Finite-range topical functions and uniformly topical functions

Thierry Bousch[*] Jean Mairesse[†]

September 2003

Abstract

We introduce a remarkable subclass of the class of topical functions, the class of *uniformly topical* functions, whose dynamical behaviour is investigated. Every uniformly topical endofunction has a spectral vector, related to some special fixed points (possibly at infinity), about which we establish various properties. In the stochastic case, we prove a multiplicative ergodic theorem, asserting that the stochastic spectral vector exists in all cases.

Résumé

Nous introduisons une sous-classe remarquable de la classe des fonctions topicales, celle des fonctions *uniformément topicales*, dont nous étudions les propriétés dynamiques. Toute endofonction uniformément topicale admet un vecteur spectral, interprétable en termes de points fixes (éventuellement à l'infini), et pour lequel on établit diverses propriétés. Dans le cas stochastique, nous démontrons un théorème ergodique multiplicatif, affirmant l'existence du vecteur spectral stochastique dans tous les cas.

Keywords: Topical function, uniformly topical function, finite range, spectral vector, cycle time vector, multiplicative ergodic theorem

AMS classification (2000): 47H09, 46T10, 37H15

Contents

[*]Laboratoire de Mathématique (UMR 8628 du CNRS), bât. 425/430, Université de Paris-Sud, 91405 Orsay Cedex, France. E-mail: `Thierry.Bousch@math.u-psud.fr`.

[†]LIAFA, CNRS et Université Paris 7, case 7014, 2 place Jussieu, 75251 Paris Cedex 05, France. E-mail: `Jean.Mairesse@liafa.jussieu.fr`.

1 Introduction

The dynamical study of "discrete event systems" leads to study the iteration of functions from $\mathbb{R}^N$ into itself, which are increasing (for the product order on coordinates) and additively homogeneous (i.e. which are commuting with some one-parameter group of translations) [Ba, Gu1]. Depending on the complexity of the system, these functions may be of various kinds, with various degrees of generality (max-plus, min-max-plus, piecewise-affine, etc.) but they always satisfy the previous two axioms. Gunawardena and Keane [GK] have coined the term *topical* for the functions satisfying these two axioms (due to Crandall & Tartar [CT]), and have investigated their common (dynamical) properties, independently of their particular form.

From the dynamical viewpoint, one of the most important questions is to determine whether a topical endofunction of $\mathbb{R}^N$ has a *spectral vector* (also called *cycle time vector*), i.e. to determine if $f^n(0)/n$ has a limit when $n \to +\infty$. It is known to hold true for certain specific classes of topical functions: in particular, for min-max-plus functions and more generally, for piecewise-affine topical functions, the existence of the spectral vector has been proved by Bewley and Kohlberg [BK, Koh], in a non-constructive way; later, various authors [ZP, ST, GG1, GG2] have rediscovered the result, and have given algorithms to compute the spectral vector in the min-max-plus case (experimentally quite efficient, but whose complexities are not yet well understood).

On the contrary, for general topical functions, the spectral vector does not always exist [Koh, GK], as soon as $N \geqslant 3$. Here is a specific counter-example (due to Kohlberg and Sparrow): let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y+1 \\ x+h(y-x) \end{bmatrix}$$

where $h : \mathbb{R} \to \mathbb{R}$ is some function. We see that f is additively homogeneous, and that it is increasing if and only if h is Lipschitz with $0 \leqslant h' \leqslant 1$. Assume this to be the case. Then we have $f^{n+1}(0) = (0, n+1, h(n))$, and the spectral vector exists if and only if $h(n)/(n+1)$ has a limit when $n \to +\infty$. However this limit may not exist: consider the function h defined by $h(x) = 0$ if $x \leqslant 0$, $h(x) = \min(x-1, 2)$ if $2 \leqslant x \leqslant 4$, and $h(2x) = 2h(x)$ everywhere (see Figure 1). The function h is indeed Lipschitz with $0 \leqslant h' \leqslant 1$ hence f is topical, however $h(n)/(n+1)$ has no limit.

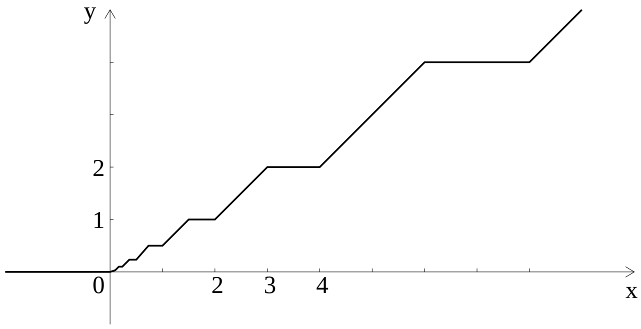

Figure 1: The function h

This phenomenon, which is completely at odds with what is known to be true for the classical families of topical functions, suggests that the class of *all* topical functions is probably too vast to have interesting properties from the dynamical viewpoint.

The above counter-example shows the typical behaviour that we would like to avoid: the orbits leave any compact (in $\mathbb{R}^3/\mathbb{R}\vec{\mathbf{u}}$), and we do not have any control on the dynamics near infinity.

Our answer to this problem is to consider the compactification $[-\infty, +\infty]^N$ of $\mathbb{R}^N$ and to study only the topical functions f having a continuous extension on this compactification. We call them *uniformly topical* functions. We study their properties in §3 and §4, via *finite-range* topical functions, which are introduced in §2 and which generalize min-max-plus functions. Topology meets algebra with Corollary 3.6, which states that uniformly topical functions are precisely the uniform limits of min-max-plus functions.

Section §5 is devoted to the iteration of a (general) uniformly topical function, and the properties of the spectral vector, which always exists.

Finally, several multiplicative ergodic theorems are proved in Section §6, about the random iterates of uniformly topical functions. The main result is the existence, in full generality, of a stochastic spectral vector (under some integrability assumption). These results include and generalize everything already known in the max-plus case [Ba, GY].

A remark about conventions: as in [AB], we use the adjectives 'increasing' and 'decreasing' (for functions) as synonyms for 'nondecreasing' and 'nonincreasing'. Similarly, for numbers and vectors, the adjectives 'positive' and 'negative' are synonyms for 'nonnegative' and 'nonpositive' ($x \geqslant 0$, $x \leqslant 0$ resp.)

2 Finite-range topical functions

2.1 Review on topical functions

Consider $\mathbb{R}^S$, where S is a finite set, viewed (preferably) as an affine space with origin $\mathbf{o}$ and dimension $N = \sharp S$. The coordinate functions are denoted by $\gamma_i : \mathbf{x} \mapsto x_i$, where i ranges over S. The set $\mathbb{R}^S$ is equipped with the product order, i.e. $\mathbf{x} \leqslant \mathbf{y}$ iff $x_i \leqslant y_i$ for all $i \in S$.

For $\mathbf{x} \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$, define $\mathbf{x} + \lambda$ (or $\lambda + \mathbf{x}$) as the point $\mathbf{x} + \lambda\vec{\mathbf{u}}$, where $\vec{\mathbf{u}} = (1, \ldots, 1)$ so that $\gamma_i(\mathbf{x} + \lambda) = \gamma_i(\mathbf{x}) + \lambda$. The function $(\lambda, \mathbf{x}) \mapsto \lambda + \mathbf{x}$ is increasing in λ and in $\mathbf{x}$, and it is a group action, i.e. $0 + \mathbf{x} = \mathbf{x}$ and $\lambda + (\lambda' + \mathbf{x}) = (\lambda + \lambda') + \mathbf{x}$.

Given a vector $\vec{\mathbf{v}} \in \mathbb{R}^S$, define (when S is non-empty) the floor $\lfloor \vec{\mathbf{v}} \rfloor = \inf v_i$ and the ceiling $\lceil \vec{\mathbf{v}} \rceil = \sup v_i$, and for two points $\mathbf{x}, \mathbf{y} \in E$, define

$$\begin{aligned} \lfloor \mathbf{x}, \mathbf{y} \rfloor &= \lfloor \mathbf{y} - \mathbf{x} \rfloor = \sup \{\lambda \in \mathbb{R} : \mathbf{x} + \lambda \leqslant \mathbf{y}\}, \\ \lceil \mathbf{x}, \mathbf{y} \rceil &= \lceil \mathbf{y} - \mathbf{x} \rceil = \inf \{\lambda \in \mathbb{R} : \mathbf{y} \leqslant \mathbf{x} + \lambda\} = -\lfloor \mathbf{y}, \mathbf{x} \rfloor. \end{aligned}$$

A function $f : \mathbb{R}^A \to \mathbb{R}^B$ (where A, B are finite sets) is said to be *topical* if it is increasing and "additively homogeneous", i.e. such that $f(\mathbf{x} + \lambda) = f(\mathbf{x}) + \lambda$. If f is topical, then $\lceil f\mathbf{x}, f\mathbf{y} \rceil \leqslant \lceil \mathbf{x}, \mathbf{y} \rceil$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^A$. (The converse is also true.) Similarly $\lfloor f\mathbf{x}, f\mathbf{y} \rfloor \geqslant \lfloor \mathbf{x}, \mathbf{y} \rfloor$. Hence, topical functions lower the floors and raise the ceilings [GK]. The distance $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$, where $\|\vec{\mathbf{v}}\| = \max |v_i|$, plays a particular role with respect to topical functions: the formula $d(\mathbf{x}, \mathbf{y}) = \lceil \mathbf{x}, \mathbf{y} \rceil \vee \lceil \mathbf{y}, \mathbf{x} \rceil$ shows that $d(f\mathbf{x}, f\mathbf{y}) \leqslant d(\mathbf{x}, \mathbf{y})$ for any topical function f. Similarly, the distance $\tilde{d}$ on the quotient space $\mathbb{R}^A/\mathbb{R}\vec{\mathbf{u}}$, defined by $\tilde{d}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \inf_{\lambda \in \mathbb{R}} d(\mathbf{x}, \lambda + \mathbf{y}) = \frac{1}{2}\big(\lceil \mathbf{x}, \mathbf{y} \rceil + \lceil \mathbf{y}, \mathbf{x} \rceil\big)$, where $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ are the images in $\mathbb{R}^A/\mathbb{R}\vec{\mathbf{u}}$ of two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^A$, is lowered by the quotient function $\tilde{f} : \mathbb{R}^A/\mathbb{R}\vec{\mathbf{u}} \to \mathbb{R}^B/\mathbb{R}\vec{\mathbf{u}}$.

Denote by $\mathbf{Tp}(\mathbb{R}^A, \mathbb{R}^B)$ the set of topical functions from $\mathbb{R}^A$ to $\mathbb{R}^B$. This set has a natural order which makes it a distributive lattice, whose operations $\vee$ and $\wedge$ are given pointwise, i.e. $(f \vee g)(\mathbf{x}) = f(\mathbf{x}) \vee g(\mathbf{x})$ and similarly for $\wedge$. In particular, the median $\operatorname{med}(f, g, h)$ is given by $\mathbf{x} \mapsto \operatorname{med}(f\mathbf{x}, g\mathbf{x}, h\mathbf{x})$.

A topical form is a real-valued topical function. A max-plus form (resp. min-plus, min-max-plus) on $\mathbb{R}^A$ is a supremum (resp. an infimum, a combination by the operations $\vee$ and $\wedge$) of a finite number of topical forms $\gamma_i + \mathrm{Cst}$. A max-plus function (resp. min-plus, min-max-plus) is a function $\mathbb{R}^A \to \mathbb{R}^B$ whose components are max-plus forms (resp. min-plus, min-max-plus). Denote by $\mathbf{Maxp}(\mathbb{R}^A, \mathbb{R}^B)$ and $\mathbf{Minp}(\mathbb{R}^A, \mathbb{R}^B)$ the spaces of max-plus and min-plus functions from $\mathbb{R}^A$ to $\mathbb{R}^B$.

The natural topology on $\mathbf{Tp}(\mathbb{R}^A, \mathbb{R}^B)$ is the topology of pointwise convergence; it coincides with the topology of uniform convergence on bounded subsets of $\mathbb{R}^A$, since topical functions are 1-Lipschitz. This topology is completely metrizable, and locally compact (the compact sets of $\mathbf{Tp}(\mathbb{R}^A, \mathbb{R}^B)$ are the closed and bounded sets). Besides, $\mathbf{Tp}(\mathbb{R}^A, \mathbb{R}^B)$ is separable, since min-max-plus functions are dense in it.

The interested reader will find many more details in [GK, BM].

2.2 Finite-range topical functions

Let $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ be the "sign" function and, by extension, we also denote by $\operatorname{sgn} : \mathbb{R}^S \to \{-1, 0, 1\}^S$ the function which gives the "sign" of a vector, component-wise.

Definition 2.1. *Let $L \geqslant 0$ be given. We say that a topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ has range at most L if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^A$,*

$$\Big[\forall \lambda \in [-L, L] \quad \operatorname{sgn}(\lambda + \mathbf{x}) = \operatorname{sgn}(\lambda + \mathbf{y})\Big] \implies \operatorname{sgn} f(\mathbf{x}) = \operatorname{sgn} f(\mathbf{y}). \tag{2.1}$$

The function f is said to have finite range if it has range at most L for some $L \geqslant 0$. Then there exists a smallest L satisfying this condition, which we call the range of f.

It is convenient to introduce the equivalence relation $\overset{L}{\approx}$ on $\mathbb{R}^S$, defined by $\mathbf{x} \overset{L}{\approx} \mathbf{y}$ iff $\operatorname{sgn}(\lambda + \mathbf{x}) = \operatorname{sgn}(\lambda + \mathbf{y})$ for all $\lambda \in [-L, L]$. The condition (2.1) can then be rewritten as $\mathbf{x} \overset{L}{\approx} \mathbf{y} \implies f\mathbf{x} \overset{0}{\approx} f\mathbf{y}$.

As an important special case, we have null-range topical functions, which satisfy (2.1) with $L = 0$, that is, $\operatorname{sgn} \mathbf{x} = \operatorname{sgn} \mathbf{y} \implies \operatorname{sgn} f\mathbf{x} = \operatorname{sgn} f\mathbf{y}$. In words, the signs of the coordinates of $f\mathbf{x}$ depend only on the signs of the coordinates of $\mathbf{x}$.

We denote by $\mathbf{FrTp}(\mathbb{R}^A, \mathbb{R}^B)$ and $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B)$ the set of topical functions with finite range and with range at most L, respectively (for $L \geqslant 0$). We also use the notation $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^B)$ instead of $\mathbf{FrTp}_0(\mathbb{R}^A, \mathbb{R}^B)$.

The above definition calls for some obvious remarks. A topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ has range at most L if and only if all its components $f_i = \gamma_i \circ f$ have range at most L (for i ranging over B). We thus have the natural bijections

$$\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B) \simeq \mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R})^B,$$
$$\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B \times \mathbb{R}^C) \simeq \mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B) \times \mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^C).$$

Observe also that if $f, g : \mathbb{R}^A \to \mathbb{R}^B$ are topical with range at most L, then it is also the case for $f \vee g$ and $f \wedge g$. Finally, a composition of finite-range topical functions also has finite range: if $f : \mathbb{R}^A \to \mathbb{R}^B$ and $g : \mathbb{R}^B \to \mathbb{R}^C$ have range at most L, L', then $g \circ f : \mathbb{R}^A \to \mathbb{R}^C$ has range at most $L + L'$; this follows from the implications $\mathbf{x} \overset{L+L'}{\approx} \mathbf{y} \implies f\mathbf{x} \overset{L'}{\approx} f\mathbf{y} \implies gf\mathbf{x} \overset{0}{\approx} gf\mathbf{y}$.

Besides, a translation $\mathbf{x} \mapsto \mathbf{x} + \vec{\mathbf{v}}$ is a finite-range topical endofunction, with range $\|v\| = \max |v_i|$. Consequently, any min-max-plus function $\mathbb{R}^A \to \mathbb{R}^B$ has finite range.

It is easy to see that a topical form $\alpha : \mathbb{R}^S \to \mathbb{R}$ has range at most L if and only if $\alpha^{-1}(0)$ is saturated for the equivalence relation $\overset{L}{\approx}$ (since all the classes are connected). This enables us to check, for instance, that the form $\alpha(x, y) = \max\bigl(x - \frac{1}{2}, y - \frac{1}{2}, \frac{x+y}{2} + \frac{1}{4}\bigr)$ has range 1 (see Figure 2), whereas the form $\alpha(x, y) = (x + y)/2$ does *not* have finite range. One can also check that a max-plus form

$$\alpha(x_1, \dots, x_N) = (x_{i_1} + a_1) \vee \cdots \vee (x_{i_K} + a_K) \qquad (i_1, \dots, i_K \text{ distinct})$$

has range $|a_1| \vee \cdots \vee |a_K|$. (To prove this last result, one can also use Lemma 2.2 below.)

Lemma 2.2. *Let $\alpha : \mathbb{R}^S \to \mathbb{R}$ be a topical form with range at most L. For all $\mathbf{x} = (x_i) \in \mathbb{R}^S$, there exists an index $i \in S$ such that $|\alpha(\mathbf{x}) - x_i| \leqslant L$. In particular, $|\alpha(\mathbf{o})| \leqslant L$.*

Proof. We may assume $\alpha\mathbf{x} = 0$, up to replacing $\mathbf{x}$ with $-\lambda + \mathbf{x}$, where $\lambda = \alpha\mathbf{x}$. Assume now that $K = \min |\alpha\mathbf{x} - x_i| = \min |x_i|$ is strictly larger than L. It would imply $\mathbf{x} \overset{L}{\approx} \mathbf{x} + \epsilon$ for $\epsilon = (K - L)/2 > 0$ and therefore $\operatorname{sgn}(\alpha\mathbf{x}) = \operatorname{sgn}(\alpha\mathbf{x} + \epsilon)$, i.e. $0 = 1$, which is absurd. $\square$

Letting $L = 0$ in the above lemma, we see that $\alpha(\mathbf{x})$ takes its values among the coordinates of $\mathbf{x}$, when α has null range.

Lemma 2.3. *A topical form of $\mathbb{R}^S$, which is a pointwise limit of topical forms with range at most L, also has range at most L.*

Proof. We have to prove that $\mathbf{FrTp}_L(\mathbb{R}^S, \mathbb{R})$ is closed in $\mathbf{Tp}(\mathbb{R}^S, \mathbb{R})$ for the topology of pointwise convergence. We shall prove that its complement is open. Let $\alpha : \mathbb{R}^S \to \mathbb{R}$ be a topical form which has not range at most L, so that there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^S$ with $\mathbf{x} \overset{L}{\approx} \mathbf{y}$ and $\operatorname{sgn} \alpha(\mathbf{x}) \neq \operatorname{sgn} \alpha(\mathbf{y})$. If $\alpha(\mathbf{x})$ and $\alpha(\mathbf{y})$ are both non-zero, then the condition $\operatorname{sgn} \beta(\mathbf{x}) \neq \operatorname{sgn} \beta(\mathbf{y})$ is obviously satisfied for all β close enough to α. Otherwise, we can revert to the previous situation, observing that $\lambda + \mathbf{x} \overset{L}{\approx} \lambda + \mathbf{y}$ when λ is small enough. $\square$

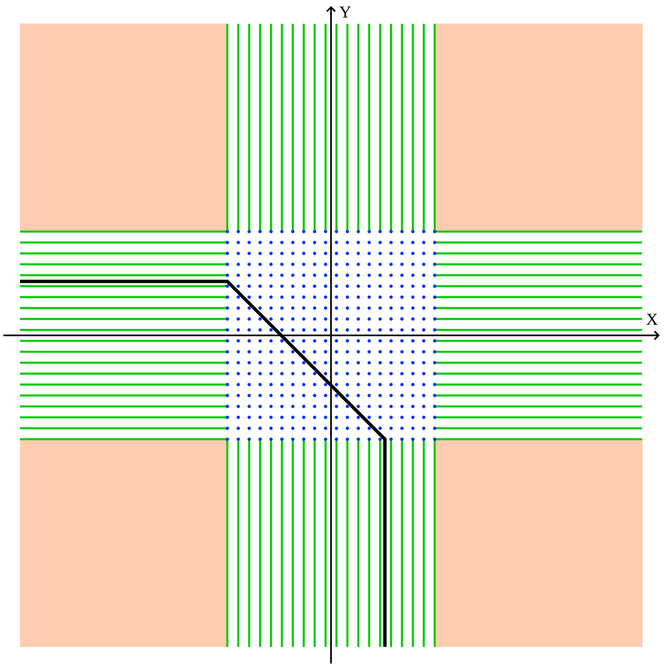

Figure 2: The set $\alpha^{-1}(0)$ and the equivalence classes of the relation $\stackrel{L}{\approx}$

Lemma 2.2 asserts that any non-empty subset of $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B)$ has an infimum and a supremum in $\mathbf{Tp}(\mathbb{R}^A, \mathbb{R}^B)$; these two functions can be obtained as pointwise limits of finite infima or suprema, so they also have range at most L, by Lemma 2.3. Hence $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B)$ is closed under arbitrary (non-empty) infima and suprema.

Theorem 2.4. *The topologies of pointwise convergence and uniform convergence coincide on* $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B)$*, and make it a compact space.*

Proof. We can restrict the study to the case of topical forms, i.e. $B = 1$.

The set $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R})$ is a compact subset of $\mathbf{Tp}(\mathbb{R}^A, \mathbb{R})$, for the topology of pointwise convergence, since it is closed and bounded, by Lemmas 2.2 and 2.3.

Since the topologies to be compared are both metrizable, it is sufficient to prove that they have the same convergent sequences. Let $\alpha_n \to \alpha$ be a pointwise convergent sequence in $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R})$. Assume the convergence not to be uniform; there will then exist $0 < \epsilon < 1$, a sequence $(\mathbf{x}_n)$ of points in $\mathbb{R}^A$, and an infinite number of indices n such that $|\alpha_n(\mathbf{x}_n) - \alpha(\mathbf{x}_n)| > \epsilon$; say $\alpha_n(\mathbf{x}_n) > \alpha(\mathbf{x}_n) + \epsilon$ for instance, and by extracting a subsequence if necessary, we can assume that the inequality holds for all $n \in \mathbb{N}$. Then let $\mathbf{y}_n = \mathbf{x}_n - \alpha(\mathbf{x}_n)$, so that $\alpha(\mathbf{y}_n) = 0$ and $\alpha_n(\mathbf{y}_n) > \epsilon$.

Then we can find a sequence of points $(\mathbf{z}_n)$ such that $\|\mathbf{z}_n\| < L + 1$ and $\mathbf{z}_n \stackrel{L+\epsilon}{\approx} \mathbf{y}_n$. It implies $\alpha(\mathbf{z}_n) \stackrel{\epsilon}{\approx} \alpha(\mathbf{y}_n)$, that is, $\alpha(\mathbf{z}_n) = 0$, and $\alpha_n(\mathbf{z}_n) \stackrel{\epsilon}{\approx} \alpha_n(\mathbf{y}_n)$, i.e. $\alpha_n(\mathbf{z}_n) > \epsilon$. Besides, we know that the pointwise convergence $\alpha_n \to \alpha$ implies the uniform convergence on compact subsets of $\mathbb{R}^A$, and since the sequence $(\mathbf{z}_n)$ is bounded, we would have $(\alpha_n - \alpha)(\mathbf{z}_n) \to 0$, a contradiction. The convergence $\alpha_n \to \alpha$ is indeed uniform. $\square$

2.3 Approximation by min-max-plus functions

Minimal max-plus upper-bounds. Let $f : \mathbb{R}^A \to \mathbb{R}^B$ be a topical function. Let

$$W(f) \;=\; \left\{ g \in \mathbf{Maxp}(\mathbb{R}^A, \mathbb{R}^B) : g \geqslant f \right\}$$

be the set of max-plus upper-bounds of f, and $V(f)$ the set of minimal elements of $W(f)$.

We have $f = \inf W(f)$, since for all $\mathbf{x} \in \mathbb{R}^A$ one can find a max-plus function $g_{\mathbf{x}}$ with $g_{\mathbf{x}} \geqslant f$ and $g_{\mathbf{x}}(\mathbf{x}) = f(\mathbf{x})$, namely $g_{\mathbf{x}} : \mathbf{y} \mapsto f(\mathbf{x}) + \lceil \mathbf{x}, \mathbf{y} \rceil$.

Besides, $V(f)$ is non-empty according to Zorn's Lemma and, more precisely, any element of $W(f)$ has a lower-bound in $V(f)$. It follows that $f = \inf V(f)$.

Consequently f is min-max-plus if $V(f)$ is a finite set. The converse statement is also true, and follows easily from the next lemma.

Lemma 2.5. *Let $\alpha_1, \ldots, \alpha_n : \mathbb{R}^S \to \mathbb{R}$ be topical forms. One has*

$$W(\alpha_1 \vee \cdots \vee \alpha_n) = W(\alpha_1) \cap \cdots \cap W(\alpha_n) \tag{2.2}$$

$$W(\alpha_1 \wedge \cdots \wedge \alpha_n) = W(\alpha_1) \cup \cdots \cup W(\alpha_n) \tag{2.3}$$

Proof. The equality (2.2) is obvious, as well as the inclusion $\supseteq$ in (2.3). We have to show that, if β is a max-plus form such that $\beta \geqslant \alpha_1 \wedge \cdots \wedge \alpha_n$, then β is an upper bound for some α_k.

Set $\beta(\mathbf{x}) = (x_1 + b_1) \vee \cdots \vee (x_N + b_N)$, where $-\infty \leqslant b_i < \infty$, and consider first the case where all the b_i's are finite. The set $\{\mathbf{x} \in \mathbb{R}^S : \beta(\mathbf{x}) \leqslant 0\}$ has a greatest element, namely, the point $\mathbf{p} = (-b_1, \ldots, -b_N)$. Evaluating the inequality $\beta \geqslant \bigwedge \alpha_k$ at this point, we see that there exists some $k \in [1, n]$ such that $\alpha_k(\mathbf{p}) \leqslant 0$. Then, for all $\mathbf{x} \in \mathbb{R}^S$ we have $\alpha_k(\mathbf{x}) \leqslant \lceil \alpha_k(\mathbf{p}), \alpha_k(\mathbf{x}) \rceil \leqslant \lceil \mathbf{p}, \mathbf{x} \rceil = \beta(\mathbf{x})$, that is, $\alpha_k \leqslant \beta$.

In the general case, where the b_i are not assumed to be finite, we apply the previous result to the forms $\beta_s = \beta \vee (s + \gamma_+)$, with $s \in \mathbb{R}$ and $\gamma_+ = \gamma_1 \vee \cdots \vee \gamma_N$. For all $s \in \mathbb{R}$, β_s is in $W(\alpha_1) \cup \cdots \cup W(\alpha_n)$, and since this set is closed in $\mathbf{Maxp}(\mathbb{R}^S, \mathbb{R})$ for the topology of pointwise convergence (as a finite union of closed sets), it contains the limit of the β_s when $s \to -\infty$, which is β. This concludes the proof. □

Theorem 2.6. *The function f has range at most L if and only if every element of $V(f)$ has range at most L.*

Proof. The condition is clearly sufficient, since $f = \inf V(f)$ and $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R}^B)$ is closed under arbitrary infima. Conversely, consider a topical form $f : \mathbb{R}^A \to \mathbb{R}$ with range at most L, and let

$$g(\mathbf{x}) = (x_{i_1} + \theta_{i_1}) \vee \cdots \vee (x_{i_K} + \theta_{i_K}) \qquad (i_1, \ldots, i_K \text{ distinct})$$

be any element of $V(f)$. Its range is $\sup |\theta_i|$, and we want to prove that it is bounded by L. Assume, by way of contradiction, that $\sup |\theta_i| > L$. We write

$$g(\mathbf{x}) = (x_s + \theta) \vee h(\mathbf{x})$$

where $|\theta| > L$ and h is a max-plus form which does not depend on the coordinate of index s. (We set $h(\mathbf{x}) = -\infty$ for all x if g has a single term.)

Choose $\theta' < \theta$ with $\theta' \overset{L}{\approx} \theta$ (that is, $\theta' < \theta$ if θ is negative, and $L < \theta' < \theta$ otherwise). The max-plus form

$$g'(\mathbf{x}) = (x_s + \theta') \vee h(\mathbf{x})$$

is a *strict* lower bound for g and hence cannot belong to $W(f)$, by the minimality of g. Hence there exists a point $\mathbf{p} \in \mathbb{R}^A$ such that $0 = g'(\mathbf{p}) < f(\mathbf{p}) \leqslant g(\mathbf{p})$, i.e.

$$0 = (p_s + \theta') \vee h(\mathbf{p}) < f(\mathbf{p}) \leqslant (p_s + \theta) \vee h(\mathbf{p}),$$

implying $h(\mathbf{p}) \leqslant 0$ and $-\theta < p_s \leqslant -\theta'$. Now let $\mathbf{q} \in \mathbb{R}^A$ be the point defined by $q_s = -\theta$ and $q_i = p_i$ for all $i \neq s$, so that $h(\mathbf{q}) = h(\mathbf{p})$, and $f(\mathbf{q}) \leqslant g(\mathbf{q}) = 0$. But we also have $f(\mathbf{p}) > 0$ and $\mathbf{p} \overset{L}{\approx} \mathbf{q}$, contradicting the assumption that f has range at most L. □

An equivalent statement for this theorem is the following.

Theorem 2.7. *A topical function $\mathbb{R}^A \to \mathbb{R}^B$ has range at most L if and only if it is a (possibly infinite) infimum of max-plus functions with range at most L.*

Such a function is therefore a pointwise limit of min-max-plus functions with range at most L. Combining this with Theorem 2.4, we get:

Corollary 2.8. *A topical function $\mathbb{R}^A \to \mathbb{R}^B$ has range at most L if and only if it is a uniform limit of min-max-plus functions with range at most L.*

2.4 Null-range topical functions

We have seen that any max-plus form $\alpha : \mathbb{R}^S \to \mathbb{R}$, written as $\alpha(\mathbf{x}) = \max_{i \in S_\alpha}(x_i + \alpha_i)$, where S_α is a non-empty subset of S, has a finite range, namely $\max_{i \in S_\alpha} |\alpha_i|$. In particular, the max-plus forms with null range can be obtained by combining some of the coordinate forms using the $\vee$ operator (in other words, it is the sup-semi-lattice (freely) generated by the coordinate forms); their number is equal to the number of non-empty subsets of S, that is $2^N - 1$ (where N is the cardinality of S). Coupling this observation with Theorem 2.7, we obtain:

Theorem 2.9. *Every null-range topical form on $\mathbb{R}^S$ is obtained as a finite combination of the coordinate forms γ_i with the operators $\vee$ and $\wedge$. In other terms, $\mathbf{NrTp}(\mathbb{R}^S, \mathbb{R})$ is the distributive lattice (freely) generated by the coordinate forms.*

The fact that $\mathbf{NrTp}(\mathbb{R}^S, \mathbb{R})$ is *freely* generated by the coordinate forms will be proved later on, in §3.3, Theorem 3.9.
In particular, there is only a finite number $\mathfrak{D}(N)$ (the cardinality of $\mathrm{Fdl}(N)$, the free distributive lattice with N generators) of null-range topical forms, but this number grows rapidly (doubly exponentially!) as a function of N. The numbers $\mathfrak{D}(N)$ are called *Dedekind numbers* [Ded]. The set $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^B)$ is in bijection with $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R})^B$, and hence has cardinality $\mathfrak{D}(\sharp A)^{(\sharp B)}$.

N	0	1	2	3	4	5	6
$\mathfrak{D}(N)$	0	1	4	18	166	7 579	7 828 352

Observe also that a null range topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ is multiplicatively positively homogeneous, i.e. $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$ for all $\lambda \geqslant 0$ and $\mathbf{x} \in \mathbb{R}^A$. Besides, we have the identities $f(\mathbf{x} \vee \mathbf{o}) = f(\mathbf{x}) \vee \mathbf{o}$ and $f(\mathbf{x} \wedge \mathbf{o}) = f(\mathbf{x}) \wedge \mathbf{o}$, to be used in §5.2.

3 Uniformly topical functions

3.1 Uniformly topical functions

Let $\overline{\mathbb{R}} = [-\infty, +\infty]$ with its usual topology.

Definition 3.1. *Say that a function $f : \mathbb{R}^A \to \mathbb{R}^B$ is uniformly topical, if it is topical and has a continuous extension $\bar{f} : \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$.*

The above definition can be restated with a slightly different formulation. Endow $\mathbb{R}^A$ and $\mathbb{R}^B$ with the uniform structures (see [AB], §3.13, or [Bou]) induced by the inclusions $\mathbb{R}^A \subseteq \overline{\mathbb{R}}^A$ and $\mathbb{R}^B \subseteq \overline{\mathbb{R}}^B$; then $f : \mathbb{R}^A \to \mathbb{R}^B$ is uniformly topical iff it is topical and uniformly continuous (for these particular uniform structures).

Several properties follow directly from this definition.
(i) A function $f : \mathbb{R}^A \to \mathbb{R}^B$ is uniformly topical iff all its coordinate functions $f_i : \mathbb{R}^A \to \mathbb{R}$ are uniformly topical;
(ii) If $f, g : \mathbb{R}^A \to \mathbb{R}^B$ are uniformly topical, so are $f \vee g$ and $f \wedge g$;
(iii) If $f : \mathbb{R}^A \to \mathbb{R}^B$ and $g : \mathbb{R}^B \to \mathbb{R}^C$ are uniformly topical, so is $g \circ f$;
(iv) If a sequence (or a net) of uniformly topical functions converges uniformly to some function $f : \mathbb{R}^A \to \mathbb{R}^B$, then f is uniformly topical.

A result of Burbanks and Sparrow [BS] states that any topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ has a continuous extension $f_\perp : \mathbb{R}_\perp^A \to \mathbb{R}_\perp^B$, where $\mathbb{R}_\perp = [-\infty, +\infty)$. We quickly recall the argument. If a topical form $f : \mathbb{R}^A \to \mathbb{R}$ has a continuous extension $f_\perp : \mathbb{R}_\perp^A \to \mathbb{R}_\perp$, this extension must be given by the formula

$$f_\perp(\mathbf{p}) \;=\; \inf_{\substack{\mathbf{x} \in \mathbb{R}^A \\ \mathbf{x} \geqslant \mathbf{p}}} f(\mathbf{x}) \;=\; \inf_{\lambda \in \mathbb{R}} f(\mathbf{p} \vee \lambda \vec{\mathbf{u}})$$

which indeed defines a function $f_\perp : \mathbb{R}_\perp^A \to \mathbb{R}_\perp$ which is increasing, additively homogeneous, and upper semi-continuous (as an infimum of continuous functions).

On the other hand, any increasing and additively homogeneous function $f_\perp : \mathbb{R}_\perp^A \to \mathbb{R}_\perp$ is automatically lower semi-continuous, which means that, for all $M \in \mathbb{R}$, the set $\{\mathbf{x} : f(\mathbf{x}) > M\}$ is open. Indeed, let $\mathbf{x} \in \mathbb{R}_\perp^A$ be such that $f(\mathbf{x}) > M$, and set $\epsilon = (f(\mathbf{x}) - M)/2$; then $f(\mathbf{y}) > M$ for all $\mathbf{y}$ in the set $\{\mathbf{y} \in \mathbb{R}_\perp^A : \mathbf{y} \geqslant -\epsilon + \mathbf{x}\}$, which is a neighbourhood of $\mathbf{x}$. Hence the function $f_\perp$ is indeed a continuous extension of f.

By duality, f also has a continuous extension $f_\top : \mathbb{R}_\top^A \to \mathbb{R}_\top^B$, where $\mathbb{R}_\top = (-\infty, +\infty]$. Accordingly, if $\mathbf{p} \in \overline{\mathbb{R}}^A$ is in $\mathbb{R}_\perp^A$ or in $\mathbb{R}_\top^A$, we know that f has a limit (in $\overline{\mathbb{R}}^B$) at $\mathbf{p}$. On the other hand, no conclusion can be reached for the points $\mathbf{p} \in \overline{\mathbb{R}}^A$ which are neither in $\mathbb{R}_\perp^A$, nor in $\mathbb{R}_\top^A$, that is, with at least one coordinate being $+\infty$ and another one $-\infty$; the limit may exist or not. We call these points *critical*. The critical set (i.e. the set of critical points) has codimension two in $\overline{\mathbb{R}}^A$; we have represented, in Figure 3, the critical set in dimension 3.

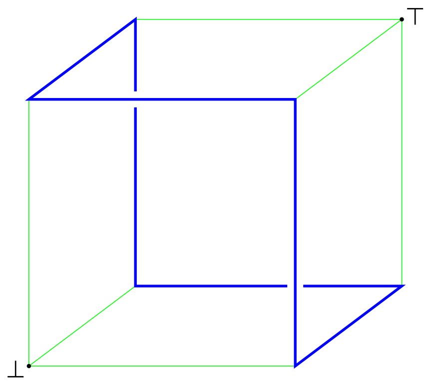

Figure 3: The cube $[-\infty, +\infty]^3$ and its critical set

The topical form $\mathbf{x} \mapsto (\sum x_i)/(\sharp A)$ has no limit (in $\overline{\mathbb{R}}$) at *any* critical point; hence it is not a uniformly topical form on $\mathbb{R}^A$ (for $\sharp A > 1$).

As examples of uniformly topical forms which do not have finite range (in general), we mention the conjugate-linear functions $f : \mathbb{R}^A \to \mathbb{R}^B$, defined by

$$\exp \kappa f_i = \sum_{j \in A} a_{ij} \exp \kappa x_j$$

where $\kappa > 0$, and (a_{ij}) is a matrix with positive entries and no line being identically zero. These functions are uniform approximations of max-plus functions, and we will come back to them later in the article.

The Definition 3.1 suggests that it is more natural to consider a uniformly topical function as a function $\overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$. From this viewpoint, we can give the following equivalent definition:

Definition 3.2. *Say that a function $\bar{f} : \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$ is uniformly topical if it is continuous, increasing, additively homogeneous, and such that $\bar{f}(\bot) = \bot$ and $\bar{f}(\top) = \top$.*

The symbols $\bot$ and $\top$ stand for the smallest and the largest element of $\overline{\mathbb{R}}^A$ or $\overline{\mathbb{R}}^B$, and the condition of additive homogeneity means that $\bar{f}(\lambda + \mathbf{x}) = \lambda + \bar{f}(\mathbf{x})$ for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \overline{\mathbb{R}}^A$, where $\lambda + \mathbf{x}$ is defined in the obvious way. The conditions $\bar{f}(\bot) = \bot$, $\bar{f}(\top) = \top$ are finiteness conditions; they are equivalent (modulo the other hypothesis) to the requirement that $\bar{f}(\mathbb{R}^A) \subseteq \mathbb{R}^B$, as can be checked by the reader.

Theorem 3.3. *Any finite-range topical function is uniformly topical.*

Proof. Consider a topical form $f : \mathbb{R}^S \to \mathbb{R}$ with range at most L. Let $(\mathbf{x}_n)$ be a sequence of points of $\mathbb{R}^S$ having a limit $\mathbf{p} \in \overline{\mathbb{R}}^S$; we have to prove that $f(\mathbf{x}_n)$ converges in $\overline{\mathbb{R}}$. Consider the sequence $(\mathbf{y}_n)$ of points of $\mathbb{R}^S$ defined by

$$\gamma_i(\mathbf{y}_n) = \begin{cases} \gamma_i(\mathbf{p}) & \text{if } \gamma_i(\mathbf{p}) \in \mathbb{R}, \\ \gamma_i(\mathbf{x}_n) & \text{otherwise} \end{cases}$$

for $i \in S,\ n \in \mathbb{N}$. We have $d(\mathbf{x}_n, \mathbf{y}_n) \to 0$ and so $|f(\mathbf{x}_n) - f(\mathbf{y}_n)| \to 0$. Hence the convergence (in $\overline{\mathbb{R}}$) of $f(\mathbf{x}_n)$ is equivalent to the convergence of $f(\mathbf{y}_n)$.

The sequence $(\mathbf{y}_n)$ satisfies the following property: for all $K \geqslant 0$, there exists $n_0 \geqslant 0$ such that $\mathbf{y}_n \overset{L+K}{\approx} \mathbf{y}_p$ for all $n, p \geqslant n_0$. Thus $f(\mathbf{y}_n) \overset{K}{\approx} f(\mathbf{y}_p)$ for n, p large enough. In particular, the sequence $\operatorname{med}(-K, f(\mathbf{y}_n), K)$ is stationary, a fortiori convergent; in terms of $\ell_1 = \liminf f(\mathbf{y}_n)$ and $\ell_2 = \limsup f(\mathbf{y}_n)$, it means that

$$\operatorname{med}(-K, \ell_1, K) = \operatorname{med}(-K, \ell_2, K)$$

and this for all $K \geqslant 0$, which is not possible unless $\ell_1 = \ell_2$. The theorem follows. $\square$

The above theorem, which is the main motivation behind Definition 3.1, can also be obtained by a more abstract argument, explaining why the cube $\overline{\mathbb{R}}^S$ turns out to be the "right" compactification of $\mathbb{R}^S$, from the viewpoint of finite-range topical functions. Let $f : \mathbb{R}^A \to \mathbb{R}^B$ be a topical function with range at most L. For every $K \geqslant 0$, there exists a (continuous) quotient function f_K between the non-separated topological spaces $\big(\mathbb{R}^A / \overset{K+L}{\approx}\big)$ and $\big(\mathbb{R}^B / \overset{K}{\approx}\big)$. By taking the projective limit for $K \to \infty$ (in the category of topological spaces), we obtain a continuous function

$$\bar{f} : \varprojlim_K \big(\mathbb{R}^A / \overset{K}{\approx}\big) \longrightarrow \varprojlim_K \big(\mathbb{R}^B / \overset{K}{\approx}\big)$$

But, it is easy to see that these limit spaces can precisely be identified with $\overline{\mathbb{R}}^A$ and $\overline{\mathbb{R}}^B$, and that $\bar{f}$ is an extension of f (modulo this identification).

The set $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$ of uniformly topical functions $\mathbb{R}^A \to \mathbb{R}^B$ can be viewed, in accordance with Definition 3.2, as a subset of the set $C(\overline{\mathbb{R}}^A, \overline{\mathbb{R}}^B)$ of continuous functions $\overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$; now, this space is endowed with a natural uniform structure (and its corresponding topology), given by the distance

$$\delta(f,g) \;=\; \sup_{i\in B}\ \sup_{\mathbf{x}\in\overline{\mathbb{R}}^A} |\tanh g_i(\mathbf{x}) - \tanh f_i(\mathbf{x})| \tag{3.1}$$

and $\mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$ is a closed subset of it. However, it is easier for the computations, and more natural from the viewpoint of topical functions, to equip $\mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$ with the usual uniform distance, that we denote by d:

$$d(f,g) \;=\; \sup_{i\in B}\ \sup_{\mathbf{x}\in\mathbb{R}^A} |g_i(\mathbf{x}) - f_i(\mathbf{x})| \tag{3.2}$$

This distance is related to the induced distance δ by the formula

$$\frac{\delta(f,g)}{2} \;=\; \tanh \frac{d(f,g)}{2}$$

which holds for all $f,g\in\mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$, as can be shown by a simple argument (indication: replace $\mathbf{x}$ by $\lambda{+}\mathbf{x}$ in the formula (3.1), and maximize over λ). Thus d and δ endow $\mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$ with the same uniform structure, and a fortiori with the same topology.

Theorem 3.4. *The set $\mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$ of uniformly topical functions from $\mathbb{R}^A$ to $\mathbb{R}^B$, equipped with the uniform distance d given by formula* (3.2)*, is a complete and separable metric space.*

Observe also that a subset $\mathcal{F}\subseteq\mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$ is precompact if and only if it is precompact as a subset of $C(\overline{\mathbb{R}}^A,\overline{\mathbb{R}}^B)$, that is, if the extensions $\bar f:\overline{\mathbb{R}}^A\to\overline{\mathbb{R}}^B$, where f ranges over $\mathcal{F}$, form an equicontinuous family.

3.2 Approximation by finite-range topical functions

By the results of the previous section, any uniform limit of finite-range topical functions is uniformly topical. The converse is also true:

Theorem 3.5. *Any uniformly topical function is a uniform limit of finite-range topical functions.*

And since we also know (Corollary 2.8) that any finite-range topical function is a uniform limit of min-max-plus functions, we get:

Corollary 3.6. *Any uniformly topical function is a uniform limit of min-max-plus functions.*

We begin with two preliminary lemmas.

Lemma 3.7. *Let C be a class of $\overset{L}{\approx}$ in $\mathbb{R}^S$, different from the class $\{\mathbf{x} : x_i > L\ \forall i\}$, and let $\mathbf{c}=(c_i)$ be any point of C. There exists a greatest topical form taking negative values on C. This form, which we denote by $\mathbf{x}\mapsto\lceil C,\mathbf{x}\rceil$, is given by the formula*

$$\lceil C,\mathbf{x}\rceil \;=\; \max_{\substack{i\in S\\ c_i\leqslant L}} \big(x_i - \max(c_i,-L)\big) \tag{3.3}$$

In particular, it is max-plus, with range at most L.

This is easy to check. Observe that $\lceil C,\mathbf{x}\rceil = \inf_{\mathbf{c}\in C}\lceil\mathbf{c},\mathbf{x}\rceil$, which justifies the notation. Besides:

Lemma 3.8. *Let $f : \mathbb{R}^S \to \mathbb{R}$ be uniformly topical. For all $\epsilon > 0$, there exists $L \geqslant 0$ such that*

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^S \qquad \mathbf{x} \overset{L}{\approx} \mathbf{y} \text{ and } f(\mathbf{y}) = 0 \implies |f(\mathbf{x})| \leqslant \epsilon. \tag{3.4}$$

Proof. The function $f : \mathbb{R}^S \to \mathbb{R}$ is uniformly continuous if we equip $\mathbb{R}^S$ with the distance $\delta(\mathbf{x}, \mathbf{y}) = \sup_i |\tanh y_i - \tanh x_i|$ and $\mathbb{R}$ with the distance $\delta(x, y) = |\tanh y - \tanh x|$. Hence, there exists $\eta > 0$ such that

$$\delta(\mathbf{x}, \mathbf{y}) \leqslant \eta \implies |\tanh f(\mathbf{y}) - \tanh f(\mathbf{x})| \leqslant \tanh \epsilon.$$

Now let $L \geqslant 0$ be such that $1 - \tanh L \leqslant \eta$. Then for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^S$ such that $\mathbf{x} \overset{L}{\approx} \mathbf{y}$, we have $\delta(\mathbf{x}, \mathbf{y}) < 1 - \tanh L \leqslant \eta$ and consequently $|\tanh f(\mathbf{y}) - \tanh f(\mathbf{x})| \leqslant \tanh \epsilon$. This last inequality reduces to $|f(\mathbf{x})| \leqslant \epsilon$ when $f(\mathbf{y}) = 0$. $\square$

Proof of Theorem 3.5. Consider $f : \mathbb{R}^S \to \mathbb{R}$ uniformly topical, and an arbitrary $\epsilon > 0$. Choose $L \geqslant 0$ as in Lemma 3.8. Then the class $C_+ = \{\mathbf{x} : x_i > L \ \forall i\}$ does not intersect $f^{-1}(0)$. Indeed, if there exists $\mathbf{y} \in C_+$ such that $f(\mathbf{y}) = 0$, then $f(\mathbf{x}) = 2\epsilon$ for $\mathbf{x} = 2\epsilon + \mathbf{y}$, contradicting (3.4) since $\mathbf{x} \overset{L}{\approx} \mathbf{y}$.

Therefore we can define the function $\alpha : \mathbb{R}^S \to \mathbb{R}$ as follows:

$$\alpha(\mathbf{x}) \;=\; \inf_{\substack{C \text{ class of } \overset{L}{\approx} \\ C \cap f^{-1}(0) \neq \varnothing}} \lceil C, \mathbf{x} \rceil \;=\; \inf_{\mathbf{y} \in f^{-1}(0)} \lceil \mathrm{cl}_L\, \mathbf{y}, \mathbf{x} \rceil$$

where $\mathrm{cl}_L\, \mathbf{y}$ is the class of $\overset{L}{\approx}$ containing $\mathbf{y}$. This formula is well defined since $\mathrm{cl}_L\, \mathbf{y}$ is never equal to C_+. Furthermore, all the topical forms appearing in the infimum have range at most L, hence the infimum exists as a real-valued function, and is a topical form with range at most L.

For all $\mathbf{y} \in \mathbb{R}^S$ such that $f(\mathbf{y}) = 0$, we have $\alpha(\mathbf{y}) \leqslant \lceil \mathrm{cl}_L\, \mathbf{y}, \mathbf{y} \rceil \leqslant 0$. It shows that $\alpha \leqslant f$.

On the other hand, for all $\mathbf{y} \in \mathbb{R}^S$ such that $f(\mathbf{y}) = 0$, we know that $f(\mathbf{x}) \leqslant \epsilon$ for all $\mathbf{x} \overset{L}{\approx} \mathbf{y}$. The topical form $f - \epsilon$ is negative on $\mathrm{cl}_L\, \mathbf{y}$, hence upper bounded by $\mathbf{x} \mapsto \lceil \mathrm{cl}_L\, \mathbf{y}, \mathbf{x} \rceil$ on $\mathbb{R}^S$. In other terms,

$$\forall \mathbf{x} \in \mathbb{R}^S \quad \forall \mathbf{y} \in \mathbb{R}^S \qquad f(\mathbf{y}) = 0 \implies (f - \epsilon)(\mathbf{x}) \leqslant \lceil \mathrm{cl}_L\, \mathbf{y}, \mathbf{x} \rceil$$

which is equivalent to

$$\forall \mathbf{x} \in \mathbb{R}^S \qquad (f - \epsilon)(\mathbf{x}) \;\leqslant\; \inf_{\mathbf{y} \in f^{-1}(0)} \lceil \mathrm{cl}_L\, \mathbf{y}, \mathbf{x} \rceil$$

that is $f - \epsilon \leqslant \alpha$. Hence we have the inequalities $f - \epsilon \leqslant \alpha \leqslant f$, with α of finite range and ϵ arbitrarily small, which completes the proof of the theorem. $\square$

3.3 Asymptotes

The "cube" $\overline{\mathbb{R}}^S$ is endowed with an action of $\mathbb{R}$, denoted by $(\lambda, \mathbf{x}) \mapsto \lambda + \mathbf{x}$ and defined in the natural way. This action has some fixed points: the *vertices* of $\overline{\mathbb{R}}^S$, that is the points whose coordinates are all infinite in absolute value. If $\mathbf{x}$ is a vertex, then $\lambda + \mathbf{x} = \mathbf{x}$ for all $\lambda \in \mathbb{R}$; if on the contrary $\mathbf{x}$ is not a vertex, then $\lambda + \mathbf{x}$ depends on λ in an injective way.

A uniformly topical (or, more generally, additively homogeneous) function $\bar{f} : \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$ must preserve the fixed points of the action: if $\mathbf{x}$ is a vertex of $\overline{\mathbb{R}}^A$, then $\bar{f}(\mathbf{x})$ is a vertex of $\overline{\mathbb{R}}^B$.

The function $\hat{f} : \{\pm\infty\}^A \to \{\pm\infty\}^B$ obtained by restricting $\bar{f}$ is called the *asymptote function* of f (or $\bar{f}$), and is denoted by $\hat{f}$ or $\mathrm{as}(f)$. This function is increasing and satisfies $\hat{f}(\bot) = \bot$, $\hat{f}(\top) = \top$.

It is obvious that $\mathrm{as}(f \vee g) = \mathrm{as}(f) \vee \mathrm{as}(g)$ and $\mathrm{as}(f \wedge g) = \mathrm{as}(f) \wedge \mathrm{as}(g)$, when f, g have the same domain and co-domain, and that $\mathrm{as}(g \circ f) = \mathrm{as}(g) \circ \mathrm{as}(f)$ when f, g can be composed.

Theorem 3.9. *Let $\theta : \{\pm\infty\}^A \to \{\pm\infty\}^B$ be an increasing function such that $\theta(\bot) = \bot$, $\theta(\top) = \top$. There exists exactly one null-range topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ such that $\hat{f} = \theta$.*

Proof. We can restrict ourselves to the case of topical forms, i.e. $B = 1$.

Recall that the *free distributive lattice* on a set S consists of a distributive lattice $\mathrm{Fdl}(S)$ and a function $S \xrightarrow{I} \mathrm{Fdl}(S)$ which is an "universal arrow" (cf. [Mcl], §III.1) in the following sense: for any distributive lattice T and any function $S \xrightarrow{J} T$, there exists exactly one lattice morphism $\mathrm{Fdl}(S) \xrightarrow{E} T$ making the following diagram commute

$$\begin{tikzcd} S \arrow[r, "I"] \arrow[d, equal] & \mathrm{Fdl}(S) \arrow[d, "E"] \\ S \arrow[r, "J"] & T \end{tikzcd}$$

As usual, "universal arrows" are unique up to isomorphism: if I, J are both universal then E is a lattice isomorphism.

Let T_1 be the distributive lattice $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R})$, and T_2 the distributive lattice of increasing functions $\{\pm\infty\}^A \to \{\pm\infty\}$ which preserve $\bot$ and $\top$. We have two natural functions $I_1 : S \to T_1$ and $I_2 : S \to T_2$, defined by $I_1 : a \mapsto \gamma_a$ and $I_2 : a \mapsto \hat{\gamma}_a$ (they are simply the natural projections from $\mathbb{R}^A$ to $\mathbb{R}$ and from $\{\pm\infty\}^A$ to $\{\pm\infty\}$). By the universal property of $\mathrm{Fdl}(A)$, there exists a (unique) lattice morphism E making the following diagram commute

$$\begin{tikzcd} A \arrow[r, "I"] \arrow[d, equal] & \mathrm{Fdl}(A) \arrow[d, "E"] \\ A \arrow[r, "I_1"] \arrow[d, equal] & T_1 \arrow[d, "\mathrm{as}"] \\ A \arrow[r, "I_2"] & T_2 \end{tikzcd}$$

Now, a classical theorem of Skolem (see [Bir], §II.5 and §III.4, or [Joh],§I.4, as well as [Ded, Sko]), states that $A \xrightarrow{I_2} T_2$ is a universal arrow, or equivalently, that the composed function $\mathrm{as} \circ E : \mathrm{Fdl}(A) \to T_2$ is a lattice isomorphism. On the other hand, E is surjective by Theorem 2.9, which implies that $\mathrm{as} : T_1 \to T_2$ and E both are lattice isomorphisms.

Hence $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R})$ is *freely* generated (as a distributive lattice) by the functions γ_a, as announced in Theorem 2.9. □

By this theorem we can *identify* the asymptote function $\hat{f} : \{\pm\infty\}^A \to \{\pm\infty\}^B$ with the unique null-range topical function $\mathbb{R}^A \to \mathbb{R}^B$, or its continuous extension $\overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$, whose asymptote is $\hat{f}$. Below, we will often make these identifications (which are compatible with the lattice operations, with composition, etc.).

Proposition 3.10. *Let $f : \mathbb{R}^A \to \mathbb{R}^B$ be a topical function with range at most L, and $\hat{f} : \mathbb{R}^A \to \mathbb{R}^B$ its asymptote (considered as a null range topical function). Then $d(f, \hat{f}) \leqslant L$.*

Proof. Let $\mathfrak{L}$ be the set of forms $\alpha \in \mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R})$ such that $\hat{\alpha} - L \leqslant \alpha \leqslant \hat{\alpha} + L$. Observe that $\mathfrak{L}$ contains all the topical forms $\gamma_i + c$, where $i \in A$ and $|c| \leqslant L$, and that $\mathfrak{L}$ is closed under the lattice operations; so it contains all the min-max-plus forms with range at most L. Hence $\mathfrak{L}$ is dense in $\mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R})$, and obviously closed (for the topology of uniform convergence), hence $\mathfrak{L} = \mathbf{FrTp}_L(\mathbb{R}^A, \mathbb{R})$ and the proposition follows. □

Corollary 3.11. *Any uniformly topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ satisfies $d(f, \hat{f}) < \infty$.*

Since we also know that $\hat{f}$ is multiplicatively positively homogeneous, it follows that $\hat{f}$ can be obtained by the formula

$$\hat{f}(\mathbf{x}) = \lim_{\lambda \to 0^+} \lambda f(\mathbf{x}/\lambda) \tag{3.5}$$

where the convergence is uniform in $\mathbf{x} \in \mathbb{R}^A$. This formula, which could have been used to define $\hat{f}$, explains why $\hat{f}$ is sometimes called the *recession function* of f (what we see of f when viewing it from very far away).

As another consequence, two uniformly topical functions $f, g : \mathbb{R}^A \to \mathbb{R}^B$ satisfy $d(f, g) < \infty$ if and only if $\hat{f} = \hat{g}$. So, we can interpret $\hat{f}$ as the class of f in $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$ for the equivalence relation $d(f, g) < \infty$.

Proposition 3.12. *Let $f_0, f_1 : \mathbb{R}^A \to \mathbb{R}^B$ be uniformly topical, and $0 < \lambda < 1$. The topical function $f = (1 - \lambda) f_0 + \lambda f_1$ is uniformly topical if and only if $\hat{f}_0 = \hat{f}_1$.*

Proof. We restrict ourselves to the case $B = 1$.

Assume that $\hat{f}_0 = \hat{f}_1$, and denote by $\bar{f}_0, \bar{f}_1 : \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}$ the continuous extensions of f_0 and f_1. We can now define $\bar{f} : \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}$ by the formula $\bar{f} = (1 - \lambda)\bar{f}_0 + \lambda \bar{f}_1$, which is well-defined, and defines a continuous function, since we never have simultaneously $f_0(\mathbf{x}) = -\infty$ and $f_1(\mathbf{x}) = +\infty$ or the opposite, as the difference $f_1 - f_0$ is bounded on $\mathbb{R}^A$.

Conversely, assume f to be uniformly topical. It results from formula (3.5) that $\hat{f} = (1 - \lambda)\hat{f}_0 + \lambda \hat{f}_1$. Let $\mathbf{x} \in \mathbb{R}^A$ be a point whose coordinates are ± 1. Then $y_0 = \hat{f}_0(\mathbf{x})$, $y_1 = \hat{f}_1(\mathbf{x})$, and $y = \hat{f}(\mathbf{x})$ are all ± 1, according to the remark following Lemma 2.2. It requires $y_0 = y_1$, otherwise $y = (1 - \lambda) y_0 + \lambda y_1$ would be strictly between -1 and 1. Hence $\hat{f}_0(\mathbf{x}) = \hat{f}_1(\mathbf{x})$ for all $\mathbf{x} \in \{\pm 1\}^A$. But any vertex $\mathbf{p} \in \{\pm\infty\}^A$ can be written as $\lim\limits_{\lambda \to +\infty} \lambda \mathbf{x}$ for some well-chosen $\mathbf{x} \in \{\pm 1\}^A$, implying $\hat{f}_0(\mathbf{p}) = \hat{f}_1(\mathbf{p})$. So we have indeed $\hat{f}_0 = \hat{f}_1$. □

In particular, f_0 and f_1 are in the same connected component of $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$ if and only if they have the same asymptote. So we can interpret $\hat{f}_0$ as the connex component of f_0 in $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$. More formally, the function as : $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B) \to \mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^B)$ is the unique continuous retraction of the canonical injection $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^B) \hookrightarrow \mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$.

4 The configurational space

4.1 Review on the permutahedron $\mathfrak{P}_S$

Let $N = \sharp S$. Recall that the *permutahedron* $\mathfrak{P}_S$ on the set S (or $\mathfrak{P}_N$, permutahedron of order N) is the convex hull in $\mathbb{R}^S$ of the $N!$ points whose coordinates are some permutation of $(1, 2, \ldots, N)$. It is a polytope of dimension $N - 1$, having $N!$ vertices. It is known that the set of its cells of dimension $N - k$ (for $1 \leqslant k \leqslant N$) is in natural bijection with the set of ordered partitions of S into k classes, that is the set of surjections from S onto $\{1, 2, \ldots, k\}$. See [GG, Sch, Zie]. $\mathfrak{P}_1$ is a point, $\mathfrak{P}_2$ a segment, $\mathfrak{P}_3$ an hexagon, and $\mathfrak{P}_4$ a truncated octahedron (cf Figures 4 and 5). We shall see that the permutahedron also appears as some compactification of $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$.

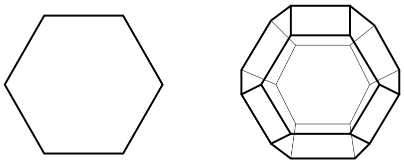

Figure 4: The permutahedra $\mathfrak{P}_3$ and $\mathfrak{P}_4$

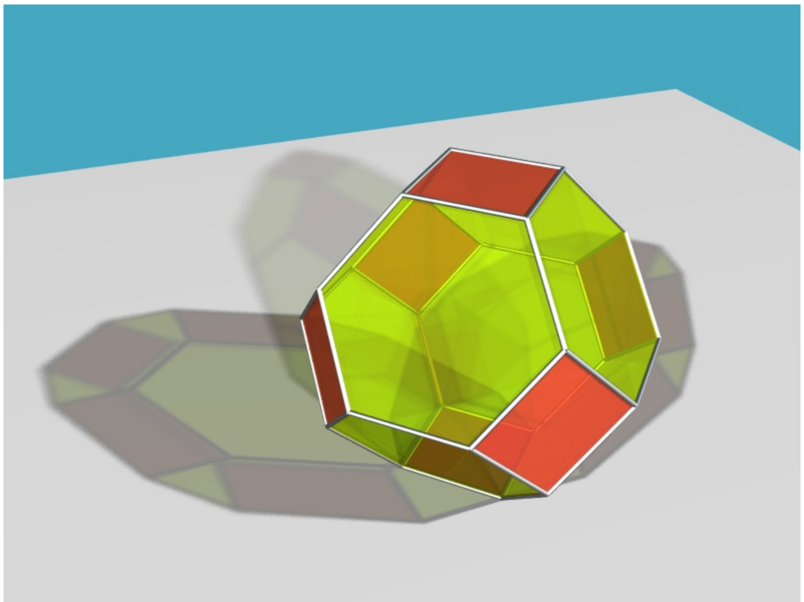

Figure 5: The permutahedron $\mathfrak{P}_4$, or truncated octahedron

The permutahedron may be described as follows. Let $\mathbf{x}$ be a point of $\mathbb{R}^S$, and let $x_1, x_2, \ldots, x_N$ be its coordinates arranged in increasing order. According to the Hardy-Littlewood-Polya Theorem [Mir, GG], the point $\mathbf{x}$ is in $\mathfrak{P}_S$ if and only if

$$x_1 + \cdots + x_N = 1 + \cdots + N \tag{4.1}$$
$$x_1 + \cdots + x_k \geqslant 1 + \cdots + k \qquad (1 \leqslant k < N) \tag{4.2}$$

The relative interior $\mathfrak{P}_S^i$ is obtained by replacing the inequalities (4.2) by strict inequalities. We can identify $\mathfrak{P}_S^i$ with $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ according to the following result, which we admit:

Proposition 4.1. *The function* $\mathfrak{z} : \mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}} \to \mathbb{R}^S$ *defined by* $\mathfrak{z}(x_1 \ldots x_N) = (y_1 \ldots y_N)$, *with*

$$y_i = \frac{N+1}{2} + \frac{1}{2}\sum_{j=1}^{N} \tanh(x_i - x_j) \qquad (1 \leqslant i \leqslant N) \tag{4.3}$$

is a diffeomorphism from $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ *to* $\mathfrak{P}_S^i$.

4.2 The configurational space $\mathrm{Cf}(S) \simeq \mathfrak{P}_S$

We now define the configurational space $\mathrm{Cf}(S)$ as the set of N^2-tuples $(x_{ij})_{i,j \in S}$ with values in $[-\infty, +\infty]$, such that

$$x_{ij} + x_{jk} + x_{ki} = 0 \tag{4.4}$$

for all $i, j, k \in S$, with the convention that the condition (4.4) is considered satisfied if the left-hand side is indeterminate, i.e. when one term is $+\infty$ and another one $-\infty$. With this convention, $\mathrm{Cf}(S)$ is closed in $[-\infty, +\infty]^{S \times S}$. This set can also be seen as the closure of the range of the function $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}} \to [-\infty, +\infty]^{S \times S}$ defined by $(x_i)_{i \in S} \mapsto (x_i - x_j)_{i,j \in S}$. The space $\mathrm{Cf}(S)$ is therefore a compactification of $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$, which intuitively allows some of the gaps between coordinates $x_i - x_j$ (denoted by x_{ij}) to be infinite.

More rigorously, let $k \in [1, N]$, a surjective function $c : S \to \{1 \dots k\}$, and $\mathbf{x} = (x_1 \dots x_N) \in \mathbb{R}^S$. Set $x_{ij} = x_i - x_j$ if $c(i) = c(j)$, $+\infty$ if $c(i) > c(j)$, and $-\infty$ if $c(i) < c(j)$. We check easily that the conditions (4.4) are satisfied, and conversely that any point of $\mathrm{Cf}(S)$ can be obtained by this procedure. The function c can be viewed as an ordered partition of the set S, and the ordered pairs $\big(x_i, c(i)\big)$ can be viewed as living in k disjoint copies of $\mathbb{R}$, arranged in the order given by c, and "infinitely distant" from one another.

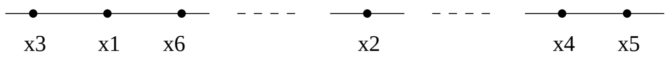

Figure 6: A point of Cf(6), for $c(1,3,6) = 1, c(2) = 2, c(4,5) = 3$

We denote by $\mathrm{Cf}(S, c)$ the set of elements of $\mathrm{Cf}(S)$ obtained in this way, which is clearly in bijection with $(\mathbb{R}^{N_1}/\mathbb{R}\vec{\mathbf{u}}) \times \cdots \times (\mathbb{R}^{N_k}/\mathbb{R}\vec{\mathbf{u}})$, where N_i is the cardinality of $c^{-1}(i)$. The sets $\mathrm{Cf}(S, c)$, for c ranging over the set of ordered partitions of S, form a partition of $\mathrm{Cf}(S)$, and we shall see that it is a cell partition, equivalent to the one of the polytope $\mathfrak{P}_S$.

The function $\mathfrak{z}$ can be extended in an obvious way to a function from $\mathrm{Cf}(S)$ to $\mathfrak{P}_S$, by the formula $(x_{ij}) \mapsto (y_i)$ with

$$y_i = \frac{N+1}{2} + \frac{1}{2} \sum_{j=1}^{N} \tanh x_{ij} \qquad (1 \leqslant i \leqslant N) \tag{4.5}$$

In illustration, Figure 7 shows how the level curves of (x_{ij}) in Cf(3) are sent to $\mathfrak{P}_3$.

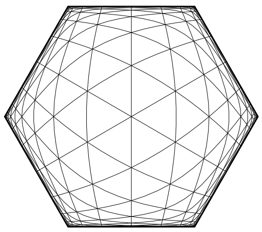

Figure 7: Level curves of x_{12}, x_{23}, x_{31} in $\mathfrak{P}_3$

We admit the following result:

Proposition 4.2. *The function* $\mathfrak{z} : \mathrm{Cf}(S) \to \mathfrak{P}_S$ *is an homeomorphism. For any ordered partition c of the set S in k classes, the subset* $\mathrm{Cf}(S, c)$ *is sent diffeomorphically to one of the cells of dimension $N - k$ of the polytope* $\mathfrak{P}_S$. *The function* $\mathfrak{z}$ *is therefore a cell equivalence between* $\mathrm{Cf}(S)$ *and* $\mathfrak{P}_S$.

Thus we can identify the configurational space $\mathrm{Cf}(S)$ with the permutahedron $\mathfrak{P}_S$, this identification being compatible with the cell structure. The interior points of the permutahedron correspond to the points of $\mathrm{Cf}(S)$ with finite coordinates, that is $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$.

4.3 Quotient by $\mathbb{R}\vec{\mathbf{u}}$ of a uniformly topical function

For $L \geqslant 0$, let $\overset{L}{\approxeq}$ be the equivalence relation on $\mathbb{R}^S$ defined as follows: $\mathbf{x}, \mathbf{y} \in \mathbb{R}^S$ satisfy $\mathbf{x} \overset{L}{\approxeq} \mathbf{y}$ if and only if $x_i - x_j \overset{2L}{\approx} y_i - y_j$ for all $i, j \in S$. Of course, we can also consider $\overset{L}{\approxeq}$ as an equivalence relation on $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$, and we will rather adopt this last point of view. An important property (which should be checked by the reader) is the following one:

$$\mathbf{x} \overset{L}{\approxeq} \mathbf{y} \quad \Longleftrightarrow \quad \big[\forall \lambda \in \mathbb{R} \quad \exists \lambda' \in \mathbb{R} \quad \lambda + \mathbf{x} \overset{L}{\approx} \lambda' + \mathbf{y}\big].$$

As an immediate consequence, if $f : \mathbb{R}^A \to \mathbb{R}^B$ is a topical function with range at most L, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^A$ are such that $\mathbf{x} \overset{K+L}{\approxeq} \mathbf{y}$, then $f(\mathbf{x}) \overset{K}{\approxeq} f(\mathbf{y})$.

Proposition 4.3. *Let $\alpha, \beta : \mathbb{R}^S \to \mathbb{R}$ be two uniformly topical forms. The function $\beta - \alpha$: $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}} \to \mathbb{R}$ has a (unique) continuous extension $\mathrm{Cf}(S) \to \overline{\mathbb{R}}$.*

Proof. Consider any $\mathsf{p} \in \mathrm{Cf}(S)$, let $c : S \to \{1, \dots, k\}$ be the surjection associated with p, as in §4.2 (such that $c(i) > c(j)$ iff $p_{ij} = +\infty$), and let $\sigma : \{1, \dots, k\} \to S$ be a section of c. Let $\mathcal{R}$ be the function from $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ into itself defined by $\mathcal{R} : \mathsf{x} \mapsto \mathsf{y}$, where

$$\begin{cases} y_i - y_j = p_{ij} & \text{if } c(i) = c(j), \\ y_{\sigma(r)} - y_{\sigma(s)} = x_{\sigma(r)} - x_{\sigma(s)} & \text{for all } r, s \in \{1, \dots, k\}, \end{cases}$$

that is,

$$\begin{aligned} y_i - y_j &= [y_i - y_{\sigma c(i)}] - [y_j - y_{\sigma c(j)}] + [y_{\sigma c(i)} - y_{\sigma c(j)}] \\ &= [p_{i,\sigma c(i)} - p_{j,\sigma c(j)}] + [x_{\sigma c(i)} - x_{\sigma c(j)}] \end{aligned}$$

for all $i, j \in S$. This function is idempotent and satisfies $\tilde{d}(\mathsf{x}, \mathcal{R}\mathsf{x}) \to 0$ when $\mathsf{x} \to \mathsf{p}$. We also see that, for all $L \geqslant 0$, we have $\mathcal{R}\mathsf{x} \overset{L}{\approxeq} \mathcal{R}\mathsf{y}$ as soon as x, y are close enough to p. Set $\ell_1 = \liminf_{\mathsf{x} \to \mathsf{p}} (\beta - \alpha)(\mathsf{x})$ and $\ell_2 = \limsup_{\mathsf{x} \to \mathsf{p}} (\beta - \alpha)(\mathsf{x})$.

Consider an arbitrary $\epsilon > 0$. According to the density Theorem 3.5, there exist α', β' with range at most L, for some appropriate $L \geqslant 0$, such that $d(\alpha, \alpha') \leqslant \epsilon$ and $d(\beta, \beta') \leqslant \epsilon$. Choose $K \geqslant 0$ such that $1 - \tanh 2K \leqslant \epsilon$. Let $\mathcal{V}$ be the intersection with $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ of a neighbourhood of p in $\mathrm{Cf}(S)$ such that $\tilde{d}(\mathsf{x}, \mathcal{R}\mathsf{x}) \leqslant \epsilon$ and $\mathcal{R}\mathsf{x} \overset{K+L}{\approxeq} \mathcal{R}\mathsf{y}$ for all $\mathsf{x}, \mathsf{y} \in \mathcal{V}$. Finally let us denote by $f : \mathbb{R}^S \to \mathbb{R}^2$ the topical function, with range at most L, defined by $f(\mathbf{x}) = (\alpha'\mathbf{x}, \beta'\mathbf{x})$.

For all $\mathsf{x}, \mathsf{y} \in \mathcal{V}$, we have $\mathsf{x}' \overset{K+L}{\approxeq} \mathsf{y}'$, where $\mathsf{x}' = \mathcal{R}\mathsf{x}$ and $\mathsf{y}' = \mathcal{R}\mathsf{y}$, and so $\tilde{f}(\mathsf{x}') \overset{K}{\approxeq} \tilde{f}(\mathsf{y}')$, that is, $(\beta' - \alpha')(\mathsf{x}') \overset{2K}{\approx} (\beta' - \alpha')(\mathsf{y}')$. In particular

$$\big|\tanh(\beta' - \alpha')(\mathsf{y}') - \tanh(\beta' - \alpha')(\mathsf{x}')\big| \leqslant 1 - \tanh 2K \leqslant \epsilon \tag{4.6}$$

From the inequalities $\tilde{d}(\mathsf{x}, \mathsf{x}') \leqslant \epsilon$, $\tilde{d}(\mathsf{y}, \mathsf{y}') \leqslant \epsilon$, $d(\alpha, \alpha') \leqslant \epsilon$, $d(\beta, \beta') \leqslant \epsilon$, we get

$$\begin{aligned} &\big|(\beta' - \alpha')(\mathsf{x}') - (\beta - \alpha)(\mathsf{x})\big| \leqslant 4\epsilon \\ &\big|(\beta' - \alpha')(\mathsf{y}') - (\beta - \alpha)(\mathsf{y})\big| \leqslant 4\epsilon \end{aligned}$$

and consequently

$$\begin{aligned}\left|\tanh(\beta'-\alpha')(\mathsf{x}')-\tanh(\beta-\alpha)(\mathsf{x})\right| &\leqslant 2\tanh 2\epsilon \leqslant 4\epsilon\\ \left|\tanh(\beta'-\alpha')(\mathsf{y}')-\tanh(\beta-\alpha)(\mathsf{y})\right| &\leqslant 2\tanh 2\epsilon \leqslant 4\epsilon\end{aligned}$$

which, when combined with (4.6), yields

$$\left|\tanh(\beta-\alpha)(\mathsf{y})-\tanh(\beta-\alpha)(\mathsf{x})\right| \leqslant 9\epsilon$$

for all x,y close enough to p. Therefore $0 \leqslant \tanh\ell_2 - \tanh\ell_1 \leqslant 9\epsilon$, and this for all $\epsilon > 0$, implying $\ell_1 = \ell_2$. □

By the previous proposition, the expression $(\beta-\alpha)(\mathsf{p})$ is well-defined, for all $\alpha,\beta \in \mathbf{UTp}(\mathbb{R}^S,\mathbb{R})$ and $\mathsf{p} \in \mathrm{Cf}(S)$; it is an element of $\overline{\mathbb{R}}$, which depends continuously on α, β and p.

More generally: for any $f,g \in \mathbf{UTp}(\mathbb{R}^A,\mathbb{R}^B)$, the difference $g-f : \mathbb{R}^A/\mathbb{R}\vec{\mathbf{u}} \to \mathbb{R}^B$ has a continuous extension $\mathrm{Cf}(A) \to \overline{\mathbb{R}}^B$, which will also be denoted by $g-f$, justifying the notation $(g-f)(\mathsf{p})$ where $\mathsf{p} \in \mathrm{Cf}(A)$.

Corollary 4.4. *Let $f : \mathbb{R}^A \to \mathbb{R}^B$ be uniformly topical. The quotient function $\tilde{f} : \mathbb{R}^A/\mathbb{R}\vec{\mathbf{u}} \to \mathbb{R}^B/\mathbb{R}\vec{\mathbf{u}}$ has a (unique) continuous extension* $\mathrm{Cf}(A) \to \mathrm{Cf}(B)$.

Indeed, the differences $f_{ij} = f_i - f_j : \mathbb{R}^A/\mathbb{R}\vec{\mathbf{u}} \to \mathbb{R}$ (with $i,j \in B$) satisfy $f_{ij}+f_{jk}+f_{ki} = 0$ and have continuous extensions $\mathrm{Cf}(A) \to \overline{\mathbb{R}}$, by the previous proposition, thus defining a continuous function $\mathrm{Cf}(A) \to \mathrm{Cf}(B)$ extending $\tilde{f}$, which we still denote by $\tilde{f}$, with some abuse of notation.

Let us provide another proof of the above corollary, under the additional assumption that f has finite range (at most L). There exists for all $K \geqslant 0$ a quotient function $\tilde{f}_K$ from $\big(\mathbb{R}^A/\overset{K+L}{\approxeq}\big)$ into $\big(\mathbb{R}^B/\overset{K}{\approxeq}\big)$. By taking the projective limit for $K \to \infty$, in the category of topological spaces, we obtain a continuous function

$$\dot{f} :\ \varprojlim_K \big(\mathbb{R}^A/\overset{K}{\approxeq}\big) \ \longrightarrow\ \varprojlim_K \big(\mathbb{R}^B/\overset{K}{\approxeq}\big)$$

It is not very difficult to see that the limit spaces can be identified with $\mathrm{Cf}(A)$ and $\mathrm{Cf}(B)$, and that $\dot{f}$ is an extension of $\tilde{f}$ (for this identification).

5 Existence and properties of the spectral vector

We say that a topical endofunction $f : \mathbb{R}^S \to \mathbb{R}^S$ has a spectral vector if $(f^n(\mathbf{p})-\mathbf{p})/n$ has a limit when $n \to \infty$, where $\mathbf{p}$ is an arbitrary point of $\mathbb{R}^S$; the existence of the limit, and its value, do obviously not depend on the choice of $\mathbf{p}$. The limit, when it exists, is the *spectral vector* of f, denoted by $\vec{\chi}(f)$.

We say that $\mathbf{p} \in \mathbb{R}^S$ is a *gliding point* (in French *point glissant*) of f if the sequence $f^{n+1}(\mathbf{p}) - f^n(\mathbf{p})$ is constant ($n \geqslant 0$). In this case, f has the spectral vector $\vec{\chi}(f) = (f-\mathrm{Id})(\mathbf{p})$, and $f^n(\mathbf{p}) = \mathbf{p} + n\vec{\chi}(f)$ for all $n \geqslant 0$.

The goal of this section is to prove that $\vec{\chi}(f)$ exists if f is uniformly topical, and to interpret it using the fixed points of $\tilde{f}$ in the configurational space $\mathrm{Cf}(S)$.

5.1 A fixed point theorem

Lemma 5.1. *Let $f : \mathbb{R}^S \to \mathbb{R}^S$ be a uniformly topical function, $\tilde{f} : \mathrm{Cf}(S) \to \mathrm{Cf}(S)$ its quotient function (extended continuously), $\mathsf{p} = (p_{ij}) \in \mathrm{Cf}(S)$ a fixed point of $\tilde{f}$, and $\vec{\mathbf{v}} = (f - \mathrm{Id})(\mathsf{p})$. Then $\vec{\mathbf{v}}$ is finite, in fact $\|\vec{\mathbf{v}}\| \leqslant d(f, \hat{f})$, and $v_i = v_j$ for all $i, j \in S$ such that $|p_{ij}| < \infty$.*

Proof. Observe first that if $i, j \in S$ are two indices such that $p_{ij} = (\gamma_i - \gamma_j)(\mathsf{p}) = (f_i - f_j)(\mathsf{p})$ is finite, we can write

$$\begin{aligned} v_i = (f_i - \gamma_i)(\mathsf{p}) &= (f_i - f_j)(\mathsf{p}) + (f_j - \gamma_j)(\mathsf{p}) - (\gamma_i - \gamma_j)(\mathsf{p}) \\ &= (f_j - \gamma_j)(\mathsf{p}) = v_j. \end{aligned}$$

Number the elements of S from 1 to N so that $p_{ij} \leqslant 0$ when $i \leqslant j$. Thus p is in the closure of $\mathbb{R}^N_{\text{inc}}/\mathbb{R}\vec{\mathbf{u}}$, where $\mathbb{R}^N_{\text{inc}}$ is the set of points $\mathbf{x} \in \mathbb{R}^N$ such that $x_1 \leqslant \cdots \leqslant x_N$.

Each coordinate of $f(\mathbf{x})$ is at distance at most $K = d(f, \hat{f})$ from the set of coordinates of $\mathbf{x}$; if $\mathbf{x} \in \mathbb{R}^N_{\text{inc}}$, it gives in particular (i) $f_1(\mathbf{x}) \geqslant x_1 - K$, and (ii) $f_s(\mathbf{x}) \geqslant x_s - K$ or $f_s(\mathbf{x}) \leqslant x_{s-1} + K$ for all $s > 1$.

We want to prove that $(f_s - \gamma_s)(\mathsf{p}) \geqslant -K$ for all $s \in S$. This inequality holds for $s = 1$, as we can see by taking the limit in (i). We shall now prove that, for any $s > 1$, we cannot have simultaneously $(f_{s-1} - \gamma_{s-1})(\mathsf{p}) \geqslant -K$ and $(f_s - \gamma_s)(\mathsf{p}) < -K$.

The inequality $(f_s - \gamma_s)(\mathsf{p}) < -K$ implies that, for all $\mathbf{x} \in \mathbb{R}^N_{\text{inc}}$ with $\tilde{\mathbf{x}}$ close enough to p, we have $(f_s - \gamma_s)(\mathbf{x}) < -K$, and so $(f_s - \gamma_{s-1})(\mathbf{x}) \leqslant K$, by (ii). Hence $(f_s - \gamma_{s-1})(\mathsf{p}) \leqslant K$, and by combining with $(f_{s-1} - \gamma_{s-1})(\mathsf{p}) \geqslant -K$, we get $p_{s,s-1} = (f_s - f_{s-1})(\mathsf{p}) \leqslant 2K$. Thus $p_{s,s-1}$ is finite, which implies $v_s = v_{s-1}$ according to the above remark, a contradiction.

So we have proved that $v_s \geqslant -K$ for all $s \in S$; the inequalities $v_s \leqslant K$ are proved similarly. □

With the notations of the lemma, we say that p is a *stable* fixed point of $\tilde{f}$ if $v_i \leqslant v_j$ for all $i, j \in S$ such that $p_{ij} < \infty$. (Note that this condition remains unchanged when f is replaced with $\lambda + f$; so it is really a condition on $\tilde{f}$, rather than f.) Intuitively, the stability condition expresses some "repulsion" between the different clusters of coordinates of p.

Lemma 5.2. *Let $f : \mathbb{R}^S \to \mathbb{R}^S$ be uniformly topical, and $\mathsf{p} \in \mathrm{Cf}(S)$. If and only if p is a stable fixed point of $\tilde{f}$, there exists a sequence $(\mathbf{x}_n)_{n \geqslant 0}$ of points of $\mathbb{R}^S$ such that $\tilde{\mathbf{x}}_n \to \mathsf{p}$ in $\mathrm{Cf}(S)$ and $d(f\mathbf{x}_n, \mathbf{x}_{n+1}) \to 0$. In this case, $(f - \mathrm{Id})(\mathsf{p})$ is the spectral vector of f.*

Proof. Assume that $\mathsf{p} = \lim \tilde{\mathbf{x}}_n$, where $(\mathbf{x}_n)$ is a sequence of points of $\mathbb{R}^S$ satisfying $d(f(\mathbf{x}_n), \mathbf{x}_{n+1}) \to 0$. Thus $\tilde{d}(\tilde{f}(\tilde{\mathbf{x}}_n), \tilde{\mathbf{x}}_{n+1}) \to 0$, so the sequences $(\tilde{f}(\tilde{\mathbf{x}}_n))$ and $(\tilde{\mathbf{x}}_{n+1})$ have the same accumulation points in $\mathrm{Cf}(S)$, which implies $\tilde{f}(\mathsf{p}) = \mathsf{p}$. Since p is fixed by $\tilde{f}$, the vector $\vec{\mathbf{v}} = (f - \mathrm{Id})(\mathsf{p})$ has finite coordinates. Therefore $\mathbf{x}_{n+1} - \mathbf{x}_n = (f - \mathrm{Id})(\mathbf{x}_n) + o(1) = \vec{\mathbf{v}} + o(1)$, and consequently $\mathbf{x}_n - \mathbf{x}_0 = n\vec{\mathbf{v}} + o(n)$. But the hypothesis $f(\mathbf{x}_n) = \mathbf{x}_{n+1} + o(1)$ also implies $f^n(\mathbf{x}_0) = \mathbf{x}_n + o(n)$, hence $(f^n - \mathrm{Id})(\mathbf{x}_0) = n\vec{\mathbf{v}} + o(n)$, i.e. $\vec{\mathbf{v}}$ is the spectral vector of f. If $i, j \in S$ are any two indices,

$$(\gamma_i - \gamma_j)(\mathbf{x}_n) = n(v_i - v_j) + o(n)$$

and the left-hand side tends to p_{ij} when $n \to \infty$. So $p_{ij} < \infty$ implies $v_i \leqslant v_j$, and p is indeed a stable fixed point of $\tilde{f}$.

Conversely, let p be a stable fixed point of $\tilde{f}$, and $\vec{\mathbf{v}} = (f - \mathrm{Id})(\mathsf{p})$. Let $c : S \to \{1, \ldots, k\}$ be the ordered partition associated to the point p, and $\sigma : \{1, \ldots, k\} \to S$ a section of c.

Let $\mathbf{x}_0$ be the point defined by $(\gamma_i - \gamma_j)(\mathbf{x}_0) = p_{ij}$ if $c(i) = c(j)$, and $\gamma_{\sigma(r)}(\mathbf{x}_0) = 0$ for each $r \in \{1, \ldots, k\}$. Denote by $\vec{\mathbf{c}}$ the vector whose components are the $c(i)$. It is easily verified that the sequence $\mathbf{x}_n = \mathbf{x}_0 + n^{1/2}\vec{\mathbf{c}} + n\vec{\mathbf{v}}$ satisfies $\tilde{\mathbf{x}}_n \to \mathsf{p}$. On the other hand $f(\mathbf{x}_n) - \mathbf{x}_{n+1} = (f - \mathrm{Id})(\mathbf{x}_n) - (\mathbf{x}_{n+1} - \mathbf{x}_n)$, which tends to $(f - \mathrm{Id})(\mathsf{p}) - \vec{\mathbf{v}} = 0$. □

Theorem 5.3. *For any uniformly topical function $f : \mathbb{R}^S \to \mathbb{R}^S$, the quotient function $\tilde{f} : \mathrm{Cf}(S) \to \mathrm{Cf}(S)$ has a least one stable fixed point. If p is any of them, $(f - \mathrm{Id})(\mathsf{p})$ is the spectral vector of f.*

Proof. For each $s \in [0, 1)$, the function $\mathbf{x} \mapsto f(s\mathbf{x})$ is contracting as an endofunction of $\mathbb{R}^S$, and so has a unique fixed point, which we denote by $\mathbf{x}(s)$. Since $\mathrm{Cf}(S)$ is compact, the net $\tilde{\mathbf{x}}(s)$ has at least one limit point in $\mathrm{Cf}(S)$ when $s \to 1^-$. Let p be any of them, limit of some subsequence $\tilde{\mathbf{x}}(s_n)$, where $s_n \to 1^-$.

It is easily checked that $(s_n\mathbf{x}(s_n))^\sim \to \mathsf{p}$ in $\mathrm{Cf}(S)$, and therefore $\tilde{f}(\mathsf{p}) = \mathsf{p}$. Let $\vec{\mathbf{v}} = (f - \mathrm{Id})(\mathsf{p}) \in \mathbb{R}^S$. From the formula $(f - \mathrm{Id})\big(s\mathbf{x}(s)\big) = (1 - s)\mathbf{x}(s)$ we get $(1 - s_n)\mathbf{x}(s_n) \to \vec{\mathbf{v}}$ and so $(1 - s_n)x_{ij}(s_n) \to v_i - v_j$ for all $i, j \in S$. On the other hand $x_{ij}(s_n) \to p_{ij}$, from which we deduce that $p_{ij} < \infty \implies v_i \leqslant v_j$. So p is indeed a stable fixed point of $\tilde{f}$, and $\vec{\mathbf{v}}$ is the spectral vector of f. □

So we obtain a new proof of the existence of the spectral vector for min-max-plus functions (cf. the introduction). Conversely, if we know that the spectral vector exists for all min-max-plus functions, then we can immediately extend this result to all uniformly topical functions, because the existence of $\vec{\chi}(f)$ is a closed property in f for the topology of uniform convergence — this fact (noted independently by Gaubert & Gunawardena) is of no use to us in the deterministic case, but we will need its stochastic analogue, Proposition 6.3, in an essential way. As one can see, the existence of $\vec{\chi}(f)$ is not really a new result; the point of the above theorem is the link between $\vec{\chi}(f)$ and the fixed points of $\tilde{f}$.

Lemma 5.4. *If $f : \mathbb{R}^S \to \mathbb{R}^S$ is topical with range at most L, and $\mathsf{x} \in \mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ is such that $\mathsf{x} \overset{L}{\approxeq} \tilde{f}(\mathsf{x})$, then $\|(f - \mathrm{Id})(\mathsf{x})\| \leqslant L$.*

Proof. The proof is very similar to the proof of Lemma 5.1. Let $\mathbf{x} \in \mathbb{R}^S$ be such that $\mathbf{y} = f(\mathbf{x}) \overset{L}{\approxeq} \mathbf{x}$, and number $1, \ldots, N$ the elements of S so that $x_1 \leqslant \cdots \leqslant x_N$. By Lemma 2.2, we have (i) $y_1 \geqslant x_1 - L$ and (ii) $y_s \geqslant x_s - L$ or $y_s \leqslant x_{s-1} + L$ for all $s > 1$. Finally, the condition $\mathbf{y} \overset{L}{\approxeq} \mathbf{x}$ gives us (iii) $y_s - y_{s-1} \overset{2L}{\approx} x_s - x_{s-1}$ for all $s > 1$.

We want to prove that $y_s \geqslant x_s - L$ for all $s \in S$. This is true for $s = 1$, by (i). Now let us prove the inequality for an arbitrary $s > 1$, assuming that it holds for $s - 1$.

We distinguish two cases. If $x_s - x_{s-1} \leqslant 2L$, then $y_s - y_{s-1} = x_s - x_{s-1}$ by (iii) and consequently $y_s - x_s = y_{s-1} - x_{s-1} \geqslant -L$. On the other hand, if $x_s - x_{s-1} > 2L$, then $y_s - y_{s-1} > 2L$ by (iii), which yields $y_s > 2L + y_{s-1} \geqslant x_{s-1} + L$ and therefore $y_s \geqslant x_s - L$, by (ii).

We have just proved that $\mathbf{y} \geqslant -L + \mathbf{x}$; the proof of $\mathbf{y} \leqslant L + \mathbf{x}$ is similar. □

Lemma 5.5. *Let $\alpha, \gamma : \mathbb{R}^S \to \mathbb{R}$ be topical forms with range at most L and zero, respectively. If $\mathsf{x}, \mathsf{y} \in \mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ are such that $\mathsf{x} \overset{L}{\approxeq} \mathsf{y}$, then $(\alpha - \gamma)(\mathsf{x}) \overset{L}{\approx} (\alpha - \gamma)(\mathsf{y})$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^S$ be such that $\mathbf{x} \overset{L}{\approxeq} \mathbf{y}$. For any $\lambda \in [-L, L]$, the topical function $f_\lambda : \mathbb{R}^S \to \mathbb{R}^2$ defined by $f_\lambda(\mathbf{x}) = (\alpha\mathbf{x}, -\lambda + \gamma\mathbf{x})$ has range at most L, so that $f_\lambda(\mathbf{x}) \overset{0}{\approxeq} f_\lambda(\mathbf{y})$. It means that $\lambda + (\alpha - \gamma)(\mathbf{x}) \overset{0}{\approx} \lambda + (\alpha - \gamma)(\mathbf{y})$ for any $\lambda \in [-L, L]$, that is, $(\alpha - \gamma)(\mathbf{x}) \overset{L}{\approx} (\alpha - \gamma)(\mathbf{y})$. □

Theorem 5.6. *Let $f : \mathbb{R}^S \to \mathbb{R}^S$ be any topical endofunction with range at most L. There exists in $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ at least one equivalence class of $\overset{L}{\approxeq}$ which is stable under $\tilde{f}$ and, if C is any such class, one has $(f - \mathrm{Id})(\mathsf{x}) = \vec{\chi}(f)$ for each $\mathsf{x} \in C$.*

In particular, f has gliding points, i.e. points $\mathbf{x} \in \mathbb{R}^S$ such that $f^n(\mathbf{x}) = \mathbf{x} + n\vec{\chi}(f)$ for all $n \geqslant 0$; indeed, every stable class of $\overset{L}{\approxeq}$, seen as an equivalence class in $\mathbb{R}^S$, consists entirely of gliding points.

Proof. Suppose that C is an equivalence class of $\overset{L}{\approxeq}$ in $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ such that $\tilde{f}(C) \subseteq C$. If x is an arbitrary point of C, one has $\|(f - \mathrm{Id})(\mathsf{x})\| \leqslant L$ by Lemma 5.4. If y is another point of C, one has $(f_s - \gamma_s)(\mathsf{x}) \overset{L}{\approx} (f_s - \gamma_s)(\mathsf{y})$ for all $s \in S$ by Lemma 5.5, which, combined with the inequalities $|(f_s - \gamma_s)(\mathsf{x})| \leqslant L$, gives us $(f - \mathrm{Id})(\mathsf{x}) = (f - \mathrm{Id})(\mathsf{y})$. This proves that $f - \mathrm{Id}$ is constant on C; let $\vec{\mathbf{v}}$ be its value.
The set $\check{C} = \{\mathbf{x} \in \mathbb{R}^S : \tilde{\mathbf{x}} \in C\}$ satisfies $f(\check{C}) \subseteq \check{C}$ and $f(\mathbf{x}) = \mathbf{x} + \vec{\mathbf{v}}$ for all $\mathbf{x} \in \check{C}$, which shows that $\vec{\mathbf{v}}$ is the spectral vector of f and all the points of $\check{C}$ are gliding points.
Existence of a stable class. Let $\mathsf{p} \in \mathrm{Cf}(S)$ be a stable fixed point of $\tilde{f}$, and $\vec{\mathbf{v}} = (f - \mathrm{Id})(\mathsf{p})$ the spectral vector of f, which satisfies $\|\vec{\mathbf{v}}\| \leqslant L$ by Lemma 5.1. Choose $\mathsf{x} \in \mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ such that $x_i - x_j \overset{2(L+1)}{\approx} p_{ij}$ for all $i, j \in S$ (which could be written $\mathsf{x} \overset{L+1}{\approxeq} \mathsf{p}$ with an obvious abuse of notation).

Note that, for any $0 < \epsilon \leqslant 1$, the class of x for the relation $\overset{L+\epsilon}{\approxeq}$ contains p in its closure; so one can find a point y such that $\mathsf{y} \overset{L+\epsilon}{\approxeq} \mathsf{x}$ and sufficiently close to p so that $\|(f - \mathrm{Id})(\mathsf{y})\| \leqslant L + \epsilon$. Combining this inequality with the relations $(f_s - \gamma_s)(\mathsf{x}) \overset{L+\epsilon}{\approx} (f_s - \gamma_s)(\mathsf{y})$ given by Lemma 5.5, we obtain $\|(f - \mathrm{Id})(\mathsf{x})\| \leqslant L + \epsilon$, and this holds for ϵ arbitrarily small, therefore $\|(f - \mathrm{Id})(\mathsf{x})\| \leqslant L$.

Let C be the class of x for the relation $\overset{L}{\approxeq}$. For any $\mathsf{y} \in C$, one has $(f_s - \gamma_s)(\mathsf{x}) \overset{L}{\approx} (f_s - \gamma_s)(\mathsf{y})$ for all $s \in S$, which implies $(f - \mathrm{Id})(\mathsf{x}) = (f - \mathrm{Id})(\mathsf{y})$ since $\|(f - \mathrm{Id})(\mathsf{x})\| \leqslant L$. So $f - \mathrm{Id}$ is constant on C, and since C contains p in its closure, this constant is merely $\vec{\mathbf{v}} = (f - \mathrm{Id})(\mathsf{p})$; in other words, f coincides on $\check{C}$ with the translation of vector $\vec{\mathbf{v}}$. Finally, the stability conditions $p_{ij} < \infty \implies v_i \leqslant v_j$ easily imply $\tilde{f}(C) \subseteq C$. ☐

5.2 Miscellaneous properties of the spectral vector

Proposition 5.7. *Let $f : \mathbb{R}^S \to \mathbb{R}^S$ be uniformly topical, with S non-empty. For all $\mathbf{p} \in \mathbb{R}^S$, one has*

$$\lfloor \mathbf{p}, f\mathbf{p} \rfloor \leqslant \lfloor \vec{\chi}(f) \rfloor \leqslant \lceil \vec{\chi}(f) \rceil \leqslant \lceil \mathbf{p}, f\mathbf{p} \rceil. \tag{5.1}$$

More precisely,

$$\lfloor \vec{\chi}(f) \rfloor = \sup_{\mathbf{p} \in \mathbb{R}^S} \lfloor \mathbf{p}, f\mathbf{p} \rfloor, \qquad \lceil \vec{\chi}(f) \rceil = \inf_{\mathbf{p} \in \mathbb{R}^S} \lceil \mathbf{p}, f\mathbf{p} \rceil, \tag{5.2}$$

and these bounds are attained if f has finite range.

Proof. For all $n \geqslant 0$, one has $n\lfloor \mathbf{p}, f\mathbf{p} \rfloor \leqslant \sum_{k=0}^{n-1} \lfloor f^k\mathbf{p}, f^{k+1}\mathbf{p} \rfloor \leqslant \lfloor \mathbf{p}, f^n\mathbf{p} \rfloor$ and similarly $n\lceil \mathbf{p}, f\mathbf{p} \rceil \geqslant \lceil \mathbf{p}, f^n\mathbf{p} \rceil$. Taking the limit $n \to \infty$ in

$$\lfloor \mathbf{p}, f\mathbf{p} \rfloor \leqslant n^{-1}\lfloor \mathbf{p}, f^n\mathbf{p} \rfloor \leqslant n^{-1}\lceil \mathbf{p}, f^n\mathbf{p} \rceil \leqslant \lceil \mathbf{p}, f\mathbf{p} \rceil,$$

we obtain (5.1). On the other hand, $(f - \mathrm{Id})(\mathbf{p})$ tends to $\vec{\chi}(f)$ when $\tilde{\mathbf{p}}$ tends to a stable fixed point of $\tilde{f} : \mathrm{Cf}(S) \to \mathrm{Cf}(S)$, which proves (5.2); and the limit is attained if $\mathbf{p}$ is a gliding point of f. ☐

Lemma 5.8. *Let $f, g : \mathbb{R}^S \to \mathbb{R}^S$ be two uniformly topical functions. If $\tilde{f}$ and $\tilde{g}$ have a common stable fixed point $\mathsf{p} \in \mathrm{Cf}(S)$, then $\vec{\chi}(f \vee g) = \vec{\chi}(f) \vee \vec{\chi}(g)$ and $\vec{\chi}(f \wedge g) = \vec{\chi}(f) \wedge \vec{\chi}(g)$.*

Proof. The formulas $(f\vee g-\mathrm{Id})(\mathsf{p})=(f-\mathrm{Id})(\mathsf{p})\vee(g-\mathrm{Id})(\mathsf{p})$ and $(f\wedge g-\mathrm{Id})(\mathsf{p})=(f-\mathrm{Id})(\mathsf{p})\wedge(g-\mathrm{Id})(\mathsf{p})$ show that p is a stable fixed point of $(f\vee g)^{\sim}$ and $(f\wedge g)^{\sim}$, with the spectral vectors of $f\vee g$ and $f\wedge g$ as indicated. $\square$

Corollary 5.9. *Let f be a uniformly topical endofunction of $\mathbb{R}^S$, and T the translation with the same spectral vector. Then*

$$\vec{\chi}(f\vee \mathrm{Id})=\vec{\chi}(f)\vee 0,\quad \vec{\chi}(f\wedge \mathrm{Id})=\vec{\chi}(f)\wedge 0,$$
$$\vec{\chi}(f\vee T)=\vec{\chi}(f\wedge T)=\vec{\chi}(f)=\vec{\chi}(T).$$

Lemma 5.10. *Let $g:\mathbb{R}^S\to\mathbb{R}^S$ be a finite-range topical function, with $Z=\{i\in S:\chi_i(g)=0\}$ non-empty and $g\geqslant \mathrm{Id}$. There exists a max-plus form $\alpha:\mathbb{R}^S\to\mathbb{R}$ such that $\alpha g=\alpha$ and $\hat{\alpha}=\bigvee_{i\in Z}\gamma_i$.*

Proof. Let $\mathbf{p}=(p_i)$ be a gliding point of g, and $\mathbf{p}^n=g^n(\mathbf{p})=\mathbf{p}+n\vec{\chi}(g)$ its iterates ($n\geqslant 0$). For all $n\in\mathbb{N}$, let $\alpha_n:\mathbb{R}^S\to\mathbb{R}$ be the max-plus form defined by $\alpha_n(\mathbf{x})=\lceil\mathbf{p}^n,\mathbf{x}\rceil$. The inequalities $\lceil\mathbf{p}^{n+1},g\mathbf{x}\rceil\leqslant\lceil\mathbf{p}^n,\mathbf{x}\rceil$ show that $\alpha_{n+1}\,g\leqslant\alpha_n$ for all $n\in\mathbb{N}$, and therefore $\alpha g\leqslant\alpha$ when taking the limit $n\to\infty$, where $\alpha(\mathbf{x})=\max_{i\in Z}(x_i-p_i)$ is the pointwise limit of the sequence α_n. On the other hand, $\alpha g\geqslant\alpha$ is an obvious consequence of $g\geqslant\mathrm{Id}$. $\square$

Proposition 5.11. *Let $f:\mathbb{R}^S\to\mathbb{R}^S$ be a finite-range topical function, $\pi:\mathbb{R}^S\to\mathbb{R}$ uniformly topical, and $\lambda\in\mathbb{R}$. Then*

$$\hat{\pi}\big[\vec{\chi}(f)\big]\leqslant\lambda \iff \exists\alpha\in W(\pi)\quad \alpha f\leqslant\alpha+\lambda \tag{5.3}$$

where $W(\pi)$ is the set of all max-plus upper bounds of π.

Proof. We can assume $\lambda=0$, up to replacing f with $f-\lambda$.
If there exists $\alpha\in W(\pi)$ such that $\alpha f\leqslant\alpha$, then $\pi f^n(\mathbf{o})\leqslant\alpha f^n(\mathbf{o})\leqslant\alpha(\mathbf{o})$ for all $n\geqslant 0$, and since $\pi f^n(\mathbf{o})=n\hat{\pi}\big[\vec{\chi}(f)\big]+o(n)$ when n is big, this implies $\hat{\pi}\big[\vec{\chi}(f)\big]\leqslant 0$.
Conversely, let us assume $\hat{\pi}\big[\vec{\chi}(f)\big]\leqslant 0$, and let $g=f\vee\mathrm{Id}$. The equivalences

$$\hat{\pi}\big[\vec{\chi}(f)\big]\leqslant 0\iff\hat{\pi}\big[\vec{\chi}(f)\big]\vee 0=0\iff\hat{\pi}\big[\vec{\chi}(f)\vee 0\big]=0\iff\hat{\pi}\big[\vec{\chi}(g)\big]=0$$

(cf. §2.4, and Corollary 5.9) show that g satisfies the hypotheses of Lemma 5.10. So, let $Z=\{i\in S:\chi_i(g)=0\}$, and α a max-plus form such that $\alpha g=\alpha$ and $\hat{\alpha}=\bigvee_{i\in Z}\gamma_i$. Let us write $\hat{\pi}$ as an infimum of null-range max-plus forms: one of them, say $\beta=\bigvee_{i\in T}\gamma_i$, must vanish on $\vec{\chi}(g)$. This requires $T\subseteq Z$, so $\hat{\pi}\leqslant\beta\leqslant\hat{\alpha}$. Up to the addition of a suitable constant to α, we have $\alpha\geqslant\pi$, and besides, $\alpha=\alpha(f\vee\mathrm{Id})=\alpha f\vee\alpha$ (since α is max-plus), i.e. $\alpha f\leqslant\alpha$. $\square$

Corollary 5.12. *If $f:\mathbb{R}^S\to\mathbb{R}^S$ and $\pi:\mathbb{R}^S\to\mathbb{R}$ are uniformly topical, then*

$$\hat{\pi}\big[\vec{\chi}(f)\big]=\inf_{\alpha\in W(\pi)}\ \sup_{\mathbf{x}\in\mathbb{R}^S}\ \alpha f(\mathbf{x})-\alpha(\mathbf{x}) \tag{5.4}$$

Indeed, Formula (5.4) is an immediate consequence of (5.3) if f has finite range; by density, the formula is true for any uniformly topical f.
Semi-conjugacies. Let f,g be uniformly topical functions such that the diagram

$$\begin{array}{ccc}\mathbb{R}^A & \xrightarrow{\ f\ } & \mathbb{R}^A\\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow\\ \mathbb{R}^B & \xrightarrow{\ g\ } & \mathbb{R}^B\end{array}$$

commutes, i.e. $\pi f = g\pi$. Then $\pi f^n = g^n \pi$ for all $n \geqslant 0$; evaluating this at the origin (of $\mathbb{R}^A$), we obtain $n\hat{\pi}\big[\vec{\chi}(f)\big] + o(n) = n\vec{\chi}(g) + o(n)$ when $n \to \infty$, hence

$$\hat{\pi}\big[\vec{\chi}(f)\big] = \vec{\chi}(g) \tag{5.5}$$

at the limit. In particular:

Proposition 5.13. *For any uniformly topical $f : \mathbb{R}^S \to \mathbb{R}^S$, the spectral vector $\vec{\chi}(f)$ is a fixed point of $\hat{f}$:*

$$\hat{f}\big[\vec{\chi}(f)\big] = \vec{\chi}(f). \tag{5.6}$$

Conversely, if $h : \mathbb{R}^S \to \mathbb{R}^S$ is a null-range topical function and $\vec{\mathbf{v}} \in \mathbb{R}^S$ a fixed point of h, one can find $f : \mathbb{R}^S \to \mathbb{R}^S$ uniformly topical such that $\hat{f} = h$ and $\vec{\chi}(f) = \vec{\mathbf{v}}$.

In other words, Formula (5.6) is the only constraint relating $\hat{f}$ and $\vec{\chi}(f)$.

Proof. Formula (5.6) is obtained by putting $f = g = \pi$ in (5.5). For the second statement, we note that the function $f : \mathbf{x} \mapsto h(\mathbf{x}) + \vec{\mathbf{v}}$ satisfies $f^n(\mathbf{o}) = \mathbf{o} + n\vec{\mathbf{v}}$ for all $n \geqslant 0$, and thus answers the problem. □

5.3 The selection principle

Proposition 5.14 (Selection principle). *Let $f_1, \ldots, f_n, g : \mathbb{R}^S \to \mathbb{R}^S$ be uniformly topical functions such that, for all $\mathbf{x} \in \mathbb{R}^S$, there exists $i \in [1, n]$ such that $f_i(\mathbf{x}) = g(\mathbf{x})$. Then there exists $i \in [1, n]$ such that $\vec{\chi}(f_i) = \vec{\chi}(g)$.*

Proof. The set $\bigcup_{i=1}^{n} \big\{\mathsf{p} \in \mathrm{Cf}(S) : (f_i - g)(\mathsf{p}) = 0\big\}$ is closed in $\mathrm{Cf}(S)$, as a finite union of closed sets; it contains $\mathbb{R}^S/\mathbb{R}\vec{\mathbf{u}}$ by hypothesis, and therefore coincides with the whole space $\mathrm{Cf}(S)$. In particular, for p a stable fixed point of $\tilde{g}$, there exists $i \in [1, n]$ such that $(f_i - g)(\mathsf{p}) = 0$. It implies $(f_i - \mathrm{Id})(\mathsf{p}) = (g - \mathrm{Id})(\mathsf{p})$, so p is a stable fixed point of $\tilde{f}_i$, with $\vec{\chi}(f_i) = \vec{\chi}(g)$. □

Corollary 5.15. *Let $f_1, f_2 : \mathbb{R}^S \to \mathbb{R}^S$ be two uniformly topical functions such that $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are comparable for all $\mathbf{x} \in \mathbb{R}^S$. Then $\vec{\chi}(f_1)$ and $\vec{\chi}(f_2)$ are comparable, and*

$$\vec{\chi}(f_1 \vee f_2) = \vec{\chi}(f_1) \vee \vec{\chi}(f_2), \quad \vec{\chi}(f_1 \wedge f_2) = \vec{\chi}(f_1) \wedge \vec{\chi}(f_2).$$

Proof. Applying the selection principle with f_1, f_2 and $g = f_1 \vee f_2$, we see that $\vec{\chi}(g)$ is equal to $\vec{\chi}(f_1)$ or $\vec{\chi}(f_2)$. On the other hand, we obviously have $\vec{\chi}(g) \geqslant \vec{\chi}(f_1)$ and $\vec{\chi}(g) \geqslant \vec{\chi}(f_2)$, so the spectral vectors $\vec{\chi}(f_1)$ and $\vec{\chi}(f_2)$ are comparable, and $\vec{\chi}(g)$ is the greater one. This proves the first formula; the second one is similar. □

Corollary 5.16. *Let G be a subset of $\mathbb{R}^S \times \mathbb{R}^S$ such that $\{\mathbf{y} \in \mathbb{R}^S : (\mathbf{x}, \mathbf{y}) \in G\}$ is totally ordered for all $\mathbf{x} \in \mathbb{R}^S$, and let $\mathcal{F}$ be the set of all uniformly topical functions $\mathbb{R}^S \to \mathbb{R}^S$ whose graph is contained in G. The set $\mathcal{F}$ is a lattice, and the function $\vec{\chi} : \mathcal{F} \to \mathbb{R}^S$ is a lattice morphism, whose range $\vec{\chi}(\mathcal{F})$ is totally ordered.*

This result is typically used as follows. Let $g : \mathbb{R}^S \to \mathbb{R}^S$ be a uniformly topical function, $s \in S$ some distinguished index, and G the set of all $(\mathbf{x}, \mathbf{y})$ such that $\gamma_i(\mathbf{y}) = g_i(\mathbf{x})$ for all $i \neq s$. Then $\mathcal{F}$ consists of all uniformly topical functions $f : \mathbb{R}^S \to \mathbb{R}^S$ such that $f_i = g_i$ for all $i \neq s$, whereas the component f_s can be anything (so $\mathcal{F}$ is isomorphic to $\mathbf{UTp}(\mathbb{R}^S, \mathbb{R})$ as an ordered set). The corollary then claims that the function $f_s \mapsto \vec{\chi}(f)$ is a lattice morphism from $\mathbf{UTp}(\mathbb{R}^S, \mathbb{R})$ to $\mathbb{R}^S$. This statement is essentially equivalent to the "duality theorem" by Gaubert and Gunawardena about "rectangular" families of min-max-plus functions [Gu2, Gu3, GG2].

6 Multiplicative ergodic theorems

6.1 Statement of the problem

Consider a probability space $(\Omega, \mathcal{T}, \mu)$ with a measurable endofunction $\tau : \Omega \to \Omega$ which preserves μ.

Consider a function $u : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ such that $u_\omega : \mathbb{R}^A \to \mathbb{R}^B$ is uniformly topical for all $\omega \in \Omega$. It can equivalently be represented by a function $\bar{u} : \Omega \times \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$ (since every u_ω has a continuous extension $\bar{u}_\omega : \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$), or by a function $\dot{u} : \Omega \to \mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$.

First we observe that the following properties are equivalent:
(i) the function $u : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ is measurable;
(ii) the functions $\omega \mapsto u_\omega(\mathbf{x})$ are measurable for all $\mathbf{x} \in \mathbb{R}^A$;
(iii) the function $\bar{u} : \Omega \times \overline{\mathbb{R}}^A \to \overline{\mathbb{R}}^B$ is measurable;
(iv) the functions $\omega \mapsto \bar{u}_\omega(\mathbf{x})$ are measurable for all $\mathbf{x} \in \overline{\mathbb{R}}^A$;
(v) the function $\dot{u} : \Omega \to \mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$ is measurable, when $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$ is endowed with the Borel σ-algebra associated with the topology of uniform convergence.

The equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) $\iff$ (v) result from the general theory of Carathéodory functions, see [AB], §4.10. Obviously (iv) $\Longrightarrow$ (ii), whereas (ii) $\Longrightarrow$ (iv) follows from the fact that, for all $\mathbf{p} \in \overline{\mathbb{R}}^A$, the function $\omega \mapsto \bar{u}_\omega(\mathbf{p})$ is the pointwise limit of the functions $\omega \mapsto u_\omega(\mathbf{x})$ when $\mathbf{x} \in \mathbb{R}^A$ tends to $\mathbf{p}$.

If u satisfies any of the above measurability conditions, we shall say that u is a *stochastic uniformly topical function.*

Semimetrics. Let $p \in [1, \infty]$. If $u, v : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ are any two stochastic uniformly topical functions, we define $D_p(u, v)$ as the L^p-norm of the uniform distance $d(u_\omega, v_\omega)$, qua function of ω, i.e.

$$D_p(u,v) = \begin{cases} \left[\int_\Omega d(u_\omega, v_\omega)^p \, \mu\right]^{1/p} & \text{if } p < \infty, \\ \operatorname*{sup\,ess}_{\omega \in \Omega} d(u_\omega, v_\omega) & \text{if } p = \infty. \end{cases}$$

These D_p are obviously semimetrics, i.e. $D_p(u, u) = 0$, $D_p(u, v) = D_p(v, u)$ and $D_p(u, w) \leqslant D_p(u, v) + D_p(v, w)$. They are "almost" distances, in the sense that $D_p(u, v) = 0$ if and only if $u_\omega = v_\omega$ for almost all $\omega \in \Omega$.

Ergodic products. Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. For all $n \in \mathbb{N}$, define $u^n_\omega : \mathbb{R}^S \to \mathbb{R}^S$ as the composition

$$u^n_\omega = u_\omega \circ u_{\tau\omega} \circ \cdots \circ u_{\tau^{n-1}\omega}$$

so that $u^0_\omega = \mathrm{Id}$ and $u^{m+n}_\omega = u^m_\omega \circ u^n_{\tau^m\omega}$ for all $m, n \in \mathbb{N}$. As in the deterministic case, we would like to know if the sequence $u^n_\omega(\mathbf{o})/n$ converges in $\mathbb{R}^S$ when $n \to \infty$, for almost every $\omega \in \Omega$. If this is the case, we shall say that u has a *stochastic spectral vector*, which is the function $\vec{\chi}_u : \Omega \to \mathbb{R}^S$, almost everywhere defined by

$$\vec{\chi}_u(\omega) = \lim_{n\to\infty} \frac{u^n_\omega(\mathbf{p}) - \mathbf{p}}{n} \tag{6.1}$$

where $\mathbf{p}$ is an arbitrary point of $\mathbb{R}^S$. (Here again, the existence of the limit, and its value, do not depend on the choice of $\mathbf{p}$.) This is automatically a measurable function, as an a.e. limit of measurable functions (more precisely, it is the restriction of a measurable function to a full-measure subset of Ω).

From the formula $u_\omega^{n+1} = u_\omega \circ u_{\tau\omega}^n$, we deduce that the stochastic spectral vector $\vec{\chi}_u$, if it exists, has to satisfy

$$\vec{\chi}_u(\omega) = \hat{u}_\omega\big[\vec{\chi}_u(\tau\omega)\big] \tag{6.2}$$

for a.e. $\omega \in \Omega$, which can be seen as a stochastic analogue of Formula (5.6).

The goal of this chapter is to prove the following theorem.

Theorem 6.1. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. If $D_1(u, \hat{u}) = \int_\Omega d(u_\omega, \hat{u}_\omega)\, \mu$ is finite, then u has a stochastic spectral vector, that is, the limit* (6.1) *exists (in $\mathbb{R}^S$) for almost every $\omega \in \Omega$.*

We recall the following well-known result (see [Par], §1.2, ex. 16, p. 27).

Lemma 6.2. *If $f \in L^1(\Omega)$, then $f(\tau^n\omega)/n \to 0$ when $n \to \infty$, for almost every $\omega \in \Omega$.*

Proof. Let $\epsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$, let $E_n \subseteq \Omega$ be the set of all ω such that $n\epsilon \leqslant |f(\omega)| < (n+1)\epsilon$. Let us also define $F_n = \bigsqcup_{m \geqslant n} E_m = \{\omega : |f(\omega)| \geqslant n\epsilon\}$ and $G_n = \{\omega : \tau^n\omega \in F_n\}$. Then

$$\begin{aligned} \|f\|_1 &= \int_\Omega |f|\,\mu \;=\; \sum_{n\in\mathbb{N}} \int_{E_n} |f|\,\mu \\ &\geqslant \epsilon \sum_{n\in\mathbb{N}} n\,\mu(E_n) \;=\; \epsilon \sum_{n\geqslant 1} \mu(F_n) \;=\; \epsilon \sum_{n \geqslant 1} \mu(G_n), \end{aligned}$$

which shows, by the Borel-Cantelli lemma, that for almost every $\omega \in \Omega$ the set $\{n : \omega \in G_n\}$ is finite, i.e., $|f(\tau^n\omega)| < n\epsilon$ when n is big enough.

Thus $\limsup_{n\to\infty} n^{-1}\,|f(\tau^n\omega)| \leqslant \epsilon$ almost everywhere, for any $\epsilon > 0$, hence the result. □

Proposition 6.3. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. Suppose that, for every $\epsilon > 0$, there exists a stochastic uniformly topical function $v : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ such that $D_1(u, v) \leqslant \epsilon$ and having a stochastic spectral vector. Then u has a stochastic spectral vector.*

Proof. Define $\vec{\chi}_-(\omega) = \liminf_{n\to\infty} u_\omega^n(\mathbf{o})/n$ and $\vec{\chi}_+(\omega) = \limsup_{n\to\infty} u_\omega^n(\mathbf{o})/n$, with the lower and upper limits computed coordinate-wise. Define also $\vec{\chi}_v(\omega) = \lim_{n\to\infty} v_\omega^n(\mathbf{o})/n$ and $\eta(\omega) = d(u_\omega, v_\omega)$. From the formula

$$\begin{aligned} d(u_\omega^{n+1}, v_\omega^{n+1}) &= d(u_\omega u_{\tau\omega}^n, v_\omega v_{\tau\omega}^n) \\ &\leqslant d(u_\omega, v_\omega) + d(u_{\tau\omega}^n, v_{\tau\omega}^n) \end{aligned}$$

we deduce $d(u_\omega^n, v_\omega^n) \leqslant S_n\eta(\omega) = \sum_{k=0}^{n-1} \eta(\tau^k\omega)$ for all $n \geqslant 0$ and $\omega \in \Omega$, and therefore

$$\frac{v_\omega^n(\mathbf{o})}{n} - \frac{S_n\eta(\omega)}{n} \;\leqslant\; \frac{u_\omega^n(\mathbf{o})}{n} \;\leqslant\; \frac{v_\omega^n(\mathbf{o})}{n} + \frac{S_n\eta(\omega)}{n}$$

By Birkhoff's theorem, $S_n\eta(\omega)/n$ converges a.e. to $\eta'(\omega) = \mathbb{E}(\eta \mid \mathcal{T}')$, where $\mathcal{T}'$ is the σ-algebra of τ-invariant $\mathcal{T}$-measurable sets. The function η' is L^1, in particular it is a.e. finite. Taking the lower and upper limits in the above inequalities, we get

$$\vec{\chi}_v(\omega) - \eta'(\omega) \;\leqslant\; \vec{\chi}_-(\omega) \;\leqslant\; \vec{\chi}_+(\omega) \;\leqslant\; \vec{\chi}_v(\omega) + \eta'(\omega)$$

almost everywhere. This shows that $\vec{\chi}_-$ and $\vec{\chi}_+$ are a.e. finite. Moreover $\|\vec{\chi}_+ - \vec{\chi}_-\|_1 \leqslant 2\mathbb{E}(\eta') \leqslant 2\epsilon$, with $\epsilon > 0$ arbitrarily small, hence $\vec{\chi}_+ = \vec{\chi}_-$ a.e. □

Proposition 6.4. *Let $u, v : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be two stochastic uniformly topical functions, having stochastic spectral vectors $\vec{\chi}_u$ and $\vec{\chi}_v$. For all $1 \leqslant p \leqslant \infty$,*

$$\|\vec{\chi}_v - \vec{\chi}_u\|_p \ \leqslant \ D_p(u, v). \tag{6.3}$$

Proof. With the notations of the previous proof, we have $\|v^n_\omega(\mathbf{o})/n - u^n_\omega(\mathbf{o})/n\| \leqslant S_n\eta(\omega)/n$ which gives us $\|\vec{\chi}_v(\omega) - \vec{\chi}_u(\omega)\| \leqslant \eta'(\omega)$ a.e., when taking the limit $n \to \infty$. Since $\|\eta'\|_p \leqslant \|\eta\|_p = D_p(u, v)$, the result follows. □

Proposition 6.5. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function with a stochastic spectral vector $\vec{\chi}_u$, and let $\zeta(\omega) = \|u_\omega(\mathbf{o})\| = d(u_\omega(\mathbf{o}), \mathbf{o})$. For all $1 \leqslant p \leqslant \infty$,*

$$\|\vec{\chi}_u\|_p \ \leqslant \ \|\zeta\|_p \ \leqslant \ D_p(u, \hat{u}). \tag{6.4}$$

If we also assume that $p < \infty$ and $\|\zeta\|_p < \infty$, then the sequence of functions $u^n_\omega(\mathbf{o})/n$ converges to $\vec{\chi}_u$ in L^p norm.

Proof. The inequalities $d(u^n_\omega(\mathbf{o}), \mathbf{o}) \leqslant S_n\zeta(\omega)$ yield $\|u^n_\omega(\mathbf{o})/n\|_p \leqslant \|\zeta\|_p$ and consequently $\|\vec{\chi}_u\|_p \leqslant \|\zeta\|_p$ by Fatou's lemma.
We shall now assume that $1 \leqslant p < \infty$, and prove that the norm $\|u^n_\omega(\mathbf{o})/n - \vec{\chi}_u\|_p$ tends to 0. Let $\epsilon > 0$ be arbitrary. One can find $L \geqslant 0$ such that $\int_{\zeta(\omega)>L} \zeta(\omega)^p \leqslant \epsilon^p$. Now define

$$v_\omega(\mathbf{x}) = \begin{cases} u_\omega(\mathbf{x}) & \text{if } \zeta(\omega) \leqslant L, \\ u_\omega(\mathbf{x}) - u_\omega(\mathbf{o}) & \text{otherwise,} \end{cases}$$

so that

$$d(u_\omega, v_\omega) = \xi(\omega) = \begin{cases} 0 & \text{if } \zeta(\omega) \leqslant L, \\ \zeta(\omega) & \text{otherwise.} \end{cases}$$

We have $\|v_\omega(\mathbf{o})\| \leqslant L$ for all ω, and consequently $\|v^n_\omega(\mathbf{o})\| \leqslant nL$ for all n, ω. Define

$$\vec{\chi}_-(\omega) = \liminf_{n\to\infty} \frac{v^n_\omega(\mathbf{o})}{n} = \lim_{n\to\infty} \vec{\chi}^n_-(\omega), \qquad \vec{\chi}_+(\omega) = \limsup_{n\to\infty} \frac{v^n_\omega(\mathbf{o})}{n} = \lim_{n\to\infty} \vec{\chi}^n_+(\omega),$$

where $\vec{\chi}^n_-(\omega) = \inf\limits_{m\geqslant n} v^m_\omega(\mathbf{o})/m$ and $\vec{\chi}^n_+(\omega) = \sup\limits_{m\geqslant n} v^m_\omega(\mathbf{o})/m$. Define also $\vec{\chi}_{\text{av}} = (\vec{\chi}_+ + \vec{\chi}_-)/2$ and $\vec{\chi}^n_{\text{av}} = (\vec{\chi}^n_+ + \vec{\chi}^n_-)/2$. From the inequalities $d(u^n_\omega, v^n_\omega) \leqslant S_n\xi(\omega)$ we get

$$\frac{u^n_\omega(\mathbf{o})}{n} - \frac{S_n\xi(\omega)}{n} \ \leqslant \ \frac{v^n_\omega(\mathbf{o})}{n} \ \leqslant \ \frac{u^n_\omega(\mathbf{o})}{n} + \frac{S_n\xi(\omega)}{n}$$

and, taking the limit,

$$\vec{\chi}_u(\omega) - \xi'(\omega) \ \leqslant \ \vec{\chi}_-(\omega) \ \leqslant \ \vec{\chi}_+(\omega) \ \leqslant \ \vec{\chi}_u(\omega) + \xi'(\omega)$$

almost everywhere, with $\xi' = \mathbb{E}(\xi \mid \mathcal{T}')$. In particular $\|\vec{\chi}_{\text{av}} - \vec{\chi}_u\|_p \leqslant \|\xi'\|_p \leqslant \|\xi\|_p \leqslant \epsilon$, and $\|\vec{\chi}_+ - \vec{\chi}_-\|_p \leqslant 2\,\|\xi'\|_p \leqslant 2\epsilon$.
The functions $\vec{\chi}^n_\pm$ are uniformly bounded, and converge to $\vec{\chi}_\pm$ almost everywhere, so they also converge in L^p norm. Let $R \geqslant 1$ be such that $\left\|\vec{\chi}^R_- - \vec{\chi}_-\right\|_p \leqslant \epsilon$ and $\left\|\vec{\chi}^R_+ - \vec{\chi}_+\right\|_p \leqslant \epsilon$. This implies $\left\|\vec{\chi}^R_+ - \vec{\chi}^R_-\right\|_p \leqslant 4\epsilon$ and $\left\|\vec{\chi}^R_{\text{av}} - \vec{\chi}_u\right\|_p \leqslant 2\epsilon$.
For all $n \geqslant R$, we have $\vec{\chi}^R_-(\omega) \leqslant v^n_\omega(\mathbf{o})/n \leqslant \vec{\chi}^R_+(\omega)$ and consequently

$$\vec{\chi}^R_-(\omega) - \frac{S_n\xi(\omega)}{n} \ \leqslant \ \frac{u^n_\omega(\mathbf{o})}{n} \ \leqslant \ \vec{\chi}^R_+(\omega) + \frac{S_n\xi(\omega)}{n}$$

hence

$$\left\| \frac{u_\omega^n(\mathbf{o})}{n} - \vec{\chi}_{\mathrm{av}}^R(\omega) \right\| \;\leqslant\; \left\| \frac{\vec{\chi}_+^R(\omega) - \vec{\chi}_-^R(\omega)}{2} \right\| + \frac{S_n\xi(\omega)}{n}$$

which yields $\left\| u_\omega^n(\mathbf{o})/n - \vec{\chi}_{\mathrm{av}}^R \right\|_p \leqslant 3\epsilon$. Combining this with $\left\| \vec{\chi}_{\mathrm{av}}^R - \vec{\chi}_u \right\|_p \leqslant 2\epsilon$, we obtain $\| u_\omega^n(\mathbf{o})/n - \vec{\chi}_u \|_p \leqslant 5\epsilon$, for all $n \geqslant R$. This actually shows that $u_\omega^n(\mathbf{o})/n$ converges in L^p norm to the stochastic spectral vector. $\square$

Proposition 6.6. *Let $v : \Omega \times \mathbb{R}^A \to \mathbb{R}^A$, $u : \Omega \times \mathbb{R}^B \to \mathbb{R}^B$ and $h : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ be stochastic uniformly topical functions such that $h_\omega v_\omega = u_\omega h_{\tau\omega}$ for a.e. $\omega \in \Omega$. Suppose also that $\int \|h_\omega(\mathbf{o})\| \, \mu < \infty$ and v has a stochastic spectral vector $\vec{\chi}_v$. Then u has a stochastic spectral vector, given by $\vec{\chi}_u(\omega) = \hat{h}_\omega\big[\vec{\chi}_v(\omega)\big]$.*

Proof. An easy calculation gives $h_\omega v_\omega^n = u_\omega^n h_{\tau^n\omega}$ for all $n \in \mathbb{N}$ and almost every $\omega \in \Omega$, and in particular

$$h_\omega\big[v_\omega^n(\mathbf{o})\big] = u_\omega^n\big[h_{\tau^n\omega}(\mathbf{o})\big]. \tag{6.5}$$

For a.e. $\omega \in \Omega$, we have $v_\omega^n(\mathbf{o}) = n\vec{\chi}_v(\omega) + o(n)$, and $h_{\tau^n\omega}(\mathbf{o}) = \mathbf{o} + o(n)$ by applying Lemma 6.2 to the function $f(\omega) = d\big[\mathbf{o}, h_\omega(\mathbf{o})\big]$. Substituting these estimates into (6.5), we get $u_\omega^n(\mathbf{o}) = n\hat{h}_\omega\big[\vec{\chi}_v(\omega)\big] + o(n)$ for almost every $\omega \in \Omega$, q.e.d. $\square$

6.2 Stochastic products of max-plus functions

Theorem 6.7. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. Suppose that $u_\omega : \mathbb{R}^S \to \mathbb{R}^S$ is max-plus for all ω, and $\int \|u_\omega(\mathbf{o})\| \, \mu < \infty$. Then u has a stochastic spectral vector.*

It should be noted that the integrability condition which appears here is slightly weaker than the one required in the general case; so the above result is more than a special case of Theorem 6.1. It is also interesting because its proof is very simple.

Proof. Let $u_{ij}(\omega) \in [-\infty, +\infty)$ be the elements of the max-plus matrix of u_ω (no line being identically $-\infty$), so that we can write $u_\omega : \mathbf{x} \mapsto \mathbf{y}$, with $y_i = \max_{j \in S}(u_{ij}(\omega) + x_j)$ for all $i, j \in S$. Saying that u is measurable means that all the functions $u_{ij} : \Omega \to [-\infty, +\infty)$ are measurable. Define $\zeta(\omega) = \|u_\omega(\mathbf{o})\| = \max_i \big|\max_j u_{ij}(\omega)\big|$.

For any $\kappa > 0$, let $v : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be the stochastic uniformly topical function defined by $v_\omega : \mathbf{x} \mapsto \mathbf{y}$, where

$$\exp \kappa y_i \;=\; \sum_{j \in S} \exp \kappa\big(u_{ij}(\omega) + x_j\big)$$

Note that $u_\omega \leqslant v_\omega \leqslant \kappa^{-1}\log(N) + u_\omega$. The point $v_\omega^n(\mathbf{o}) = \mathbf{x}^n(\omega) = \big(x_1^n(\omega), \ldots, x_N^n(\omega)\big)$ is given by

$$\begin{pmatrix} \exp \kappa x_1^n(\omega) \\ \vdots \\ \exp \kappa x_N^n(\omega) \end{pmatrix} = A(\omega)A(\tau\omega)\cdots A(\tau^{n-1}\omega) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

where $A(\omega)$ is the matrix with coefficients $A_{ij}(\omega) = \exp \kappa u_{ij}(\omega)$. Thus, for all $s \in S$,

$$\begin{aligned} \exp \kappa x_s^n(\omega) &= \big\langle \vec{\mathbf{e}}_s^*, A(\omega)A(\tau\omega)\cdots A(\tau^{n-1}\omega)\vec{\mathbf{u}} \big\rangle \\ &= \big\langle \vec{\mathbf{u}}^*, A^*(\tau^{n-1}\omega)\cdots A^*(\tau\omega)A^*(\omega)\vec{\mathbf{e}}_s \big\rangle \\ &= \mathcal{N}\big[A^*(\tau^{n-1}\omega)\cdots A^*(\tau\omega)A^*(\omega)\vec{\mathbf{e}}_s\big] \end{aligned}$$

for the norm $\mathcal{N}(\vec{\mathbf{v}}) = \sum |v_i|$. On the other hand

$$\|A^*(\omega)\| = \max_{i,j\in S} |A_{ij}(\omega)| = \exp\ \kappa \max_{i,j\in S} u_{ij}(\omega) = \exp\kappa\,\lceil \mathbf{o}, u_\omega(\mathbf{o})\rceil \leqslant \exp\kappa\zeta(\omega)$$

so $\log^+ \|A^*(\omega)\|$ is summable, which implies, by the Oseledets-Raghunathan multiplicative ergodic theorem [Ose, Rag], that the sequence $x_s^n(\omega)/n$ has a limit in $[-\infty, +\infty)$ for μ-almost every $\omega \in \Omega$. This limit is actually a.e. finite, because $v_\omega \geqslant u_\omega$ and therefore

$$\left\lfloor \mathbf{o}, \frac{v_\omega^n(\mathbf{o})}{n} \right\rfloor \geqslant \left\lfloor \mathbf{o}, \frac{u_\omega^n(\mathbf{o})}{n} \right\rfloor \geqslant \frac{1}{n}\sum_{i=0}^{n-1} \lfloor \mathbf{o}, u_{\tau^i\omega}(\mathbf{o})\rfloor \geqslant -\frac{1}{n}\sum_{i=0}^{n-1}\zeta(\tau^i\omega)$$

and the rightmost expression converges a.e. to $\mathbb{E}(-\zeta \mid \mathcal{T}')$, which is a.e. finite.

So v has a stochastic spectral vector, for all $\kappa > 0$. When $\kappa \to \infty$, the semi-distance $D_1(u,v) \leqslant D_\infty(u,v) \leqslant \kappa^{-1}\log N$ tends to zero, which proves, by Proposition 6.3, that u has a stochastic spectral vector. $\square$

6.3 Proof of Theorem 6.1

6.3.1 The norms $\mathcal{N}_K$

Recall that a *stochastic* matrix is a matrix with positive elements, whose sum in every line is equal to 1. A matrix will be called *co-stochastic* when its transpose is stochastic.

Lemma 6.8. *Let $K \geqslant 1$, and $A = (a_{ij})$ be a co-stochastic $N\times N$ matrix such that $a_{ii} \geqslant 1/K$ and $a_{ij} \leqslant K a_{ji}$ for all $i,j \in [1,N]$. Let $\mathcal{P}_N(K)$ be the convex hull of the $N!$ points with coordinates $(1/Z, K/Z, \ldots, K^{N-1}/Z)$ up to a permutation, where $Z = 1 + K + \cdots + K^{N-1}$. The set $\mathcal{P}_N(K)$ is stable under A, i.e. $A\mathbf{x} \in \mathcal{P}_N(K)$ for all $\mathbf{x} \in \mathcal{P}_N(K)$.*

The permutahedron $\mathcal{P}_N(K)$ is clearly contained, and relatively compact, in the simplex defined by $x_i > 0$, $\sum x_i = 1$, as illustrated by Figure 8 (for $N = 3$ and $K = 2$).

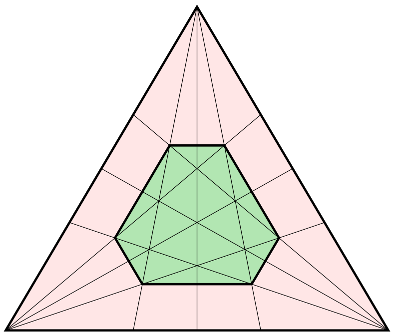

Figure 8: The set $\mathcal{P}_3(2)$, inside the unit simplex

Proof. Let $\mathbf{x}$ be the point with coordinates $x_i = K^{i-1}/Z$, for $1 \leqslant i \leqslant N$. Because of the symmetries of the problem, it is sufficient to prove that $\mathbf{y} = A\mathbf{x}$ belongs to $\mathcal{P}_N(K)$. First note that

$$y_1 + \cdots + y_N = x_1 + \cdots + x_N \tag{6.6}$$

since A is co-stochastic. Besides, we claim that

$$y_1 + \cdots + y_s \geqslant x_1 + \cdots + x_s \tag{6.7}$$

for all $0 \leqslant s \leqslant N$. Indeed,

$$\begin{aligned}\sum_{i\leqslant s} y_i &= \sum_{i\leqslant s}\sum_j a_{ij}x_j = \sum_j x_j\Big(\sum_{i\leqslant s} a_{ij}\Big)\\ &= \sum_{j\leqslant s} x_j\Big(1-\sum_{i>s} a_{ij}\Big) + \sum_{j>s} x_j\Big(\sum_{i\leqslant s} a_{ij}\Big)\end{aligned}$$

and consequently,

$$\begin{aligned}\sum_{i\leqslant s} y_i - \sum_{i\leqslant s} x_i &= \sum_{i\leqslant s}\sum_{j>s} a_{ij}x_j - \sum_{j\leqslant s}\sum_{i>s} a_{ij}x_j\\ &= \sum_{i\leqslant s}\sum_{j>s} a_{ij}x_j - \sum_{i\leqslant s}\sum_{j>s} a_{ji}x_i\\ &= \sum_{i\leqslant s}\sum_{j>s} x_i(a_{ij}K^{j-i} - a_{ji})\end{aligned}$$

and this last expression is obviously positive, since $a_{ij} \geqslant a_{ji}/K$; this proves (6.7).

Now, let $1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant N$ be any sequence of s distinct indices, sorted by increasing order. As the sequence $i_k - k$ is increasing, there exists $t \in [0, s]$ such that $i_k = k$ for all $k \leqslant t$ and $i_k \geqslant k+1$ for all $k > t$. Note that $y_i \geqslant a_{ii}x_i \geqslant x_i/K$ for all i. Combining this with Inequality (6.7), we obtain

$$\begin{aligned}y_{i_1} + \cdots + y_{i_s} &= (y_1 + \cdots + y_t) + (y_{i_{t+1}} + \cdots + y_{i_s})\\ &\geqslant (x_1 + \cdots + x_t) + K^{-1}(x_{i_{t+1}} + \cdots + x_{i_s})\\ &\geqslant (x_1 + \cdots + x_t) + K^{-1}(x_{(t+1)+1} + \cdots + x_{s+1})\\ &= (x_1 + \cdots + x_t) + (x_{t+1} + \cdots + x_s)\\ &= x_1 + \cdots + x_s\end{aligned}$$

By the Hardy-Littlewood-Polya Theorem, these inequalities, combined with (6.6), show that $\mathbf{y}$ is indeed in $\mathcal{P}_N(K)$. ☐

Fix $K \geqslant 1$, and let $Z = 1 + K + \cdots + K^{N-1}$. To each vector $\vec{\mathbf{v}}$, associate the number

$$\mathcal{N}_K(\vec{\mathbf{v}}) = \max_{\sigma\in\mathfrak{S}(N)} Z^{-1}\sum_{i=1}^{N} K^{\sigma(i)-1}\,|v_i| \tag{6.8}$$

Obviously $\mathcal{N}_K : \mathbb{R}^N \to \mathbb{R}$ is a norm on $\mathbb{R}^N$; we have inserted the factor Z^{-1} so that $\vec{\mathbf{u}} = (1, \ldots, 1)$ has norm 1, but this is not essential.

By construction, this norm is invariant under coordinate permutations. Assuming that the coordinates are numbered so that $|v_1| \leqslant |v_2| \leqslant \cdots \leqslant |v_N|$, the norm is given by:

$$\mathcal{N}_K(\vec{\mathbf{v}}) = \frac{|v_1| + K\,|v_2| + \cdots + K^{N-1}\,|v_N|}{Z} \tag{6.9}$$

It is also a lattice norm, i.e. $|\vec{\mathbf{v}}| \leqslant |\vec{\mathbf{w}}|$ implies $\mathcal{N}_K(\vec{\mathbf{v}}) \leqslant \mathcal{N}_K(\vec{\mathbf{w}})$. In particular, $\vec{\mathbf{v}}$ and $|\vec{\mathbf{v}}|$ have the same norm; see [AB], §8.1.

Lemma 6.9. *Let A be a stochastic $N \times N$ matrix such that $a_{ii} \geqslant 1/K$ and $a_{ij} \leqslant K a_{ji}$ for all i, j. Then A decreases the norm $\mathcal{N}_K$, i.e. $\mathcal{N}_K(A\vec{\mathbf{v}}) \leqslant \mathcal{N}_K(\vec{\mathbf{v}})$ for all $\vec{\mathbf{v}} \in \mathbb{R}^N$.*

Proof. We treat the case $\vec{\mathbf{v}} \geqslant 0$ first. For any permutation $\sigma \in \mathfrak{S}(N)$, let $\ell(\sigma)$ be the linear form with coordinates $\ell(\sigma)_i = K^{\sigma(i)-1}/Z$. We can write

$$\mathcal{N}_K(\vec{\mathbf{v}}) = \sup_{\sigma\in\mathfrak{S}(N)} \langle \ell(\sigma), \vec{\mathbf{v}}\rangle = \sup_{\ell\in\mathcal{P}_N(K)} \langle \ell, \vec{\mathbf{v}}\rangle$$

for all $\vec{\mathbf{v}} \geqslant 0$. In particular,

$$\begin{aligned}\mathcal{N}_K(A\vec{\mathbf{v}}) &= \sup_{\ell\in\mathcal{P}_N(K)} \langle \ell, A\vec{\mathbf{v}}\rangle = \sup_{\ell\in\mathcal{P}_N(K)} \langle A^*\ell, \vec{\mathbf{v}}\rangle = \sup_{\ell\in A^*[\mathcal{P}_N(K)]} \langle \ell, \vec{\mathbf{v}}\rangle \\ &\leqslant \sup_{\ell\in\mathcal{P}_N(K)} \langle \ell, \vec{\mathbf{v}}\rangle = \mathcal{N}_K(\vec{\mathbf{v}})\end{aligned}$$

since $A^*\big[\mathcal{P}_N(K)\big] \subseteq \mathcal{P}_N(K)$ by Lemma 6.8.

If $\vec{\mathbf{v}}$ is no longer supposed positive, we proceed as follows. Since A is positive, $|A\vec{\mathbf{v}}| \leqslant A\,|\vec{\mathbf{v}}|$, implying $\mathcal{N}_K(A\vec{\mathbf{v}}) \leqslant \mathcal{N}_K(A\,|\vec{\mathbf{v}}|)$ since $\mathcal{N}_K$ is a lattice norm. Combining this with $\mathcal{N}_K(A\,|\vec{\mathbf{v}}|) \leqslant \mathcal{N}_K(|\vec{\mathbf{v}}|) = \mathcal{N}_K(\vec{\mathbf{v}})$, we obtain the desired inequality. □

Lemma 6.10. *Let $K_1, \dots, K_N \geqslant 1$ and $r_1, \dots, r_N \geqslant 0$, with the K_i being all distinct. The system of equations $\mathcal{N}_{K_1}(\vec{\mathbf{x}}) = r_1$, …, $\mathcal{N}_{K_N}(\vec{\mathbf{x}}) = r_N$ has a finite number of solutions in $\mathbb{R}^N$, not exceeding $2^N N!$.*

Proof. Because of the symmetries of the problem, it is sufficient to find the positive solutions $\vec{\mathbf{x}}$ with $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$; the other solutions are then obtained by permuting the coordinates or changing their signs. For such a vector, $\mathcal{N}_K(\vec{\mathbf{x}}) = Z^{-1}(x_1 + Kx_2 + \cdots + K^{N-1}x_N)$, so we have to solve the linear system

$$\begin{pmatrix} 1 & K_1 & K_1^2 & \cdots & K_1^{N-1} \\ 1 & K_2 & K_2^2 & \cdots & K_2^{N-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & K_N & K_N^2 & \cdots & K_N^{N-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} Z_1 r_1 \\ Z_2 r_2 \\ \vdots \\ Z_N r_N \end{pmatrix}$$

This system has a non-zero determinant, by the Vandermonde formula, and therefore has at most one solution in the region being considered. By permuting the coordinates and changing the signs, we can obtain at most $2^N N!$ solutions in the whole space. □

Lemma 6.11. *Let $(\mathbf{x}_n)_{n\geqslant 0}$ be a sequence of points in $\mathbb{R}^N$, whose differences $\mathbf{x}_{n+1} - \mathbf{x}_n$ are bounded. Assume that there exist at least N different values of K such that the sequence $\mathcal{N}_K(\mathbf{x}_n - \mathbf{x}_0)/n$ is convergent. Then the sequence $(\mathbf{x}_n - \mathbf{x}_0)/n$ is convergent.*

Proof. Define $\vec{\mathbf{v}}_n = (\mathbf{x}_n - \mathbf{x}_0)/n$, for all $n \geqslant 1$. By our hypotheses, this sequence is bounded, so we have to prove that the set A of all its limit points, is a singleton.

The sequence $\vec{\mathbf{v}}_n$ also satisfies $\vec{\mathbf{v}}_{n+1} - \vec{\mathbf{v}}_n \to 0$, which implies, as is well-known, that A is connected.

Let $K_1, \dots, K_N$ be N different values of K, such that all the sequences $\mathcal{N}_{K_i}(\vec{\mathbf{v}}_n)$ converge, and let $r_1, \dots, r_N$ be their limits. For all $\vec{\mathbf{x}} \in A$, one has simultaneously $\mathcal{N}_{K_i}(\vec{\mathbf{x}}) = r_i$ for all $i = 1, \dots, N$. By the previous lemma, A must be finite. As it must also be connected, it is indeed a singleton. □

6.3.2 The case of $\hat{u}_\omega$ permutations

A topical function $f : \mathbb{R}^A \to \mathbb{R}^B$ is Lipschitz, so its partial derivatives $\partial f_i/\partial x_j$, in the sense of distributions, are L^∞ functions; moreover, f is almost everywhere differentiable, and its derivative in the usual sense coincides with its derivative in the sense of distributions. We have the relations

$$0 \leqslant \frac{\partial f_i}{\partial x_j} \leqslant 1, \qquad \sum_{j \in S} \frac{\partial f_i}{\partial x_j} = 1,$$

expressing that f is increasing and additively homogeneous. In other words, the jacobian matrix $(\partial f_i/\partial x_j)_{ij}$ is a.e. a stochastic matrix.

For finite-range topical functions, there are additional constraints. Indeed, a topical form $\alpha : \mathbb{R}^S \to \mathbb{R}$ with range at most L satisfies

$$|\alpha(\mathbf{x}) - x_j| > L \implies \frac{\partial \alpha}{\partial x_j} = 0. \tag{6.10}$$

This immediately follows from the definition when $\alpha(\mathbf{x}) = 0$, and otherwise, we revert to this situation by an appropriate translation of $\mathbf{x}$.

Let $E : \mathbb{R}^S \to \mathbb{R}^S$ be the uniformly topical function defined by

$$\begin{aligned} E_i(\mathbf{x}) &= x_i - \frac{1}{N} \sum_{j \in S} \frac{e^{x_i}}{e^{x_i} + e^{x_j}} \\ &= x_i - \frac{1}{N} \sum_{j \in S} \left(1 + e^{x_j - x_i}\right)^{-1} \end{aligned}$$

Its partial derivatives are

$$\frac{\partial E_i}{\partial x_j} = \begin{cases} N^{-1} \kappa_{ij}^{-2} & \text{si } i \neq j, \\ 1 - N^{-1} \displaystyle\sum_{s \neq i} \kappa_{is}^{-2} & \text{si } i = j, \end{cases}$$

where $\kappa_{ij} = 2 \cosh \frac{x_i - x_j}{2} = \exp \frac{x_i - x_j}{2} + \exp \frac{x_j - x_i}{2}$. We note that the jacobian matrix is symmetric, and $\partial E_i/\partial x_i \geqslant 3/4$. On the other hand, $\mathbf{x} - 1 \leqslant E(\mathbf{x}) \leqslant \mathbf{x}$, so $\hat{E} = \mathrm{Id}$ and $d(E, \mathrm{Id}) \leqslant 1$.

Lemma 6.12. *Let $f : \mathbb{R}^S \to \mathbb{R}^S$ be a topical function with range at most 1, whose asymptote is the identity map, and $0 < \lambda \leqslant 1$. The (uniformly topical) function $g = \lambda E + (1 - \lambda) f$ is 1-Lipschitz for the distance associated with the norm $\mathcal{N}_K$, provided that*

$$K \ \geqslant \ \frac{4N \cosh^2 1}{\lambda}. \tag{6.11}$$

Proof. It is sufficient to prove that the tangent linear function $Df_{\mathbf{x}}$ decreases the norm $\mathcal{N}_K$, for almost all $\mathbf{x}$. Let $\mathbf{x}$ be a differentiability point for f (and thus also for g), and define

$$a_{ij} = \frac{\partial g_i}{\partial x_j} = \lambda \frac{\partial E_i}{\partial x_j} + (1 - \lambda) \frac{\partial f_i}{\partial x_j}$$

where the partial derivatives are evaluated at $\mathbf{x}$. By Lemma 6.9, it suffices to prove the inequalities $a_{ii} \geqslant 1/K$ and $a_{ij} \leqslant K a_{ji}$, for all $i, j \in S$.

First, $a_{ii} \geqslant \lambda(\partial E_i/\partial x_i) \geqslant 3\lambda/4$, which is greater than $1/K$ provided that $K \geqslant 4/(3\lambda)$, a condition which is weaker than (6.11).

For the second inequality, we have to distinguish two cases, according to the value of $|x_i - x_j|$. If $|x_i - x_j| > 2$, then $|f_i(\mathbf{x}) - x_j| \geqslant |x_i - x_j| - |f_i(\mathbf{x}) - x_i| > 1$ and consequently $\partial f_i/\partial x_j = 0$ by (6.10). We have also $\partial f_j/\partial x_i = 0$ for the same reason, and since $\partial E_i/\partial x_j = \partial E_j/\partial x_i$, we obtain $a_{ij} = a_{ji}$ in this case. If, on the contrary, $|x_i - x_j| \leqslant 2$, then $a_{ij} \leqslant 1$ and

$$a_{ji} \geqslant \lambda \frac{\partial E_j}{\partial x_i} = \frac{\lambda}{N\kappa_{ij}^2} \geqslant \frac{\lambda}{4N\cosh^2 1}$$

which yields $a_{ij} \leqslant K a_{ji}$ indeed, provided that K satisfies (6.11). □

Linear topical functions, permutations. To each function $\varphi : B \to A$, one can associate an "induced" function $\varphi^* : \mathbb{R}^A \to \mathbb{R}^B$ defined in the obvious way; this is a topical function with null range, and it is also linear (Gaubert and Gunawardena [GG2] call "substitution" such a function). Conversely, it is easily seen that every linear null-range topical function is obtained in this way. Going from φ to φ^* is contravariant, i.e. $(\varphi\psi)^* = \psi^*\varphi^*$ and $(\mathrm{Id})^* = \mathrm{Id}$.

If $A = B$ and φ is a permutation (i.e., a bijection), then φ^* is an invertible element of $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^A)$, and we shall also call it a (coordinate) "permutation", with some abuse of language. As is easily verified, these are the only invertible elements of $\mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^A)$.

Proposition 6.13. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. Assume that all u_ω have range at most L, for some constant $L \geqslant 0$ independent of ω, and that the asymptotes $\hat{u}_\omega$ are permutations. Then u has a stochastic spectral vector.*

Proof. We can assume $L = 1$ without loss of generality.

Fix $0 < \lambda \leqslant 1$, and consider the stochastic uniformly topical functions $u', v, v' : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ defined by $u'_\omega = \hat{u}_\omega^{-1} u_\omega$ (so $\hat{u}'_\omega = \mathrm{Id}$), $v'_\omega = \lambda E + (1-\lambda) u'_\omega$, and $v_\omega = \hat{u}_\omega v'_\omega = \lambda(\hat{u}_\omega E) + (1-\lambda) u_\omega$. In particular $\hat{u}_\omega = \hat{v}_\omega$, and $d(u_\omega, v_\omega) = d(u'_\omega, v'_\omega) = \lambda d(u'_\omega, E) \leqslant 2\lambda$.

We want to prove that the sequence $\mathbf{x}^n(\omega)/n$, where $\mathbf{x}^n(\omega) = v^n_\omega(\mathbf{o})$, is convergent for almost every $\omega \in \Omega$. First, we note that

$$d\big(\mathbf{x}^n(\omega), \mathbf{x}^{n+1}(\omega)\big) = d\big[v^n_\omega(\mathbf{o}), v^n_\omega(v_{\tau^n\omega}\mathbf{o})\big] \leqslant d\big(\mathbf{o}, v_{\tau^n\omega}(\mathbf{o})\big) \leqslant 1$$

and so, by Lemma 6.11, we have to find N different values of K such that $h_n(\omega)/n$ converges, where $h_n(\omega) = \mathcal{N}_K\big(\mathbf{x}^n(\omega)\big)$. We shall take $K_1 = (4N\cosh^2 1)/\lambda$, and $K_t = tK_1$ for $1 \leqslant t \leqslant N$. For each of these values of K, the functions v'_ω are 1-Lipschitz relatively to the norm $\mathcal{N}_K$, by Lemma 6.12, and it is also the case for the functions v_ω, since $\hat{u}_\omega$ is isometric. As a consequence of this, the sequence of functions h_n is (ergodically) subadditive: for all $n, p \geqslant 0$,

$$\begin{aligned} h_{n+p}(\omega) &= \mathcal{N}_K\big(v^{n+p}_\omega(\mathbf{o})\big) = \mathcal{N}_K\big(v^n_\omega v^p_{\tau^n\omega}(\mathbf{o})\big) \\ &\leqslant \mathcal{N}_K\big(v^n_\omega(\mathbf{o})\big) + \mathcal{N}_K\big(v^n_\omega v^p_{\tau^n\omega}(\mathbf{o}) - v^n_\omega(\mathbf{o})\big) \\ &\leqslant \mathcal{N}_K\big(v^n_\omega(\mathbf{o})\big) + \mathcal{N}_K\big(v^p_{\tau^n\omega}(\mathbf{o}) - \mathbf{o}\big) \\ &= h_n(\omega) + h_p(\tau^n\omega) \end{aligned}$$

and therefore $h_n(\omega)/n$ converges for almost every $\omega \in \Omega$, by Kingman's Theorem [Kin, Der].

So v has a stochastic spectral vector. But $D_1(u, v) \leqslant 2\lambda$, with λ arbitrarily small, and this implies, by Proposition 6.3, that u has a stochastic spectral vector. □

6.3.3 The case of linear $\hat{u}_\omega$

In this section, we suppose that $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ is a stochastic uniformly topical function, whose asymptotes $\hat{u}_\omega$ are linear; we shall write $\hat{u}_\omega = \phi^*_\omega$, with $\phi : \Omega \to S^S$ measurable. Then $\hat{u}^n_\omega = [\phi^{(n)}_\omega]^*$, where

$$\phi^{(n)}_\omega = \phi_{\tau^{n-1}\omega} \circ \cdots \circ \phi_{\tau\omega} \circ \phi_\omega$$

Let $\mathrm{Eqr}(S)$ be the set of all equivalence relations on S, ordered by graph inclusion ($r \leqslant r'$ means that r is finer than r'), and Δ_S its smallest element, the equality relation, whose graph is merely the diagonal of $S \times S$. Define $R^0_\omega = \Delta_S$ and $R^n_\omega = \phi_\omega^{-1} R^{n-1}_{\tau\omega}$, so that

$$R^n_\omega = \phi_\omega^{-1}\phi_{\tau\omega}^{-1}\cdots\phi_{\tau^{n-1}\omega}^{-1}\Delta_S = [\phi^{(n)}_\omega]^{-1}\Delta_S \tag{6.12}$$

that is, $(i,j) \in R^n_\omega$ iff $\phi^{(n)}_\omega(i) = \phi^{(n)}_\omega(j)$. We note that $R^n_\omega \geqslant R^{n-1}_\omega$, because $\phi^{-1}_{\tau^{n-1}\omega}\Delta_S \geqslant \Delta_S$. So we have, for every ω, an increasing (and therefore stationary) sequence $\Delta_S = R^0_\omega \leqslant R^1_\omega \leqslant R^2_\omega \leqslant \cdots$ of equivalence relations on S; let $R_\omega = \lim_n R^n_\omega = \sup_n R^n_\omega$ be its limit. Obviously, $R_\omega = \phi_\omega^{-1} R_{\tau\omega}$.

Proposition 6.14. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. Assume that all u_ω have range at most L, for some constant $L \geqslant 0$ independent of ω, and that the asymptotes $\hat{u}_\omega$ are linear. Suppose also that the function $\omega \mapsto R_\omega$ is constant, with its value R being an equivalence relation whose classes all have the same cardinality. Then u has a stochastic spectral vector.*

Proof. For every $\mathbf{x} \in \mathbb{R}^S$, let $s_R(\mathbf{x}) = \max_{iRj} |x_i - x_j|$, and let $P_R(K)$ be the set of all $\mathbf{x} \in \mathbb{R}^S$ such that $s_R(\mathbf{x}) \leqslant K$.

We shall first prove the proposition under the additional assumption that there exists a constant $K \geqslant 0$ such that $u^n_\omega(\mathbf{o}) \in P_R(K)$ for all $n \in \mathbb{N}$, $\omega \in \Omega$.

Since all the classes of R have the same cardinality, we can write $S = C \times H$, where $C = S/R$ is the set of equivalence classes, H is equipotent to every class, and the relation R is given by $iRj \iff p(i) = p(j)$, where $p : S \to C$ is the natural projection.

The hypothesis $\phi_\omega^{-1}R = R$ shows that the functions $\phi_\omega : S \to S$ have quotient maps (by p) $\kappa_\omega : C \to C$, which are bijective. The functions $\psi_\omega : S \to S$ defined by $\psi_\omega = \kappa_\omega \times \mathrm{Id}_H$ are bijections of S, such that $\psi_\omega(s) \, R \, \phi_\omega(s)$ for all $s \in S$, $\omega \in \Omega$. Now define

$$v_\omega = \mathrm{med}\big(\psi^*_\omega - (K+L),\, u_\omega,\, \psi^*_\omega + (K+L)\big)$$

Since $d(u_\omega, \phi^*_\omega) \leqslant L$ and $d(\phi^*_\omega\mathbf{x}, \psi^*_\omega\mathbf{x}) \leqslant s_R(\mathbf{x})$, we see that $v_\omega(\mathbf{x}) = u_\omega(\mathbf{x})$ for all $\mathbf{x} \in P_R(K)$. By induction on n, we deduce $v^n_\omega(\mathbf{o}) = u^n_\omega(\mathbf{o})$. Now, v satisfies the conditions of Proposition 6.13 (because $\hat{v}_\omega = \psi^*_\omega$), and therefore has a stochastic spectral vector, which shall also be a stochastic spectral vector for u.

Let us handle the general case now, without the assumption that the $s_R(u^n_\omega\mathbf{o})$ are bounded. Let $\epsilon > 0$ be arbitrary. One can find an integer $N \geqslant 0$ such that $\mu\{\omega : R^N_\omega \neq R\} \leqslant \epsilon$. Then, define

$$v_\omega = \begin{cases} u_\omega & \text{if } R^N_\omega = R, \\ \rho \circ u_\omega & \text{otherwise,} \end{cases}$$

where $\rho : \mathbb{R}^S \to \mathbb{R}^S$ is the topical function, with range L, defined by

$$\rho_i(\mathbf{x}) = (x_i - L) \vee \big(L + \min_{jRi} x_j\big)$$

Note that $\hat{\rho} = \mathrm{Id}$ and $s_R(\rho\mathbf{x}) = (s_R(\mathbf{x}) - 2L)^+ \leqslant s_R(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^S$; on the other hand, $s_R(\phi^*_\omega\mathbf{x}) \leqslant s_R(\mathbf{x})$, because $\phi_\omega^{-1}R = R$, hence $s_R(u_\omega\mathbf{x}) \leqslant s_R(\mathbf{x}) + 2L$ since $d(u_\omega, \phi^*_\omega) \leqslant L$.

We claim that

$$v^n_\omega(\mathbf{o}) \in P_R(4NL) \tag{6.13}$$

for all $n \geqslant 0$, $\omega \in \Omega$. This is obvious if $n \leqslant 2N$, because $s_R(v_\omega\mathbf{x}) \leqslant s_R(u_\omega\mathbf{x}) \leqslant s_R(\mathbf{x}) + 2L$ for all $\mathbf{x}$, hence $s_R(v^n_\omega\mathbf{o}) \leqslant 2nL \leqslant 4NL$.

Now fix $n > 2N$; we want to prove (6.13) for all ω, assuming that the property is true for $n-1$. We shall distinguish two cases. If $R_\omega^N \neq R$, then $v_\omega = \rho u_\omega$ and $v_\omega^n(\mathbf{o}) = \rho u_\omega(\mathbf{y})$ where $\mathbf{y} = v_\omega^{n-1}(\mathbf{o})$. By the induction hypothesis, $s_R(\mathbf{y}) \leqslant 4NL$, hence $s_R(u_\omega \mathbf{y}) \leqslant 2L + s_R(\mathbf{y}) \leqslant (4N+2)L$ and $s_R(v_\omega^n \mathbf{o}) = (s_R(u_\omega \mathbf{y}) - 2L)^+ \leqslant 4NL$. On the contrary, if $R_\omega^N = R$, we write $v_\omega^n(\mathbf{o}) = v_\omega^N(\mathbf{y})$ with $\mathbf{y} = v_{\tau^N\omega}^{n-N}(\mathbf{o})$. Since the functions v_ω have range at most $2L$, we have $d(v_\omega^N, \hat{v}_\omega^N) \leqslant 2NL$. But $s_R(\hat{v}_\omega^N \mathbf{y}) = s_R\big([\phi_\omega^{(N)}]^* \mathbf{y}\big) = 0$ because $[\phi_\omega^{(N)}]^{-1}\Delta_S = R$, and therefore $s_R(v_\omega^N \mathbf{y}) \leqslant 4NL$.

So v, which has the same asymptote as u, satisfies the additional condition (6.13), and so has a stochastic spectral vector, by the first part of this proof. On the other hand, $D_1(u, v) \leqslant \epsilon L$, with ϵ arbitrarily small, which proves, by Proposition 6.3, that u has a stochastic spectral vector. □

Proposition 6.15. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. Assume that all u_ω have range at most L, for some constant $L \geqslant 0$ independent of ω, and that the asymptotes $\hat{u}_\omega$ are linear. Then u has a stochastic spectral vector.*

Proof. Let $r(\omega)$ be the number of classes of R_ω. From the formula $R_\omega = \phi_\omega^{-1} R_{\tau\omega}$ we deduce that $r(\tau\omega) \geqslant r(\omega)$ everywhere, which implies $r(\tau\omega) = r(\omega)$ almost everywhere, by a standard argument. By slicing Ω into a finite number of τ-invariant pieces, up to a negligible set, we can assume that r is a constant function.

In this case, the function R_ω takes its values in $\mathrm{Eqr}(r, S)$, the set of equivalence classes with r classes. Then define $S' = C \times H$, with $\sharp C = r$ and $\sharp H = N - r + 1$; let $p' : S' \to C$ be the natural projection, and R' the equivalence relation on S' defined by $iR'j \iff p'(i) = p'(j)$.

For all $R \in \mathrm{Eqr}(r, S)$, one can find two functions $S \xrightarrow{j} S' \xrightarrow{p} S$ such that $pj = \mathrm{Id}_S$ and $p^{-1}R = R'$. From this observation, we can construct two measurable functions $j : \Omega \to (S')^S$ and $p : \Omega \to S^{S'}$, such that $p_\omega j_\omega = \mathrm{Id}_S$ and $p_\omega^{-1} R_\omega = R'$ for all $\omega \in \Omega$.

Let $v : \Omega \times \mathbb{R}^{S'} \to \mathbb{R}^{S'}$ be the stochastic uniformly topical function defined by $v_\omega = p_\omega^* u_\omega j_{\tau\omega}^*$. If $\hat{u}_\omega = \phi_\omega^*$, then $\hat{v}_\omega = \psi_\omega^*$, where $\psi_\omega = j_{\tau\omega}\phi_\omega p_\omega$, and an easy calculation yields $\psi_\omega^{(n)} = j_{\tau^n\omega}\phi_\omega^{(n)} p_\omega$. The relations ${R'_\omega}^n = (\psi_\omega^{(n)})^{-1}\Delta_{S'}$ are given by

$$\begin{aligned}
(\psi_\omega^{(n)})^{-1}\Delta_{S'} &= p_\omega^{-1}(\phi_\omega^{(n)})^{-1} j_{\tau^n\omega}^{-1}\Delta_{S'} \\
&= p_\omega^{-1}(\phi_\omega^{(n)})^{-1}\Delta_S \qquad (j_{\tau^n\omega} \text{ injective}) \\
&= p_\omega^{-1} R_\omega^n
\end{aligned}$$

hence $R'_\omega = p_\omega^{-1} R_\omega = R'$. So v satisfies the hypotheses of Proposition 6.14, and therefore has a stochastic spectral vector. Finally, the semi-conjugacy relation $j_\omega^* v_\omega = u_\omega j_{\tau\omega}^*$ implies, by Proposition 6.6, that u also has a stochastic spectral vector. □

6.3.4 The general case — end of the proof

Let $\mathfrak{D}(S) = \mathbf{NrTp}(\mathbb{R}^S, \mathbb{R})$. There exists a natural injection $I_S : \mathbb{R}^S \to \mathbb{R}^{\mathfrak{D}(S)}$, defined by $\mathbf{x} \mapsto [\alpha \mapsto \alpha\mathbf{x}]$; actually, I_S is a bijection between the points of $\mathbb{R}^S$ and the lattice morphisms $\mathfrak{D}(S) \to \mathbb{R}$. The function I_S is topical, with null range, and non-linear as soon as $\sharp S \geqslant 2$. Every $f \in \mathbf{NrTp}(\mathbb{R}^A, \mathbb{R}^B)$ induces a function $f^* : \mathfrak{D}(B) \to \mathfrak{D}(A)$, defined by $f^*\beta = \beta \circ f$, which itself induces a linear topical function $(f^*)^* = \mathfrak{D}f : \mathbb{R}^{\mathfrak{D}(A)} \to \mathbb{R}^{\mathfrak{D}(B)}$, which makes the following diagram commute:

$$\begin{array}{ccc}
\mathbb{R}^A & \xrightarrow{f} & \mathbb{R}^B \\
{\scriptstyle I_A}\downarrow & & {\scriptstyle I_B}\downarrow \\
\mathbb{R}^{\mathfrak{D}(A)} & \xrightarrow{\mathfrak{D}f} & \mathbb{R}^{\mathfrak{D}(B)}
\end{array}$$

The matrix element of $\mathfrak{D}f$ located in line $\beta \in \mathfrak{D}(B)$ and column $\alpha \in \mathfrak{D}(A)$ is 1 if $\alpha = \beta f$, and 0 otherwise.

On the other hand, the canonical injection $\gamma : S \to \mathfrak{D}(S)$, which associates to each $s \in S$ the corresponding coordinate form γ_s, induces a (linear, null-range topical) function $\gamma^* : \mathbb{R}^{\mathfrak{D}(S)} \to \mathbb{R}^S$ which is a retraction of I_S, i.e. $\gamma^* \circ I_S = \mathrm{Id}_{\mathbb{R}^S}$.

Proposition 6.16. *Let $u : \Omega \times \mathbb{R}^S \to \mathbb{R}^S$ be a stochastic uniformly topical function. Assume that all u_ω have range at most L, for some constant $L \geqslant 0$ independent of ω. Then u has a stochastic spectral vector.*

Proof. Let $v : \Omega \times \mathbb{R}^{\mathfrak{D}(S)} \to \mathbb{R}^{\mathfrak{D}(S)}$ be the stochastic uniformly topical function defined by

$$v_\omega = \mathrm{med}(\mathfrak{D}\hat{u}_\omega - L,\ I_S\, u_\omega\, \gamma^*,\ \mathfrak{D}\hat{u}_\omega + L)$$

The functions v_ω have range at most L, with $\hat{v}_\omega = \mathfrak{D}\hat{u}_\omega$. In particular, v has a stochastic spectral vector, by Proposition 6.15. On the other hand,

$$\begin{aligned} v_\omega\, I_S &= \mathrm{med}\big[(\mathfrak{D}\hat{u}_\omega)I_S - L,\ (I_S\, u_\omega)(\gamma^*\, I_S),\ (\mathfrak{D}\hat{u}_\omega)I_S + L\big] \\ &= \mathrm{med}\big[I_S\, \hat{u}_\omega - L,\ I_S\, u_\omega,\ I_S\, \hat{u}_\omega + L\big] \\ &= I_S\, u_\omega \end{aligned}$$

since $\hat{u}_\omega - L \leqslant u_\omega \leqslant \hat{u}_\omega + L$. From this, we deduce $v_\omega^n\, I_S = I_S\, u_\omega^n$ and in particular $v_\omega^n(\mathbf{o}) = I_S\big[u_\omega^n(\mathbf{o})\big]$, hence

$$\frac{u_\omega^n(\mathbf{o})}{n} = \gamma^*\Big[\frac{v_\omega^n(\mathbf{o})}{n}\Big] \ \longrightarrow\ \gamma^*\big[\vec{\chi}_v(\omega)\big]$$

as $n \to \infty$, for almost every $\omega \in \Omega$. ☐

Proposition 6.17. *Let $u : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ be a stochastic uniformly topical function, and $1 \leqslant p < \infty$. The following two properties are equivalent:*
(i) $D_p(u, \hat{u}) < \infty$;
(ii) For every $\epsilon > 0$, there exists $L \geqslant 0$ and a stochastic uniformly topical function $v : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ with all v_ω having range at most L, and $D_p(u, v) \leqslant \epsilon$.

Proof. (ii) $\Longrightarrow$ (i): if $D_p(u, v) < \infty$, then $\hat{u}_\omega = \hat{v}_\omega$ for almost all ω, and consequently $D_p(u, \hat{u}) = D_p(u, \hat{v}) \leqslant D_p(u, v) + D_p(v, \hat{v}) \leqslant \epsilon + L$.

(i) $\Longrightarrow$ (ii): suppose $D_p(u, \hat{u}) < \infty$, and let $\epsilon > 0$. Let $g_0, g_1, g_2 \ldots$ be a sequence of finite-range topical functions $\mathbb{R}^A \to \mathbb{R}^B$, dense in $\mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$. For all $f \in \mathbf{UTp}(\mathbb{R}^A, \mathbb{R}^B)$, let $\kappa(f)$ be the smallest integer k such that $d(f, g_k) \leqslant \epsilon/2$.

Let $v, v^0, v^1, v^2, \ldots : \Omega \times \mathbb{R}^A \to \mathbb{R}^B$ be the stochastic uniformly topical functions defined as follows: $v_\omega = g_{\kappa(u_\omega)}$, so that $D_p(u, v) \leqslant \epsilon/2$ and therefore $D_p(v, \hat{v}) < \infty$, and

$$v_\omega^n = \begin{cases} g_{\kappa(u_\omega)} & \text{if } \kappa(u_\omega) < n, \\ \hat{u}_\omega & \text{otherwise,} \end{cases}$$

for $n \geqslant 0$. The sequence of functions $h_n : \omega \mapsto d(v_\omega^n, v_\omega)^p$ is decreasing, and converges pointwise to zero; as h_0 is integrable, we have $D_p(v^n, v)^p = \mathbb{E}(h_n) \to 0$ by the dominated convergence theorem. We can choose n such that $D_p(v^n, v) \leqslant \epsilon/2$, so that $D_p(v^n, u) \leqslant \epsilon$, and the functions v_ω^n have bounded range. ☐

By combining Propositions 6.16 and 6.17 via Proposition 6.3, we finally obtain Theorem 6.1.

6.4 An inverse ergodic theorem

Theorem 6.18. *We take the same hypotheses and notations as in Theorem 6.1. We also assume that τ has a measurable inverse $\theta = \tau^{-1} : \Omega \to \Omega$. Then, for almost every $\omega \in \Omega$,*

$$u_{\theta^{n-1}\omega}\, u_{\theta^{n-2}\omega} \cdots u_{\theta\omega}\, u_\omega(\mathbf{o}) \;=\; n\,\vec{\chi}(\theta^{n-1}\omega) + o(n) \tag{6.14}$$

as $n \to \infty$.

Remarks. If $\tau : \Omega \to \Omega$ is not invertible, taking the natural extension will yield an invertible $\tau' : \Omega' \to \Omega'$ on some bigger probability space.

Theorem 6.18 takes a slightly simpler form if we know that $\vec{\chi} \circ \tau = \vec{\chi}$; then the right-hand side of (6.14) reduces to $n\,\vec{\chi}(\omega) + o(n)$. This is the case, in particular, for "augmenting" dynamics, that is, when $u_\omega(\mathbf{x}) \geqslant \mathbf{x}$ for all ω and $\mathbf{x}$. Indeed, we have in this case $\vec{\chi}(\tau\omega) \leqslant \hat{u}_\omega \vec{\chi}(\tau\omega) = \vec{\chi}(\omega)$, which implies $\vec{\chi} \circ \tau = \vec{\chi}$ by a standard argument.

Proof. Let $\mathbf{x}^n_\omega = u^n_{\theta^{n-1}\omega}(\mathbf{o})$. We have to prove that the function

$$G(\omega) = \limsup_{n\to\infty} \left\| \frac{\mathbf{x}^n_\omega}{n} - \vec{\chi}(\theta^{n-1}\omega) \right\|$$

is almost everywhere zero. Let $\eta(\omega) = d(u_\omega, \hat{u}_\omega)$ and $\eta'(\omega) = \mathbb{E}(\eta \mid \mathcal{T}')$. The inequality $\|u^n_\omega(\mathbf{o})\| \leqslant S_n\eta(\omega)$ implies $\|\vec{\chi}(\omega)\| \leqslant \eta'(\omega)$ almost everywhere.

Let $0 < \epsilon < 1$ be arbitrary. By Egorov's theorem, there exists a measurable $F_1 \subseteq \Omega$, with $\mu(F_1) \leqslant \epsilon^2$, such that $u^n_\omega(\mathbf{o})/n$ converges uniformly to $\vec{\chi}$ on $\Omega - F_1$. So, let $n_0 > 0$ be such that

$$d\big[u^s_\omega(\mathbf{o}), s\vec{\chi}(\omega)\big] \leqslant \epsilon s$$

for all $s \geqslant n_0$ and $\omega \notin F_1$.

Let h be the characteristic function of F_1, and $h' = \mathbb{E}(h \mid \mathcal{T}')$. The set $F_2 = \{\omega : h'(\omega) \geqslant \epsilon\}$ satisfies $\mu(F_2) \leqslant \epsilon$, because $\epsilon^2 \geqslant \mu(F_1) = \mathbb{E}(h) = \mathbb{E}(h') \geqslant \epsilon\mu(F_2)$.

Let $v_\omega(\mathbf{x}) = \hat{u}_\omega(\mathbf{x}) + \vec{\chi}(\omega)$, so that $d(u_\omega, v_\omega) \leqslant (\eta + \eta')(\omega)$. On the other hand, an easy calculation yields $v_\omega\big[s\vec{\chi}(\tau\omega)\big] = (s+1)\vec{\chi}(\omega)$ and consequently $v^k_\omega\big[s\vec{\chi}(\tau^k\omega)\big] = (s+k)\vec{\chi}(\omega)$ for all $s, k \geqslant 0$.

Now, for all $\omega \in \Omega$ and $n \in \mathbb{N}$, and $0 \leqslant s \leqslant n$ arbitrary,

$$\begin{aligned}
d\big[\mathbf{x}^n_\omega, n\vec{\chi}(\theta^{n-1}\omega)\big] &= d\Big[u^{n-s}_{\theta^{n-1}\omega}(\mathbf{x}^s_\omega), v^{n-s}_{\theta^{n-1}\omega}\big(s\vec{\chi}(\theta^{s-1}\omega)\big)\Big] \\
&\leqslant d\Big[u^{n-s}_{\theta^{n-1}\omega}(\mathbf{x}^s_\omega), u^{n-s}_{\theta^{n-1}\omega}\big(s\vec{\chi}(\theta^{s-1}\omega)\big)\Big] + d(u^{n-s}_{\theta^{n-1}\omega}, v^{n-s}_{\theta^{n-1}\omega}) \\
&\leqslant d\big[\mathbf{x}^s_\omega, s\vec{\chi}(\theta^{s-1}\omega)\big] + \sum_{k=s}^{n-1} (\eta + \eta')(\theta^k\omega)
\end{aligned}$$

Assume now $\omega \notin F_2$. Almost surely, $h(\omega) + \cdots + h(\theta^{n-1}\omega) = nh'(\omega) + o(n)$ and therefore $h(\omega) + \cdots + h(\theta^{n-1}\omega) \leqslant \epsilon n$ for all n big enough, since $h'(\omega) < \epsilon$. In this case, there exists s such that $(1-\epsilon)n \leqslant s \leqslant n$ and $\theta^{s-1}\omega \notin F_1$, so that

$$\begin{aligned}
d\big[\mathbf{x}^n_\omega, n\vec{\chi}(\theta^{n-1}\omega)\big] &\leqslant \epsilon n + \sum_{k=\lceil(1-\epsilon)n\rceil}^{n-1} (\eta + \eta')(\theta^k\omega) \\
&= \epsilon n\big[1 + 2\eta'(\omega)\big] + o(n)
\end{aligned}$$

when n is big. Thus, $G(\omega) \leqslant \epsilon\big[1 + 2\eta'(\omega)\big]$ for almost every $\omega \notin F_2$. Consequently

$$\mu\Big\{\omega : G(\omega) > \epsilon\big[1 + 2\eta'(\omega)\big]\Big\} \leqslant \epsilon$$

and this holds with ϵ arbitrarily small, hence $G = 0$ a.e. $\square$

When comparing this result with Theorem 6.7, one could be tempted to believe that, when the functions u_ω are max-plus, the a.e. convergence (6.14) would also hold under the weaker integrability hypothesis $\int \|u_\omega(\mathbf{o})\| \, \mu < \infty$. This is not the case, and we invite the reader to devise his own counterexamples.

References

[AB] C. D. Aliprantis & K. C. Border, *Infinite dimensional analysis: a hitchhiker's guide*, second edition, Springer (1999)

[Ba] F. Baccelli, G. Cohen, G. J. Olsder & J. P. Quadrat, *Synchronization and linearity*, Wiley (1992)

[BK] T. Bewley & E. Kohlberg, *The asymptotic theory of stochastic games*, Math. Oper. Res. **1** (1976), pp. 197–208

[Bir] G. Birkhoff, *Lattice theory*, third edition, AMS (1967)

[Bou] N. Bourbaki, *Éléments de mathématique, livre III: Topologie générale*, fourth edition, Hermann (1965)

[BM] T. Bousch & J. Mairesse, *Asymptotic height optimization for topical IFS, Tetris heaps, and the finiteness conjecture*, J. Amer. Math. Soc. **15** (2002), pp. 77–111

[BS] A. D. Burbanks & C. T. Sparrow, *All monotone homogeneous functions (on the positive cone) admit continuous extension*, preprint 1999-13, Statistical Laboratory, Cambridge University (1999)

[CT] M. Crandall & L. Tartar, *Some relations between nonexpansive and order preserving mappings*, Proc. Amer. Math. Soc. **78** (1980), pp. 385–390

[Ded] R. Dedekind, *Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler*, Festschrift der Technischen Hochschule zu Braunschweig bei Gelegenheit der 69. Versammlung Deutscher Naturforscher und Ärzte (1897), pp. 1–40

[Der] Y. Derriennic, *Sur le théorème ergodique sous-additif*, CRAS Paris A **281** (1975), pp. 985–988

[GG] P. Gaiha & S. K. Gupta, *Adjacent vertices on a permutohedron*, SIAM J. Appl. Math. **32** (1977), pp. 323–327

[GG1] S. Gaubert & J. Gunawardena, *A nonlinear hierarchy for discrete event dynamical systems*, in A. Giua, R. Smedinga & M. Spathopoulos (eds.), *Proceedings of WODES 98*, IEE (1998)

[GG2] S. Gaubert & J. Gunawardena, *The duality theorem for min-max functions*, CRAS Paris A **326** (1998), pp. 43–48

[GY] P. Glasserman & D. D. Yao, *Monotone structure in discrete-event systems*, Wiley (1994)

[Gu1] J. Gunawardena (ed.), *Idempotency*, Cambridge University Press (1998)

[Gu2] J. Gunawardena, *Cycle times and fixed points of min-max functions*, in G. Cohen & J.-P. Quadrat (eds.), *11th international conference on analysis and optimization of systems*, LNCIS **199**, Springer (1994), pp. 266–272

[Gu3] J. Gunawardena, *A dynamic approach to timed behaviour*, in B. Jonsson & J. Parrow (eds.), *CONCUR'94: Concurrency theory*, LNCS **836**, Springer (1994), pp. 178–193

[GK] J. Gunawardena & M. Keane, *On the existence of cycle times for some non-expansive maps*, preprint HPL-BRIMS-95-003, Hewlett-Packard (1995)

[Joh] P. T. Johnstone, *Stone spaces*, Cambridge University Press (1982)

[Kin] J. F. C. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. B **30** (1968), pp. 499–510

[Koh] E. Kohlberg, *Invariant half-lines of nonexpansive piecewise-linear transformations*, Math. Oper. Res. **5** (1980), pp. 366–372

[Mcl] S. Mac Lane, *Categories for the working mathematician*, second edition, Springer (1997)

[Mir] L. Mirsky, *Results and problems in the theory of doubly-stochastic matrices*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete **1** (1963), pp. 319–334

[Ose] V. I. Oseledets, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trudy Moskov. Math. Obsc. **19** (1968), pp. 179–210, english translation in Trans. Moscow Math. Soc. **19** (1968), pp. 197–231

[Par] W. Parry, *Topics in ergodic theory*, Cambridge University Press (1981)

[Rag] M. S. Raghunathan, *A proof of Oseledec's multiplicative ergodic theorem*, Isr. J. Math. **32** (1979), pp. 356–362

[Sch] P. H. Schoute, *Analytical treatment of the polytopes regularly derived from regular polytopes*, Johannes Müller, Amsterdam (1911)

[ST] U. Schwiegelshohn & L. Thiele, *Dynamic min-max problems*, Discrete Event Dynamic Systems **9** (1999), pp. 111–134

[Sko] T. Skolem, *Om konstitutionen av den identiske kalkuls grupper*, Proc. Third Scand. Math. Congr., Almqvist & Wiskell, Uppsala (1913), pp. 149–163

[Zie] G. M. Ziegler, *Lectures on polytopes*, Graduate texts in Math. **152**, Springer (1995)

[ZP] U. Zwick & M. Paterson, *The complexity of mean payoff games on graphs*, Theoret. Comp. Sci. **158** (1996), pp. 343–359