On the existence of fixed points for the $\cdot/\mathrm{GI}/1/\infty$ queue

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Abstract

A celebrated theorem of Burke's asserts that when a Poisson process with mean interarrival time equal to α is input to a single server queue whose service times are i.i.d. exponentials (symbolically $\cdot/\mathrm{M}/1/\infty$) with mean $\beta < \alpha$, the equilibrium departure process from the queue is also a mean α Poisson process. In this sense mean α Poisson processes are "fixed points" for the $\cdot/\mathrm{M}/1/\infty$ queueing operator.

We consider the following question: For queues which dispense i.i.d. services of finite mean, but otherwise of arbitrary distribution (i.e. $\cdot/\mathrm{GI}/1/\infty$ queues), do fixed points exist? We show that the answer is in the affirmative if the service time s satisfies: $P\{s \neq E[s]\} > 0$ and $\exists a > 0$, such that $E\left[s^{3+a}\right] < \infty$. More precisely, there are values $\alpha > E[s]$ such that there exists a unique ergodic fixed point of mean α. The set of such values $\mathcal{S}$ is infinite and satisfies $\inf\{u \in \mathcal{S}\} = E[s]$ and $\sup\{u \in \mathcal{S}\} = \infty$.

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1 Introduction

Consider a single server first-come-first-served (FCFS) queue with infinite waiting room, at which the service times are i.i.d. exponentials of mean β (a $\cdot/M/1/\infty$ queue, following Kendall's notation). A celebrated theorem of Burke's [9] asserts that if the arrival process at the queue is a Poisson process with mean inter-arrival time $\alpha > \beta$, then so is the equilibrium departure process.

In this paper we consider the question: Do single server FCFS queues with i.i.d. services of mean β, but otherwise of arbitrary distribution (i.e. $\cdot/GI/1/\infty$ queues), possess fixed points? Under some mild moment assumptions on the service times, we shall show that fixed points exist for $\cdot/GI/1/\infty$ queues.

The question of the existence of fixed points is intimately related to the limiting behavior of the distribution of departure processes from a tandem of queues. More precisely, consider the effect of passing an arrival process through an infinite tandem of $\cdot/GI/1/\infty$ FCFS queues. The queues are indexed by $k \in \mathbb{N}$ and the customers are indexed by $n \in \mathbb{Z}$. At queue k, write $s(n,k)$ for the service time of customer n and $t(n,k)$ for the inter-arrival time between customers n and $n+1$. We assume that the initial inter-arrival process, $t_0 = \{t(n,0), n \in \mathbb{Z}\}$, is ergodic and independent of the service times $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$. We also assume that the service variables $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ are i.i.d. and that $E[s(0,0)] = \beta < E[t(0,0)] = \alpha < \infty$. To avoid trivialities we assume that the service times are nonconstant; i.e. $P\{s(0,0) \neq \beta\} > 0$. It is well-known that under the stability assumption, $\beta < \alpha$, the equilibrium departure process from the first queue $t_1 = \{t(n,1), n \in \mathbb{Z}\}$ is ergodic of mean α. By induction, each of the equilibrium departure processes $t_k = \{t(n,k), n \in \mathbb{Z}\}$, $k \in \mathbb{N}$, is ergodic of mean α. We are now ready to state the following fixed point problems.

Existence For a given service distribution, does there exist a mean α stationary and ergodic arrival process, t_0, such that the corresponding departure process from the queue, t_1, is distributed identically as t_0? If it exists, we shall call it an *ergodic fixed point* of mean α.

Uniqueness Among ergodic processes of mean α, if a fixed point exists then is it unique (in distribution)?

Convergence Consider an infinite tandem of independent copies of $./GI/1/\infty$ queues. The problem of convergence pertains to the following question: If an ergodic process of mean α is input to the first queue, then does the departure process from the k-th queue converge weakly to the mean α ergodic fixed point as $k \to \infty$? We shall say that the fixed point is *an attractor* if the answer to this question is in the affirmative. Note that if a fixed point of mean α is attractive, then it is also unique.

A strand of research in stochastic network theory has pursued these questions for quite some time. Perhaps the earliest and best-known result is the previously-mentioned theorem of Burke's, which shows that the Poisson process of rate $1/\alpha$ is a fixed point for the $\cdot$/M/k/∞ queue with mean service time $\beta < \alpha$. Anantharam [1] established it's uniqueness, and Mountford and Prabhakar [23] established that it is an attractor.

Another well-studied queueing system in this context is the infinite server queue dispensing i.i.d. services (symbolically, the $\cdot$/GI/∞ queue). Mirasol [21] showed that a mean α Poisson process is a fixed point for the $\cdot$/GI/∞ queue. Vere-Jones [29] showed that it is an attractor, assuming that the initial arrival process is weakly mixing. Prabhakar et al [25] use a natural coupling argument to generalize Vere-Jones' result for arbitrary ergodic input processes.

For $\cdot$/GI/1/∞ queues, the subject of this paper, Chang [10] established the uniqueness of an ergodic fixed point, should it exist, assuming that the services have a finite mean and an unbounded support. Prabhakar [24] provides a complete solution to the problems of uniqueness and convergence assuming a finite first moment for the service time and, of course, the existence of an ergodic fixed point.

However, the existence of such fixed points is only known for exponential and geometric service times. This paper aims to establish the existence of fixed points for a large class of service time distributions, which includes the exponential and geometric service distributions. More precisely, we establish the following result: If the service time s has mean β and there exists an $a > 0$ such that $E[s^{3+a}] < \infty$, then there is a set $\mathcal{S}$ closed in (β, ∞), with $\inf\{u \in \mathcal{S}\} = \beta$, $\sup\{u \in \mathcal{S}\} = \infty$ and such that

(a) For $\alpha \in \mathcal{S}$, there exists a mean α ergodic fixed point for the queue. Given this, [24] implies the attractiveness of the fixed point for all ergodic initial inter-arrival processes, t_0, of mean $\alpha \in \mathcal{S}$.

(b) For $\alpha \notin \mathcal{S}$, consider the stationary (but not ergodic) process F of mean α obtained as the convex combination of the ergodic fixed points of means $\underline{\alpha}$ and $\overline{\alpha}$ where $\underline{\alpha} = \sup\{u \in \mathcal{S}, u \leq \alpha\}$ and $\overline{\alpha} = \inf\{u \in \mathcal{S}, \alpha \leq u\}$. It is clear that F is a fixed point for the queue. If the inter-arrival times of the input process have a mean α, then the Cesaro average of the law of the first k inter-departure processes converges weakly to F as $k \to \infty$.

These results rely heavily on a weak continuity property of the queueing operator obtained by Borovkov [7] and a strong law of large numbers for the total time spent by a customer in a tandem of queues proved in Baccelli, Borovkov and Mairesse [3].

Some remarks on the organization of the paper: Sections 2 and 3 set up the mathematical preliminaries and introduce the problem that will be studied. Section 4 introduces the "queueing operator" as a mapping from inter-arrival processes into inter-departure processes. As mentioned previously, the questions of existence, uniqueness and attractiveness of fixed points are closely connected. Thus Section 6, which establishes the existence of fixed points, utilizes the results and constructions in [10] and [24] pertaining to uniqueness and attractiveness. To keep the paper self-contained, we therefore recall the essentials of [10, 24] in Section 5.

2 Preliminaries

2.1 Notations and conventions

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(K, \mathcal{K})$ be an arbitrary measurable space. Denote by $\mathcal{L}(K)$ the set of K-valued random variables; and by $\mathcal{M}(K)$ the set of probability measures on $(K, \mathcal{K})$. Throughout lowercase letters, e.g. u, v or x, will be used to denote random variables and uppercase letters, e.g. U, V or X, will denote the corresponding distribution. This convention is often used without notice. The symbol $\sim$ is used for 'is distributed as', for instance $u \sim U$ and $u \sim v$ imply $U = V$.

When the measurable space $(K, \mathcal{K})$ is metrizable, we equate $\mathcal{K}$ to the the Borel σ-algebra induced by the metric. For $u_n, n \in \mathbb{N}$, and u belonging to $\mathcal{L}(K)$, the notation $u_n \xrightarrow{w} u$ and $U_n \xrightarrow{w} U$ will be used to denote the weak convergence of U_n to U. On $\mathcal{M}(K)$, we always and often implicitly consider the topology generated by the weak convergence (*weak topology*).

We use the notations: $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $x^+ = \max(x, 0)$.

2.2 Stationarity and ergodicity

Let $\theta : \Omega \to \Omega$ be a measurable transformation. We say that θ is *stationary* if $P\{\theta^{-1}(\mathcal{A})\} = P\{\mathcal{A}\}, \forall \mathcal{A} \in \mathcal{F}$. A stationary transformation θ is said to be *ergodic* if $\mathcal{A} \in \mathcal{F}$ and $\mathcal{A} = \theta^{-1}(\mathcal{A})$ implies $P\{\mathcal{A}\} = 0$ or 1. We also say that the quadruple $(\Omega, \mathcal{F}, P, \theta)$ is stationary (or ergodic).

Let $(K, \mathcal{K})$ be a measurable space. Given x in $\mathcal{L}(K)$, define $x \circ \theta$ by $x \circ \theta(\omega) = x(\theta\omega), \omega \in \Omega$. Below, the index set I is either equal to $\mathbb{N}$ or to $\mathbb{Z}$. A random sequence $\{x(n), n \in I\}$ in $\mathcal{L}(K^I)$ is said to be *compatible* with θ if $x(n) = x(0) \circ \theta^n$. When θ is stationary (respectively ergodic), a compatible sequence $\{x(n)\}$ is said to be *stationary* (respectively *ergodic*).

There is an alternative way to define stationary and ergodic processes. Let x in $\mathcal{L}(K^I)$ be a random process (with distribution X). We say that x is *stationary* (respectively *ergodic*) if the quadruple $(K^I, \mathcal{K}^I, X, L)$ is stationary (respectively ergodic), where L is the shift transformation defined by

$$L: \quad K^I \to K^I, \ \ [Lx](n) = x(n+1) \ . \tag{1}$$

In this case, we also say that the distribution X is stationary (resp. ergodic). It is clear that the first definition implies the second one. Barring some technicalities, the converse is also true (see Doob [13], Chapter X, for details). It is possible for an ergodic process (second definition) to be compatible with a stationary and *non-ergodic* θ. On the other hand, a stationary non-ergodic process (second definition) can not be compatible with an ergodic θ.

It is convenient for us to play with the two definitions. Throughout the paper, we work on a single probability quadruple $(\Omega, \mathcal{F}, P, \theta)$ where θ is assumed to be stationary. The probability space is assumed to be large enough to define all the random variables that we consider. We say that a process x in $\mathcal{L}(K^I)$ is stationary if it is compatible with θ. We say that x is ergodic if it is compatible with θ and if $(K^I, \mathcal{K}^I, X, L)$ is ergodic.

This convention has a double advantage. First, it enables to simplify many statements. For instance, several stationary processes are always jointly stationary. Second, it enables to define on the same probability quadruple processes which are either stationary or ergodic.

We conclude the discussion of stationarity and ergodicity by recalling two ergodic theorems. Let u in $\mathcal{L}(K^I)$ be stationary and let $f : K^I \to \mathbb{R}$ be measurable and integrable with respect to U: $E[f(u)] < \infty$. Then Birkhoff's Ergodic Theorem states that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} f(u \circ \theta^i) = E[f(u) \mid \mathcal{T}] \ \ P-a.s. \text{ and in } L^1 \ , \tag{2}$$

where $\mathcal{T}$ is the invariant σ-algebra defined by $\mathcal{T} = \left\{\mathcal{A} \in \mathcal{F} \mid \theta^{-1}(\mathcal{A}) = \mathcal{A}\right\}$. The r.v. $E[u(0) \mid \mathcal{T}]$ is called the *pathwise mean* of u. Let U be in $\mathcal{M}(K^I)$. The Ergodic Decomposition Theorem states that there is a unique decomposition of U, given by

$$U = \int_{\mathcal{M}(K^I)} X\,\Gamma(dX)\ , \tag{3}$$

where $\Gamma\{X \in \mathcal{M}(K^I) \mid (K^I, \mathcal{K}^I, X, L) \text{ ergodic}\} = 1$. That is, Γ is a distribution whose mass is concentrated on the set of ergodic distributions on K^I.

2.3 The spaces $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{Z}}$

Let $I = \mathbb{N}$ or $\mathbb{Z}$ and denote by $\mathbb{R}^I$ the set of real sequences $u = \{u(i), i \in I\}$. For any finite subset S of I, π_S is the projection of $\mathbb{R}^I$ into $\mathbb{R}^S$, defined by $\pi_S(u) = (u(i), i \in S)$.

We consider the product topology, or topology of coordinatewise convergence, on $\mathbb{R}^I$: If $u_n, n \in \mathbb{N}$, and u belong to $\mathbb{R}^I$, then $\lim_n u_n = u$ if $\lim_n u_n(i) = u(i), \forall i \in I$. This topology is metrizable using the metric

$$d(u,v) = \sum_{i \in I} \frac{1}{2^{|i|}} \frac{|u(i) - v(i)|}{1 + |u(i) - v(i)|}\ .$$

The metric space $(\mathbb{R}^I, d)$ is Polish. In this paper, we work on the measurable space $(\mathbb{R}^I, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra generated by the product topology. On $\mathcal{M}(\mathbb{R}^I)$, the topology of weak convergence is metrizable (by the Prohorov metric, for instance), and $\mathcal{M}(\mathbb{R}^I)$ is a Polish space. Weak convergence can be characterized as follows: For $U_n, n \in \mathbb{N}$, and U in $\mathcal{M}(\mathbb{R}^I)$, $U_n \xrightarrow{w} U$ if and only if $U_n \pi_S^{-1} \xrightarrow{w} U \pi_S^{-1}$ for all finite subsets S of I. For more details see, for example, Billingsley [5], Dudley [14] or Ethier and Kurtz [15].

On the space $(\mathbb{R}^I, \mathcal{B})$ write $\mathcal{M}_s(\mathbb{R}^I)$ for the set of stationary probability measures, and $\mathcal{M}_e(\mathbb{R}^I)$ for the set of ergodic probability measures. For $\alpha \in \mathbb{R}$, denote by $\mathcal{M}^\alpha(\mathbb{R}^I)$ (respectively $\mathcal{M}_s^\alpha(\mathbb{R}^I)$, $\mathcal{M}_e^\alpha(\mathbb{R}^I)$) the set of (respectively stationary, ergodic) probability measures whose one-dimensional marginals have a mean of α. Let $\mathcal{L}_s(\mathbb{R}^I)$, $\mathcal{L}_e(\mathbb{R}^I), \ldots,$ denote the corresponding sets of random variables.

Let $u = \{u(n), n \in \mathbb{Z}\} \in \mathcal{L}(\mathbb{R}^{\mathbb{Z}})$ be a random process with distribution U. For $l \in \mathbb{Z}$, let ${}_l u$ and ${}_l U$ denote the truncated process $\{u(n), n \geq l\}$ and its distribution, respectively.

3 The Model

3.1 Tandem model

Consider a bi-infinite sequence of customers passing through an infinite number of queues in tandem. Customers are indexed by $\mathbb{Z}$ and queues are indexed by $\mathbb{N}$. The numbering of each customer is fixed at the first queue and remains the same as he/she passes through the tandem. For $n \in \mathbb{Z}, k \in \mathbb{N}$, let $A(n,k) \in \mathbb{R}$ be the arrival time of customer n at station k (with the convention $A(0,0) = 0$), $D(n,k) \in \mathbb{R}$ be the departure time of customer n from station k, $t(n,k) \in \mathbb{R}_+$ be the n-th inter-arrival time at station k defined by

$t(n,k) = A(n+1,k) - A(n,k)$, and $s(n,k) \in \mathbb{R}_+$ be the service time of the n-th customer at station k.

The stations are single server queues with an infinite buffer and a FCFS (First-Come-First-Served) service policy. Using Kendall's notation, each queue is of type $./ ./1/\infty$ FCFS. The equation $D(n,k) = \max(A(n,k), D(n-1,k)) + s(n,k)$ relates departure times to arrival and service times. The stations are in tandem, a customer departing from station k immediately enters station $k+1$: $A(n,k+1) = D(n,k)$.

Given to us initially are the service times $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ and the initial arrival process $\{A(n,0), n \in \mathbb{Z}\}$. The question is whether one can compute all the variables $A(n,k), D(n,k), n \in \mathbb{Z}, k \in \mathbb{N}$, using the two equations presented above. Since there is a bi-infinite sequence of customers, the answer to this question is not immediate. In fact, in order for the setting to be constructive, we need to introduce a backward limiting procedure (§3.2) and to impose probabilistic assumptions (§3.3).

Let $w(n,k)$ be the *waiting time* of the n-th customer at the k-th station, equal to the amount of time from its arrival to the beginning of its service. Thus $w(n,k) = D(n,k) - A(n,k) - s(n,k)$.

¿From the definitions, one can easily obtain the following equations:

$$\begin{aligned} w(n,k) &= [w(n-1,k) + s(n-1,k) - t(n-1,k)]^+ && (4)\\ t(n,k) &= s(n+1,k-1) + [t(n,k-1) - s(n,k-1) - w(n,k-1)]^+ && (5)\\ t(n,k+1) &= t(n,k) + w(n+1,k) - w(n,k) + s(n+1,k) - s(n,k)\,. && (6) \end{aligned}$$

The first two are known as Lindley's equations. The third one is less well-known. It is used, for instance, in [24, 28] and follows from the equalities

$$\begin{aligned} w(n+1,k) - w(n,k) &= (\ A(n+1,k+1) - A(n+1,k) - s(n+1,k)\ ) \\ &\quad - (\ A(n,k+1) - A(n,k) - s(n,k)\ ) \\ &= (A(n+1,k+1) - A(n,k+1)\ ) \\ &\quad + (A(n,k) - A(n+1,k)) - s(n+1,k) + s(n,k). \end{aligned}$$

Equations (4)-(5) imply that a queue is monotone in the following sense. Fix a service process $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ and consider two inter-arrival processes $t_A(0) = \{t_A(n,0), n \in \mathbb{Z}\}$ and $t_B(0) = \{t_B(n,0), n \in \mathbb{Z}\}$. We use the subscript A (resp. B) for the quantities associated with the network under inter-arrival process $t_A(0)$ (resp. $t_B(0)$). Suppose that either of the following conditions holds:

$$\begin{aligned} & t_A(n,0) \geq t_B(n,0), && \forall n \in \mathbb{Z}, \\ \text{or,}\quad & w_A(0,0) \leq w_B(0,0),\ t_A(n,0) \geq t_B(n,0), && \forall n \in \mathbb{N}\,. \end{aligned}$$

Then, assuming the variables are defined and depending on which of the two conditions holds, we have, either $\forall n \in \mathbb{Z}, \forall k \in \mathbb{N}$, or $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}$,

$$w_A(n,k) \leq w_B(n,k), \quad \text{and} \quad t_A(n,k) \geq t_B(n,k)\,. \tag{7}$$

Single station. We shall often use or prove results for a single station instead of for the whole tandem. At such times we shall use lighter notations by supressing the dependence on k. Hence, we might consider passing the inter-arrival process $t = \{t(n), n \in \mathbb{Z}\}$ through a queue with service process $s = \{s(n), n \in \mathbb{Z}\}$. The corresponding waiting time process will be denoted by $\{w(n), n \in \mathbb{Z}\}$ (or by $\{w(n,t), n \in \mathbb{Z}\}$, if we wish to emphasize the inter-arrival process).

It is convenient to think of a tandem model (resp. a single station) as being defined by the data $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ (resp. $\{s(n), n \in \mathbb{Z}\}$), and to think of the inter-arrival process $\{t(n,0), n \in \mathbb{Z}\}$ (resp. $\{t(n), n \in \mathbb{Z}\}$) as the initial condition of the model. This allows us to consider the same model subject to different initial conditions.

3.2 Loynes' scheme

Consider a single server queue of type $G/G/1/\infty$ FCFS. The inter-arrival process $t = \{t(n), n \in \mathbb{Z}\}$ and the service process $s = \{s(n), n \in \mathbb{Z}\}$ are assumed to be jointly ergodic and $t(0)$ and $s(0)$ are assumed to have finite means. The following results are standard and due to Loynes [20] (see also [4] for a detailed presentation). Our purpose is to state the results in a form that is suitable for us.

First, consider the truncated inter-arrival process ${}_kt = \{t(n), n \geq k\}$ and suppose that $w(k, {}_kt) = 0$. For $n \geq k$, the variables $D(n, {}_kt)$ and $w(n, {}_kt)$ can be obtained using the recursions in §3. The inter-departure times are given by ${}_kt(n,1) = D(n+1, {}_kt) - D(n, {}_kt), n \geq k$.

Second, consider the station operating on the bi-infinite process $\{t(n), n \in \mathbb{Z}\}$. In this case, possible candidates for the waiting and departure times are (as may be seen by developing the recursion in (4))

$$\begin{aligned} w(n) &= \lim_{k \to +\infty} w(n, {}_{-k}t) = [\, \sup_{i=1}^{\infty} (\sum_{j=1}^{i} s(n-j) - t(n-j))\,]^+ \, ; \\ D(n) &= \lim_{k \to +\infty} D(n, {}_{-k}t) \, . \end{aligned} \tag{8}$$

These are the limits, as k goes to infinity, of the corresponding quantities for the queueing systems starting empty and with inter-arrival processes ${}_{-k}t$. The limits exist in $\mathbb{R} \cup \{+\infty\}$ since the sequences $w(n, {}_kt)$ and $D(n, {}_kt)$ are increasing functions of k. Consider the following two cases.

- The unstable case, $E[s(0)] > E[t(0)]$: One obtains that $P\{w(n) = \infty\} = P\{D(n) = \infty\} = 1$, and the action of the station on the bi-infinite arrival process is not well defined. Nevertheless, since $\{{}_{-k}t(n,1),\ h \leq n \leq H\} \xrightarrow{k} \{s(n+1,0),\ h \leq n \leq H\}$ P-a.s., for all $h \leq H \in \mathbb{Z}$, we may (and will) say that the "departure process" equals the service process s.
- The stable case, $E[s(0)] < E[t(0)]$: Here one has that $P\{w(n) < \infty\} = P\{D(n) < \infty\} = 1$. Further, the processes $\{w(n)\}$ and $\{t(n,1)\}$ are ergodic and $E[t(0,1)] = E[t(0)]$.

The main results of this paper assume that the services are i.i.d., that t and s are independent, and that $P\{s(0) \neq E[s(0)]\} > 0$. Under these assumptions if $E[s(0)] =$

$E[t(0)]$, then the queue is unstable in the above sense. In particular, we shall say that the departure process equals the service process s.

Suppose now that the processes t and s are jointly stationary (but not ergodic). The limit processes in (8) are still defined. However, the system can be either stable or unstable depending on $\omega \in \Omega$. On the event $\mathcal{E}_1 = \{E[s(0) \mid \mathcal{T}] > E[t(0) \mid \mathcal{T}]\}$ (or $\mathcal{E}_1 = \{E[s(0) \mid \mathcal{T}] \geq E[t(0) \mid \mathcal{T}]\}$ if s is i.i.d., not degenerate and independent of t), the system is unstable. On the event $\mathcal{E}_2 = \{E[s(0) \mid \mathcal{T}] < E[t(0) \mid \mathcal{T}]\}$, it is stable. Conditioned on $\mathcal{E}_1$ and $\mathcal{E}_2$, the previously mentioned results hold. In particular, given $\alpha \in \mathbb{R}_+$ if $\mathcal{I}$ is the event $\{E[t(0) \mid \mathcal{T}] = \alpha\} \cap \mathcal{E}_2$, then on $\mathcal{I}$ one has that $E[t(0,1) \mid \mathcal{I}] = E[t(0) \mid \mathcal{I}] = \alpha$. Thus, conditioned on the set of stable paths $\mathcal{E}_2$, the station preserves pathwise means.

3.3 Probabilistic assumptions

We work on a stationary probability quadruple $(\Omega, \mathcal{F}, P, \theta)$, see §2.2. We list the probabilistic assumptions that will be in force at one point of the paper or another, as specified at the appropriate place. They are stated in the context of the tandem model of §3.1. Occasionally they will also be used in the context of a single station model, in which case one has to adapt the notations accordingly.

(i) The inter-arrival process $\{t(n,0), n \in \mathbb{Z}\}$ is ergodic, with $E[t(0,0)] < \infty$.

(ii) The services $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ are i.i.d., with $E[s(0,0)] < \infty$.

(iii) The processes $\{t(n,0), n \in \mathbb{Z}\}$ and $\{s(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ are independent.

(iv) $E[s(0,0)] < E[t(0,0)]$.

(v) $P\{s(0,0) \neq E[s(0,0)]\} > 0$.

(vi) $\exists\, a > 0$ such that $E[s(0,0)^{3+a}] < \infty$.

(vii) $\exists\, \nu,\ E[s(0,0)] < \nu < E[t(0,0)],\ \ E\left[\ \sup_{n \in \mathbb{N}^*} [\sum_{i=-n}^{-1} \nu - t(i,0)]^+\ \right] < \infty$.

Condition (iv) ensures stability (see §3.2). If condition (v) is not verified, the model becomes trivial: If $E[t(0,0)] > E[s(0,0)]$, then $P\{t(n,1) \geq E[s(0,0)]\} = 1$ for every $n \in \mathbb{Z}$; and it is easy to see that for all $k \geq 1$ and $n \in \mathbb{Z}$, $t(n,k) = t(n,1)$. In other words, every departure process from a stable deterministic server queue is a fixed point for the queue. Conditions (vi) and (vii) are required by our method of proof.

We note that condition (vii) is slightly stronger than $E[w(0,0)] < \infty$. In general, a model satisfying conditions (i)-(vi) need not verify $E[w(0,0)] < \infty$. Counter-examples can be obtained by adapting the model of Example 7.4. The following results, relating waiting times in a $G/GI/1/\infty$ queue and a $G/D/1/\infty$ queue (D for Deterministic), are proved in [11]. If $E[s(0,0)^2] < \infty$, then, setting $\beta = E[s(0.0)]$,

$$E[w(0,0)] < \infty \quad \Longrightarrow \quad E[\sup_{n\geq 1}[\sum_{i=-n}^{-1} \beta - t(i,0)]^+ \,] < \infty \tag{9}$$

$$\exists \nu > \beta,\; E[\sup_{n\geq 1}[\sum_{i=-n}^{-1} \nu - t(i,0)]^+ \,] < \infty \quad \Longrightarrow \quad E[w(0,0)] < \infty\,.$$

Condition (vii) is satisfied, for example, by the deterministic process: $P\{t(0,0) = \alpha\} = 1$. The main result of the paper (that of the existence of fixed points) will be proved under assumptions (i)-(vi). However, in the course of the proof, we need assumption (vii). This is because we get at fixed points by considering a tandem network whose inter-departure processes are required to have a waiting time of finite mean. Thus, we shall see that condition (vii) is not required to establish the existence of fixed points; rather, it is only needed for our method of proof.

4 The Queueing Operator

We consider a single server station with a random service process s in $\mathcal{L}(\mathcal{R}_+^{\mathbb{Z}})$. The station can be viewed as an operator, say Φ, from $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ into itself, mapping an inter-arrival process into an inter-departure process. When the service and the arrival processes are independent, it can also be viewed as an operator of $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$ into itself, taking distributions of inter-arrival processes into distributions of inter-departure processes. Indeed, given the independence between arrivals and services, the distribution of the inter-departure process does not depend on the representatives of the service and inter-arrival processes. Thus we shall use Φ for both the mapping of random processes and of their corresponding distributions. (From now on we shall always assume that condition (iii) of §3.3 is in effect.) Denote by Φ^k the k-th iterate of the operator Φ.

It follows from (4)-(5) that Φ is measurable on $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$. Further, if S is ergodic, then Loynes' results assert that Φ preserves stationarity: $\Phi(\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})) \subset \mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$, and ergodicity: $\Phi(\mathcal{M}_e(\mathbb{R}_+^{\mathbb{Z}})) \subset \mathcal{M}_e(\mathbb{R}_+^{\mathbb{Z}})$. And Φ preserves means in the following sense (β is the mean service time):

$$\begin{cases} \forall \alpha > \beta, & \Phi:\ \mathcal{M}_e^{\alpha}(\mathbb{R}_+^{\mathbb{Z}}) \longrightarrow \mathcal{M}_e^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})\,. \\ \forall \alpha < \beta, & \Phi:\ \mathcal{M}_e^{\alpha}(\mathbb{R}_+^{\mathbb{Z}}) \longrightarrow \{S\}\ . \end{cases} \tag{10}$$

If S is i.i.d. and nonconstant, we also have $\Phi(\mathcal{M}_e^{\beta}(\mathbb{R}_+^{\mathbb{Z}})) = \{S\}$.

Having defined the queueing operator Φ, we investigate some of its properties in the next few sections. In particular, we isolate a compact and convex set $\mathcal{D}^\alpha$ on which Φ is linear and weakly continuous, and such that $\Phi(\mathcal{M}_e^{\alpha}(\mathbb{R}_+^{\mathbb{Z}}))) \subset \mathcal{D}^\alpha$. This is an essential step in establishing the existence of fixed points.

4.1 A convex and compact set: $\mathcal{D}^\alpha$

Let $S \in \mathcal{M}_e(\mathbb{R}_+^{\mathbb{Z}})$ be a service process distribution. Let $\mathcal{D}^\alpha$ be the set of probability measures over $\mathbb{R}_+^{\mathbb{Z}}$ which are stationary, have one-dimensional marginals of mean less or equal to α, and which strongly dominate the service process distribution (see §8). That is:

$$\mathcal{D}^\alpha = \bigcup_{0 \le x \le \alpha} \mathcal{M}_s^x(\mathbb{R}_+^{\mathbb{Z}}) \cap \left\{ X \in \mathcal{M}(\mathbb{R}_+^{\mathbb{Z}}) \mid X \ge_{st} S \right\} . \tag{11}$$

Lemma 4.1. *The set $\mathcal{D}^\alpha$ is convex and compact with respect to the weak topology. Furthermore, if the service process is ergodic and if $\alpha > \beta$, where β is the mean service time, then $\Phi(\mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})) \subset \mathcal{D}^\alpha$.*

Proof. We prove compactness first. The set $\mathcal{D} = \left\{ U \mid U \in \mathcal{M}(\mathbb{R}_+^{\mathbb{Z}}),\ U \ge_{st} S \right\}$ is closed in the weak topology by a direct application of Proposition 8.2. Let us prove that the set $\mathbb{M} = \cup_{0 \le x \le \alpha} \mathcal{M}_s^x(\mathbb{R}_+^{\mathbb{Z}})$ is compact. First of all, $\mathbb{M}$ is tight. Indeed, fix $\varepsilon > 0$ and consider the set $K = \prod_{i \in \mathbb{Z}} [0, 2^{|i|+2}/\varepsilon]$ which is compact in the product topology according to Tychonov's Theorem. It is immediate to check that for all $U \in \mathbb{M}$, we have $U\{K\} \ge 1 - \alpha\varepsilon$. We conclude that $\mathbb{M}$ is tight, hence sequentially compact according to Prohorov's Theorem. Any limit of elements of $\mathbb{M}$ is in $\mathbb{M}$: indeed, it is stationary (easy verification) and has a one dimensional mean which is less than α by Fatou's Lemma. We conclude that $\mathbb{M}$ is compact. Now, $\mathcal{D}^\alpha$ is compact since it is the intersection of a closed set and a compact set.

For convexity, consider $U_1, \ldots, U_k \in \mathcal{D}^\alpha$ and reals $\alpha_1, \ldots, \alpha_k \in [0, 1]$ with $\sum_i \alpha_i = 1$. It suffices to show that $U = \sum_i \alpha_i U_i \ge_{st} S$, since the rest is obvious. But, for $i = 1, \ldots, k$, and for all increasing functions $f : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}_+$,

$$\int f dU_i \ge \int f dS \implies \sum_{i=1}^{k} \alpha_i \int f dU_i \ge \int f dS \implies \int f dU \ge \int f dS ,$$

which completes the proof.

Let T be in $\mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$ and let $\{t(n), n \in \mathbb{Z}\}$ be distributed as T. If S is ergodic and α is strictly greater than the mean service time, then by Loynes' results, $\Phi(T) \in \mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$. From Equation (5), we have $t(n,1) = s(n+1) + [t(n) - s(n) - w(n)]^+$. Thus inter-departure times strongly dominate service times; i.e. $\Phi(T) \ge_{st} S$. Hence $\Phi(\mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})) \subset \mathcal{D}^\alpha$. $\square$

As a remark, we note that Φ also maps the set $\tilde{\mathcal{D}}^\alpha = \cup_{0 \le x \le \alpha} \mathcal{M}_e^x(\mathbb{R}_+^{\mathbb{Z}}) \cap \left\{ X \in \mathcal{M}(\mathbb{R}_+^{\mathbb{Z}}) \mid X \ge_{st} S \right\}$ into itself. However the set $\tilde{\mathcal{D}}^\alpha$ is neither convex nor compact. Indeed, a linear combination of ergodic measures is not ergodic, and a weak limit of ergodic processes is not necessarily ergodic (see Example 7.3). This is the reason why we have to work on the set $\mathcal{D}^\alpha$.

4.2 Linearity and continuity of the queueing operator

Since the queueing action is defined on sample paths, the fact that Φ is 'linear' on $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$ is immediate. However, it needs a careful definition and an interpretation. For $U_1, U_2 \in$

$\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$ and $p \in [0,1]$, we have

$$\Phi(pU_1 + (1-p)U_2) = p\Phi(U_1) + (1-p)\Phi(U_2) .$$

The interpretation is as follows. Let s, u_1 and u_2 be independent random processes distributed respectively as S, U_1 and U_2. Take δ to be a Bernouilli r.v., independent of s, u_1, u_2, such that $P\{\delta = 1\} = p = 1 - P\{\delta = 2\}$. Then the process c, defined as $c(\omega) = u_{\delta(\omega)}(\omega)$, $\omega \in \Omega$, has the distribution $pU_1 + (1-p)U_2$ and $\Phi(c)$ has the distribution $p\Phi(U_1) + (1-p)\Phi(U_2)$.

We now investigate the continuity properties of Φ with respect to the weak topology. First, we prove that Φ is continuous on one-sided infinite inter-arrival processes (adapting the definitions and notations of §4 in an obvious way).

Lemma 4.2. *Let u and $u_k, k \in \mathbb{N}$ in $\mathcal{L}(\mathbb{R}_+^{\mathbb{N}})$ be inter-arrival processes such that $u_k \xrightarrow{w} u$. Let $s \in \mathcal{L}(\mathbb{R}_+^{\mathbb{N}})$ be the service process. Suppose that the initial workloads verify $w(0, u_k) \xrightarrow{w} w(0, u)$. Then $\Phi(u_k) \xrightarrow{w} \Phi(u)$ and $\{w(n, u_k), n \in \mathbb{N}\} \xrightarrow{w} \{w(n, u), n \in \mathbb{N}\}$.*

Proof. For any inter-arrival process $v = \{v(n), n \in \mathbb{N}\}$ the recursion in Eqn (4) implies that, for all $n > 0$,

$$w(n, v) \quad = \quad \max \Big(\max_{j=1}^{n-1} \sum_{i=j}^{n-1} (s(i) - v(i)) \ , \ w(0, v) + \sum_{i=0}^{n-1} (s(i) - v(i)), 0 \Big) .$$

The form of the above expression implies that the mapping of $\{w(0, v), v(0), \cdots v(n-1)\}$ into $\{w(0, v), \cdots, w(n, v)\}$ is continuous. (Observe that for this to hold it is necessary to work with an arrival process indexed by $\mathbb{N}$ and not by $\mathbb{Z}$.) Since $u_k \xrightarrow{w} u$ and $w(0, u_k) \xrightarrow{w} w(0, u)$, the continuous mapping theorem (Theorem 5.2 in [5]) implies that $\{w(0, u_k), \cdots, w(n, u_k)\} \xrightarrow{w} \{w(0, u), \cdots, w(n, u)\}$ for every $n \in \mathbb{N}$. Thus $\{w(n, u_k), n \in \mathbb{N}\} \xrightarrow{w} \{w(n, u), n \in \mathbb{N}\}$.

For any arrival process v, Eqn (5) states that $\Phi(v)(n) = s(n+1) + [v(n) - s(n) - w(n)]^+$. Again by invoking the continuous mapping theorem, we conclude that $\Phi(u_k) \xrightarrow{w} \Phi(u)$. □

The weak continuity of Φ for two-sided inter-arrival processes requires extra assumptions. These are stated in the next theorem.

Theorem 4.3. *Assume the following conditions hold: (1) the service process s in $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ and the inter-arrival processes $u_k, k \in \mathbb{N}$ and u in $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ are all stationary, (2) u_k and u are independent of s, (3) $E[s(0)] < \infty$ and $E[s(0)] < E[u_k(0)]$, (4) $E[s(0) | \mathcal{T}] < E[u(0) | \mathcal{T}]$ a.s., where $\mathcal{T}$ is the invariant σ-algebra, and (5) $u_k \xrightarrow{w} u$. Then $\Phi(u_k) \xrightarrow{w} \Phi(u)$ and $\{w(n, u_k), n \in \mathbb{Z}\} \xrightarrow{w} \{w(n, u), n \in \mathbb{Z}\}$.*

Theorem 4.3 is a slight generalization of a result due to Borovkov, see Chapter 1.11 in [6] or Chapter 4 in [7]. Borovkov proves his result assuming that u and s are ergodic and that the stability condition $E[s(0)] < E[u(0)]$ holds. As our proof follows the original argument very closely, we have postponed it until the Appendix, see §7. We also show in the Appendix, Example 7.3, that Theorem 4.3 is not true if we do not assume that $E[s(0) \mid \mathcal{T}] < E[u(0) \mid \mathcal{T}]$ a.s. We are now ready to prove next theorem.

Theorem 4.4. *Let s be an i.i.d. service process of finite mean such that $P\{s(0) \neq E[s(0)]\} > 0$. Then the operator Φ is weakly continuous on $\mathcal{D}^\alpha$, for $\alpha > E[s(0)]$.*

Proof. Let $u_k, k \in \mathbb{N}$, and u be inter-arrival processes in $\mathcal{D}^\alpha$, independent of s, and such that $u_k \xrightarrow{w} u$. We prove that $\Phi(u_k) \xrightarrow{w} \Phi(u)$.

Let us first consider the case $U = S$. Since $\Phi(S) = S$, we need to show that $\Phi(u_k) \xrightarrow{w} s$. Now for $n \in \mathbb{Z}$ and $K \in \mathbb{N}^*$, Eqn (8) gives

$$w(n, u_k) = [\ \sup_{i\geq 1}(\sum_{j=1}^{i} s(n-j) - u_k(n-j))\ ]^+ \geq [\ \sup_{1\leq i\leq K}(\sum_{j=1}^{i} s(n-j) - u_k(n-j))\ ]^+ .$$

Let $\tilde{s}$ be distributed as s and independent of s. Using the continuous mapping theorem we get that for each n, as k goes to ∞,

$$[\ \sup_{1\leq i\leq K} \sum_{j=1}^{i} s(n-j) - u_k(n-j)\ ]^+ \ \xrightarrow{w}\ [\ \sup_{1\leq i\leq K} \sum_{j=1}^{i} s(n-j) - \tilde{s}(n-j)\ ]^+ .$$

Since $(s - \tilde{s})$ is i.i.d., symmetric and nonconstant, $\sum_{i=1}^{n}(s(i) - \tilde{s}(i))$ is a null-recurrent random walk. Therefore $[\ \sup_{1\leq i\leq K}(\sum_{j=1}^{i} s(n-j) - \tilde{s}(n-j))\ ]^+$ converges to ∞ almost surely, as K goes to ∞. We conclude that $\lim_k w(n, u_k) = +\infty$ almost surely. Equation (5) implies that $\{\Phi(u_k)(n), n \in \mathbb{Z}\} \to \{s(n+1), n \in \mathbb{Z}\}$ almost surely, hence $\Phi(U_k) \xrightarrow{w} S$.

Now consider an arbitrary $U \in \mathcal{D}^\alpha$ and suppose that $u_k \xrightarrow{w} u$. Since U belongs to $\mathcal{D}^\alpha$, we have $U \geq_{st} S$. According to Proposition 8.4, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and two random processes $\tilde{u} \sim U$ and $\tilde{s} \sim S$ such that $\tilde{u} \geq \tilde{s}$, $\tilde{P}$-a.s. In particular,

$$\lim_n \frac{1}{n} \sum_{i=1}^{n} \tilde{s}(i) \leq \lim_n \frac{1}{n} \sum_{i=1}^{n} \tilde{u}(i) .$$

But the left and right hand sides of the above inequality have the same distribution as $E[s(0)]$ and $E[u(0) \mid \mathcal{T}]$ respectively. We conclude that

$$E[s(0)] \leq E[u(0) \mid \mathcal{T}] \quad P - a.s. \tag{12}$$

Define the disjoint θ-invariant events $A = \{E[s(0)] = E[u(0) \mid \mathcal{T}]\}$ and $B = \{E[s(0)] < E[u(0) \mid \mathcal{T}]\}$. Write $\Phi 1\!\!1_A$ (resp. $\Phi 1\!\!1_B$) for the operator from $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ into $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ which equals Φ on A (resp. on B) and equals the identity on B (resp. on A). From Theorem 4.3, $\Phi 1\!\!1_B(u_k) \xrightarrow{w} \Phi 1\!\!1_B(u)$. And from the first part of the proof, $\Phi 1\!\!1_A(u_k) \xrightarrow{w} \Phi 1\!\!1_A(u)$. We conclude that $\Phi(u_k) \xrightarrow{w} \Phi(u)$. $\square$

The set of all finite signed measures on $\mathbb{R}^{\mathbb{Z}}$ is a $\mathbb{R}$-vector space. Equipped with the topology of weak convergence, it is a locally convex topological vector space. The set of probability measures $\mathcal{D}^\alpha$ is a compact and convex subset of this vector space. Under the assumptions of Theorem 4.4, Φ is continuous on $\mathcal{D}^\alpha$. This allows one to apply the Schauder-Tychonov Fixed Point Theorem ([26], Chapter 5): There exists U in $\mathcal{D}^\alpha$ such that $\Phi(U) = U$. However, we already know that S belongs to $\mathcal{D}^\alpha$ and verifies $\Phi(S) = S$. The problem is to show that Φ admits other fixed points. This is done in §6.

Theorems 4.3 and 4.4 state weak convergence results. We now investigate the problem of convergence in expectation. Under the conditions of Theorem 4.3, Loynes' results tell us that $E[\Phi(u)(0)] = E[u(0)]$. Hence, $\lim_k E[\Phi(u_k)(0)] = E[\Phi(u)(0)]$ if, and only if, $\lim_k E[u_k(0)] = E[u(0)]$. For the workload, the situation is more complex. In the case of a $GI/GI/1/\infty$ queue (that is the inter-arrival process is i.i.d.), it is proved in [2] that if $E[u_k(0)^2] \to E[u(0)^2] < \infty$, then $\lim_k E[w(0, u_k)] = E[w(0, u)]$. In Lemma 4.5, we prove a result on the lim inf. For the sake of completeness, the Example 7.4 of the Appendix presents a situation in which $\lim_k E[\Phi(u_k)(0)] = E[\Phi(u)(0)]$, but $\lim_k E[w(0, u_k)] > E[w(0, u)]$.

Lemma 4.5. *Consider a stationary service process s and stationary inter-arrival processes $u_k, k \in \mathbb{N}$, and u. Suppose that $u_k \xrightarrow{w} u$. Then,*

$$\liminf E[w(0, u_k)] \geq E[w(0, u)] \,. \tag{13}$$

The result holds even if $E[w(0, u)] = +\infty$.

Proof. When the required assumptions are met, this is a direct consequence of Theorem 4.3 and Theorem 5.3 in [5]. Let us prove the result in the general case.

First, observe that the truncated processes ${}_{-l}u_k = \{u_k(n), n \geq -l\}$ converge weakly to ${}_{-l}u$ for any $l \in \mathbb{N}$. Setting $w(-l, {}_{-l}u_k) = w(-l, {}_{-l}u) = 0$, one obtains (via Lemma 4.2) for a given l, that $w(0, {}_{-l}u_k) \xrightarrow{w} w(0, {}_{-l}u)$ as k goes to $+\infty$. Using Theorem 5.3 in [5], we get

$$\liminf_k E[w(0, {}_{-l}u_k)] \geq E[w(0, {}_{-l}u)] \,. \tag{14}$$

Now, for any inter-arrival process v, Loynes' result asserts that $w(0, {}_{-l}v) \nearrow w(0, v)$ as l goes to ∞. Therefore, by monotone convergence, we get that

$$E[w(0, {}_{-l}u_k)] \nearrow_l E[w(0, u_k)] \text{ and } E[w(0, {}_{-l}u)] \nearrow_l E[w(0, u)] \,. \tag{15}$$

Together, (14) and (15) give

$$\liminf_k E[w(0, u_k)] \geq \liminf_k E[w(0, {}_{-l}u_k)] \geq E[w(0, {}_{-l}u)] \xrightarrow{l} E[w(0, u)] \,.$$

This concludes the proof. $\square$

5 Uniqueness and Convergence

This section recalls several relevant results about the uniqueness and attractiveness of fixed points, should they exist. There are two purposes this section serves. First, its inclusion gives a more complete picture of fixed point theorems for the queueing operator. Second, and more important, some of the techniques used in the proofs of uniqueness and attractiveness will be needed in Section 6, especially for characterizing the set of values α for which ergodic fixed points of mean α exist.

All results in this section are based upon the papers [10, 24], except for Theorem 5.3 of Section 5.2.

5.1 Uniqueness

Write $\mathcal{M}_s^{(1)}(\mathbb{R}^{\mathbb{Z}})$ for the set of stationary measures whose one dimensional marginals are integrable. Let X and Y be in $\mathcal{M}_s^{(1)}(\mathbb{R}^{\mathbb{Z}})$ and let $\mathcal{D}(X,Y) = \{(x,y) \in \mathcal{L}_s(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}) : x \sim X, y \sim Y\}$. That is, $\mathcal{D}(X,Y)$ is the set of jointly stationary processes whose marginals are distributed as X and Y.

The $\bar{\rho}$ distance between X and Y is given by

$$\bar{\rho}(X,Y) = \inf_{(x,y)\in\mathcal{D}(X,Y)} E[\,|x(0) - y(0)|\,], \tag{16}$$

where $x = \{x(n), n \in \mathbb{Z}\}$ and $y = \{y(n), n \in \mathbb{Z}\}$. See Gray [17], Chapter 8, for a proof that $\bar{\rho}$ is indeed a distance and that the infimum in (16) is attained in $\mathcal{D}(X,Y)$.

Convergence in the $\bar{\rho}$ distance implies weak convergence, but the converse is not true. In particular, the set $\mathcal{D}^{\alpha}$ defined in §4.1 is not compact in the $\bar{\rho}$ topology. To see this, consider the ergodic distributions $U_n, n \in \mathbb{N}^*$, defined as follows. With L denoting the left shift operator (see (1)) and v_n denoting the periodic sequence

$$(\cdots, 2, \underbrace{0, \cdots, 0}_{n}, \underbrace{2, \cdots, 2}_{n}, 0, \cdots)$$

of period $2n$, let $U_n\{\ L^i(v_n)\ \} = 1/2n,\ i = 0, \ldots, 2n-1$. Thus the sample paths of the process u_n are uniformly shifted versions of the sequence v_n. It is immediate that $U_n \xrightarrow{w} U$, where $U\{\ (\ldots,0,0,\ldots)\ \} = U\{\ (\ldots,2,2,\ldots)\ \} = 1/2$, but $\bar{\rho}(U_n, U) = 1, \forall n$.

Proposition 5.1. *Consider a queue with service process s and two different inter-arrival processes: $t_A = \{t_A(n), n \in \mathbb{Z}\}$ and $t_B = \{t_B(n), n \in \mathbb{Z}\}$ possibly of different means. For both arrivals suppose that conditions (i)-(v) of §3.3 are verified. Then,*

$$\bar{\rho}(\Phi(T_A), \Phi(T_B)) \ \le\ \bar{\rho}(T_A, T_B)\,. \tag{17}$$

Furthermore, if $E[t_A(0)] = E[t_B(0)]$ and $T_A \neq T_B$, there exists $k \in \mathbb{N}^*$ such that

$$\bar{\rho}(\Phi^k(T_A), \Phi^k(T_B)) \ <\ \bar{\rho}(T_A, T_B)\,. \tag{18}$$

Proof. We recall the elegant proof of (17), due to Chang [10]. For u, v in $\mathcal{L}(\mathbb{R}^{\mathbb{Z}})$, define $u \vee v = \{\max(u(n), v(n)), n \in \mathbb{Z}\}$. The monotonicity property of (7) implies that

$$(\Phi(t_A) \vee \Phi(t_B))(n) \le \Phi(t_A \vee t_B)(n), \ \ \forall n \in \mathbb{Z}\,.$$

In particular, the above inequality holds in expectation. Since $E[t_A \vee t_B(0)] \ge \max(E[t_A(0)], E[t_B(0)]) > E[s(0,0)]$, by Loynes' results one obtains that $E[\Phi(t_A \vee t_B)(0)] = E[t_A \vee t_B(0)]$, $E[\Phi(t_A)(0)] = E[t_A(0)]$, and $E[\Phi(t_B)(0)] = E[t_B(0)]$. Hence,

$$\begin{aligned} 2E[t_A \vee t_B(0)] \ &-\ E[t_A(0)] - E[t_B(0)] \\ &=\ 2E[\Phi(t_A \vee t_B)(0)] - E[\Phi(t_A)(0)] - E[\Phi(t_B)(0)] \\ &\ge\ 2E[\Phi(t_A) \vee \Phi(t_B)(0)] - E[\Phi(t_A)(0)] - E[\Phi(t_B)(0)] \\ E[\,|t_A(0) - t_B(0)|\,] \ &\ge\ E[\,|\Phi(t_A)(0) - \Phi(t_B)(0)|\,]\,. \end{aligned} \tag{19}$$

If the joint distribution of (t_A, t_B) is such that $\bar{\rho}(T_A, T_B) = E[\,|t_A(0) - t_B(0)|\,]$, then

$$\bar{\rho}(T_A, T_B) = E[\,|t_A(0) - t_B(0)|\,] \geq E[\,|\Phi(t_A)(0) - \Phi(t_B)(0)|\,] \geq \bar{\rho}(\Phi(T_A), \Phi(T_B)) \,.$$

This proves (17). Eqn (18) was proved in [10] for services with unbounded support. This restriction is removed in [24]. Section 5.3 outlines the method of proof of [24]. $\square$

Corollary 5.2. *Consider a queue verifying assumptions (i)-(v). The associated operator Φ has at most one fixed point in $\mathcal{M}_e^\alpha(\mathbb{R}_+^\mathbb{Z})$, for $\alpha > E[s(0)]$.*

Proof. Assume that T_A and T_B are two different ergodic fixed points of mean α. Since $\Phi^k(T_A) = T_A, \Phi^k(T_B) = T_B$, we have $\bar{\rho}(\Phi^k(T_A), \Phi^k(T_B)) = \bar{\rho}(T_A, T_B)$ for all k, which contradicts (18). $\square$

5.2 Uniqueness implies ergodicity

Consider the set (sp stands for 'stationary pathwise'):

$$\mathcal{L}_{sp}^\alpha(\mathbb{R}_+^\mathbb{Z}) = \left\{ x \in \mathcal{L}_s^\alpha(\mathbb{R}_+^\mathbb{Z}) \;:\; \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x(i) = \alpha \;\; P-a.s. \right\}$$

and define $\mathcal{M}_{sp}^\alpha(\mathbb{R}_+^\mathbb{Z})$ accordingly. If $X \in \mathcal{M}_{sp}^\alpha(\mathbb{R}_+^\mathbb{Z})$ is not ergodic, then its ergodic components all have one-dimensional mean α.

Proposition 5.1 and Corollary 5.2 continue to hold for inter-arrival processes in $\mathcal{L}_{sp}^\alpha(\mathbb{R}_+^\mathbb{Z})$ instead of $\mathcal{L}_e^\alpha(\mathbb{R}_+^\mathbb{Z})$ as their proofs are entirely based on pathwise arguments. In particular, there is at most one fixed point in $\mathcal{M}_{sp}^\alpha(\mathbb{R}_+^\mathbb{Z})$. In fact, we have the following stronger result.

Theorem 5.3. *Under the assumptions of Corollary 5.2, if $C \in \mathcal{M}_{sp}^\alpha(\mathbb{R}_+^\mathbb{Z})$ is a fixed point for the queue, then it is necessarily ergodic; that is, $C \in \mathcal{M}_e^\alpha(\mathbb{R}_+^\mathbb{Z})$.*

Proof. Suppose that the ergodic decomposition of C is given by

$$C = \int U \,\Gamma(dU),$$

where Γ is some distribution on $\mathcal{M}_e^\alpha(\mathbb{R}_+^\mathbb{Z})$. That is, $\Gamma \in \mathcal{M}(\mathcal{M}_e^\alpha(\mathbb{R}_+^\mathbb{Z}))$.

Now consider two independent random variables x and y in $\mathcal{L}(\mathcal{M}_e^\alpha(\mathbb{R}_+^\mathbb{Z}))$ distributed as Γ. If C is not ergodic, Γ is not a point mass. By the independence of x and y, $P\{x(\omega) \neq y(\omega)\} > 0$. Therefore, by the contractiveness of the $\bar{\rho}$ metric, there is a finite random variable k such that on the event $\{x \neq y\}$, we have

$$\bar{\rho}(x(\omega),\; y(\omega)) > \bar{\rho}\left(\Phi^{k(\omega)}(x(\omega)),\; \Phi^{k(\omega)}(y(\omega))\right) \,.$$

Let $K \in \mathbb{N}$ be such that $P\{k \leq K\} > 0$. Then,

$$E\left[\bar{\rho}(x, y)\right] > E\left[\bar{\rho}\left(\Phi^K(x),\; \Phi^K(y)\right)\right] \,. \tag{20}$$

The finiteness of the above expectations is not an issue since for any two measures P and Q in $\mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$, we have $\bar{\rho}(P,Q) \leq E[p(0)] + E[q(0)] = 2\alpha$.

Let us prove that $\Phi(x)$ is distributed as Γ. This is immediate if we show that $\Gamma = \Gamma \circ \Phi^{-1}$. Consider the implications

$$\begin{aligned} C = \Phi(C) \quad & \overset{(a)}{\Longleftrightarrow} \quad \int U \ \Gamma(dU) = \int \Phi(U) \ \Gamma(dU) = \int U \ \Gamma \circ \Phi^{-1}(dU) \\ & \overset{(b)}{\Longrightarrow} \quad \Gamma = \Gamma \circ \Phi^{-1} . \end{aligned}$$

Equivalence (a) follows from the linearity of Φ and (b) follows from the uniqueness of ergodic decompositions. Hence $\Phi(x)$ and $\Phi(y)$ are both distributed as Γ, and are *independent* (because, given the service process, Φ is a measurable map from $\mathcal{M}_e^\alpha$ into itself). Therefore $E\left[\bar{\rho}(x,y)\right] = E\left[\bar{\rho}\left(\Phi^k(x),\ \Phi^k(y)\right)\right]$ for every k, contradicting (20). □

5.3 Convergence

Theorem 5.4. *Consider the tandem model of §3.1 under assumptions (i)-(v) of §3.3. Let $\alpha > E[s(0,0)]$ be such that there exists $U \in \mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$ such that $\Phi(U) = U$. Then for any initial inter-arrival process $T \in \mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$, $\bar{\rho}(\Phi^k(T), U) \longrightarrow 0$ (and hence $\Phi^k(T) \xrightarrow{w} U$) as k goes to infinity.*

We outline the idea of the proof. For details, see Prabhakar [24]. Consider two inter-arrival processes t_A and t_B such that $E[t_A(0)] = E[t_B(0)]$ and $T_A \neq T_B$. Using $E[t_A(0)] = E[t_B(0)]$ and Birkhoff's Ergodic Theorem, we obtain

$$\begin{aligned} \lim_n \frac{1}{n} \sum_{i=1}^n (t_A(i) - t_B(i)) \mathbb{1}_{\{t_A(i) \geq t_B(i)\}} &= \lim_n \frac{1}{n} \sum_{i=1}^n (t_B(i) - t_A(i)) \mathbb{1}_{\{t_B(i) \geq t_A(i)\}} \\ E[|t_A(0) - t_B(0)|] &= 2 \times \lim_n \frac{1}{n} \sum_{i=1}^n (t_A(i) - t_B(i)) \mathbb{1}_{\{t_A(i) \geq t_B(i)\}} \end{aligned}$$

A similar relationship holds for $E[|\Phi(t_A)(0) - \Phi(t_B)(0)|]$. Given this, we can reinterpret (19) as follows:

$$\begin{aligned} & \lim_n \frac{1}{n} \sum_{i=1}^n (t_A(i) - t_B(i)) \mathbb{1}_{\{t_A(i) \geq t_B(i)\}} \\ & \qquad \geq \lim_n \frac{1}{n} \sum_{i=1}^n (\Phi(t_A)(i) - \Phi(t_B)(i)) \mathbb{1}_{\{\Phi(t_A)(i) \geq \Phi(t_B)(i)\}} . \end{aligned} \tag{21}$$

In [24], a pathwise coupling argument is used to prove (21). The argument is precise enough to prove both (18) and Theorem 5.4.

6 Existence

Throughout this section we consider the tandem model of §3.1 under conditions (i)-(v) of §3.3. Recall that S denotes the distribution of $\{s(n,k), n \in \mathbb{Z}\}$. We use the notations

$$\alpha = E[t(0,0)] > E[s(0,0)] = \beta\,.$$

Our aim is to prove Theorem 6.4. We do this by considering Cesaro averages of the inter-departure processes from the tandem. We prove that a weak subsequential limit of these averages is a non-trivial fixed point for the queueing operator.

Given the initial inter-arrival process T, define the measures $C_k \in \mathcal{M}_s^\alpha(\mathbb{R}_+^{\mathbb{Z}})$, $k \in \mathbb{N}^*$, as the following Cesaro averages

$$C_k = \frac{1}{k}\sum_{i=1}^{k} \Phi^i(T). \tag{22}$$

For any event $A \in \mathcal{B}$, we have $C_k\{A\} = \sum_{i=1}^{k} \Phi^i(T)\{A\}/k$. Note that C_k is not necessarily ergodic. It is ergodic if, and only if, the distributions $\Phi^i(T)$ are all equal. (For an interpretation of C_k, see the first paragraph of §4.2.)

Lemma 6.1. *The sequence C_n admits weak subsequential limits in $\mathcal{D}^\alpha$. If C is such a weak limit then we have $\Phi(C) = C$.*

Proof. Since the distributions $\Phi^i(T)$ belong to $\mathcal{D}^\alpha$ and $\mathcal{D}^\alpha$ is convex, C_k belongs to $\mathcal{D}^\alpha$. By the compactness of $\mathcal{D}^\alpha$, there exists C in $\mathcal{D}^\alpha$ and a subsequence $C_{\sigma(n)}$ such that $C_{\sigma(n)} \xrightarrow{w} C$. The weak continuity of Φ on $\mathcal{D}^\alpha$ (Theorem 4.4) implies that $\Phi(C_{\sigma(n)}) \xrightarrow{w} \Phi(C)$. Now, by the 'linearity' of Φ,

$$\Phi(C_{\sigma(n)}) = \frac{1}{\sigma(n)}\sum_{i=1}^{\sigma(n)} \Phi^{i+1}(T) = C_{\sigma(n)} + \frac{\Phi^{\sigma(n)+1}(T)}{\sigma(n)} - \frac{\Phi(T)}{\sigma(n)}\,.$$

The above expression makes sense as an equality between signed measures. Denoting the total variation norm of signed measures by $\|.\|$, one obtains

$$\|\frac{\Phi^{\sigma(n)+1}(T)}{\sigma(n)} - \frac{\Phi(T)}{\sigma(n)}\| = \sup_{A\in\mathcal{B}} \mid \frac{\Phi^{\sigma(n)+1}(T)}{\sigma(n)}\{A\} - \frac{\Phi(T)}{\sigma(n)}\{A\} \mid \;\le\; \frac{2}{\sigma(n)}\,.$$

It follows that $(\Phi^{\sigma(n)+1}(T) - \Phi(T))/\sigma(n)$ converges to zero in total variation, and hence also weakly. We conclude that $\Phi(C_{\sigma(n)}) \xrightarrow{w} C$.

Now the uniqueness of the weak limit implies $\Phi(C) = C$. □

As was already discussed in §4.2, the service process S is a trivial fixed point for Φ. The next and most technical step is to show that the fixed point C obtained in Lemma 6.1 is different from S.

Let $s = \{s(n), n \in \mathbb{Z}\}$ in $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ be distributed as S. Below, we denote by $\{w(n,u), n \in \mathbb{Z}\}$ the waiting time process obtained by sending the inter-arrival process $u \in \mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ through the station with service process s. Let $c = \{c(n), n \in \mathbb{Z}\}$ in $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ be distributed as C and independent of s. By Birkhoff's Ergodic Theorem, we have

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} s(i) = \beta\,, \quad \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} c(i) = E[c(0) \mid \mathcal{T}] \quad a.s. \tag{23}$$

Since $C \in \mathcal{D}^\alpha$, we have $C \geq_{st} S$. Using the same argument as in the proof of Theorem 4.4 (see (12)), we obtain that $E[c(0) \mid \mathcal{T}] \geq \beta$, P-a.s. We want to rule out the possibility that $P\{E[c(0) \mid \mathcal{T}] = \beta\} > 0$. This is a consequence of the next few results.

Lemma 6.2. *We have $P\{w(0,c) = +\infty\} \geq P\{E[c(0) \mid \mathcal{T}] = \beta\}$.*

Proof. For $u \in \mathcal{L}(\mathbb{R}^I), I = \mathbb{N}$ or $\mathbb{Z}$, we use the notation $E[u \mid \mathcal{T}]$ for the random process $\{E[u(n) \mid \mathcal{T}], n \in I\}$. Let us first prove that

$$w(0,c) \geq_{icx} w\left(0, E[c \mid \mathcal{T}]\right)\,. \tag{24}$$

Using (37), we have $-c \geq_{cx} -E[c \mid \mathcal{T}]$. As c is independent of s, we obtain that $E[c \mid \mathcal{T}]$ is also independent of s (the r.v. $E[c \mid \mathcal{T}]$ is measurable with respect to the σ-algebra generated by c). Hence, we have $(s, -c) \geq_{cx} (s, -E[c|\mathcal{T}])$, see [27] Section 1.10. It implies that

$$s - c \geq_{cx} s - E[c \mid \mathcal{T}]\,.$$

It is known that $s - c \geq_{cx} s - E[c \mid \mathcal{T}]$ implies $w(0,c) \geq_{icx} w(0, E[c \mid \mathcal{T}])$, see for instance Section 4.2.7 in [4]. (This is a similar statement as the folk theorem "given the services, waiting times are minimized by deterministic inter-arrivals".)

For the sake of completeness, we recall the argument. Suppose that $E[w(n,c) \mid \mathcal{T}] \geq w(n, E[c \mid \mathcal{T}])$ for some n. Since the function $h(u,v) = [u+v]^+$ is convex, one obtains

$$\begin{aligned}
E[w(n+1,c) \mid \mathcal{T}] &= E\left[\,[w(n,c) + s(n) - c(n)]^+ \mid \mathcal{T}\,\right] \\
&\geq \left[\,E[w(n,c) \mid \mathcal{T}] + E[s(n) - c(n) \mid \mathcal{T}]\,\right]^+ \\
&\geq \left[\,w(n, E[c \mid \mathcal{T}]) + s(n) - E[c(n) \mid \mathcal{T}]\,\right]^+ \\
&= w(n+1, E[c \mid \mathcal{T}])\,,
\end{aligned}$$

where we have successively used Lindley's equation (4), convexity and Jensen's conditional inequality, the recursion assumption and Lindley's equation again.

For $l \in \mathbb{N}$, we consider the truncated processes ${}_{-l}c$ and $E[{}_{-l}c \mid \mathcal{T}]$. Suppose that $E[w(-l, {}_{-l}c) \mid \mathcal{T}] = w(-l, E[{}_{-l}c \mid \mathcal{T}]) = 0$. From the above argument one deduces that $E[w(0, {}_{-l}c) \mid \mathcal{T}] \geq w(0, E[{}_{-l}c \mid \mathcal{T}])$. Now, using (37), $w(0, {}_{-l}c) \geq_{cx} E[w(0, {}_{-l}c) \mid \mathcal{T}]$. By pasting these two results together, one obtains that $w(0, {}_{-l}c) \geq_{icx} w(0, E[{}_{-l}c|\mathcal{T}])$. Hence, for all increasing and convex function f such that the integrals are well defined, we have

$$\int f\left(w(0, {}_{-k}c)\right) dP \geq \int f\left(w(0, E[{}_{-k}c \mid \mathcal{T}])\right) dP\,. \tag{25}$$

Since this holds for each l and as l goes to infinity, $w(0, {}_{-l}c) \nearrow w(0,c)$ and $w(0, E[{}_{-l}c|\mathcal{T}]) \nearrow w(0, E[c\,|\,\mathcal{T}])$, the monotone convergence theorem implies that Eqn (25) holds in the limit. We conclude that (24) holds.

Proceeding, on the event $\{E[c(0) \mid \mathcal{T}] = \beta\}$, the stationarity of the process $E[c \mid \mathcal{T}]$ implies that $E[c\,|\,\mathcal{T}] = \{\cdots, \beta, \beta, \cdots\}$. Thus, on the event $\{E[c(0)\,|\,\mathcal{T}] = \beta\}$, Eqn (8) gives

$$w(0, E[c \mid \mathcal{T}]) = [\ \sup_{i=1}^{\infty} \sum_{j=1}^{i} (s(-j) - \beta)\ ]^+ .$$

Now, $E[s(0) - \beta] = 0$ and condition (v) in §3.3 states that $P\{s(0) - \beta \neq 0\} > 0$. Hence $\sum_{i=1}^{n}(s(-i) - \beta)$ is a null-recurrent random walk. We conclude that on the event $\{E[c(0) \mid \mathcal{T}] = \beta\}$, we have

$$w(0, E[c \mid \mathcal{T}]) = \sup_{i=1}^{\infty} \sum_{j=1}^{i} (s(-j) - \beta) = +\infty \ \ a.s.$$

But $w(0,c) \geq_{icx} w(0, E[c\,|\,\mathcal{T}])$. Therefore the previous result implies that $P\{w(0,c) = +\infty\} \geq P\{E[c(0) \mid \mathcal{T}] = \beta\}$. □

Suppose, for the moment, that $P\{E[c(0)\,|\,\mathcal{T}] = \beta\} > 0$. Then the above lemma implies that $E[w(0,c)] = \infty$. Let $c_{\sigma(l)}$ be distributed as $C_{\sigma(l)}$. Then Lemma 4.5 implies

$$E[w(0, c_{\sigma(l)})] \longrightarrow +\infty . \tag{26}$$

Now consider the previously-defined processes $\{t(n,k), n \in \mathbb{Z}\}$ and $\{s(n,k), n \in \mathbb{Z}\}$, $k \in \mathbb{N}$, distributed as $\Phi^k(T)$ and S respectively.

Define the average delay, W_l, experienced by customer 0 in going from station 1 to l as $W_l = \sum_{k=1}^{l} w(0,k)/l$. By linearity of expectations and the fact that $c_{\sigma(l)} \sim \sum_{k=1}^{\sigma(l)} \Phi^k(T)/\sigma(l)$, we get that

$$E[W_{\sigma(l)}] = \frac{1}{\sigma(l)} \sum_{k=1}^{\sigma(l)} E[w(0,k)] = E[w(0, c_{\sigma(l)})] .$$

It follows from (26) that $E[W_{\sigma(l)}] \to +\infty$. We proceed by recalling a needed result.

Theorem 6.3. *Consider the tandem model defined in §3. Suppose that conditions (i)-(vi) of §3.3 are verified. Then there exists a finite constant $M(\alpha)$ such that*

$$\frac{1}{n} \sum_{k=1}^{n} w(0,k) \overset{n \to \infty}{\longrightarrow} M(\alpha), \quad P - a.s. ; \tag{27}$$

where $M(\alpha) = \sup_{x \geq 0}(\gamma(x) - \alpha x)$, and the function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ depends only on the service process, is continuous, strictly increasing, concave and verifies $\gamma(0) = 0$.

If we assume furthermore that the initial inter-arrival process verifies assumption (vii) of §3.3, then the convergence in (27) also holds in L_1.

Theorem 6.3 is proved in [3]. For further properties of γ, please refer to [3, 16, 18]. Note that the limit $M(\alpha)$ depends on the initial inter-arrival process only through its mean.

In addition to the assumptions that have been made since the beginning of the section, Theorem 6.3 requires assumptions (vi) and (vii). To carry on, let us assume that these two conditions are also in force.

Theorem 6.3 implies that

$$\lim_l E[W_{\sigma(l)}] = \lim_l E[w(0, c_{\sigma(l)})] = M(\alpha) \ < \ \infty \, .$$

This contradicts (26). We conclude that $P\{E[c(0) \mid \mathcal{T}] = \beta\} = 0$. We are now ready to prove the following result.

Theorem 6.4. *Consider a queue verifying assumptions (i)-(vi) of §3.3 and let $\mathcal{S} \subset (\beta, \infty)$ be the set of values α for which the queue admits an ergodic fixed point of mean α. Then $\mathcal{S}$ is not empty.*

Proof. Write *supp* u for the support of the distribution of the random variable u. Consider the process c as defined in Lemma 6.1. By the ergodic decomposition theorem and the linearity of Φ, we have

$$C = \int_{\mathcal{M}_e(\mathbb{R}_+^{\mathbb{Z}})} X\Gamma(dX), \quad \Phi(C) = \int_{\mathcal{M}_e(\mathbb{R}_+^{\mathbb{Z}})} \Phi(X)\Gamma(dX) \, .$$

But C is a fixed point (from Lemma 6.1). Therefore, the uniqueness of ergodic decompositions and the mean preservation property of stable queues imply that

$$C_\alpha = \int_{\mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})} X\Gamma(dX)$$

is a fixed point for every α in the support of $E[c(0) \,|\, \mathcal{T}]$. By Theorem 5.3, the distributions C_α are ergodic. We deduce that

$$\textit{supp } E[c(0) \mid \mathcal{T}] \setminus \{\beta\} \ \subset \ \mathcal{S} \, . \tag{28}$$

At last, we can choose the initial inter-arrival process to satisfy condition (vii) of §3.3. In this case, we have *supp* $E[c(0) \,|\, \mathcal{T}] \cap (\beta, \infty) \neq \emptyset$ and we conclude that $\mathcal{S}$ is not empty. $\square$

To the best of our knowledge, this provides the first positive answer (apart from the case of exponential or geometric service times) to the intriguing question of the existence of non-trivial fixed points for the queueing operator.

6.1 Characterization of the set $\mathcal{S}$

Throughout the section, we consider a model verifying assumptions (i)-(vi) of §3.3. It is important to notice that we do not require condition (vii) to be verified.

Let $\mathcal{S}$ be the set of values α for which there exists an ergodic fixed point of mean α. In this section, we establish that $\mathcal{S}$ is unbounded and closed in the set (β, ∞), Theorem 6.7. We believe that $\mathcal{S} = (\beta, +\infty)$ but we have not been able to prove this last point, see Conjecture 6.9.

¿From now on, for $\alpha \in \mathcal{S}$, we denote by C_α the unique ergodic fixed point of mean α. We denote by $c_\alpha = \{c_\alpha(n), n \in \mathbb{Z}\}$ a process distributed as C_α and independent of the service process s.

When $\alpha \in \mathcal{S}$, we recall the result presented in Theorem 5.4: If U is an ergodic inter-arrival process of mean α, then $\overline{\rho}(\Phi^k(U), C_\alpha) \xrightarrow{k\to\infty} 0$ and $\Phi^k(U) \xrightarrow{w} C_\alpha$. Proposition 6.8 describes the limiting behavior from inputing an ergodic arrival process when the mean inter-arrival time does not belong to $\mathcal{S}$.

Lemma 6.5. *Let T be an ergodic inter-arrival process of mean $\alpha > \beta$. Let C be a subsequential weak limit of the Cesaro averages of $\Phi^k(T)$. Suppose that there exists a $\delta \in \mathcal{S} \cap (\beta, \alpha]$ (respectively $\delta \in \mathcal{S} \cap [\alpha, \infty)$), then we have $C_\delta \leq_{st} C$ and $w(0, c_\delta) \geq_{st} w(0, c)$ (respectively $C_\delta \geq_{st} C$ and $w(0, c_\delta) \leq_{st} w(0, c)$). Further, if $\mathcal{S} \cap [\alpha, \infty) \neq \emptyset$, then $E[c(0)] = \alpha$.*

Remark: In Theorem 6.7 we shall prove that both the sets $\mathcal{S} \cap (\beta, \alpha]$ and $\mathcal{S} \cap [\alpha, \infty)$ are not empty for every $\alpha \in (\beta, \infty)$.

Proof. Consider the case $\delta \in \mathcal{S} \cap [\alpha, \infty)$. The other case can be treated similarly. Let $t = \{t(n,0), n \in \mathbb{Z}\}$ be distributed as T. The process t_δ, equal to $\{(\delta/\alpha)t(n,0), n \in \mathbb{Z}\}$, is ergodic and of mean δ. At this mean Φ admits the ergodic fixed point C_δ. By Theorem 5.4, we have $\Phi^k(T_\delta) \xrightarrow{w} C_\delta$, and by construction $t(n,0) \leq t_\delta(n,0)$ for all n. Using the monotonicity properties of a queue (see (7)) we get that for all $n \in \mathbb{Z}$ and all $k \in \mathbb{N}$,

$$t(n,k) \leq t_\delta(n,k) \;\text{ and }\; w(n, \Phi^k(t)) \geq w(n, \Phi^k(t_\delta)) \,.$$

If σ is the subsequence along which C is defined, then

$$\frac{1}{\sigma(k)} \sum_{i=1}^{\sigma(k)} \Phi^i(T) \leq_{st} \frac{1}{\sigma(k)} \sum_{i=1}^{\sigma(k)} \Phi^i(T_\delta) \implies C \leq_{st} C_\delta \,. \tag{29}$$

The last implication is obtained by going to the limit and using Proposition 8.2. Similarly, one obtains that $w(0,c) \geq_{st} w(0,c_\delta)$.

Now, Fatou's Lemma implies $E[c(0)] \leq \alpha$. We wish to show that $E[c(0)] = \alpha$. Since we have $1/k \sum_{i=1}^k \Phi^i(T_\delta) \xrightarrow{w} C_\delta$ and since the one dimensional marginals converge in L_1 (trivially, since $E[1/k \sum_{i=1}^k \Phi^i(t_\delta)(0)] = \delta = E[c_\delta(0)]$), by Theorem 5.4 of [5] the family $\{1/k \sum_{i=1}^k \Phi^i(t_\delta)(0), k \in \mathbb{N}^*\}$ is uniformly integrable. But (29) shows that a subsequence of this family stochastically dominates the family $\{1/\sigma(k) \sum_{i=1}^{\sigma(k)} \Phi^i(t)(0), k \in \mathbb{N}^*\}$. Therefore $\{1/\sigma(k) \sum_{i=1}^{\sigma(k)} \Phi^i(t)(0), k \in \mathbb{N}^*\}$ is also uniformly integrable. Since it converges to $c(0)$ in distribution, we conclude (Theorem 5.4 of [5]) that $\lim_k E[1/\sigma(k) \sum_{i=1}^{\sigma(k)} \Phi^i(t)(0)] = E[c(0)] = \alpha$. ☐

Lemma 6.6. *The following statements are true:*
(a) For $\alpha_1, \alpha_2 \in \mathcal{S}$ and $\alpha_1 < \alpha_2$, $C_{\alpha_1} \leq_{st} C_{\alpha_2}$ and $w(0, c_{\alpha_1}) \geq_{st} w(0, c_{\alpha_2})$.
(b) For $\alpha \in \mathcal{S}$, we have $E[w(0, c_\alpha)] = M(\alpha)$, where $M(.)$ is defined in Theorem 6.3.

Proof. Part (a) is a direct consequence of Lemma 6.5.

Consider part (b). We first argue that $E[w(0, c_\alpha)] \leq M(\alpha)$. Let t_α be an ergodic inter-arrival process of mean α and suppose it verifies condition (vii) of §3.3. ¿From Theorem 6.3 we get that $\lim_k 1/k \sum_{i=1}^k E[w(0, \Phi^i(t_\alpha))] = M(\alpha)$. And from Theorem 5.4 we get that $\Phi^k(T_\alpha) \xrightarrow{w} C_\alpha$. Hence $1/k \sum_{i=1}^k \Phi^i(T_\alpha) \xrightarrow{w} C_\alpha$. Now, Lemma 4.5 implies

$$\liminf_k \frac{1}{k} \sum_{i=1}^k E[w(0, \Phi^i(t_\alpha))] \;\geq\; E[w(0, c_\alpha)] \, .$$

Thus $E[w(0, c_\alpha)] \leq M(\alpha)$. We digress to prove the following claim.

Claim: For any $\delta > \beta$, $\mathcal{S} \cap (\beta, \delta) \neq \emptyset$.
Proof of Claim: Consider an initial ergodic inter-arrival process t_δ of mean $\delta > \beta$. Suppose that t_δ satisfies condition (vii) of §3.3. Let C be a weak subsequential limit of $\Phi^k(T_\delta)$ and let c be distributed as C. The analysis of the previous section shows that *supp* $E[c(0) \,|\, \mathcal{T}] \subset (\beta, \infty)$. By Fatou's Lemma, $E[c(0)] \leq \delta$. Together with $E[c(0)] = E[E[c(0) \,|\, \mathcal{T}]]$ these two facts imply that *supp* $E[c(0) \mid \mathcal{T}] \cap (\beta, \delta] \neq \emptyset$. From (28), we conclude that $\mathcal{S} \cap (\beta, \delta] \neq \emptyset$ and prove the claim.

Consider a δ in $\mathcal{S} \cap (\beta, \alpha)$. The process t_α defined by $t_\alpha(n, 0) = (\alpha/\delta) c_\delta(n), n \in \mathbb{Z}$, is an ergodic inter-arrival process of mean α. Furthermore, t_α verifies condition (vii) of §3.3. Indeed, from (9) we obtain the following implication

$$E[w(0, c_\delta)] \leq M(\delta) < \infty \implies E\,[\,\sup_{n \geq 1} [\sum_{i=-n}^{-1} \beta - c_\delta(i)]^+\,] < \infty.$$

In turn this implies

$$E\,[\,\sup_{n \geq 1} [\sum_{i=-n}^{-1} \frac{\alpha}{\delta}\beta - t_\alpha(i, 0)]^+\,] < \infty. \tag{30}$$

Since $\alpha\beta/\delta \in (\beta, \alpha)$, Eqn (30) shows that t_α satisfies condition (vii). Given this, Theorem 6.3 implies that $E[w(0, c_k)] \to M(\alpha)$, where c_k is distributed as $C_k = 1/k \sum_{i=1}^k \Phi^i(T_\alpha)$.

On the other hand, from Theorem 5.4 $c_k \xrightarrow{w} c_\alpha$ and from Theorem 4.3 $w(0, c_k) \xrightarrow{w} w(0, c_\alpha)$. To prove part (b) it suffices to show that $E[w(0, c_k)] \to E[w(0, c_\alpha)]$.

Now, since $t_\alpha(n, 0) \geq c_\delta(n)$ for every n, we get $\forall k \in \mathbb{N}$ that

$$w(0, \Phi^k(t_\alpha)) \leq w(0, \Phi^k(c_\delta)) \implies w(0, \Phi^k(t_\alpha)) \leq_{st} w(0, c_\delta) \implies w(0, c_k) \leq_{st} w(0, c_\delta) \, .$$

In the above we have sucessively used the monotonicity property (7), and the fact that $\Phi^k(c_\delta) \sim c_\delta$. Part (a) gives $w(0, c_\alpha) \leq_{st} w(0, c_\delta)$, and as shown above we have $E[w(0, c_\delta)] \leq M(\delta) < \infty$. The dominated convergence theorem allows us to conclude that $E[w(0, c_k)] \to E[w(0, c_\alpha)]$. $\square$

Theorem 6.7. *The set $\mathcal{S}$ is closed in (β, ∞) and* $\inf\{u \in \mathcal{S}\} = \beta$, $\sup\{u \in \mathcal{S}\} = +\infty$.

Proof. In the proof of Lemma 6.6, it was shown that for any $\alpha > \beta$, we must have $\mathcal{S} \cap (\beta, \alpha] \neq \emptyset$. It implies that $\inf\{u \in \mathcal{S}\} = \beta$.

To prove $\sup\{u \in \mathcal{S}\} = +\infty$, suppose for contradiction that $\sup\{u \in \mathcal{S}\} < \infty$ and consider $\alpha > \sup\{u \in \mathcal{S}\}$. Let T be an ergodic inter-arrival process of mean α verifying condition (vii) of §3.3. Let C be a weak subsequential limit of the Cesaro averages of $\Phi^k(T)$, defined as in Lemma 6.1. By Lemma 6.5, $C_\delta \leq_{st} C$ for any $\delta \in \mathcal{S}$. Using an argument identical to the one in the proof of Theorem 4.4 (Eqn (12)), we obtain that $\delta \leq E[c(0) \mid \mathcal{T}]$, P-a.s. Since *supp* $E[c(0) \mid \mathcal{T}] \subset \mathcal{S}$, see (28), we conclude that $E[c(0) \mid \mathcal{T}] = \sup\{u \in \mathcal{S}\}$, P-a.s. and that $\sup\{u \in \mathcal{S}\} \in \mathcal{S}$.

Suppose $\zeta = \sup\{u \in \mathcal{S}\}$. Since C is a fixed point (Lemma 6.1), Theorem 5.3 allows us to conclude that $C = C_\zeta$. Hence $1/\sigma(k) \sum_{i=1}^{\sigma(k)} \Phi^i(T) \xrightarrow{w} C_\zeta$. Lemma 4.5, Theorem 6.3 and Lemma 6.6 together imply

$$\begin{aligned} \liminf_k \frac{1}{\sigma(k)} \sum_{i=1}^{\sigma(k)} E[w(0, \Phi^i(t))] &\geq E[w(0, c_\zeta)] \\ \implies M(\alpha) &\geq M(\zeta) . \end{aligned}$$

Now a corollary of Theorem 6.3 is that $M(x)$ is a positive and decreasing function that is *strictly decreasing* on the interval $\{x \mid M(x) > 0\}$. We conclude that $M(\alpha) = M(\zeta) = 0$. Thus $E[w(0, c_\zeta)] = M(\zeta) = 0$, or $P\{w(0, c_\zeta) = 0\} = 1$. Let us input the process c_ζ into the tandem network. Using Eqn (6) recursively, we obtain

$$\begin{aligned} \Phi^k(c_\zeta)(0) &= c_\zeta(0) + \sum_{i=0}^{k-1}(s(1,i) - s(0,i)) + \sum_{i=0}^{k-1}(w(1, \Phi^i(c_\zeta)) - w(0, \Phi^i(c_\zeta))) \\ &= c_\zeta(0) + \sum_{i=0}^{k-1}(s(1,i) - s(0,i)) . \end{aligned}$$

Since the service times are i.i.d. and nonconstant (assumption (v) of §3.3), the partial sums $\sum_{i=0}^{k-1}(s(1,i) - s(0,i))$ form a null-recurrent random walk. Hence there is a k for which $\Phi^k(c_\zeta)(0) < 0$ with strictly positive probability, which is impossible. Hence we cannot have $M(\zeta) = 0$. In turn, this implies that $\sup\{u \in \mathcal{S}\} = \infty$. Note that this implies via Lemma 6.5 that $E[c(0)] = \alpha$.

We now prove that $\mathcal{S}$ is closed in (β, ∞). Consider a sequence α_k of elements of $\mathcal{S}$ that increases to $\alpha \in (\beta, \infty)$. Let T and C be defined as above (for the mean α). Using Lemma 6.5, we have $C_{\alpha_k} \leq_{st} C$ and $\alpha_k \leq E[c(0) \mid \mathcal{T}]$ P-a.s. Since $E[c(0)] = E[E[c(0) \mid \mathcal{T}]] = \alpha$, we conclude that *supp* $E[c(0) \mid \mathcal{T}] \cap (\alpha_k, \alpha] \neq \emptyset$. Given (28), this further implies $\mathcal{S} \cap (\alpha_k, \alpha] \neq \emptyset$. Passing to the limit, we conclude that $\alpha \in \mathcal{S}$. The proof works similarly when α_k is a decreasing sequence. ∎

Proposition 6.8. *Consider an ergodic inter-arrival process T of mean $\alpha > \beta$. There are two possible situations:*

1. *If $\alpha \in \mathcal{S}$, we have $\Phi^k(T) \xrightarrow{w} C_\alpha$;*

2. *If $\alpha \notin \mathcal{S}$, we have $1/k \sum_{i=1}^k \Phi^k(T) \xrightarrow{w} pC_{\underline{\alpha}} + (1-p)C_{\overline{\alpha}}$ where*

$$\underline{\alpha} = \sup\{u \in \mathcal{S} \ ; \ u \leq \alpha\}, \quad \overline{\alpha} = \inf\{u \in \mathcal{S} \ ; \ \alpha \leq u\}, \ \text{and } p = \frac{\overline{\alpha} - \alpha}{\overline{\alpha} - \underline{\alpha}} . \tag{31}$$

In words, the weak limit is a linear combination of the largest ergodic fixed point of mean less than α and of the smallest ergodic fixed point of mean more than α. The weak limit is always of mean α.

Proof. The case $\alpha \in \mathcal{S}$ follows from Theorem 5.4. Consider an $\alpha \notin \mathcal{S}$. Denote by C a subsequential weak limit of the Cesaro averages of $\Phi^k(T)$, as in Lemma 6.1. By Lemma 6.5, we have $C_u \leq_{st} C \leq_{st} C_v$ for any $u, v \in \mathcal{S}$ such that $u < \alpha < v$. Therefore, $u \leq E[c(0) \mid \mathcal{T}] \leq v$, P-a.s. Since *supp* $E[c(0) \mid \mathcal{T}] \subset \mathcal{S}$ (see (28)) and $E[c(0)] = \alpha$ (Lemma 6.5) we conclude that *supp* $E[c(0) \mid \mathcal{T}] = \{\underline{\alpha}, \overline{\alpha}\}$, where $\underline{\alpha}$ and $\overline{\alpha}$ are defined in (31).

Now we know from Lemma 6.1 that C is a fixed point. Given that *supp* $E[c(0) \mid \mathcal{T}] = \{\underline{\alpha}, \overline{\alpha}\}$, Theorem 5.3 tells us that C must equal $pC_{\underline{\alpha}} + (1-p)C_{\overline{\alpha}}$. Therefore $E[c(0)] = p\underline{\alpha} + (1-p)\overline{\alpha}$ and from $E[c(0)] = \alpha$, we conclude that $p = (\overline{\alpha} - \alpha)/(\overline{\alpha} - \underline{\alpha})$.

A consequence of the above argument is that any converging subsequence of $1/k \sum_{i=1}^k \Phi^i(T)$ must converge to $pC_{\underline{\alpha}} + (1-p)C_{\overline{\alpha}}$. Since we are in a weakly compact space, it implies that $1/k \sum_{i=1}^k \Phi^i(T) \xrightarrow{w} pC_{\underline{\alpha}} + (1-p)C_{\overline{\alpha}}$. □

The previous proposition characterizes the set $\mathcal{S}$ to a certain extent. But we believe that more is true.

Conjecture 6.9. *For any $\alpha > \beta = E[s(0,0)]$, there exists an ergodic fixed point of mean α. In other words, $\mathcal{S} = (\beta, +\infty)$.*

It is possible to show that $\mathcal{S}$ is equal to the image of the derivative of γ defined in Theorem 6.3 (as γ is concave, its derivative γ' is continuous except at a countable number of points. At the points of discontinuity, we consider that both the left and the right-hand limits belong to the image). Hence, the conjecture is true if the function γ has a continuous derivative. However, we have not been able to prove this.

7 Appendix I

7.1 Proof of Theorem 4.3

First, we state the following preliminary lemma (for a proof see Lemmas 1 and 2 in Chapter 4.1 of [7]).

Lemma 7.1. *Let $\{u(n), n \in \mathbb{Z}\}$ be a stationary sequence in $\mathcal{L}(\mathbb{R}^\mathbb{Z})$. For any $c > 0$,*

$$P\left\{\sup_{k \geq 1} \sum_{i=-k}^{-1} u(i) > 0\right\} \ \leq \ \frac{1}{c} E[u(0)^+] + P\{u(0) > -c\} .$$

Fix an l in $\mathbb{N}^*$ and consider random vectors $u_k, k \in \mathbb{N}$, and u in $\mathcal{L}(\mathbb{R}^l)$. If $u_k \xrightarrow{w} u$ and $E[u_k(i)^+] \to E[u(i)^+]$ for $i = 1, \cdots, l$, then

$$E[\ (\sum_{i=1}^{l} u_k(i))^+\ ] \xrightarrow{k} E[\ (\sum_{i=1}^{l} u(i))^+\ ] .$$

We prove the following result and show that it implies Theorem 4.3.

Theorem 7.2. *Suppose that $v_k, k \in \mathbb{N}$, and v are stationary sequences in $\mathcal{L}(\mathbb{R}^{\mathbb{Z}})$ and define $S_k(n) = \sup_{r\geq 0}[\ \sum_{i=n-r}^{n-1} v_k(i)\ ]^+$, $S(n) = \sup_{r\geq 0}[\ \sum_{i=n-r}^{n-1} v(i)\ ]^+$ and $S_k = \{S_k(n), n \in \mathbb{Z}\}, S = \{S(n), n \in \mathbb{Z}\}$. Under the assumptions*

(1) $v_k \xrightarrow{w} v$;

(2) $E[v_k(0)^+] \xrightarrow{k} E[v(0)^+]$ and $E[v(0)^+] < \infty$;

(3) $\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} v(i) = E[v(0) \mid \mathcal{T}] < 0, \quad P-a.s.$,

it holds that $S_k \xrightarrow{w} S$.

Proof. This result, originally due to Borovkov, was proved assuming that v is ergodic and under the following version of assumption (3):

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} v(i) = E[v(0)] < 0, \quad P-a.s.$$

Except for the last part of the proof, the arguments used here are exactly the same as Borovkov's. The original proof can be found in Theorem 1 of Chapter 4.1 in [7]. We closely follow the presentation of Theorems 8.5.1 and A.2.9 of [8].

The result is first proved under the following (soon-to-be-validated) assumption

$$\limsup_{k\to+\infty} P\left\{\sup_{l\geq L}\sum_{i=-l}^{-1} v_k(i) \geq 0\right\} \xrightarrow{L\to\infty} 0 . \tag{32}$$

Fix an $\varepsilon > 0$ and choose L_1 such that for all $L \geq L_1$, the limsup in (32) is less than ε. Assumption (3) allows us to choose L_2 such that for all $L \geq L_2$, we have $P\{\sup_{l\geq L}\sum_{i=-l}^{-1} v(i) \geq 0\} \leq \varepsilon$. let $L = \max(L_1, L_2)$. By the Continuous Mapping Theorem,

$$Z_k = \max_{1\leq l\leq L}[\ \sum_{i=-l}^{-1} v_k(i)\ ]^+ \xrightarrow{w} \max_{1\leq l\leq L}[\ \sum_{i=-l}^{-1} v(i)\ ]^+ = Z .$$

Let x in $\mathbb{R}_+$ be a point of continuity of the distribution of Z for all $L \geq 1$ (the set of discontinuity points is countable). By the Portmanteau Theorem we obtain that $\lim_k P\{Z_k \geq x\} = P\{Z \geq x\}$. But, since $L \geq L_1$,

$$\begin{aligned}\limsup_k P\{S_k(0) \geq x\} &\leq \limsup_k P\{\sup_{l\geq L}\sum_{i=-l}^{-1} v_k(i) \geq x\} + \limsup_k P\{Z_k \geq x\} \\ &\leq \varepsilon + P\{Z \geq x\} \ \leq\ \varepsilon + P\{S(0) \geq x\} .\end{aligned}$$

And since $L \geq L_2$,

$$\begin{aligned}
\liminf_k P\{S_k(0) \geq x\} &\geq \liminf_k P\{Z_k \geq x\} = P\{Z \geq x\} \\
&\geq P\{S(0) \geq x\} - P\{\sup_{l \geq L}[\sum_{i=-l}^{-1} v(i)\,]^+ \geq x\} \geq P\{S(0) \geq x\} - \varepsilon .
\end{aligned}$$

Therefore one obtains that $S_k(0) \xrightarrow{w} S(0)$. The same argument shows that all finite dimensional marginals of S_k converge in distribution to the corresponding marginals of S. Hence, we conclude that $S_k \xrightarrow{w} S$ ([5], Section 3).

We are left with having to prove that (32) holds. Set $Z_k(l) = \sum_{i=-l}^{-1} v_k(i)$ and $Z(l) = \sum_{i=-l}^{-1} v(i)$. For arbitrary s in $\mathbb{R}_+^*$ and j in $\mathbb{N}^*$,

$$\begin{aligned}
P\{\sup_{l \geq L} Z_k(l) \geq 0\} &\leq P\left\{\sup_{l \geq L} Z_k(l) \geq 0 \ ; \ \frac{Z_k(L)}{L} \leq -s \ ; \ \bigcap_{i=1}^{\infty}\{\frac{Z_k(L+ij) - Z_k(L)}{ij} \leq -s\}\right\} \\
&+P\left\{\frac{Z_k(L)}{L} > -s\right\} + P\left\{\bigcup_{i=1}^{\infty}\{\frac{Z_k(L+ij) - Z_k(L)}{ij} > -s\}\right\} .
\end{aligned}$$

Denote by A_1, A_2 and A_3 the three terms on the right-hand side of the inequality.

By the Continuous Mapping Theorem, $v_k(0)^+ \xrightarrow{w} v(0)^+$ and, by assumption (2), $E[v_k(0)^+] \to E[v(0)^+]$. Therefore (see Theorem 5.4 in [5]) the family $\{v_k(0)^+\}$ is uniformly integrable, i.e.

$$\limsup_u \ (\ \sup_k \int_u^{\infty} P\{v_k(0)^+ > t\}dt \ ) = \limsup_u \ (\ \sup_k \int_u^{\infty} P\{v_k(0) > t\}dt \ ) = 0 .$$

We use this to upper-bound A_1:

$$\begin{aligned}
A_1 &\leq P\{ \sup_{l \geq L} Z_k(l) \geq 0 \ ; \bigcap_{i=0}^{\infty}\{Z_k(L+ij) \leq -s(L+ij)\} \ \} \\
&\leq P\{\exists i \geq 0, \exists \ 0 < l \leq j \mid v_k(-(L+ij+l)) \geq Ls/j + si\} \\
&\leq j \sum_{i=0}^{\infty} P\{v_k(0) \geq Ls/j + si\} \ \leq \ \frac{j}{s} \int_{Ls/j-s}^{\infty} P\{v_k(0) \geq t\}dt .
\end{aligned}$$

The above inequality can be rewritten as $A_1 \leq (j/s) f_k(Ls/j)$, where the functions f_k are such that $\lim_L(\ \sup_k (j/s) f_k(Ls/j) \ ) = 0$ for any given j and s. Let $L(j,s)$ be such that $A_1 \leq \varepsilon$ for all $L \geq L(j,s)$ and all k. Set $L = \max(L_1, L_2, L(j,s))$.

Using successively the stationarity of v_k (to infer that $Z_k(L+ij) - Z_k(L) \sim Z_k(ij)$), Lemma 7.1 with $c = sj$ and the Portmanteau Theorem (since $[-2s, +\infty[$ is a closed set), one can bound A_3 as follows:

$$\begin{aligned}
A_3 \ \leq \ P\left\{\sup_{i \geq 1}(Z_k(ij) + sij) > 0\right\} &\leq P\left\{\frac{Z_k(j)}{j} \geq -2s\right\} + \frac{1}{s}E\left[\left(\frac{Z_k(j)}{j} + s\right)^+\right] \\
&\leq P\left\{\frac{Z(j)}{j} \geq -2s\right\} + \frac{1}{s}E\left[\left(\frac{Z_k(j)}{j} + s\right)^+\right] .
\end{aligned}$$

Now for k large enough, the second part of Lemma 7.1 implies

$$\frac{1}{s}E\left[\left(\frac{Z_k(j)}{j}+s\right)^+\right] \;\leq\; \frac{1}{s}E\left[\left(\frac{Z(j)}{j}+s\right)^+\right]+\varepsilon.$$

Collecting the bounds obtained so far, we can write for all k large enough that

$$\begin{aligned} P\left\{\sup_{l\geq L} Z_k(l)\geq 0\right\} \;&\leq\; 2\varepsilon + P\left\{\frac{Z(j)}{j}\geq -2s\right\} \\ &\quad + P\left\{\frac{Z(L)}{L}> -s\right\}+\frac{1}{s}E\left[\left(\frac{Z(j)}{j}+s\right)^+\right]. \end{aligned} \tag{33}$$

The following development deviates from the proof in [8] and uses the new form of assumption (3): $Z(n)/n \longrightarrow E[v(0)\mid \mathcal{T}]<0$, $P-a.s.$ We fix an $s>0$ so that

$$P\{E[v(0)\mid \mathcal{T}]>-3s\}\leq \varepsilon \tag{34}$$

and obtain the bound

$$\begin{aligned} P\{Z(n)/n\geq -2s\} \;&\leq\; P\{Z(n)/n\geq -2s\,;\, E[v(0)\mid \mathcal{T}]\leq -3s\}+P\{E[v(0)\mid \mathcal{T}]>-3s\} \\ &\leq\; P\{|Z(n)/n-E[v(0)\mid \mathcal{T}]|\geq s\}+\varepsilon\,. \end{aligned}$$

Since

$$Z(n)/n \xrightarrow{\;n\;} E[v(0)\mid \mathcal{T}],\;\; P-a.s. \text{ and in } L_1\,, \tag{35}$$

we conclude that there exists N such that $\forall n\geq N, P\{Z(n)/n\geq -2s\}\leq 2\varepsilon$. Choosing j and L to be bigger than N allows us to upper-bound two of the three terms in the right-hand side of (33).

The only term which remains to be controlled is $E[(Z(j)/j+s)^+]/s$. The a.s. and L_1 convergence in (35) implies (see Theorem 5.4 in [5]) that the family $|Z(n)/n|$ is uniformly integrable. We deduce that the family $(Z(n)/n+s)^+$ is also uniformly integrable. Therefore (again, by Theorem 5.4 in [5])

$$E\left[\left(\frac{Z(n)}{n}+s\right)^+\right] \xrightarrow{\;n\;} E\left[(E[v(0)\mid \mathcal{T}]+s)^+\right]\,.$$

But,

$$\begin{aligned} E\left[(E[v(0)\mid \mathcal{T}]+s)^+\right] \;&=\; E\left[(E[v(0)\mid \mathcal{T}]+s)\,\mathbb{1}\{E[v(0)\mid \mathcal{T}]>-s\}\right] \\ &\leq\; sP\left\{E[v(0)\mid \mathcal{T}]>-s\right\}\leq s\varepsilon\,, \end{aligned}$$

and we conclude that there exists M such that $\forall n\geq M$,

$$\frac{1}{s}E\left[\left(\frac{Z(n)}{n}+s\right)^+\right]\leq \frac{1}{s}E\left[(E[v(0)\mid \mathcal{T}]+s)^+\right]+\varepsilon\leq 2\varepsilon\,.$$

Choose s as needed in (34), set $j=\max(N,M)$ and $L=\max(L_1,L_2,L(j,s),N)$. With this we obtain that for large enough k, we have $P\{\sup_{l\geq L} Z_k(l)\geq 0\}\leq 8\varepsilon$, concluding the proof of Theorem 7.2. □

We show, via the following notational translations, that Theorem 7.2 implies Theorem 4.3. Let $v_k = s - u_k$ and $v = s - u$. All the assumptions of Theorem 7.2 are verified for the processes v_k and v. In particular, the Continuous Mapping Theorem implies that $(s(0) - u_k(0))^+ \xrightarrow{w} (s(0) - u(0))^+$ and $(s(0) - u_k(0))^+ \leq s(0)$. Hence, by the Dominated Convergence Theorem, $E[v_k(0)^+] \to E[v(0)^+]$.

Now, observe that $w(n, u_k) = S_k(n) = [\sup_{r \geq 1} \sum_{i=n-r}^{n-1} v_k(i)]^+$ (Lindley's equation (8)). Theorem 7.2 tells us that $\{w(n, u_k), n \in \mathbb{Z}\} \xrightarrow{w} \{w(n, u), n \in \mathbb{Z}\}$ as k goes to infinity. From Eqn (6), we obtain that the weak convergence of the workload implies the weak convergence of the departure process. Thus, $\Phi(u_k) \xrightarrow{w} \Phi(u)$.

7.2 Counter-examples

The following two examples illustrate the tightness of the assumptions made in Theorem 4.3 and Lemma 4.5.

Example 7.3. Consider the periodic sequences $u_k \in \mathbb{R}^{\mathbb{Z}}$ of period

$$(\underbrace{0, \cdots, 0}_{k}, \underbrace{4, \cdots, 4}_{k})$$

and such that $u_k(-1) = 4, u_k(0) = 0$. Recall that L is the left shift defined in (1). Starting from these sequences we construct stationary arrival processes $t_k = \{t_k(n), n \in \mathbb{Z}\}, k \in \mathbb{N}^*$, whose distributions are given by $P\{t_k = L^i(u_k)\} = 1/2k, i = 0, \cdots, 2k-1$. The process t_k is ergodic with mean inter-arrival time $E[t_k(0)] = 2$.

Observe that $t_k \xrightarrow{w} t_\infty$ as k goes to infinity, where the process t_∞ is specified by

$$P\{t_\infty = (\cdots, 0, 0, 0, \cdots)\} = P\{t_\infty = (\cdots, 4, 4, 4, \cdots)\} = \frac{1}{2}.$$

The process t_∞ is stationary, but not ergodic, and of mean 2.

Now, consider a deterministic service process $s(n) = 1, n \in \mathbb{Z}$. All the conditions of Theorem 4.3 are met except one: $P\{E[t_\infty(0) \mid \mathcal{T}] \leq E[s(0)]\} = 1/2$. We show below that the conclusion of the Theorem is not verified.

The inter-departure processes $\Phi(t_k)$ are such that $P\{\Phi(t_k) = L^i(v_k)\} = 1/2k, i = 0, \cdots, 2k-1$, where $v_k \in \mathbb{R}_+^{\mathbb{Z}}$ is periodic of period

$$(\underbrace{1, \cdots, 1}_{k+[k/3]}, 4 - (k \mod 3), \underbrace{4, \cdots, 4}_{k-[k/3]-1})$$

This is illustrated on Figure 1, where we have represented the evolution of the workload on a trajectory for the inter-arrival process t_k.

We have $\Phi(t_k) \xrightarrow{w} \tilde{t}_\infty$, where the process $\tilde{t}_\infty$ is such that

$$P\{\tilde{t}_\infty = (\cdots, 1, 1, 1, \cdots)\} = \frac{2}{3}, \ \ P\{\tilde{t}_\infty = (\cdots, 4, 4, 4, \cdots)\} = \frac{1}{3}.$$

On the other hand, the inter-departure process $\Phi(t_\infty)$ is defined by

$$P\{\Phi(t_\infty) = (\cdots, 1, 1, 1, \cdots)\} = P\{\Phi(t_\infty) = (\cdots, 4, 4, 4, \cdots)\} = \frac{1}{2}.$$

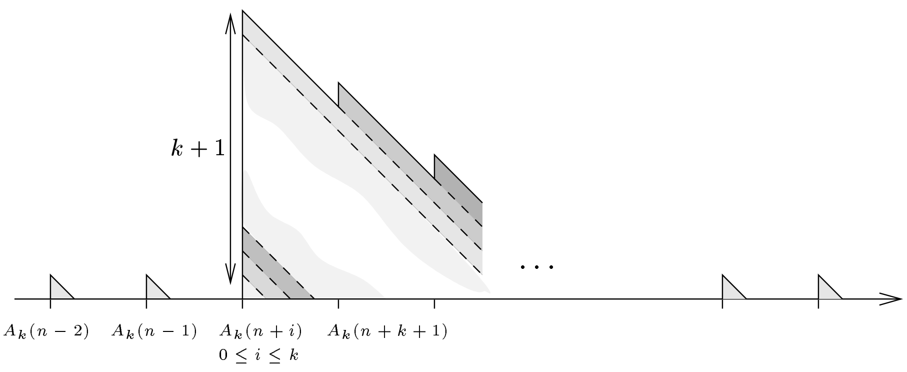

Figure 1: Evolution of the workload $w(n, t_k)$.

We conclude that $\lim_k \Phi(t_k) \neq \Phi(\lim_k t_k)$.

In the same way, we can check that $w(0, t_k) \xrightarrow{w} \tilde{w}$ with $P\{\tilde{w} = 0\} = 1/3$ and $P\{\tilde{w} = \infty\} = 2/3$. On the other hand, we have $P\{w(0, t_\infty) = 0\} = P\{w(0, t_\infty) = \infty\} = 1/2$.

Example 7.4. We adapt examples used to illustrate the workload moment problem in a single server queue (see Miyazawa [22] or Daley and Rolski [11]).

Let $c = \{c(n), n \in \mathbb{N}^*\}$ be a $\mathbb{N}^*$-valued sequence of i.i.d. and integrable random variables, called *cycles*. Let $c(0)$ be independent of c and distributed as the *recurrence time* of $c(1)$, i.e. $P\{c(0) = k\} = P\{c(1) \geq k\}/E[c(1)]$ for $k \geq 1$. Define $C(-1) = 0$ and $C(k) = \sum_{i=0}^{k} c(i)$ and consider the process $u = \{u(n), n \in \mathbb{N}^*\}$ defined as follows

$$u(n) = \begin{cases} 0 & \text{if } C(k-1) < n < C(k) \\ 2c(k) & \text{if } n = C(k) \end{cases}$$

The process u is stationary and verifies $E[u(1)] = 2$. It is a regenerative process with a cycle structure given by the cycle process c. We extend it to a bi-infinite stationary process.

In order to define the arrival processes to our queue, consider the cycle processes $c_k, k \in \mathbb{N}^*$, defined by (for $n \geq 1$)

$$c_k(n) = \begin{cases} 1 & \text{w.p. } 1 - 1/k^2 \\ k & \text{w.p. } 1/k^2 \end{cases}$$

Let c_∞ denote the deterministic process $\{1, 1, \cdots\}$. We have $c_k \xrightarrow{w} c_\infty$ and $E[c_k(1)] \to E[c_\infty(1)] = 1$. However, we do not have convergence of the second moments as $\lim_k E[c_k(1)^2] = 2$ and $E[c_\infty(1)^2] = 1$.

Consider a deterministic service process $s = \{\cdots, 1, 1, \cdots\}$ and the stationary arrival processes t_k and t_∞ in $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ associated with the cycle processes c_k and c_∞ defined above. In particular, we have $t_\infty = \{\cdots, 2, 2, \cdots\}$. We also have $t_k \xrightarrow{w} t_\infty$ and $E[t_k] = E[t_\infty] = 2$. The various conditions of Theorem 4.3 are verified.

During a cycle of length strictly greater than one, the workload builds up, until the arrival of the last customer of the cycle which is delayed long enough to empty the workload. The workload seen by an arriving customer is equal to the *backward recurrence time* of its position in the cycle. A direct computation, or an application of Theorem 4.3, shows that $w(0,t_k) \xrightarrow{w} w(0,t_\infty)$. However, a simple calculation provides the following result for $k \in \mathbb{N}^* \cup \{+\infty\}$:

$$E[w(0,t_k)] = \frac{E[c_k(0)(c_k(0)-1)]}{2E[c_k(0)]} .$$

We conclude that $\lim_k E[w(0,t_k)] = 1/2$ which differs from $E[w(0,t_\infty)] = 0$.

8 Appendix II - Stochastic Orders

A stochastic order is a partial order on a space of distributions (or on a space of random variables). In this paper three types of orders are used: (i) increasing (or strong) order, (ii) convex order, and (iii) increasing convex order. These are denoted by $\geq_{st}$, $\geq_{cx}$ and $\geq_{icx}$, respectively. We shall only recall the results that we need and refer the reader to Stoyan [27] for more details.

8.1 Strong order

Let K be a partially ordered Polish space. A function f from K to $\mathbb{R}$ is said to be *increasing* if $\forall x, y \in K, x \geq y \Rightarrow f(x) \geq f(y)$.

Definition 8.1. Consider $u, v \in \mathcal{L}(K)$ with respective distributions U and V. We say that u (resp. U) strongly dominates v (resp. V), and write $u \geq_{st} v$ (resp. $U \geq_{st} V$) if:

$$E[f(u)] \geq E[f(v)] \ \ \text{(resp. } E_U[f] \geq E_V[f]) ,$$

for all increasing real functions f such that the expectations are well defined.

The next proposition, where the second part easily follows from the first one, was proved by Whitt [30].

Proposition 8.2. *For U and V belonging to $\mathcal{M}(K)$, $U \geq_{st} V$ iff $E_U[f] \geq E_V[f]$ for all increasing and* continuous *real functions f, such that the expectations are well defined.*

For $U_n, V_n, n \in \mathbb{N}, U$ and V belonging to $\mathcal{M}(K)$, suppose that $U_n \xrightarrow{w} U$, $V_n \xrightarrow{w} V$ and that $U_n \geq_{st} V_n$. Then $U \geq_{st} V$.

Let $K_i, i \in I$, be a finite or countable family of partially ordered Polish spaces. Consider the product space $K = \prod_{i \in I} K_i$ equipped with the product topology, the associated Borel σ-algebra, and the coordinatewise partial order. Then K is a partially ordered Polish space.

For a proof of the next result, see [19] or [27], Theorem 4.1.2, for the first part and [27], Theorem 2.2.4., for the second part.

Proposition 8.3. *Let $K = \prod_{i \in I} K_i$ be the product space associated with a finite or countable family of partially ordered Polish spaces. For $U, V \in \mathcal{M}(K)$*

$$U \geq_{st} V \iff U\pi_i^{-1} \geq_{st} V\pi_i^{-1},\ i \in I, \tag{36}$$

where $U\pi_i^{-1}, V\pi_i^{-1} \in \mathcal{M}(K_i)$ are the i-th marginal of U and V respectively.

Let I be a finite family and E a partially ordered Polish space. Let ϕ be a measurable and increasing function from a subset of $K = \prod_{i \in I} K_i$ to E. Then

$$U \geq_{st} V \implies U\phi^{-1} \geq_{st} V\phi^{-1}\,.$$

Equivalently, if $u, v \in \mathcal{L}(K)$ are such that $u \sim U$ and $v \sim V$, then $u \geq_{st} v$ implies $\phi(u) \geq_{st} \phi(v)$.

The next result is proved in [19] and generalizes a classical theorem of Strassen.

Proposition 8.4. *Let $U, V \in \mathcal{M}(K)$ be such that $U \geq_{st} V$. There exists a probability triple $(\Omega, \mathcal{F}, P)$ and r.v. u and v in $\mathcal{L}(K)$ such that:*

$$u \sim U, v \sim V,\ \ u \geq v,\ P\ a.s.$$

In the same spirit, the following representation theorem is due to Skorohod (see for instance Chapter 11.7 in [14]).

Proposition 8.5. *Let K be a Polish space and let $U_n, n \in \mathbb{N}, U \in \mathcal{M}(K)$ be such that $U_n \xrightarrow{w} U$. Then there exists a probability triple $(\Omega, \mathcal{F}, P)$ and random variables u_n and u in $\mathcal{L}(K)$ such that:*

$$u_n \sim U_n, u \sim U,\ \ u_n \longrightarrow u,\ P - a.s.$$

8.2 Convex order

Let K be a partially ordered $\mathbb{R}$-linear space. We assume that the order is compatible with the linear structure. A function f from K to $\mathbb{R}$ is said to be convex if it verifies

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \quad \forall x, y \in K, \forall\, 0 \leq \alpha \leq 1\,.$$

Definition 8.6. Consider $u, v \in \mathcal{L}(K)$ with respective distributions U and V. We say that $u \geq_{cx} v$ or $u \geq_{icx} v$ (resp. $U \geq_{cx} V$ or $U \geq_{icx} V$) if

$$E[f(u)] \geq E[f(v)] \text{ (resp. } E_U[f] \geq E_V[f])\,,$$

for all convex real functions f ($\geq_{cx}$) or for all increasing and convex real functions f ($\geq_{icx}$) such that the expectations are well defined.

It is immediate that $u \geq_{icx} v$ is implied both by $u \geq_{st} v$ and $u \geq_{cx} v$. The equivalence in (36) holds for the convex order when I is finite, see Chapter 1.10 of [27].

Finally, as a consequence of the definition, we obtain the following fact: Let $\mathcal{G}$ be a sub-σ-algebra of the probability space, then $u \geq_{cx} E[u \mid \mathcal{G}]$. Indeed, using the conditional Jensen's inequality, we have for any convex function f,

$$\int f(u) dP = \int E[f(u) \mid \mathcal{G}] dP \geq \int f\left(E[u \mid \mathcal{G}]\right) dP\,. \tag{37}$$

References

[1] V. Anantharam. Uniqueness of stationary ergodic fixed point for a ./M/K node. *Annals Appl. Prob.*, 3(1):154–172, 1993. Correction published in 4(2):607, 1994.

[2] S. Asmussen and H. Johansen. Über eine stetigkeitsfrage betreffend das bedienungssystem GI/GI/s. *Elektron. Inf.verarb. Kibern.*, 22:565–570, 1986.

[3] F. Baccelli, A. Borovkov, and J. Mairesse. Asymptotic results on infinite tandem queueing networks. Liafa reseach report 99/15, Université Paris 7, 1999. To appear in *Prob. Theory Related Fields.*

[4] F. Baccelli and P. Brémaud. *Elements of Queueing Theory.* Number 26 in Applications of Mathematics. Springer Verlag, Berlin, 1994.

[5] P. Billingsley. *Convergence of Probability Measures.* John Wiley & Sons, New York, 1968.

[6] A. Borovkov. *Stochastic Processes in Queueing Theory.* Springer-Verlag, Berlin, 1976. Russian edition: Nauka, 1972.

[7] A. Borovkov. *Asymptotic Methods in Queueing Theory.* John Wiley & Sons, New York, 1984. Russian edition: Nauka, 1980.

[8] A. Brandt, P. Franken, and B. Lisek. *Stationary Stochastic Models.* Prob. and Math. Stat. Wiley, New York, 1990.

[9] P. Burke. The output of a queueing system. *Operations Research*, 4:699–704, 1956.

[10] C.S. Chang. On the input-output map of a G/G/1 queue. *J. Appl. Prob.*, 31(4):1128–1133, 1994.

[11] D. Daley and T. Rolski. Finiteness of waiting-time moments in general stationary single-server queues. *Annals Appl. Prob.*, 2(4):987–1008, 1992.

[12] D. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes.* Springer Series in Statistics. Springer-Verlag, 1988.

[13] J.L. Doob. *Stochastic Processes.* Wiley, 1953.

[14] R. Dudley. *Real Analysis and Probability.* Brooks/Cole Pub. Co., New York, 1989.

[15] S. Ethier and T. Kurtz. *Markov Processes: Characterization and Convergence.* Wiley, New-York, 1986.

[16] P. Glynn and W. Whitt. Departures from many queues in series. *Annals Appl. Prob.*, 1(4):546–572, 1991.

[17] R. Gray. *Probability, Random Processes, and Ergodic Properties.* Springer-Verlag, Berlin, 1988.

[18] A. Greenberg, O. Schlunk, and W. Whitt. Using distributed-event parallel simulation to study departures from many queues in series. *Proba. Engineering and Informational Sc.*, 7:159–186, 1993.

[19] T. Kamae, U. Krengel, and G.L. O'Brien. Stochastic inequalities on partially ordered spaces. *Annals of Probability*, 5:899–912, 1977.

[20] R. Loynes. The stability of a queue with non-independent interarrival and service times. *Proc. Camb. Philos. Soc.*, 58:497–520, 1962.

[21] N.M. Mirasol. The output of an $M/G/\infty$ queueing system is Poisson. *Operations Research*, 11(2):282–284, 1963.

[22] M. Miyazawa. A formal approach to queueing processes in the steady state and their applications. *J. Appl. Prob.*, 16:332–346, 1979.

[23] T. Mountford and B. Prabhakar. On the weak convergence of departures from an infinite sequence of ./M/1 queues. *Annals Appl. Prob.*, 5(1):121–127, 1995.

[24] B. Prabhakar. On the attractiveness of the fixed points of a ./GI/1 queueing operator. Technical report, Stanford Univ., 1999. In preparation.

[25] B. Prabhakar, T. Mountford, and N. Bambos. Convergence of departures in tandem networks of $\cdot$/GI/∞ queues. *Prob. in the Engin. and Informational Sc.*, 10:487–500, 1996.

[26] W. Rudin. *Functional Analysis, 2nd edition.* McGraw-Hill, 1991.

[27] D. Stoyan. *Comparison Methods for Queues and other Stochastic Models.* Wiley, New York, 1984.

[28] W. Szczotka and F. Kelly. Asymptotic stationarity of queues in series and the heavy traffic approximation. *Annals of Probability*, 18:1232–1248, 1990.

[29] D. Vere-Jones. Some applications of probability generating functionals to the study of input-output streams. *Journal of the Royal Statistical Society, Series B*, 30:321–333, 1968.

[30] W. Whitt. Uniform conditional stochastic order. *J. Appl. Prob.*, 17:112–123, 1992.