}\{X \in \cdot \mid \mathcal{A}\}$ for the conditional law of X given an event $\mathcal{A}$. The corresponding Laplace transform is denoted $\Psi_{X|\mathcal{A}}(s)$. The convolution product of two probability distributions μ and ν is denoted by $\mu \star \nu$. The indicator function of a subset A of a set is denoted by $\mathbb{1}_A$. It is convenient to denote by $\mathrm{Exp}(a)$ the exponential distribution of parameter a defined by $\mathrm{Exp}(a)[x,+\infty) = \exp(-ax), x \in \mathbb{R}_+$. Recall that $\Psi_{\mathrm{Exp}(a)}(s) = a/(a+s)$.

We consider an $M/M/1$ queue with the following notations. Let $(A_n)_{n\in\mathbb{Z}}$ be the arrival Poisson process of intensity λ. Let $(\tau_n)_{n\in\mathbb{Z}}$ be the inter-arrival times, with $\tau_n = A_{n+1} - A_n$. Denote by $(\sigma_n)_{n\in\mathbb{Z}}$ the service times of the customers. The sequence $(\sigma_n)_n$ is i.i.d. and $\sigma_0 \sim \mathrm{Exp}(\mu)$. We assume that the stability condition $\lambda < \mu$ is satisfied, and we consider the queue in equilibrium. Let $(Q_t)_{t\in\mathbb{R}}$ be the queue-length process, where Q_t is the number of customers either in service or in the buffer at time t.

The state of the server can be described as an alternating sequence of idle and busy periods. A *busy period* is a maximal period during which $Q_t > 0$. An idle period is a maximal period during which $Q_t = 0$. The *length* $|B|$ of a busy period B (not to be confused with its duration) is the number of customers served during the busy period. Throughout, when we consider a generic busy period B, we denote for simplicity by $\sigma_0, \dots, \sigma_{|B|-1}$ and $\tau_0, \dots, \tau_{|B|-1}$ respectively the service times and the inter-arrival times of the different customers in the busy period.

Lemma 2.1. *Let Δ_n be the event that a generic busy period consists of $n+1$ customers, then*

$$\Delta_n = \{\sum_{j=0}^{i} \tau_j < \sum_{j=0}^{i} \sigma_j,\ i = \{0,\dots,n-1\};\ \sum_{j=0}^{n} \tau_j \geq \sum_{j=0}^{n} \sigma_j\}\ . \tag{2.1}$$

The justification is easy.

The durations of successive busy periods and idle periods are independent random variables. The duration of an idle period is clearly distributed as $\mathrm{Exp}(\lambda)$. The distribution of a busy period is more complex. The next results can be found for instance in [3, Chapter II.2.2] or [11, Chapter 1.2]. The probability that a busy period B consists of $(n+1)$ customers is given by

$$\mathbb{P}\{|B| = n+1\} = C_n \frac{\lambda^n \mu^{n+1}}{(\lambda+\mu)^{2n+1}}\ , \tag{2.2}$$

where C_n is the n-th Catalan number, see §3. Let δ_n be the conditional law of the duration of a busy period, given that the length of the busy period is $(n+1)$. The Laplace transform of δ_n is given by

$$\Psi_{\delta_n}(s) = \frac{(\lambda+\mu)^{2n+1}}{(\lambda+\mu+s)^{2n+1}}\ . \tag{2.3}$$

Hence, δ_n is the distribution of the sum of $(2n+1)$ i.i.d. r.v.'s of law $\mathrm{Exp}(\lambda+\mu)$.

Given two random variables $X \sim \mathrm{Exp}(\alpha)$ and $Y \sim \mathrm{Exp}(\beta)$, where $\alpha,\ \beta \in \mathbb{R}_+^*$, recall that

$$\mathfrak{L}[X \mid X \geq Y] = \mathrm{Exp}(\alpha+\beta) \star \mathrm{Exp}(\alpha),\ \mathfrak{L}[X \mid X < Y] = \mathrm{Exp}(\alpha+\beta)\ . \tag{2.4}$$

Using elementary arguments based on the memoryless property of the exponential distribution, we get:

$$\sigma_0 \sim \mathrm{Exp}(\mu),\ \ \sigma_{|B|-1} \sim \mathrm{Exp}(\lambda+\mu)\ . \tag{2.5}$$

Furthermore, remarking that $\{|B| = 1\} = \{\sigma_0 \leq \tau_0\}$ and using (2.4), it follows that:

$$\mathfrak{L}[\sigma_0 \mid |B| = 1] = \mathrm{Exp}(\lambda+\mu),\ \mathfrak{L}[\sigma_0 \mid |B| > 1] = \mathrm{Exp}(\lambda+\mu) \star \mathrm{Exp}(\mu)\ . \tag{2.6}$$

Our goal is now to derive the law of a service in the *middle* of B, i.e. of a service which is neither the first nor the last one (assuming that $|B| > 2$).

Let σ_* be the service of a generic customer numbered $*$ and let B be the busy period it belongs to. Define the events

$$\begin{aligned}
\mathfrak{E}_{\mathrm{o}} &= \{* \text{ is the } \textbf{o}\text{nly customer of } B\} = \{|B| = 1\} \\
\mathfrak{E}_{\mathrm{f}} &= \{* \text{ is the } \textbf{f}\text{irst customer of } B \text{ and } |B| > 1\} \\
\mathfrak{E}_{\mathrm{l}} &= \{* \text{ is the } \textbf{l}\text{ast customer of } B \text{ and } |B| > 1\} \\
\mathfrak{E}_{\mathrm{m}} &= \{* \text{ is in the } \textbf{m}\text{iddle of } B \text{ and } |B| > 2\}\ .
\end{aligned}$$

Clearly the four events are disjoint and $\mathbb{P}\{\mathfrak{E}_{\mathrm{o}} \cup \mathfrak{E}_{\mathrm{f}} \cup \mathfrak{E}_{\mathrm{l}} \cup \mathfrak{E}_{\mathrm{m}}\} = 1$. Since the lengths of successive busy periods are i.i.d., we obtain immediately that

$$\begin{aligned}
\mathbb{P}\{\mathfrak{E}_{\mathrm{f}}\} &= \mathbb{P}\{\mathfrak{E}_{\mathrm{l}}\} \\
\mathbb{P}\{\mathfrak{E}_{\mathrm{f}}\} &= \mathbb{P}\{|B| > 1\}/\mathbb{E}[|B|] \\
\mathbb{P}\{\mathfrak{E}_{\mathrm{o}} \cup \mathfrak{E}_{\mathrm{f}}\} &= 1/\mathbb{E}[|B|]\ .
\end{aligned}$$

Now using (2.2), we get $\mathbb{P}\{|B| > 1\} = \lambda/(\lambda+\mu)$ and $\mathbb{E}[|B|] = \mu/(\mu-\lambda)$. It follows that

$$\mathbb{P}\{\mathfrak{E}_{\mathrm{o}}\} = \frac{\mu-\lambda}{\mu+\lambda},\quad P\{\mathfrak{E}_{\mathrm{f}}\} = \mathbb{P}\{\mathfrak{E}_{\mathrm{l}}\} = \frac{\lambda(\mu-\lambda)}{\mu(\mu+\lambda)},\ \mathbb{P}\{\mathfrak{E}_{\mathrm{m}}\} = \frac{2\lambda^2}{\mu(\mu+\lambda)}\ . \tag{2.7}$$

Clearly, $\sigma_0 \sim \mathcal{L}[\sigma_* \mid \mathfrak{E}_{\mathrm{o}} \cup \mathfrak{E}_{\mathrm{f}}]$ and $\sigma_{|B|-1} \sim \mathcal{L}[\sigma_* \mid \mathfrak{E}_{\mathrm{l}}]$. We deduce that

$$\begin{aligned}\Psi_{\sigma_*}(s) &= \mathbb{P}\{\mathfrak{E}_{\mathrm{o}} \cup \mathfrak{E}_{\mathrm{f}}\}\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{o}}\cup\mathfrak{E}_{\mathrm{f}}}(s) + \mathbb{P}\{\mathfrak{E}_{\mathrm{l}}\}\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{l}}}(s) + \mathbb{P}\{\mathfrak{E}_{\mathrm{m}}\}\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{m}}}(s) \\ &= \mathbb{P}\{\mathfrak{E}_{\mathrm{o}} \cup \mathfrak{E}_{\mathrm{f}}\}\Psi_{\sigma_0}(s) + \mathbb{P}\{\mathfrak{E}_{\mathrm{l}}\}\Psi_{\sigma_{|B|-1}}(s) + \mathbb{P}\{\mathfrak{E}_{\mathrm{m}}\}\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{m}}}(s)\,.\end{aligned}$$

That is,

$$\frac{\mu}{\mu+s} = \frac{\mu-\lambda}{\mu}\frac{\mu}{\mu+s} + \frac{\lambda(\mu-\lambda)}{\mu(\mu+\lambda)}\frac{\mu+\lambda}{\mu+\lambda+s} + \frac{2\lambda^2}{\mu(\mu+\lambda)}\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{m}}}(s)\,.$$

After simplification of the above expression, we obtain the Laplace transform of the conditional law of σ_* on the event $\mathfrak{E}_{\mathrm{m}}$:

$$\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{m}}}(s) = \frac{(2\mu+s)(\mu+\lambda)}{2(\mu+s)(\mu+\lambda+s)}\,. \tag{2.8}$$

As a by-product, we can prove that the process of instants of beginning of services is not a Poisson process, in contrast with the process of completion of services (departure instants) which is Poisson of intensity λ according to Burke Theorem [2, 7]. Let us detail the argument. Let ξ_* be the difference between the instants of beginning of services of two generic successive customers numbered $*$ and $(*+1)$. Let B be the busy period of $*$ and let ν be the first idle period following B ($\nu \sim \mathrm{Exp}(\lambda)$). Using (2.5), (2.6), we get immediately that

$$\begin{aligned}\xi_*\mathbb{1}_{\mathfrak{E}_{\mathrm{o}}} = (\sigma_0+\nu)\mathbb{1}_{\mathfrak{E}_{\mathrm{o}}} &\implies \mathcal{L}[\xi_* \mid \mathfrak{E}_{\mathrm{o}}] = \mathrm{Exp}(\lambda+\mu)\star\mathrm{Exp}(\lambda) \\ \xi_*\mathbb{1}_{\mathfrak{E}_{\mathrm{f}}} = \sigma_0\mathbb{1}_{\mathfrak{E}_{\mathrm{f}}} &\implies \mathcal{L}[\xi_* \mid \mathfrak{E}_{\mathrm{f}}] = \mathrm{Exp}(\lambda+\mu)\star\mathrm{Exp}(\mu) \\ \xi_*\mathbb{1}_{\mathfrak{E}_{\mathrm{l}}} = (\sigma_{|B|-1}+\nu)\mathbb{1}_{\mathfrak{E}_{\mathrm{l}}} &\implies \mathcal{L}[\xi_* \mid \mathfrak{E}_{\mathrm{l}}] = \mathrm{Exp}(\lambda+\mu)\star\mathrm{Exp}(\lambda)\,,\end{aligned}$$

and

$$\mathcal{L}[\xi_* \mid \mathfrak{E}_{\mathrm{m}}] = \mathcal{L}[\sigma_* \mid \mathfrak{E}_{\mathrm{m}}]\,.$$

Since we have just computed $\Psi_{\sigma_*|\mathfrak{E}_{\mathrm{m}}}(s)$, we deduce the Laplace transform of ξ_*:

$$\Psi_{\xi_*}(s) = \frac{\lambda(\mu^2(\lambda+\mu)+\mu(2\mu+\lambda)s+\lambda s^2)}{\mu(\lambda+s)(\mu+s)(\lambda+\mu+s)}\,. \tag{2.9}$$

We check on this expression that $\mathbb{E}[\xi_*] = 1/\lambda$ and we have $\mathbb{E}[\xi_*^2] = \frac{\lambda^3-\mu\lambda^2+\mu^3+\mu^2\lambda}{\lambda^2\mu^2(\lambda+\mu)}$. In particular, we have $\mathbb{E}[\xi_*^2] < \mathbb{E}[d_*^2] = 2/\lambda^2$, where d_* is a generic inter-departure time.

3 Dyck Paths

The Catalan numbers $(C_n)_{n\in\mathbb{N}}$ are defined by

$$C_n = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}\,. \tag{3.1}$$

The generating function of these numbers is given by

$$\sum_{n=0}^{+\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}\,.$$

The first Catalan numbers are $C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,\ C_6 = 132,\ C_7 = 429, \cdots$. They appear in many combinatorial contexts see for instance [5, 10]. In particular, C_n is the number of Dyck paths of length $2n$. A *Dyck path* of length $2n$ is a path in the lattice $\mathbb{N}\times\mathbb{N}$ which begins at the origin $(0,0)$ ends at $(0,2n)$ and with steps of type $(1,1)$

or $(1,-1)$. Denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, observ ethat $\mathcal{D}_0$ is a singleton whose element is the unique Dyck path of length 0.

We now define a family of polynomials related to Dyck paths. Let $\pi \in \mathcal{D}_n$ and let γ_j be the line $y = x - 2j$, for $j \in \{0, \ldots, n-1\}$ and denote by α_j the length of the intersection of γ_j with π (equivalently $\alpha_j + 1$ is the number of lattice points common to π and γ_j). We introduce two polynomials P_π and R_π defined by

$$P_\pi(y_0, y_1, \ldots, y_{n-1}) = \prod_{i=0}^{n-1} \frac{y_i^{\alpha_i}}{\alpha_i!}, \quad R_\pi(y_0, y_1, \ldots, y_{n-1}) = \prod_{i=0}^{n-1} y_i^{\alpha_i}. \tag{3.2}$$

Let $(P_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ be the two families of polynomials defined by $P_n = \sum_{\pi\in\mathcal{D}_n} P_\pi$, $R_n = \sum_{\pi\in\mathcal{D}_n} R_\pi$. Clearly P_n and R_n are homogeneous polynomials of degree n over the n variables $y_0, y_1, \cdots, y_{n-1}$.

4 The Law of the Services in a Busy Period

Recall that the queue-length process $(Q_t)_{t\in\mathbb{R}}$ is a continuous time Birth-and-Death process on $\mathbb{N}$ with generator P such that $P_{n,n+1} = \lambda,\ n \geq 0\,; P_{n,n-1} = \mu,\ n \geq 1\,; P_{n,m} = 0,\ |n-m| \geq 2$. Let $(q_n)_{n\in\mathbb{Z}}$ denote the Markov chain embedded at its jump instants. More precisely, let T be the point process obtained as the superposition of the arrival and departure processes and let $(T_n)_{n\in\mathbb{Z}}$ be its points with the convention $T_0 = A_0$. Then we set $q_n = Q_{T_n^-}$. The transition matrix of $(q_n)_{n\in\mathbb{Z}}$ is given by

$$p_{0,1} = 1;\ p_{i,i-1} = \frac{\mu}{\lambda+\mu};\ p_{i,i+1} = \frac{\lambda}{\lambda+\mu},\ i \geq 1; \tag{4.1}$$

and $p_{i,j} = 0$ otherwise.

A busy period corresponds to an excursion of $(q_n)_{n\in\mathbb{Z}}$ from 0 to its first return to 0. With the same numbering convention as in Section 2, the generic busy period B consists of $n+1$ customers if and only if

$$q_0 = 0,\ q_i > 0,\ i \in \{1, \cdots, 2n+1\},\ q_{2n+2} = 0\,. \tag{4.2}$$

On this event, the (random) path with successive edges $(i-1, q_i - 1)$, $i \in \{1, \cdots, 2n+1\}$ is a (random) Dyck path of length $2n$. We call it the Dyck path *associated with* B (see Figure 1). On the event $\{|B| = n+1\}$, all Dyck paths appear with the same probability (the probability of a given trajectory $(q_n)_n$ depends only on the number of increasing and decreasing jumps, see (4.1)). On the event that $|B| = n+1$ and that the associated Dyck path is $\pi \in \mathcal{D}_n$, the power of y_{i-1} in P_π is the number of customers which join the system between the the i-th and the $(i+1)$-th departures. Combining these observations with the fact that the time between successive transitions of $(Q_t)_t$ are independent r.v.'s of law $\text{Exp}(\lambda+\mu)$ as long as the queue is non empty, we get :

Theorem 4.1. *Given that the length of the busy period is $n+1$, the conditional density of the random vector $(\sigma_0, \ldots, \sigma_n)$ representing the service times of the successive customers is*

$$D(y_0, \ldots, y_n) = \frac{(\lambda+\mu)^{2n+1}}{C_n} e^{-(\lambda+\mu)(y_0+\cdots+y_{n-1})} P_n(y_0, \ldots, y_{n-1}) e^{-(\lambda+\mu)y_n}, \tag{4.3}$$

where P_n is the Dyck polynomial of degree n defined in Section 3.

A direct computation of the Laplace transform leads to the following :

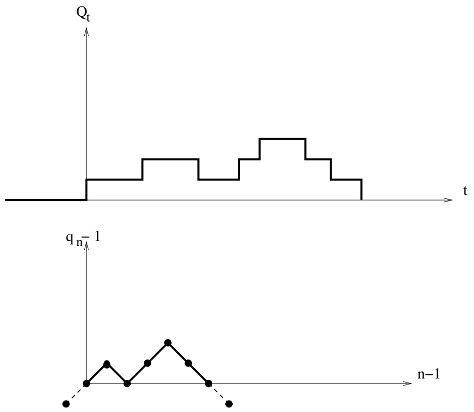

Figure 1: Dyck path associated with a busy period.

Corollary 4.2. *Consider a random vector* $(\tilde{\sigma_0},\dots,\tilde{\sigma_n}) \sim \mathcal{L}[(\sigma_0,\dots,\sigma_n) \mid |B| = n+1]$. *Its Laplace transform is given by*

$$\Psi_{(\tilde{\sigma}_0,\dots,\tilde{\sigma}_n)}(s_0,\dots,s_n) = \mathbb{E}[\prod_{i=0}^{n} e^{-s_i\tilde{\sigma}_i}] \tag{4.4}$$

$$= \frac{1}{C_n}(\prod_{i=0}^{n-1} z_i) R_n(z_0,\dots,z_{n-1}) z_n \ , \tag{4.5}$$

where $z_i = \frac{\lambda+\mu}{\lambda+\mu+s_i}$, $\forall i \in \{0,\dots,n\}$, *and* R_n *is defined in Section 3.*

Let us paraphrase the above results in a somewhat more intuitive way. In a busy period of length $(n+1)$, the conditional law of $(\sigma_0,\dots\sigma_n)$ is the same as the law of $(\tilde{\sigma}_0,\dots\tilde{\sigma}_n)$ that we now describe. The law of $\tilde{\sigma}_n$ is an $\mathrm{Exp}(\lambda+\mu)$ independent of $(\tilde{\sigma}_0,\dots,\tilde{\sigma}_{n-1})$. Let Π be a r.v. uniformly distributed over $\mathcal{D}_n = \{\pi_1,\dots,\pi_{C_n}\}$. Conditionally on $\{\Pi = \pi_i\}$, the r.v.'s $\tilde{\sigma}_j$ are independent and distributed as the sum of k_j^i random variables of law $\mathrm{Exp}(\lambda+\mu)$, where $(k_j^i - 1)$ is the exponent of y_j in P_{π_i}. This is illustrated in Table 1.

We now exploit the correspondance with Dyck paths.

Let $\mathcal{D}_n^{<i>}$ be the set of Dyck paths of length $2n$ where the first return to the axis $\{(n,0),\, n\in\mathbb{N}\}$, after the origin $(0,0)$, occurs at the point $(2i,0)$, $i \in \{1,\dots,n\}$. Clearly, the sets $\mathcal{D}_n^{<i>}$ are disjoint and $\mathcal{D}_n = \cup_{i=1}^{n}\mathcal{D}_n^{<i>}$. Furthermore

$$\mathcal{D}_n^{<i>} \simeq \mathcal{D}_{i-1} \times \mathcal{D}_{n-i} \ . \tag{4.6}$$

A consequence of the above is the very classical identity on Catalan numbers :

$$C_n = \sum_{i+j=n-1} C_i C_j \ .$$

Let $\mathcal{R}_n$ be defined by $\mathcal{R}_n(z_0,\dots,z_{n-1}) = R_n(z_0,\dots,z_{n-1}) \times z_0 \dots z_{n-1}$, going back to Corollary 4.2, we have

$$\Psi_{(\tilde{\sigma}_0,\dots,\tilde{\sigma}_n)}(s_0,\dots,s_n) = \frac{1}{C_n}\mathcal{R}_n(z_0,\dots,z_{n-1}).z_n \ .$$

where $z_i = \frac{\lambda+\mu}{\lambda+\mu+s_i}$, $\forall i \in \{0,\dots,n\}$. We also define $\mathcal{R}_n^{(i)}(z_i) = \mathcal{R}_n(1,\dots,1,z_i,1,\dots,1)$, then

$$\Psi_{\tilde{\sigma}_i}(s_i) = \frac{1}{C_n}\mathcal{R}_n^{(i)}(z_i) \ .$$

Proposition 4.3. *On the event $\{|B| = n+1\}$, we have*

$$\mathcal{R}_0^{(0)}(z_0) = z_0 \,, \forall n \geq 0 \,, \; \mathcal{R}_n^{(0)}(z_0) = \sum_{i+j=n-1} z_0 \mathcal{R}_i^{(0)}(z_0) C_j \,, \tag{4.7}$$

for, $0 < k \leq n-1$,

$$\mathcal{R}_k^{(k)}(z_k) = C_k z_k \,, \; \mathcal{R}_n^{(k)}(z_k) = \sum_{i+j=n-k-1} \mathcal{R}_i^{(k)}(z_k) C_j + \sum_{l=0}^{k-1} \mathcal{R}_{n+l-k}^{(l)}(z_k) \,. \tag{4.8}$$

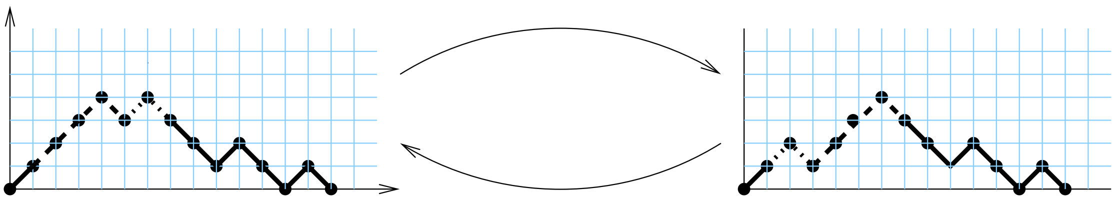

Figure 2: The mapping $\Psi \,:\, \mathcal{D}_n \longrightarrow \mathcal{D}_n$.

On Table 1, one notices a simple relation between the laws of σ_0 and σ_1, which is actually always true :

Proposition 4.4. *Let B be a generic busy period, for $n \geq 1$ we have*

$$\mathcal{L}[\sigma_0 \mid |B| = n+1] = \mathcal{L}[\sigma_1 \mid |B| = n+1] \star \mathrm{Exp}(\lambda + \mu) \,. \tag{4.9}$$

	$\mathrm{Exp}(\lambda+\mu)$	1	2	3	4	5	6	7
$\|B\| = 4$	σ_0		2	2	1			
	σ_1	2	2	1				
	σ_2	3	2					
	σ_3	1						
$\|B\| = 5$	σ_0		5	5	3	1		
	σ_1	5	5	3	1			
	σ_2	7	5	2				
	σ_3	9	5					
	σ_4	1						
$\|B\| = 6$	σ_0		14	14	9	4	1	
	σ_1	14	14	9	4	1		
$\|B\| = 7$	σ_0		42	42	28	14	5	1
	σ_1	42	42	28	14	5	1	

Table 1. Services.

The table should be read as follows. For instance, on $\{|B| = 5\}$, the law of σ_2 is $\mu_{\sigma_2} = \frac{7}{14}\mathrm{Exp}(\lambda+\mu) + \frac{5}{14}\mathrm{Exp}(\lambda+\mu) \star \mathrm{Exp}(\lambda+\mu) + \frac{2}{14}\mathrm{Exp}(\lambda+\mu) \star \mathrm{Exp}(\lambda+\mu) \star \mathrm{Exp}(\lambda+\mu)$.

Proof. The mapping $\Psi : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ is defined in Figure 2. It is clearly an involution, hence a bijection. More formally, given a Dyck path $\pi \in \mathcal{D}_n$ such that $R_\pi = y_1^k . y_0 Q(y_0, y_2, \ldots, y_{n-1})$ then $\Psi(\pi) \in \mathcal{D}_n$ is defined by $R_{\Psi(\pi)} = y_0^{k+1} Q(y_1, y_2, \ldots, y_{n-1})$. In view of Corollary 4.2, it completes the proof. □

5 Dyck Paths Polynomials

We go back to the family of polynomials $(P_n)_{n\in\mathbb{N}}$ defined in Section 3. We are going to use Theorem 4.1 to give nice expressions for the P_n's. Let Δ_n be the event that a generic busy period consists of $n+1$ customers. Let $A_0, \ldots, A_n$ be borelians of $\mathbb{R}_+^*$,

$$\mathbb{P}\{\sigma_i \in A_i,\ i = 0, \ldots, n \mid \Delta_n\} = \frac{\mathbb{P}\{\sigma_i \in A_i,\ i = 0, \ldots, n; \Delta_n\}}{\mathbb{P}\{\Delta_n\}} .$$

Let $L_n = \mathbb{P}\{\sigma_i \in A_i,\ i \in \{0, n\}; \Delta_n\}$ and for $k = 0, \ldots, n$, let $Y_k = \sum_{i=0}^k y_i$ and $X_k = \sum_{i=0}^k x_i$. Using Lemma 2.1, we have

$$\begin{aligned} L_n &= (\lambda\mu)^{n+1} \int_{A_0\times\cdots\times A_n} e^{-\mu Y_n}\, dy_0 \ldots dy_n \int_0^{Y_0} e^{-\lambda x_0} dx_0 \int_0^{Y_1 - X_0} e^{-\lambda x_1} dx_1 \cdots \\ &\qquad \cdots \int_0^{Y_{n-1}-X_{n-2}} e^{-\lambda x_{n-1}} dx_{n-1} \int_{Y_n - X_{n-1}}^{\infty} e^{-\lambda x_n} dx_n \\ &= \lambda^n \mu^{n+1} \int_{A_0\times\cdots\times A_n} e^{-(\mu+\lambda) Y_n}\, dy_0 \ldots dy_n \int_0^{Y_0} dx_0 \int_0^{Y_1 - X_0} dx_1 \cdots \int_0^{Y_{n-1}-X_{n-2}} dx_{n-1} . \end{aligned}$$

Then, using theorem 4.1, we get

$$P_n(y_0, \ldots, y_{n-1}) = \int_0^{y_0} dx_0 \int_0^{y_0 + y_1 - x_0} dx_1 \ldots \int_0^{y_0 + \cdots + y_{n-1} - (x_0 + \cdots + x_{n-2})} dx_{n-1} . \tag{5.1}$$

Simple manipulations of formula (5.1) then yield :

Lemma 5.1. *The polynomials $(P_n)_{n\in\mathbb{N}}$ satisfy the following equations*

$$P_n(y_0, \ldots, y_{n-1}) = \int_{y_1}^{y_0 + y_1} P_{n-1}(y, y_2, \ldots, y_{n-1})\, dy \tag{5.2}$$

and

$$P_n(y_0, \ldots, y_{n-1}) = \int_0^{y_0} dx_0 \int_0^{y_1 + x_0} dx_1 \ \ldots \int_0^{y_{n-1} + x_{n-2}} dx_{n-1} . \tag{5.3}$$

For completeness, here is a direct proof of (5.2) without using Theorem 4.1.

Let $\mathcal{D}_n^{(i)}$ be the set of all Dyck paths of length $2n$ starting with i steps of type $(1,1)$ followed by one step of type $(1,-1)$ and define the polynomial $P_n^{(i)}$ such that

$$\sum_{\pi \in \mathcal{D}_n^{(i)}} P_\pi = \frac{y_0^i}{i!} P_n^{(i)}(y_1, \ldots, y_{n-1}) .$$

Clearly, we have

$$P_n(y_0, \ldots, y_{n-1}) = \sum_{i=1}^n \frac{y_0^i}{i!} P_n^{(i)}(y_1, \ldots, y_{n-1}) .$$

Hence, we get

Figure 3: Proof of the equality (5.4): Paths contributing to $P_k^{(n-1)}$ (Left) and to $P_j^{(n)}$ (Right).

$$\begin{aligned}
\int_{y_1}^{y_0+y_1} P_{n-1}(y, y_2, \ldots, y_{n-1})\, dy &= \int_{y_1}^{y_0+y_1} \sum_{i=1}^{n-1} \frac{y^i}{i!} P_{n-1}^{(i)}(y_2, \ldots, y_{n-1})\, dy \\
&= \sum_{i=1}^{n-1} \frac{1}{(i+1)!}[(y_0+y_1)^{i+1} - y_1^{i+1}] P_{n-1}^{(i)}(y_2, \ldots, y_{n-1}) \\
&= \sum_{j=1}^{n} \frac{y_0^j}{j!} \sum_{k=0}^{n-j} \frac{y_1^k}{k!} P_{n-1}^{(k+j-1)}(y_2, \ldots, y_{n-1})
\end{aligned}$$

With the help of Figure 3, we notice that

$$P_n^{(j)}(y_1, \ldots, y_{n-1}) = \sum_{k=j-1}^{n-1} \frac{y_1^{k-j+1}}{(k-j+1)!} P_{n-1}^{(k)}(y_2, \ldots, y_{n-1}) = \sum_{k=0}^{n-j} \frac{y_1^k}{k!} P_{n-1}^{(k+j-1)}(y_2, \ldots, y_{n-1}) \,. \tag{5.4}$$

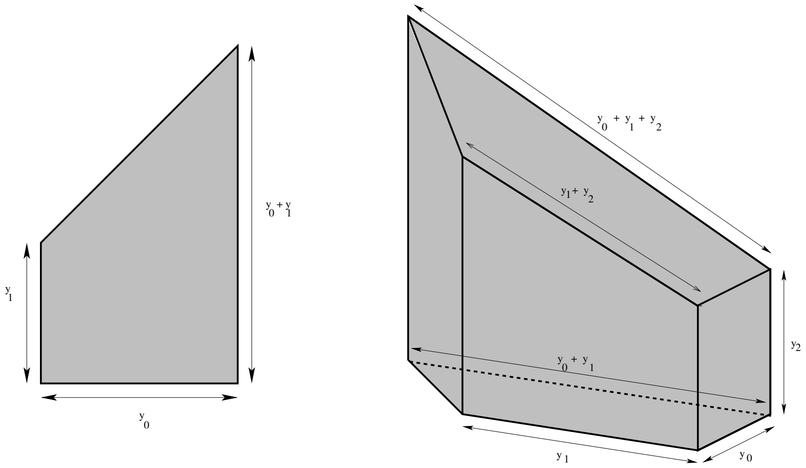

Figure 4: The volumes of the gray areas are $P_2(y_0, y_1)$ (left) and $P_3(y_0, y_1, y_2)$ (right).

It leads to

$$\int_{y_1}^{y_0+y_1} P_{n-1}(y, y_2, \ldots, y_{n-1})\, dy = \sum_{i=1}^{n} \frac{y_0^i}{i!} P_n^{(i)}(y_1, \ldots, y_{n-1}) = P_n(y_0, \ldots, y_{n-1}) .$$

This result can also be proved using the theory of species presented in [1]. Finally, using (5.3), the polynomials $(P_n)_{n\in\mathbb{N}}$ can be interpreted as volumes. We give a representation of this in Figure 4 for $n = 2$ and $n = 3$.

Conclusion. Here are several other simple models of queues for which the queue-length process is a Birth-and-Death process: the M/M/K/∞ queue, the M/M/∞ queue, or the M/M/K/L queue ($K \leq L < \infty$). In each case, if the generic busy period is of length $n+1$, we can associate with it a Dyck path of length $2n$. However, the different Dyck paths of length $2n$ are not equiprobable anymore. Hence, we do not get a simple formula for the joint law of the services as in Theorem 4.1.

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