A Graphical Approach of the Spectral Theory in the (max,+) Algebra

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Abstract— **In this paper, we study matrices in the (max,+) algebra. We introduce a new tool for describing the deterministic spectral behaviour of matrices of size 3×3. It consists in a graphical representation of eigenvectors and domains of attraction in the projective space. A more complete version of this paper is accessible via ftp, [6].**

I. Introduction

Discrete Events Dynamic Systems (DEDS) are a common framework to represent communication or manufacturing networks. Petri nets, and more precisely Event Graphs, are an example of a formalism to study DEDS. Event Graphs model phenomena such as synchronization or blocking. They have a simple interpretation in a non conventional algebra, the (max,+) algebra.

The spectral theory of matrices in the (max,+) algebra is now well known. It can be tracked back to [5] or [4] One of the main differences with the classical spectral theory is that there is a unique eigenvalue for irreducible matrices. As a consequence, the main interest and difficulty is to study eigenvectors associated with the unique eigenvalue. For a timed Event Graph, the eigenvalue is exactly the mean cycle time (inverse of the throughput rate). On the other hand, eigenvectors are associated with quantities such as : number of tokens in a place, waiting times or idle times. Multiple eigenvectors mean multiple possible regimes for these quantities.

In this paper, we present the classical spectral results under a new light. We develop a tool for describing the spectral behaviour of matrices of size 3×3. It consists in a graphical representation of asymptotic regimes in a projective space.

II. The (max,+) Algebra

We consider systems whose dynamic behaviour is driven by a recursive equation of the form :

$$x_i(n+1) \quad = \quad \max_{1 \le j \le k} (A_{ij} + x_j(n)),\ i = 1, \ldots, k\,. \qquad (1)$$

We allow A_{ij} to be equal to $-\infty$. Let us introduce some new notations.

Definition 1 ((max,+) algebra) We consider the semi-field $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. The law $\oplus$ is "max" and $\otimes$ is the usual addition. We set $\varepsilon = -\infty$ and $e = 0$. The element ε is neutral for the operation $\oplus$ and absorbing for $\otimes$. The element e is neutral for $\otimes$. The law $\oplus$ is idempotent, i.e. $a \oplus a = a$. $(\mathbb{R} \cup \{\varepsilon\}, \oplus, \otimes)$ is an idempotent semiring or dioid. It is usually referred to as the (max,+) algebra. We shall denote it by $\mathbb{R}_{max}$.

In the rest of the paper, the notations "+,×" will stand for the usual addition and multiplication. However, we will write ab for $a \otimes b$ whenever there is no possible confusion. For example, for $a \in \mathbb{R}$, $a^d = a^{\otimes d} = d \times a$.

We define the product spaces $\mathbb{R}^k_{max}, \mathbb{R}^{k \times k}_{max}$. We define the product of a vector by a scalar : $a \in \mathbb{R}_{max}, u \in \mathbb{R}^k_{max},\ (a \otimes u)_i = a \otimes u_i = a + u_i$.

This work was supported by the Direction des Recherches Etudes et Techniques (DRET) under contract n° 91 815. Supported in part by the European Grant BRA-QMIPS of CEC DG XIII.

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The matrix product is defined in a natural way, replacing + and × by $\oplus$ and $\otimes$ respectively. Let $A, B \in \mathbb{R}^{k \times k}_{max}$,

$$(A \otimes B)_{ij} = \max_l (A_{il} + B_{lj}) = \bigoplus_l A_{il} \otimes B_{lj}\ .$$

The matrix-vector product is defined in a similar way.

With these notations, the basic evolution Equation (1) takes a very simple and convenient form. It can be rewritten as :

$$x(n+1) \quad = \quad A \otimes x(n)\,. \qquad (2)$$

Here $x(n) = (x_1(n), x_2(n), \ldots, x_k(n))'$ and A is a $k \times k$ matrix.

We consider the following "eigenvalue problem". We want to find non trivial solutions of the equation $A \otimes u = \lambda \otimes u$, where $A \in \mathbb{R}^{k \times k}$ is an irreducible (see Def. 4) matrix, u is a column vector (the "eigenvector") and λ is a real constant (the "eigenvalue"). We also define periodic solutions of the eigenvalue problem.

Definition 2: A periodic solution (or regime) of period d is a set of vectors $\{u^1, \ldots, u^d\}$ of $\mathbb{R}^k$ verifying $Au^i = \lambda u^{i+1}$, $i = 1, \ldots, d-1$ and $Au^d = \lambda u^1$. It implies that the vectors $u^1, \ldots, u^d$ are eigenvectors of A^d.

The eigenvalue of a matrix A gives the asymptotic growth rate of A^n/n (see Theorem 3 for a more precise statement). On the other hand, eigenvectors and periodic regimes are related with the problem of computing differences such as $A^{n+1}u - A^n u$ or $(A^n u)_i - (A^n u)_j$. These differences are related to many quantities of interest, see Sections III and VI. In this paper, we focus essentially on eigenvectors and periodic regimes.

III. A Simple Manufacturing Model

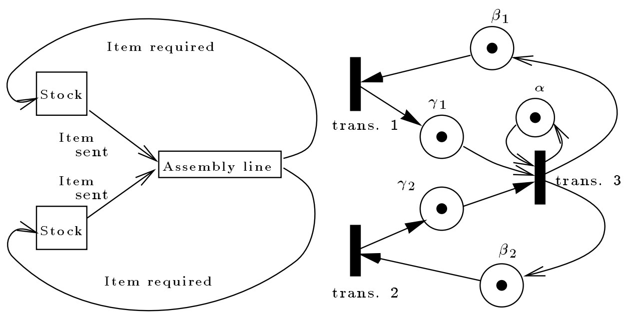

Fig. 1. A manufacturing model and its Petri net representation.

There are two types of items which have to be assembled together to form a part. There is a stock for each kind of item. We suppose that these stocks are infinite. Each time a part is completed at the assembly line, a request is sent to the storage facilities. New items are then sent to the assembly line. We denote :

- α : operating time at the assembly line.
- $\beta_i, i = 1, 2$: communication time between the assembly line and stock i.
- $\gamma_i, i = 1, 2$: transportation time between the stocks and the assembly line.

We consider three daters $(x_i(n), i = 1, 2, 3)$ associated with this system. The first two correspond to the instants at which an item is sent from the stocks. The third one corresponds to the instants at which a part is completed at the assembly line. For example, $x_3(n)$ is the n-th instant of completion of a part at the assembly line. Then the (max,+) linear system corresponding to this system is the following one :

$$x(n+1) = M \otimes x(n), \; M = \begin{pmatrix} \varepsilon & \varepsilon & \beta_1 \\ \varepsilon & \varepsilon & \beta_2 \\ \gamma_1+\alpha & \gamma_2+\alpha & \alpha \end{pmatrix} . \quad (3)$$

The eigenvalue of M is

$$\lambda = \max(\alpha, (\beta_1+\gamma_1+\alpha)/2, (\beta_2+\gamma_2+\alpha)/2)$$

(see Theorem 1). It corresponds to the mean cycle time, i.e. the inverse of the throughput of the system. One can also compute, for example, the idle time of the assembly line between the completion of a task and the beginning of the next one. Let us denote it by δ. We have :

$$\delta(n) = x_3(n) - x_3(n-1) - \alpha \, . \quad (4)$$

This example will be continued in Section VI.

This manufacturing system can be modelled using an Event Graph representation as shown in Figure 1. Event Graphs can efficiently model systems with synchronization, fork-join properties and/or blocking. It is known that all Event Graphs can be described by an evolution equation of the form of Equation (1). For more insights on all modelling aspects, the reader is referred to [1], [2] or [3].

IV. Spectral Theory in $\mathbb{R}_{max}$

The spectral theory of irreducible matrices in $\mathbb{R}_{max}$ is now classical. Theorem 1 can be tracked back to [8] and Theorems 2 and 3 were proved in [3]. A complete treatment and precise references can be found in [2], see also [7]. However, the idea of illustrating the spectral behaviours by graphical representations in a projective space, as in Section V, is new.

A. General Presentation

We recall that we want to find non trivial solutions of the equation $Au = \lambda u$, where $A \in \mathbb{R}_{max}^{k\times k}$.

Definition 3: The graph associated with a matrix A is a directed graph having a number of nodes equal to the size of A. It contains an arc from i to j iff $A_{ji} \neq \varepsilon$. The valuation of this arc is A_{ji}.

Definition 4: A matrix A is irreducible if : $\forall i,j \quad \exists n \geq 0 \;\mid\; (A^n)_{ij} > \varepsilon$ (or equivalently if its graph is strongly connected). A matrix A is aperiodic if : $\exists n \geq 0, \;\; \forall i,j \;\mid\; (A^n)_{ij} > \varepsilon$.

From now on, we consider only irreducible matrices in $\mathbb{R}_{max}^{k\times k}$.

Theorem 1: For each circuit of the graph of A, $\zeta = \{t_1, t_2, \cdots, t_j, t_{j+1} = t_1\}$, we define its average weight by $p(\zeta) = (A_{t_1 t_j} \otimes \cdots \otimes A_{t_3 t_2} \otimes A_{t_2 t_1})/j$ (here the division is the conventional one). The matrix A has a unique (non ε) eigenvalue, λ. It satisfies the relation $\lambda = \max_\zeta \; p(\zeta)$, where the maximum is taken over all the circuits of (the graph of) A.

There might be several eigenvectors. An eigenvector has all its coordinates different from ε (due to the irreducibility assumption).

Definition 5: We normalize a matrix by dividing (in $\mathbb{R}_{max}$, i.e. by subtracting in the conventional algebra) all its entries by its eigenvalue.

A normalized matrix has e as eigenvalue. Eigenvectors and periodic regimes are invariant by a translation of all the entries of a matrix by the same real constant.

Definition 6: For a matrix A, with eigenvalue λ, we define :

Critical circuit A circuit ζ of A is said to be critical if its average weight is maximal, i.e. if $p(\zeta) = \lambda$.

Critical graph It consists of the nodes and arcs of A belonging to the critical circuit(s). A critical column (resp. line) of A is a vector $A_{.i}$ (resp. $A_{i.}$) where i belongs to the critical graph. A critical term of A is a term A_{ij} where i and j belong to the critical graph.

For a general graph, we define :

Cyclicity The cyclicity of a strongly connected graph is the greatest common divisor of the lengths of all the circuits. The cyclicity of a general graph is the least common multiple of the cyclicities of its strongly connected subgraphs.

To study the spectral behaviour of a matrix A, it is enough to know :

- The number of strongly connected subgraphs (**s.c.s.**) of its critical graph.
- The cyclicity of its critical graph.

In the following, a matrix is said to be of type **SCS**j**-CYC**k if its critical graph has j s.c.s. and a cyclicity of k.

The two following theorems justify the previous assertion. For a normalized matrix A of size k, we define $A^+ = A \oplus A^2 \oplus \cdots \oplus A^k$. We check that $A^+ \oplus A^{k+1} = A^+$. We check also that A^+ has the same critical graph as A.

Theorem 2: Let A be a normalized matrix.

a. Critical columns $A^+_{.i}$ are eigenvectors.

b. For i,j belonging to the critical graph, $A^+_{.i}$ and $A^+_{.j}$ are different iff they belong to two different s.c.s. of the critical graph.

c. Every eigenvector of A writes as a linear combination (in $\mathbb{R}_{max}$, see Section IV-B) of critical columns $A^+_{.i}$.

Because of c., the vectors $A^+_{.i}$, i belonging to the critical graph, are called the extremal eigenvectors.

Theorem 3: For an irreducible matrix A of size k and whose eigenvalue is λ, there exist integers d and N such that :

$$\forall n \geq N, \quad A^{n+d} = \lambda^{\otimes d} \otimes A^n \, .$$

Furthermore the smallest d verifying the property is equal to the cyclicity of the critical graph of A. From now on, we will call it the cyclicity of A. A cyclicity of d will provide periodic regimes of period d for the eigenvalue problem.

The good interpretation is that there exists an initial transient regime for the powers of a matrix A. After the transient regime, the sequence $\{A^n\}$ becomes periodic (more rigorously, it is the sequence $\{\pi(A^n)\}$, see Definition 7, which becomes periodic).

The term A^n_{ji} can be interpreted as the heaviest path of n steps starting from i and arriving at j in the graph of A. Theorems 1 and 3 state that the asymptotic growth rate of A^n is given by the circuits of A having the maximal weight.

B. The Projective Space

A "linear combination" (in $\mathbb{R}^k_{max}$) of eigenvectors is an eigenvector, i.e. if u_1 and u_2 are eigenvectors and $\alpha_1, \alpha_2 \in \mathbb{R}$, then $(\alpha_1 \otimes u_1) \oplus (\alpha_2 \otimes u_2)$ is also an eigenvector. In particular, if u is an eigenvector and $\alpha \in \mathbb{R}$, then $\alpha \otimes u$ is also one. This motivates the introduction of an "additive" projective space $\mathbb{P}\mathbb{R}^k$.

Definition 7 ($\mathbb{P}\mathbb{R}^k$) The "additive" projective space $\mathbb{P}\mathbb{R}^k$ is defined as the quotient of $\mathbb{R}^k$ by the parallelism relation :

$$u, v \in \mathbb{R}^k \quad u \simeq v \iff \exists a \in \mathbb{R} \; such \; that \; u = a \otimes v \, .$$

Let π be the canonical projection of $\mathbb{R}^k$ into $\mathbb{P}\mathbb{R}^k$.

The projection π can be interpreted geometrically. It is the orthogonal projection on the hyperspace orthogonal to the vector $\mathbb{1} = (1, \ldots, 1)'$. The projective space $\mathbb{P}\mathbb{R}^k$ is isomorphic to $\mathbb{R}^{k-1}$. For matrices of size 2 or 3, we can represent, in

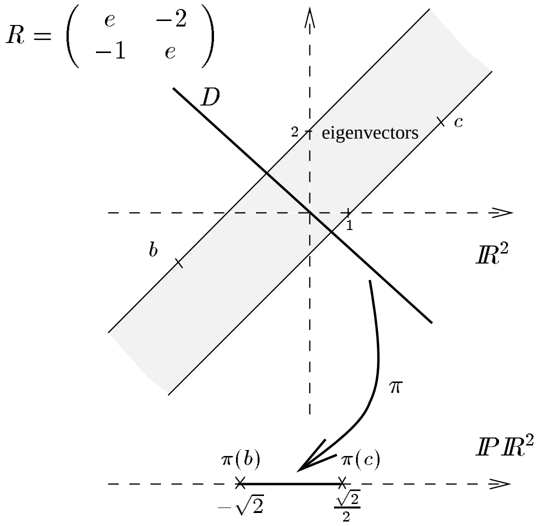

Fig. 2. The projective space $I\!PI\!R^2$.

$I\!R \simeq I\!PI\!R^2$ and $I\!R^2 \simeq I\!PI\!R^3$ respectively, eigenvectors and periodic regimes modulo the parallelism relation.

Let us consider the matrix R of Figure 2. It verifies $R^+ = R$. By Theorem 2, the extremal eigenvectors are $\pi(e,-1)'$ and $\pi(-2,e)'$ and the set of eigenvectors is the set of linear combinations of these two points. It is the strip represented in Figure 2. The line D, in Figure 2, is the hyperspace orthogonal to the first bisecting line. To obtain the set of eigenvectors of R in $I\!PI\!R^2$, we consider the orthogonal projection on D.

Let us introduce a distance $d(.,.)$ on $I\!PI\!R^k$ which we are going to call the projective distance.

Definition 8: We consider $x, y \in I\!PI\!R^k$. Let $u, v \in I\!R^k$ be two representatives of x and y, i.e. $\pi(u) = x$ and $\pi(v) = y$.

$$d(x,y) = d(u,v) = \bigoplus_i (u_i - v_i) \;\oplus\; \bigoplus_i (v_i - u_i)\,.$$

It is easy to check that $d(x,y)$ does not depend on the representatives u and v. It is also easy to check that it is a distance in $I\!PI\!R^k$. It is nothing else than the $\mathcal{L}_\infty$ norm on the projective space $I\!PI\!R^k$. We write either $d(x,y)$ or $d(u,v)$ with a little abuse of notation.

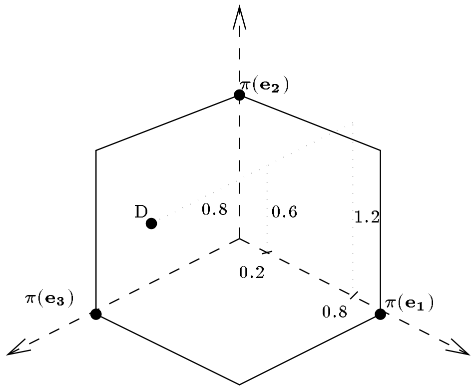

Fig. 3. Unit ball of the projective distance in $I\!PI\!R^3$.

Let us represent the unit ball of the projective distance in $I\!PI\!R^3$. The regular hexagon in Figure 3 is the section of the unit square (i.e. the unit ball of $I\!R^3$ for the $\mathcal{L}_\infty$ norm) by the projection plane. The three represented axes are the orthogonal projection of the basis of $I\!R^3$. The represented points are $\pi(\mathbf{e_1}) = \pi(1,e,e)'$, $\pi(\mathbf{e_2}) = \pi(e,1,e)'$, $\pi(\mathbf{e_3}) = \pi(e,e,1)'$ and $D = \pi(0.2, 0.6, 0.8)'$.

The practical way of representing a point X of $I\!PI\!R^3$ is to choose a vector ($\in I\!R^3$) in the parallelism class of X and to draw it in the three axes obtained by projection of the orthonormal basis of $I\!R^3$ (it is easy to check that the point we obtain does not depend on the representative in the parallelism class). The point D of Figure 3 illustrates this. We have drawn two constructions : one corresponding to (0.2, 0.6, 0.8) and the other one to $(0.8,\ 1.2,\ 1.4) = 0.6 \otimes (0.2,\ 0.6,\ 0.8)$.

Let us illustrate what the "linear combination" of two vectors means in $I\!PI\!R^k$. We consider examples of dimension 3. Let $u = (u_1, u_2, u_3)'$ and $v = (v_1, v_2, v_3)'$ be two vectors of $I\!R^3$. Let $\lambda, \mu \in I\!R$.

$$\pi(\lambda \otimes \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \oplus \mu \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}) = \pi(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \oplus (\mu - \lambda) \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}).$$

Let us assume for example that we have,

$$u_1 - v_1 \le u_2 - v_2 \le u_3 - v_3\,.$$

Depending on the value of $\alpha = \mu - \lambda$, there are four possible cases.

1. If $\alpha \le u_1 - v_1$, then $\pi(u \oplus \alpha v) = \pi(u)$.
2. If $u_1 - v_1 \le \alpha \le u_2 - v_2$, then $\pi(u \oplus \alpha v) = \pi(\alpha v_1, u_2, u_3)'$.
3. If $u_2 - v_2 \le \alpha \le u_3 - v_3$, then $\pi(u \oplus \alpha v) = \pi(\alpha v_1, \alpha v_2, u_3)'$.
4. If $u_3 - v_3 \le \alpha$, then $\pi(u \oplus \alpha v) = \pi(v)$.

This particular example corresponds to the case of points $\pi(\mathbf{e_3})$ ($\pi(u)$) and $\pi(\mathbf{e_1})$ ($\pi(v)$) in Figure 3. The "line" between $\pi(\mathbf{e_3})$ and $\pi(\mathbf{e_1})$, in Figure 3, is the set of linear combinations of the two points. When two values are equal in $\{u_i - v_i,\ i = 1,2,3\}$, the picture is degenerate (see Figure 5, matrix D).

The regular hexagon of Figure 3 is the convex hull of the points $\pi(\mathbf{e_1}), \pi(\mathbf{e_2})$ and $\pi(\mathbf{e_3})$, i.e. the set $\{\pi(\alpha \otimes \mathbf{e_1} \oplus \beta \otimes \mathbf{e_2} \oplus \gamma \otimes \mathbf{e_3}),\ \alpha, \beta, \gamma \in I\!R\}$.

C. Change of Basis

A matrix A of $I\!R_{max}^{k \times k}$ can be considered as a "linear" operator on $I\!R^k$. We want to have a formula of change of basis for the matrix associated with a given linear operator. We are only interested in permutation of the coordinates and translation of the origin.

Definition 9: Let σ be a permutation of $\{1, \ldots, k\}$. The matrix of permutation associated with σ is P defined by :

$$P_{\sigma(i),i} = e,\ P_{ji} = \varepsilon,\ \forall j \ne \sigma(i)\,.$$

Lemma 1: Let A be a $k \times k$ matrix and let $\hat{A}$ be the matrix associated with the same operator in a new basis obtained from the original one by a permutation σ of the coordinates. Matrix P is the permutation matrix associated with σ and P^{-1} is the one associated with σ^{-1}. We have $\hat{A} = P^{-1} \otimes A \otimes P$.

We consider a matrix A. We denote by $\tilde{A}$ the matrix associated with the same operator in a new basis obtained from the original one by a translation of the origin of the projective space.

Lemma 2: Let A be a $k \times k$ matrix. Let $u \in I\!R^k$ be (a representative of) the new origin written in the old basis. In the new basis, we have $\tilde{A} = P^{-1} \otimes A \otimes P$, *where* $P_{ii} = u_i,\ P_{ij} = \varepsilon,\ \forall i \ne j$.

The critical graph of a matrix is not modified by a change of basis. Lemma 3 shows that some basis have a particular interest.

Lemma 3: We consider a matrix A, irreducible, of size k. Let λ be the eigenvalue and u an eigenvector of A. Let P be the matrix of change of the origin associated with u. Let $\tilde{A} = P^{-1} \otimes A \otimes P$. We have the following property $\forall i, j \in 1, \cdots, k, \quad \tilde{A}_{ij} \leq \lambda$ and $\forall p, q$ such that (p, q) belongs to the critical graph (i.e. for all critical terms), we have $\tilde{A}_{qp} = \lambda$.

D. Classification

The set of eigenelements of a matrix is the set of eigenvectors and periodic regimes. Each spectral behaviour corresponds to a specific form for the set of eigenelements. It is possible to prove that this set is compact iff the matrix is aperiodic (Def. 4). In the following, we restrict our attention to aperiodic matrices.

We propose an algorithm to find the spectral type (scs$j$-cyc$k$) of a matrix and to determine the set of its eigenelements. We consider a matrix $A \in \mathbb{R}_{max}^{k\times k}$, irreducible and aperiodic.

Stage 1: Find the eigenvalue and normalize A.
Compute the smallest integer, denoted by N, such that :

$$\exists d \in \mathbb{N},\ \exists \lambda \in \mathbb{R} \ \mid\ \forall n \geq N,\ A^{n+d} = \lambda^{\otimes d} \otimes A^n \ .$$

The eigenvalue of A is λ. Normalize matrix A. For simplicity, we will keep the original notation, i.e. $A := A - \lambda$. We can also use Karp's algorithm to compute λ, see [3]. The set of eigenelements remains unchanged by the normalization.

Stage 2: Find an eigenvector of A.

• Consider an i_0 such that $\exists n \mid (A^n)_{i_0 i_0} = e$. By Theorem 2, this condition is verified if and only if i_0 belongs to the critical graph of A.

• Compute :

$$u = \bigoplus_{n=0}^{k} A^n \delta_{i_0}, \ \ where\ \ (\delta_{i_0})_{i_0} = e,\ (\delta_{i_0})_j = \varepsilon,\ j \neq i_0 \ .$$

From Theorem 2, u is an eigenvector of A.

Stage 3: Write A in a new basis.
Consider a new basis obtained from the original one by a translation of the origin. The new origin is u, the eigenvector of A obtained in the previous stage. For simplicity, we keep the notation A for the operator in the new basis. By Lemma 2, we have $A := P^{-1}AP$, where $P_{ii} = u_i,\ P_{ij} = \varepsilon, i \neq j$.
By Lemma 3, all critical terms of A are now equal to e and all non-critical terms are less than or equal to e. The set of eigenelements in the new basis is obtained from the one in the original basis by a translation of $-u$.

Stage 4: Determine the critical graph of A.
The critical graph is determined by drawing the graph of the terms equal to e in A and by keeping only the circuits of this graph. Compute the number of strongly connected subgraphs (j) and the cyclicity (k) of the critical graph. The spectral type of A is **scsj-cyck**.

Stage 5: Compute the projective size of A.
Consider A^N the stationary version of A (see Stage 1). Let $\mathcal{C}$ be the set of couples (i, j) such that either $A_{i.}$ is a critical line or $A_{.j}$ is a critical column (Definition 6). We set $\alpha = |\min_{(i,j)\in\mathcal{C}} A^N_{ij}|$. We call α the projective size of A. If $\alpha \neq 1$, we scale the matrix A. We set $A := A/\alpha$ (each entry of A is divided (in the conventional algebra) by α). This scaling modifies the set of eigenelements by an homothetic transformation of center $\mathbf{e} = (e, \ldots, e)'$ and of ratio α.

Stage 6: Check non-critical terms of A. Final classification.
Consider the terms A^N_{ij}, $(i, j) \in \mathcal{C}$, which are not critical terms (i.e i or j belong to the critical graph but not i and j). If they are all equal to -1, matrix A^N is said to be canonical. The set of eigenelements of A, which is the same as the one of A^N, has a canonical form. For matrices of size 3, such sets have been represented in Figures 4 to 8. If they are not all equal to -1, the sets of eigenelements are modified. For matrices of size 3, we give, in Figure 5, an example of such modifications.

V. Illustrated Spectral Theory in Dimension 3

We are ready to take a closer look at irreducible and aperiodic matrices of size 3. Using Theorems 2 and 3, it is easy to show that there are only six possible spectral behaviours, which can be sorted in four categories.

- scs1-cyc1
- scs2-cyc2
- scs3-cyc1 and scs1-cyc3.
- scs2-cyc1 and scs1-cyc2.

We propose canonical examples of some of the different spectral behaviours. By lack of space, we do not propose all the cases. For a complete treatment, see [6]. For each case considered, we are going to draw the set of eigenelements, in $\mathbb{P}\mathbb{R}^3$. We will also represent the domains of attraction of the different eigenelements. For a matrix A, we call domain of attraction of an eigenvector (resp. of a periodic regime) the set of initial conditions $\{x_0\}$ such that $\pi(A^n x_0)$ converges to that eigenvector (resp. periodic regime). By Theorem 3, this convergence occurs in finite time.

A. Scs3-Cyc1 and Scs1-Cyc3

If A is a scs1-cyc3 matrix, then A^3 is a scs3-cyc1 matrix (but the converse is false !). For example,

$$A = \begin{pmatrix} . & . & e \\ e & . & . \\ . & e & . \end{pmatrix},\ \ B = A^3 = \begin{pmatrix} e & . & . \\ . & e & . \\ . & . & e \end{pmatrix}, \qquad (5)$$

where (.) stands for -1.

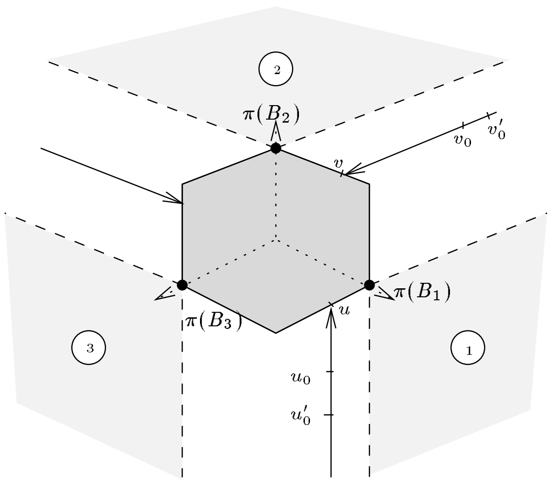

Fig. 4. Scs3-cyc1, set of eigenvectors of B.

We consider first the scs3-cyc1 case. There are three extremal eigenvectors and no periodic regime of period greater than one (Theorems 2 and 3). Let us consider more specifically the matrix B. It is a normalized matrix and we check that $B^+ = B$. By Theorem 2, the three columns B_1, B_2 and B_3 of B are the extremal eigenvectors. The set of eigenvectors is the $\mathbb{R}_{max}$ convex hull of these three eigenvectors. In Figure 4, it is represented in dark gray.
If the initial condition x_0 is in the light gray zone number i, then the limit value (of $\pi(B^n x_0)$) is $\pi(B_i)$. If the initial condition

is in one of the white strips, then the limit value is the nearest point for the projective distance (and this limit is attained in one step as $B^2 = B$). For example, for initial conditions u_0 or u_0' (resp. v_0, v_0') the limit value is u (resp. v).

We will now consider what happens if we modify the non-critical terms of the matrix B. We consider three different examples to illustrate it.

$$C = \begin{pmatrix} e & . & . \\ -\frac{1}{2} & e & . \\ . & . & e \end{pmatrix},\ D = \begin{pmatrix} e & -0.6 & . \\ . & e & 0.6 \\ . & . & e \end{pmatrix},\ (.) = -1$$

$$E = \begin{pmatrix} e & . & -\frac{1}{2} \\ -0.2 & e & . \\ -0.2 & -\frac{1}{2} & e \end{pmatrix}, (.) = -1\,.$$

We have represented the sets of eigenvectors of these matrices in Figure 5.

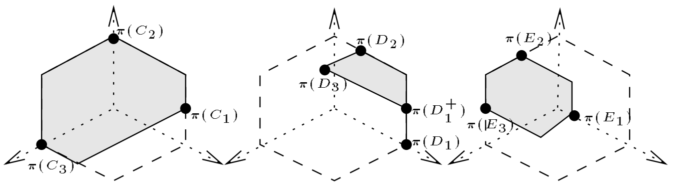

Fig. 5. Scs3-cyc1, sets of eigenvectors of C, D and E.

We can represent these sets using the procedure described in Section IV-B. Let us consider the matrix C for example. We represent the three columns of $C^+ = C$, $\pi(C_1)$, $\pi(C_2)$ and $\pi(C_3)$. The convex hull of these three points is the set of eigenvectors of C.

Now we consider the case of scs1-cyc3 matrices. There is only one eigenvector but there are periodic regimes of period 3. The set of periodic regimes of period 3 of a scs1-cyc3 matrix M is equal to the set of eigenvectors of M^3. Let us consider more specifically the matrix A defined in Equation (5) above.

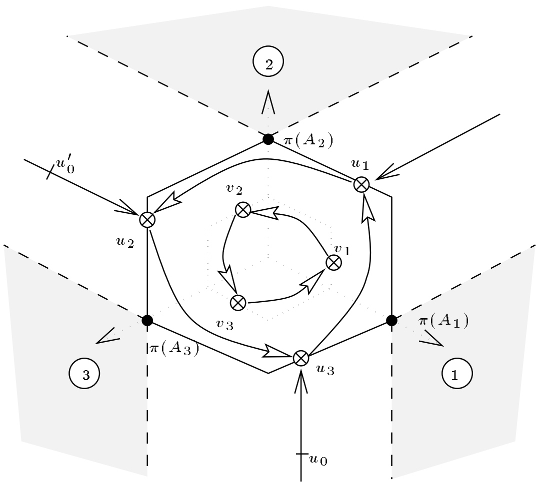

Fig. 6. Scs1-cyc3, set of periodic regimes of A.

The set Π of periodic regimes of A has been represented in Figure 6. It is easy to check that the unique eigenvector of A is $\mathbf{e} = (e, e, e)'$. If the initial condition is in one of the gray zones, then the stationary periodic regime is $\{\pi(A_1), \pi(A_2), \pi(A_3)\}$. If the initial condition is in one of the white strips, the limit regime consists of three points on the boundary of the hexagon. For example for an initial condition u_0 or u_0', the limit regime is $\{u_1, u_2, u_3\}$. If the initial condition u belongs to Π, the stationary periodic regime is $\{u, Au, A^2u\}$. We have also drawn an example of such a regim ($\{v_1, v_2, v_3\}$).

Remark In the cases we have been dealing with so far, domains of attraction had a very easy algebraic characterization. In fact for an initial condition u, the limit value of $\pi(A^n u)$ (or $\pi(B^n u)$ was the "nearest" (for the projective distance) eigenvector or periodic regime. This is a general result. It is a consequence of the synchronization phenomena occuring in $\mathbb{R}_{max}$. However we will see, in the scs1-cyc2 case for example, that the nearest eigenvector is not always unique which makes the description of domains of attraction more delicate.

B. Scs2-Cyc1 and Scs1-Cyc2

In the same way as previously, if A is a scs1-cyc2 matrix then A^2 is a scs2-cyc1 matrix, the converse being false. For example,

$$F = \begin{pmatrix} . & e & . \\ e & . & . \\ . & . & -2 \end{pmatrix},\ G = F^2 = \begin{pmatrix} e & . & . \\ . & e & . \\ . & . & -2 \end{pmatrix},\ (.) = -1\,,$$

Let us consider the scs2-cyc1 case and more precisely the matrix G. By Theorem 2, there are two extremal eigenvectors $\pi(G_1)$ and $\pi(G_2)$ (the first two columns of $G^+ = G$) and no periodic regime of period greater than 1. In Figure 7, we have represented eigenvectors and domains of attraction for G.

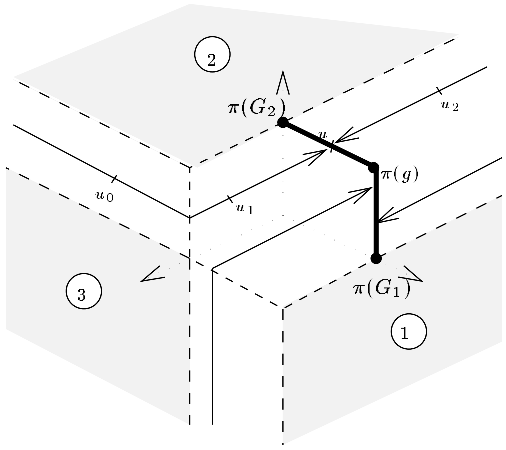

Fig. 7. Scs2-cyc1 (resp. scs1-cyc2), set of eigenelements of G (resp. F).

The set of eigenvectors is the linear combinations of $\pi(G_1)$ and $\pi(G_2)$. As opposed to the scs3-cyc1 case, no eigenvector has a domain of attraction restricted to itself. If the initial condition x_0 is in the gray zone 1 or 2, the limit value of $\pi(G^k x_0)$ will be $\pi(G_1)$ or $\pi(G_2)$ respectively. If it is in zone 3, then the limit value will be $\pi(G)$. When the initial condition is in one of the white strips, the limit value is given by the arrow. For example, if the initial condition is u_0, u_1 or u_2, the limit regime is u.

The picture remains the same for matrix F which is scs1-cyc2. There is only one eigenvector which is $\pi(G)$. The "line" between $\pi(G_1)$ and $\pi(G_2)$ is the set of periodic regimes of period 2. Two points of this set belong to the same periodic regime if they are "symmetric" with respect to $\pi(G)$. For an initial condition in zone 3, the limit regime is the eigenvector $\pi(G)$. For an initial condition in zones 1 or 2, the limit regime is $\{\pi(G_1), \pi(G_2)\}$ and so on.

C. *Scs2-Cyc2*

The canonical example of such a matrix is :

$$H = \begin{pmatrix} . & e & . \\ e & . & . \\ . & . & e \end{pmatrix}, \; (.) = -1 \, .$$

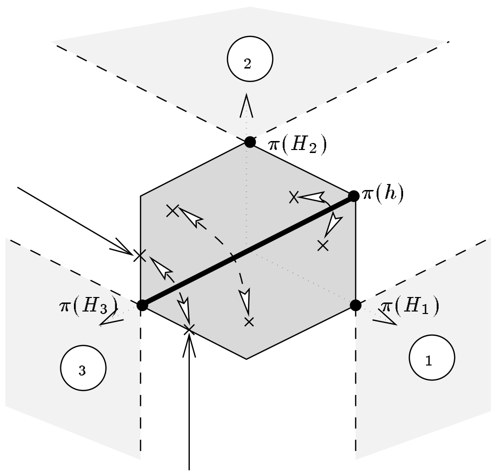

Fig. 8. Scs2-cyc2, set of eigenelements of H.

Figure 8 has to be interpreted as in the previous examples. The set of eigenvectors is $[\pi(H_3), \pi(h)]$ and the set of eigenelements is the hexagon.

VI. Application to the Manufacturing Model

We consider the manufacturing model of Figure 1. Matrix M in Equation (3) is either scs1-cyc2 or scs1-cyc1, depending on the values of α, β_i and γ_i. As an example, let $\alpha = 2$, $\beta_1 = 2$, $\beta_2 = 1$, $\gamma_1 = 2$ and $\gamma_2 = 0$. We obtain :

$$x(n+1) = M \otimes x(n), \; M = \begin{pmatrix} \varepsilon & \varepsilon & 2 \\ \varepsilon & \varepsilon & 1 \\ 4 & 2 & 2 \end{pmatrix} .$$

The eigenvalue is $\lambda = 3$. The circuit $(1,3)$ is the unique critical circuit. The set (in the projective space) of periodic regimes is $\{\pi(u_\eta), \pi(v_\eta)\} = \{\pi(1+\eta, e, 2), \pi(1, e, 2+\eta)\}$, $\eta \in [e, 1]$. The unique eigenvector is $u_0 = v_0 = u = (1, e, 2)'$.

We consider a specific periodic regime $\{u_\eta, v_\eta\}$. We suppose that the system is in this regime, i.e. $\pi(x(2n)) = \pi(u_\eta)$ and $\pi(x(2n+1)) = \pi(v_\eta)$. We consider the idle time as defined in Equation (4). We have $\delta(2n) = 1 + (u_\eta)_3 - (v_\eta)_3$ and $\delta(2n+1) = 1 + (v_\eta)_3 - (u_\eta)_3$. We see that the idle time of the assembly line depends on the periodic regime of the system.
Suppose that we want to minimize the idle time. We also want to have a control which consists in choosing the initial condition. The maximal idle time is :

$$\overline{\delta} = \max \delta(2n), \delta(2n+1)$$

$$= \max\left(3 - \eta + (u_\eta)_3 - (v_\eta)_3, 3 - \eta + (v_\eta)_3 - (u_\eta)_3\right) = 3 + \eta \, .$$

So we minimize $\overline{\delta}$ for the eigenvector u ($\eta = 0$). We conclude that we have to choose the initial condition in the domain of attraction of u. Such a domain is given by the graphical representation of matrix M (which will be close to the one of Figure 7).

VII. Conclusion

The main contribution of this paper is the graphical characterization of the domains of attraction of the eigenelements of a matrix. The drawback of the approach is that it considers only matrices of size 3. However, it should be noted that for a general matrix, if the critical circuit is of size 3 or less, then the proposed approach applies. Moreover, the study of these 3x3 matrices provides a good intuition of the general case. For example, in any dimension, the domains of attraction will be "polyhedrons" and the sets of eigenelements, compact polyhedrons.

Acknowledgments

The author would like to thank François Baccelli and Stéphane Gaubert for numerous very fruitful talks and an anonymous referee for helpful comments.

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