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RANDOM WALKS ON FREE PRODUCTS OF CYCLIC GROUPS

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Abstract

Let G be a free product of a finite family of finite groups, with the set of generators being formed by the union of the finite groups. We consider a transient nearest-neighbor random walk on G. We give a new proof of the fact that the harmonic measure is a special Markovian measure entirely determined by a finite set of polynomial equations. We show that in several simple cases of interest, the polynomial equations can be explicitly solved, to get closed form formulas for the drift. The examples considered are $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/k\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$, and the Hecke groups $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$. We also use these various examples to study Vershik's notion of extremal generators, which is based on the relation between the drift, the entropy, and the growth of the group.

1. *Introduction*

The properties of the harmonic measure associated with a nearest-neighbor random walk on a free group, or a free product of groups, have been studied by many authors, see [**5**, **24**, **13**, **15**, **28**], or the monograph [**30**] and the references therein. In this context, the Green kernel has a multiplicative structure. Consequently, the harmonic measure is Markovian, see in particular [**24**, Section 5] and [**28**, Section 6]. In [**17**], this property is viewed from a different angle. It is proved that the harmonic measure is a Markovian measure with a special combinatorial structure, called *Markovian multiplicative*. It is entirely determined by its initial distribution, which is itself characterized as the unique solution to a finite set of polynomial equations coined as the *Traffic Equations*. The result of [**17**] is proved for a whole class of pairs formed by a group (or monoid) and a finite set of generators: the so-called *0-automatic pairs*. The property of being Markovian multiplicative is very specific. For instance, in the related context of trees with finitely many cone types, the harmonic measure is Markovian but not Markovian multiplicative, see [**21**, Section 5].

In this paper, we focus on nearest-neighbor random walks (NNRW) on free products of finite groups, and in particular of finite cyclic groups. There may be several motivations for specifically studying such random walks. First, they are among the simplest non-commutative random walks. As such, they serve as a reference point, and numerous results have first been proved in this context before being extended. Second, they may be pertinent in the physics of polymers. This is discussed and argued in [**22**, **23**].

There are three types of results being proved in the paper.

Section 3 - We revisit the result of [**17**]. Consider a free product of finite groups G, the set of generators Σ being the union of the finite groups (the *natural* generators).

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The pair (G,Σ) is a special case of 0-automatic pair. Consider a transient NNRW on (G,Σ), say $(X_n)_n$. Assume that the group elements are written in reduced form. The *harmonic measure* is the law of $X_\infty = \lim_n X_n$. First, we give a short proof of the special Markovian structure of the harmonic measure. This proof is different from the one in [**17**] and would not work in the more general context of 0-automatic pairs. Second, we take advantage of the restricted setting to prove more precise results. In particular, we characterize the cases where the harmonic measure is stationary with respect to the translation shift.

Section 4 - The result on the harmonic measure has interesting computational consequences. Indeed, in many situations, the Traffic Equations can be solved "explicitly", in order to get a closed form formula for the drift, the entropy or the minimal harmonic functions. We illustrate this by explicitly computing the drift in the following cases: (i) the general NNRW on the modular group $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$, (ii) a two-parameter family of NNRWs on $\mathbb{Z}/3\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$, and (iii) the simple (with respect to minimal generators) NNRW on $\mathbb{Z}/k\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$, and on the Hecke groups $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$.

Section 5 - We investigate Vershik's notion of extremal generators, which is based on the link between drift, entropy, and growth for random walks on a group [**26**]. We prove the following. For a free product of finite groups, the set of natural generators is extremal. This uses the special structure of the harmonic measure (see Section 3). We also show that for the group $\mathbb{Z}/4\mathbb{Z}\star\mathbb{Z}/4\mathbb{Z}$, the minimal set of generators is not extremal. This is achieved by explicitly computing the drift and the entropy (as in Section 4).

In our view, the interest of the present work is to provide a large collection of tractable models, whereas very few were previously available. This is a potential source for examples or counter-examples. In particular, none of the drift computations in the paper appeared in the literature before. For the few examples of non-elementary explicit computations previously available, see [**4**, **5**, **15**, **21**, **23**, **24**]. Nevertheless, there exists an alternative potential method for the effective computation of the drift (not the entropy) which is due to Sawyer and Steger [**24**]. In this approach, the drift is expressed as a functional of the *first-passage generating series* of the random walk. The simplest of our computational results can also be retrieved using this method. We detail and discuss this approach in §4.1. The Sawyer-Steger method links the problem of computing the drift with the problem of computing the generating series of transition probabilities. Concerning the latter problem, there exists an important literature, especially for random walks on free groups and free products, see [**2**, **3**, **8**, **29**] and [**30**, Sections II.9 and III.17].

For much more material on random walks on discrete infinite groups (including aspects not even touched upon here, e.g., boundary theory, or central/local limit theorems), see [**30**, **11**, **26**] and the references there.

Part of the results presented in this paper, as well as in the companion paper [**17**], were announced without proofs in the Proceedings of the International Colloquium of Mathematics and Computer Science held in Vienna in September 2004 [**18**].

2. *Preliminaries*

Notations. Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{N}^*$ the set of positive integers. We denote the support of a random variable by supp. If μ is a measure on a group $(G,*)$, then μ^{*n} is the n-fold convolution product of μ, that is the image of the product measure $\mu^{\otimes n}$ by the product map $G\times\cdots\times G\to G,\ (g_1,\dots,g_k)\mapsto g_1*g_2*\cdots*g_k$. The symbol $\sqcup$ is used for the disjoint union of sets. Given a finite set Σ, a vector $x\in\mathbb{R}^\Sigma$, and $S\subset\Sigma$, set $x(S)=\sum_{u\in S}x(u)$.

2.1. *Random walks on groups*

Consider a finitely generated group $(G,*)$ with unit element 1_G. Let $\Sigma\subset G$ be a finite set of generators of G (with $1_G\notin\Sigma$ and $u\in\Sigma\implies u^{-1}\in\Sigma$). The *length* with respect to Σ of a group element u is:

$$|u|_\Sigma=\min\{k\mid\ u=s_1*\cdots*s_k, s_i\in\Sigma\}\,. \tag{1}$$

The *Cayley graph* $\mathfrak{X}(G,\Sigma)$ of a group G with respect to a set of generators Σ is the graph with G as set of vertices and with an edge between u and v if and only if $u^{-1}v\in\Sigma$. Observe that $|u|_\Sigma$ is the geodesic distance from 1_G to u in the Cayley graph.

Let μ be a probability distribution over Σ. Consider the Markov chain on the state space G with one-step transition probabilities given by: $\forall g\in G,\forall a\in\Sigma,\ P_{g,g*a}=\mu(a)$. This Markov chain is called the *random walk* (associated with) (G,μ). It is a *nearest neighbor* random walk: one-step moves occur between nearest neighbors in the Cayley graph $\mathfrak{X}(G,\Sigma)$. When $\mu(s)=1/|\Sigma|$ for all $s\in\Sigma$, we say that the random walk is *simple*.

Let $(x_n)_n$ be a sequence of i.i.d. r.v.'s distributed according to μ. Set

$$X_0=1,\ X_{n+1}=X_n*x_n=x_0*x_1*\cdots*x_n\,. \tag{2}$$

The sequence $(X_n)_n$ is a *realization* of the random walk (G,μ). The law of X_n is μ^{*n}. Since $|u*v|_\Sigma\le|u|_\Sigma+|v|_\Sigma$, Guivarc'h [**9**] observed that a simple corollary of Kingman's Subadditive Ergodic Theorem [**12**] is the existence of a constant $\gamma\in\mathbb{R}_+$ such that a.s. and in L^p, for all $1\le p<\infty$,

$$\lim_{n\to\infty}\frac{|X_n|_\Sigma}{n}=\gamma\,. \tag{3}$$

The constant γ is called the *drift*. Intuitively, γ is the speed of escape to infinity of the walk.

2.2. *Free products and harmonic measure*

Let $(G_i)_{i\in I}$ be a finite family of finite groups, with $|I|\ge 2$. Let 1_{G_i} be the unit of G_i. Set $\Sigma_i=G_i\backslash\{1_{G_i}\}$ and set $\Sigma=\sqcup_i\Sigma_i$. Let $\iota:\Sigma\to I$ be defined by $\iota(u)=j$ if $u\in\Sigma_j$. It is also convenient to set $\Sigma_a=\Sigma_{\iota(a)}$ for all $a\in\Sigma$.

Let Σ^* be the free monoid over the alphabet Σ and denote its unit, the empty word, by 1. Define the set of *normal form words* $L\subset\Sigma^*$ by

$$L=\{u_1\cdots u_k\in\Sigma^*,\ \forall i\in\{1,\cdots,k-1\},\iota(u_i)\neq\iota(u_{i+1})\}\,. \tag{4}$$

Hence, L consists of all words over the alphabet Σ whose consecutive letters come from different subalphabets Σ_i. Observe that $1\in L$.

The *free product* $G = \star_{i\in I} G_i$ is the group with set of elements L, unit element 1, and group law $*$ defined recursively by:

$$u_1 \cdots u_k * v_1 \cdots v_l = \begin{cases} u_1 \cdots (u_{k-1})(u_k)(v_1)(v_2)\cdots v_l & \text{if } \iota(u_k) \neq \iota(v_1) \\ u_1 \cdots (u_{k-1})(u_k * v_1)(v_2)\cdots v_l & \text{if } \iota(u_k) = \iota(v_1),\ u_k \neq v_1^{-1} \\ u_1 \cdots u_{k-1} * v_2 \cdots v_l & \text{if } u_k = v_1^{-1} \end{cases},$$

where in the second case, $(u_k * v_1)$ is the product in $G_{\iota(u_k)}$ of u_k and v_1. Roughly, the law of G is the concatenation with possible simplifications at the contact point to reach a normal form word.

The *length* of an element u of $(\star_{i\in I} G_i)$ is the length (i.e. number of letters) of the word u in L. We denote it by $|u|$. Observe that $|u| = \min\{k \mid u_1 * \cdots * u_k = u,\ u_i \in \Sigma\} = |u|_\Sigma$.

Let μ be a probability measure over Σ such that $\bigcup_{n\in\mathbb{N}^*} \operatorname{supp} \mu^{*n} = \star_{i\in I} G_i$. Let $(X_n)_n$ be a realization of the random walk $(\star_{i\in I} G_i, \mu)$ as defined above. The sequence $(X_n)_n$ can be viewed as a Markov chain on L. Below the drift is defined according to (3) with respect to the length $|\cdot|$.

The group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ is amenable, and any nearest neighbor random walk on it is recurrent. But apart from this group, all the free products considered are non-amenable. Therefore, if G is not $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, any random walk living on the whole group is transient and has a strictly positive drift (see [**9**] and [**30**, Chapter 1.B] for details).

¿From now on, the random walks considered are assumed to be transient. Equivalently, we work on a free product group G different from $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ and the support of μ generates the whole group.

Consider the set $\Sigma^{\mathbb{N}}$ equipped with the product topology. Denote by $(u_1 \cdots u_n \Sigma^{\mathbb{N}})$ the order-n cylinder in $\Sigma^{\mathbb{N}}$ defined by $u_1 \cdots u_n$. Define the set of *(right) infinite normal form words* $L^\infty \subset \Sigma^{\mathbb{N}}$ by

$$L^\infty = \{u_0 u_1 u_2 \cdots \in \Sigma^{\mathbb{N}},\ \forall i \in \mathbb{N}^*, \iota(u_i) \neq \iota(u_{i+1})\}\,. \tag{5}$$

A word belongs to L^∞ iff all its finite prefixes belong to L.

Consider the map $\Sigma \times L^\infty \to L^\infty, (a,\xi) \mapsto a\cdot\xi$, with $a\cdot\xi = a\xi_0\xi_1\cdots$ if $\iota(a) \neq \iota(\xi_0)$, $a\cdot\xi = (a*\xi_0)\xi_1\cdots$ if $\iota(a) = \iota(\xi_0), a \neq \xi_0^{-1}$, and $a\cdot\xi = \xi_1\xi_2\cdots$ if $a = \xi_0^{-1}$. Equip $\Sigma^{\mathbb{N}}$ with the Borel σ-algebra associated with the product topology. This induces a σ-algebra on L^∞. Given a measure ν^∞ on L^∞ and $a \in \Sigma$, define the measure $a\nu^\infty$ by: $\int f(\xi) d(a\nu^\infty)(\xi) = \int f(a\cdot\xi) d\nu^\infty(\xi)$. A probability measure ν^∞ on L^∞ is *μ-invariant* if

$$\nu^\infty(\cdot) = \sum_{a\in\Sigma} \mu(a)[a\nu^\infty](\cdot)\,. \tag{6}$$

Proposition 2.1. *There exists a r.v. X_∞ valued in L^∞ such that a.s.*

$$\lim_{n\to\infty} X_n = X_\infty\,,$$

meaning that the length of the common prefix between X_n and X_∞ goes to infinity a.s. Let μ^∞ be the distribution of X^∞. The probability μ^∞ is μ-invariant and is the only μ-invariant probability on L^∞. It is called the harmonic measure *of (G,μ).*

The drift of the random walk is given by:

$$\gamma = \sum_{a\in\Sigma} \mu(a)\Big[-\mu^{\infty}(a^{-1}\Sigma^{\mathbb{N}}) + \sum_{b\in\Sigma\setminus\Sigma_a} \mu^{\infty}(b\Sigma^{\mathbb{N}})\Big]\,. \tag{7}$$

In words, γ is the expected change of length of an infinite normal form distributed according to μ^{∞}, when left-multiplied by an element distributed according to μ.

In the context of the free group, this is proved for instance in [**15**, Theorem 1.12, Theorem 4.10]. The proofs adapt easily to the present setting. Several of the key arguments go back to [**6**, **7**], see [**15**] for precise references.

Intuitively, the harmonic measure μ^{∞} gives the direction in which $(X_n)_n$ goes to infinity.

3. *Free Products Have a Markovian Harmonic Measure*

In [**17**, Theorems 4.5 and 5.3], it is proved that the harmonic measure for random walks on 0-automatic pairs has a special Markovian multiplicative structure. In this section, we revisit the result. We concentrate on a subclass of 0-automatic pairs: the pairs formed by free products of finite groups with natural generators. In this setting, we get a more elementary proof of the result. We also refine the result by discussing the cases where the harmonic measure is shift-invariant (on top of being μ-invariant). To that purpose, we associate two sets of equations with the random walk: the *Traffic Equations* (as in [**17**]) but also the *Stationary Traffic Equations*. The results in Propositions 3.8 and 3.9 are new.

Define $\mathring{\mathcal{B}} = \{x \in \mathbb{R}^{\Sigma} \mid \forall u,\ x(u) > 0,\ \sum_u x(u) = 1\}$. Consider $r \in \mathring{\mathcal{B}}$. Define the matrix P of dimension $\Sigma \times \Sigma$ by

$$P_{u,v} = \begin{cases} r(v)/r(\Sigma\setminus\Sigma_u) & \text{if } v \in \Sigma\setminus\Sigma_u \\ 0 & \text{otherwise} \end{cases}\,. \tag{8}$$

It is the transition matrix of an irreducible Markov Chain on the state space Σ. Set $p = (r(a)r(\Sigma\setminus\Sigma_a), a \in \Sigma)$ and $\pi = p/p(\Sigma)$. Observe that $\pi P = \pi$. In words, π is the stationary distribution of the Markov chain defined by P.

Let $(U_n)_n$ be a realization of the Markov chain with transition matrix P and starting from U_1 such that $P\{U_1 = x\} = r(x)$. Set $U^{\infty} = \lim_n U_1 \cdots U_n$, and let ν^{∞} be the distribution of U^{∞}. Clearly the support of ν^{∞} is included in L^{∞}. For $u = u_1 \cdots u_k \in L$, we have

$$\begin{aligned} \nu^{\infty}(u\Sigma^{\mathbb{N}}) &= r(u_1)P_{u_1,u_2}\cdots P_{u_{k-1},u_k} \qquad (9)\\ &= r(u_1)\frac{r(u_2)}{r(\Sigma\setminus\Sigma_{u_1})}\cdots\frac{r(u_k)}{r(\Sigma\setminus\Sigma_{u_{k-1}})} = \frac{r(u_1)}{r(\Sigma\setminus\Sigma_{u_1})}\cdots\frac{r(u_{k-1})}{r(\Sigma\setminus\Sigma_{u_{k-1}})}r(u_k)\,. \end{aligned}$$

We call ν^{∞} the *Markovian multiplicative* probability measure associated with r.

The measure ν^{∞} is in general non-stationary with respect to the translation shift $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}, (x_n)_n \mapsto (x_{n+1})_n$. Indeed, the distribution of the first marginal is r which is different in general from the stationary distribution π. An important special case is when ν^{∞} is nevertheless stationary and ergodic, i.e. when $r = \pi$. This happens if and only if

$$\forall i \in I, \quad r(\Sigma_i) = 1/|I|\,. \tag{10}$$

Definition 3.1 (Traffic Equations). *The* Traffic Equations *associated with* (G,μ) *are defined by: for all* a *in* Σ,

$$x(a)=\mu(a)\sum_{u\in\Sigma\setminus\Sigma_a}x(u)+\sum_{u*v=a}\mu(u)x(v)+\sum_{u\in\Sigma\setminus\Sigma_a}\mu(u^{-1})\frac{x(u)}{\sum_{v\in\Sigma\setminus\Sigma_u}x(v)}x(a)\,. \quad (11)$$

The Traffic Equations are closely related to the harmonic measure of (G,μ). Next lemma is proved in a more general context in [**17**, Lemma 5.2].

Lemma 3.2. *If the harmonic measure* μ^∞ *is the Markovian multiplicative measure associated with* $r\in\mathring{\mathcal{B}}$*, then* r *is a solution to the Traffic Equations (11). Conversely, if the Traffic Equations admit a solution* $r\in\mathring{\mathcal{B}}$*, then the harmonic measure* μ^∞ *is the Markovian multiplicative measure associated with* r.

Using (10), we can complete the statement of Lemma 3.2 as follows.

Definition 3.3 (Stationary Traffic Equations). *The* Stationary Traffic Equations *associated with* (G,μ) *are defined by: for all* a *in* Σ,

$$x(a)=\mu(a)\frac{|I|-1}{|I|}+\sum_{u*v=a}\mu(u)x(v)+x(a)\frac{|I|}{|I|-1}\sum_{u\in\Sigma\setminus\Sigma_a}\mu(u^{-1})x(u)\,. \quad (12)$$

Lemma 3.4. *The harmonic measure* μ^∞ *is Markovian multiplicative associated with* r and *ergodic if and only if the Stationary Traffic Equations admit a solution* r *in* $\mathring{\mathcal{B}}$.

A corollary of Lemma 3.2, resp. Lemma 3.4, is that the Traffic Equations, resp. the Stationary Traffic Equations, have at most one solution in $\mathring{\mathcal{B}}$. The Stationary Traffic Equations do not always have solution. But the Traffic Equations do, see Theorem 3.5.

Theorem 3.5. *Let* $G=\star_{i\in I}G_i$ *be the free product of a finite family of finite groups, with* $|I|\geq 2$*, and* $\forall i,|G_i|>1$*. Let* μ *be a probability measure on* $\Sigma=\sqcup_i G_i\setminus\{1_{G_i}\}$*. Assume that* $\bigcup_{n\in\mathbb{N}^*}\mathrm{supp}\,\mu^{*n}=G$ *and that the random walk* (G,μ) *is transient. Then the Traffic Equations (11) have a unique solution* $r\in\mathring{\mathcal{B}}$*. The harmonic measure of the random walk is the Markovian multiplicative measure associated with* r.

For finitely generated free groups, the special Markovian structure of the harmonic measure is a classical result [**5**, **24**, **15**]. Theorem 3.5 is proved in [**17**] in the more general context of 0-automatic pairs. Here we give a short proof of Theorem 3.5 which is close in spirit to the proofs in [**5**, **24**, **15**, **21**]. On the other hand, the proof below is quite different from the one in [**17**] and would not work in the general context of 0-automatic pairs.

Proof. For all $a\in\Sigma$, define $q(a)=P\{\exists n\mid X_n=a\}$, the probability of ever hitting a. Clearly, $0<q(a)<1$. Besides, we have: $\forall a\in\Sigma$,

$$q(a)=\mu(a)+\sum_{u*v=a}\mu(u)q(v)+q(a)\sum_{c\in\Sigma\setminus\Sigma_a}\mu(c)q(c^{-1})\,. \quad (13)$$

The first two terms on the right-hand side of (13) are more or less obvious. Now assume that the random walk starts with an initial step of type $c \in \Sigma \backslash \Sigma_a$. Given the tree-like structure of the Cayley graph, it has to go back to 1 before possibly reaching a. Now, the probability of ever hitting 1 starting from c is equal to the probability of ever hitting c^{-1} starting from 1. This accounts for the third right-hand term in (13).

A simple rewriting of the Traffic Equations (11) gives:

$$\frac{x(a)}{x(\Sigma\backslash\Sigma_a)} = \mu(a) + \sum_{u*v=a} \mu(u)\frac{x(v)}{x(\Sigma\backslash\Sigma_v)} + \frac{x(a)}{x(\Sigma\backslash\Sigma_a)} \sum_{u\in\Sigma\backslash\Sigma_a} \mu(u^{-1})\frac{x(u)}{x(\Sigma\backslash\Sigma_u)} \,. \quad (14)$$

Hence it is natural to look for a solution r to the Traffic Equations satisfying

$$\forall a \in \Sigma, \quad \frac{r(a)}{r(\Sigma\backslash\Sigma_a)} = q(a)\,. \quad (15)$$

It remains to be proved that the Equations (15) have a solution in r. Clearly, they have a solution iff the following equations have a solution:

$$\begin{cases} r(\Sigma_i) = q(\Sigma_i) r(\Sigma\backslash\Sigma_i) & (i) \\ \sum_{j\in I} r(\Sigma_j) = 1 & (\mathrm{sum}) \end{cases} \,. \quad (16)$$

For a given i, consider the Equations: $\{(i), (\mathrm{sum})\}$. The solution is $r(\Sigma_i) = q(\Sigma_i)/(1+q(\Sigma_i))$, $r(\Sigma\backslash\Sigma_i) = 1/(1+q(\Sigma_i))$. In order to have a global solution to (16), the necessary and sufficient condition is that:

$$\sum_{i\in I} \frac{q(\Sigma_i)}{1+q(\Sigma_i)} = 1\,. \quad (17)$$

Now let us prove (17). Starting from (13) and summing over Σ_i, we get:

$$\begin{aligned} q(\Sigma_i) &= \mu(\Sigma_i) + \sum_{a\in\Sigma_i} \sum_{u*v=a} \mu(u)q(v) + q(\Sigma_i) \sum_{a\in\Sigma\backslash\Sigma_i} \mu(a^{-1})q(a) \\ &= \mu(\Sigma_i) + \mu(\Sigma_i)q(\Sigma_i) - \sum_{a\in\Sigma_i} \mu(a^{-1})q(a) + q(\Sigma_i) \sum_{a\in\Sigma\backslash\Sigma_i} \mu(a^{-1})q(a) \end{aligned}$$

It follows that

$$\frac{q(\Sigma_i)}{1+q(\Sigma_i)} = \mu(\Sigma_i) + \frac{q(\Sigma_i)}{1+q(\Sigma_i)} \sum_{a\in\Sigma\backslash\Sigma_i} \mu(a^{-1})q(a) - \frac{1}{1+q(\Sigma_i)} \sum_{a\in\Sigma_i} \mu(a^{-1})q(a)$$

$$\sum_i \frac{q(\Sigma_i)}{1+q(\Sigma_i)} = 1 + \sum_i \Big[\sum_{a\in\Sigma_i} \mu(a^{-1})q(a)\Big]\Big[\sum_{j\neq i} \frac{q(\Sigma_j)}{1+q(\Sigma_j)} - \frac{1}{1+q(\Sigma_i)}\Big]$$

$$\sum_i \frac{q(\Sigma_i)}{1+q(\Sigma_i)} - 1 = \Big[\sum_{a\in\Sigma} \mu(a^{-1})q(a)\Big]\Big[\sum_i \frac{q(\Sigma_i)}{1+q(\Sigma_i)} - 1\Big]\,.$$

Since $0 < q(a) < 1$ for all a, we have $\sum_{a\in\Sigma} \mu(a^{-1})q(a) < 1$. We conclude that we must have: $\sum_i q(\Sigma_i)/(1+q(\Sigma_i)) - 1 = 0$. Hence Equation (17) holds. It implies that the Traffic Equations have a solution which is:

$$\forall a \in \Sigma, \quad r(a) = \frac{q(a)}{1+q(\Sigma_a)}\,. \quad (18)$$

According to Lemma 3.2, such a solution is necessarily unique, and the harmonic measure is the Markovian multiplicative measure associated with r. ∎

Corollary 3.6. *Under the assumptions of Theorem 3.5, the drift is given by*

$$\gamma = \sum_{a\in\Sigma} \mu(a)\big[-r(a^{-1}) + \sum_{b\in\Sigma\setminus\Sigma_a} r(b)\big] , \tag{19}$$

where r is the unique solution in $\mathring{\mathcal{B}}$ to the Traffic Equations.

Corollary 3.7. *Under the assumptions of Theorem 3.5, we have*

$$\forall a \in \Sigma, \qquad P\{\exists n \mid X_n = a\} = r(a)/r(\Sigma\setminus\Sigma_a) ,$$

where r is the unique solution in $\mathring{\mathcal{B}}$ to the Traffic Equations.

It follows from Lemma 3.4 and Theorem 3.5 that the harmonic measure is shift-invariant iff the Stationary Traffic Equations have a solution. In Proposition 3.8, we give a sufficient condition for this to happen.

Proposition 3.8. *Let H be a finite group and let $(G_i)_{i\in I}$ be a finite family of copies of H. Let π_i be the isomorphism between G_i and H. Let ν be a probability measure on $H\setminus\{1_H\}$. Consider the free product $G = \star_{i\in I} G_i$ and let μ be the probability measure on $\Sigma = \sqcup_i G_i\setminus\{1_{G_i}\}$ defined by: $\forall g \in G_i\setminus\{1_{G_i}\}$, $\mu(g) = \nu\circ\pi_i(g)/|I|$. Then the harmonic measure of (G,μ) is stationary and ergodic with respect to the translation shift $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}, (x_n)_n \mapsto (x_{n+1})_n$.*

Proof. Set $\Sigma_i = G_i\setminus\{1_{G_i}\}$ for all i. Let $a \in \Sigma_i$ and $b \in \Sigma_j$ be such that $\pi_i(a) = \pi_j(b)$. By a symmetry argument, we have $r(a) = r(b)$, where r is the solution to the Traffic Equations. A direct consequence is that $r(\Sigma_i) = r(\Sigma_j)$ for all i, j. Hence Condition (10) is satisfied and the harmonic measure is ergodic. □

Proposition 3.9. *Consider a random walk (G,μ) where $G = G_1 \star G_2$ is the free product of two arbitrary finite groups. Then the harmonic measure μ^∞ satisfies: $\forall u \in L, \forall k \in \mathbb{N}$, $\mu^\infty(u\Sigma^{\mathbb{N}}) = \mu^\infty(\Sigma^{2k}u\Sigma^{\mathbb{N}})$. That is, μ^∞ is stationary and ergodic with respect to the shift $\tau^2 : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}, (x_n)_n \mapsto (x_{n+2})_n$.*

Proof. Applying (17), we get:

$$\frac{q(\Sigma_1)}{1+q(\Sigma_1)} + \frac{q(\Sigma_2)}{1+q(\Sigma_2)} = 1 \quad\Longrightarrow\quad q(\Sigma_1)q(\Sigma_2) = 1 . \tag{20}$$

Consider $u = u_1 \cdots \in L$ with for instance $u_1 \in \Sigma_1$. We have, using (11):

$$\begin{aligned}\mu^\infty(\Sigma^2 u\Sigma^{\mathbb{N}}) &= \sum_{v_1\in\Sigma_1}\sum_{v_2\in\Sigma_2} \mu^\infty(v_1v_2u\Sigma^{\mathbb{N}}) \;=\; \sum_{v_1\in\Sigma_1} q(v_1) \sum_{v_2\in\Sigma_2} q(v_2)\mu^\infty(u\Sigma^{\mathbb{N}}) \\ &= q(\Sigma_1)q(\Sigma_2)\mu^\infty(u\Sigma^{\mathbb{N}}) \;=\; \mu^\infty(u\Sigma^{\mathbb{N}}) .\end{aligned}$$

We prove in the same way that: $\forall u \in L, \forall k \in \mathbb{N}$, $\mu^\infty(u\Sigma^{\mathbb{N}}) = \mu^\infty(\Sigma^{2k}u\Sigma^{\mathbb{N}})$. □

There exists no simple analog of Proposition 3.9 for the free product of three or more finite groups.

The Identity (20): $q(\Sigma_1)q(\Sigma_2) = 1$, is quite unexpected. Indeed it can be rephrased as: the average number of different elements visited in Σ_1 is the inverse of the average number of different elements visited in Σ_2.

Free products of countable groups

Consider a free product $G = \star_{i\in I} G_i$ of a finite family of *countable* groups. Let μ be a probability measure on the countable set $\Sigma = \sqcup_i G_i \backslash \{1_{G_i}\}$. Assume that $\bigcup_{n\in\mathbb{N}^*} \operatorname{supp} \mu^{*n} = G$ and that the random walk (G,μ) is transient. One can define the set of Traffic Equations (resp. Stationary Traffic Equations) exactly as in (11) (resp. (12)). It is a set of infinitely many equations involving infinite sums. We have the following.

THEOREM 3.10. *The statements of Lemmas 3.2 and 3.4, Theorem 3.5, Corollaries 3.6 and 3.7, Propositions 3.8 and 3.9, remain true for a free product $G = \star_{i\in I} G_i$ of a finite family of countable groups.*

Proof. The only difference with the case of finite groups is that we have to prove that $q(\Sigma_i)$ is finite for all $i \in I$. We proceed as follows. For $u \in G$, define the r.v. $\tau(u) = \min\{n \mid X_n = u\}$ (with $\tau(u) = \infty$ if u is not reached). Define the series $q(u,z) = \sum_{n\in\mathbb{N}} P\{\tau(u) = n\} z^n$. Observe that $q(u,1) = q(u)$, the probability of ever hitting u. The family of series $(q(u,z))_{u\in\Sigma}$ satisfies the following version of (13):

$$\forall a \in \Sigma,\ q(a,z) = z\,\mu(a) + z \sum_{u*v=a} \mu(u) q(v,z) + z\, q(a,z) \sum_{c\in\Sigma\setminus\Sigma_a} \mu(c) q(c^{-1},z)\,. \quad (21)$$

For $i \in I$, we set $q(\Sigma_i,z) = \sum_{u\in\Sigma_i} q(u,z)$, $R_i(z) = \sum_{u\in\Sigma_i} \mu(u^{-1}) q(u,z)$, and $S_i(z) = \sum_{j\neq i} R_j(z)$. These series are well-defined since all the coefficients are bounded by 1. Observe that, since $q(a) < 1$, we have $R_i(1) < \mu(\Sigma_i)$ and therefore $S_i(1) < 1 - \mu(\Sigma_i)$.

Now, starting from (21) and summing over Σ_i, we get:

$$q(\Sigma_i,z) \ = \ z\,\mu(\Sigma_i) + z\,\mu(\Sigma_i) q(\Sigma_i,z) - z\,R_i(z) + z\,q(\Sigma_i,z) S_i(z)\,.$$

Therefore $q(\Sigma_i) = q(\Sigma_i,1) = (\mu(\Sigma_i) - R_i(1))/(1-\mu(\Sigma_i) - S_i(1))$ is finite. □

As an example of this situation, consider the free group $G = \mathbb{Z}\star\mathbb{Z}$. Denote by a and b the generators of the two factors, so that $\Sigma = \{a^i, i \in \mathbb{Z}\setminus\{0\}\} \cup \{b^j, j \in \mathbb{Z}\setminus\{0\}\}$. Let μ be a probability measure on Σ whose support generates G. The harmonic measure μ^∞ of the random walk (G,μ) satisfies, e.g., $\mu^\infty(a^{k_1} b^{k_2} \cdots a^{k_\ell} \Sigma^{\mathbb{N}}) = q(a^{k_1}) q(b^{k_2}) \cdots q(b^{k_{\ell-1}}) r(a^{k_\ell})$. When μ is concentrated on $\{a, a^{-1}, b, b^{-1}\}$, the following simplifications hold (see [**17**]): $\forall k > 0,\ q(a^k) = q(a)^k$, $r(a^k) = q(a)^{k-1} r(a)$, $\forall k < 0$, $q(a^k) = q(a^{-1})^{-k}$, $r(a^k) = q(a^{-1})^{-k-1} r(a^{-1})$, and the analog for b.

4. *Explicit Drift Computations*

In Theorem 3.5, the harmonic measure is completely determined via the vector r which is itself the solution of an explicit finite set of polynomial equations of degree 2. In small or simple examples, it is possible to go further, that is, to solve these equations to get closed form formulas for the harmonic measure, and therefore the drift. It is the program that we now carry out. We compute the drift for several specific and interesting cases of free products of two cyclic groups.

The details of the computations and complete proofs of the results are not given.

They can be found in an appendix posted on the Math ArXiv [**19**]. Some of the results have been obtained with the help of Maple and Mathematica.

4.1. *Comparison with other methods for computing the drift*

We first discuss alternative existing methods for computing the drift. (They do not work for computing the entropy for instance.)

Let $G = G_1 \star G_2$ be a free product of two finite groups. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$ and $\Sigma = \Sigma_1 \sqcup \Sigma_2$. Let μ be a probability measure on Σ such that: $\forall i, \forall x \in \Sigma_i,\ \mu(x) = \mu(\Sigma_i)/\#\Sigma_i$. In words, μ is uniform on each of the two groups. Consider the random walk (G, μ). Here, computing the drift becomes elementary and does not require knowing that the harmonic measure is Markovian.

Set $p = \mu(\Sigma_1), k_1 = \#\Sigma_1$, and $k_2 = \#\Sigma_2$. Denote by $i \in \{1,2\}$, the set of elements of G whose normal form representative ends with a letter in Σ_i. When we are far from the unit element 1_G, the random walk $(X_n)_n$ on G induces a Markov chain on $\{1,2\}$ with transition matrix:

$$P = \begin{bmatrix} p(k_1-1)/k_1 & p/k_1 + 1 - p \\ (1-p)/k_2 + p & (1-p)(k_2-1)/k_2 \end{bmatrix} .$$

Let π be the stationary distribution, that is $\pi P = \pi, \pi(1)+\pi(2) = 1$. By the Ergodic Theorem for Markov Chains, we have $\lim_n [\ P\{X_n \in 1\}\ ,\ P\{X_n \in 2\}\ ] = \pi$. The value of the drift follows readily:

$$\begin{aligned} \gamma = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} E[|X_{i+1}|_\Sigma - |X_i|_\Sigma] + \frac{|X_0|_\Sigma}{n} \ &= \ E_\pi[|X_1|_\Sigma - |X_0|_\Sigma] \\ = \frac{2p(1-p)(k_1k_2-1)}{(1-p)k_1 + pk_2 + k_1k_2} \ . \end{aligned} \tag{22}$$

Now assume that $G = G_1 \star \cdots \star G_k$, where the G_i are finite groups, and assume that $\forall i, \forall x \in \Sigma_i = G_i \backslash \{1_{G_i}\},\ \mu(x) = \mu(\Sigma_i)/\#\Sigma_i$. Then each of the finite groups can be collapsed into a single node, and the random walk (G, μ) projects into a nearest neighbor randow walk on a tree with k cone types, using the terminology of [**21**]. In particular the formulas for the drift given in [**21**] apply.

None of the formulas obtained in §4.2-§4.5 correspond to the above two situations.

Now let us discuss the Sawyer-Steger method [**24**]. It was developped for the free group but adapts to the present situation. Let $G = G_1 \star \cdots \star G_k$ be a free product of finite groups. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$ and $\Sigma = \sqcup_i \Sigma_i$. For $g \in G$, define the r.v. $\tau(g) = \min\{n \mid X_n = g\}$ (with $\tau(g) = \infty$ if g is not reached). Define the *first-passage generating series* $S \in \mathbb{R}[[y,z]]$ by:

$$S(y,z) = \sum_{k \in \mathbb{N}} y^k \sum_{|g|_\Sigma = k} \sum_{n \in \mathbb{N}} P\{\tau(g) = n\} z^n \ . \tag{23}$$

Let S_y and S_z denote the partial derivatives of S with respect to y and z. Adapting the results in [**24**, Theorem 2.2 and Section 6] (see also [**21**, Section 6]), one obtains the following formula for the drift:

$$\gamma = S_y(1,1)/S_z(1,1) \ . \tag{24}$$

For $u \in G$, define the series $q(u,z) = \sum_{n \in \mathbb{N}} P\{\tau(u) = n\} z^n$. Observe that $q(u,1) = q(u)$, the probability of ever hitting u, defined at the beginning of the proof of Theorem 3.5. In particular, if one encapsulates the Equations (13) as $q(a) = \Phi_a(q)$, then $q(a,z) = z\Phi_a(q(z))$. Using this last set of Equations, as well as the corresponding set of Equations for the derivatives $dq(a,z)/dz$, and playing around with the Equations (23) and (24), we get:

$$\gamma = \frac{\sum_{i=1}^{k} q_i/(1+q_i)^2}{\sum_{i=1}^{k} q_i'/(1+q_i)^2}\,, \qquad q_i = \sum_{u\in\Sigma_i} q(u,1), \quad q_i' = \sum_{u\in\Sigma_i} \Big[\frac{dq(u,z)}{dz}\Big]_{|z=1}\,. \tag{25}$$

This formula is more complicated than the one obtained in (19). Our approach centers around the knowledge that μ^∞ is Markovian multiplicative, which is a more direct path. Consequently, it gives more chances to solve the equations to get a closed form formula. As an exercise, we tried to retrieve the results for γ in §4.2-§4.5 using (25). We succeeded in two cases: formulas (26) and (28). On the other hand, the results in §4.4-§4.5 seem out of reach.

4.2. *Random walks on* $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$

The group $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$ is isomorphic to the modular group $\mathrm{PSL}(2,\mathbb{Z})$, i.e. the group of 2x2 matrices with integer entries and determinant 1, quotiented by $\pm\mathrm{Id}$.

Consider a general nearest neighbor random walk $(\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z},\mu)$. Set $\mu(a) = 1-p-q, \mu(b) = p, \mu(b^2) = q$. In Figure 1, we have represented the Cayley graph of $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$ and the one-step transitions of the random walk.

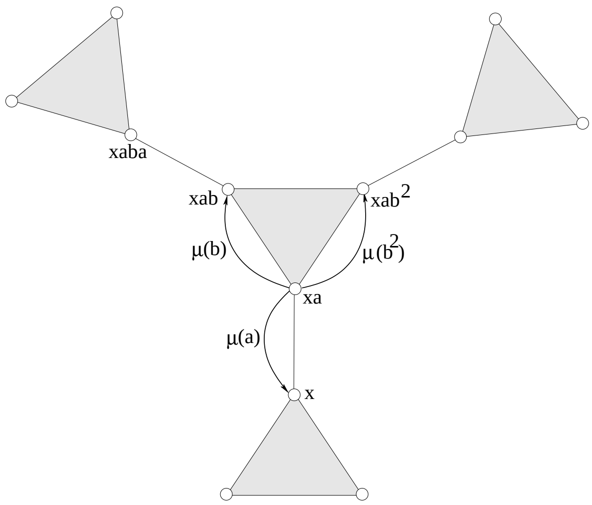

FIGURE 1. *A nearest neighbor random walk on* $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z}$.

Using Theorem 3.5 and solving the Traffic Equations explicitly, we get :

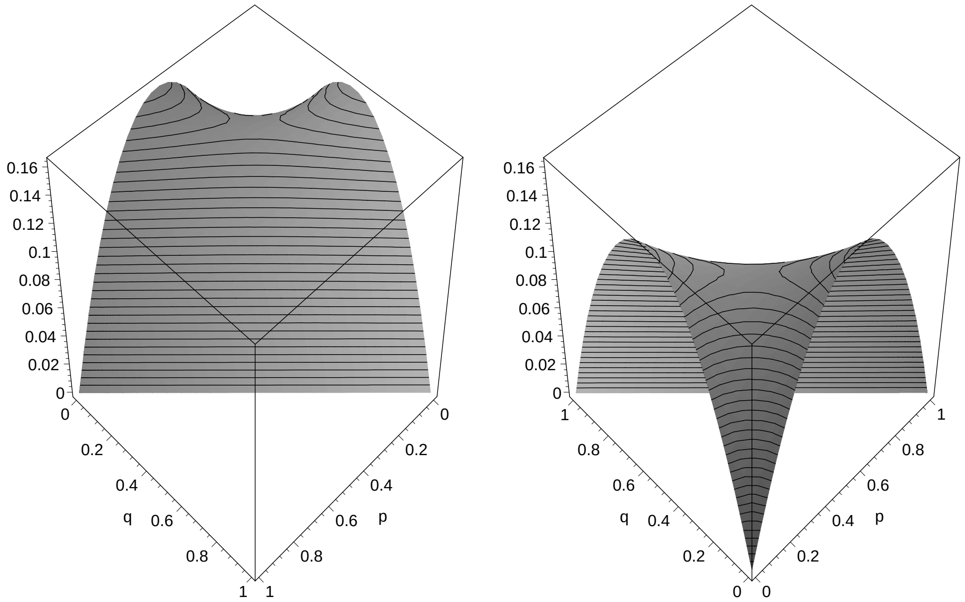

FIGURE 2. *The drift of the random walk* $(\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/3\mathbb{Z},\mu)$ *as a function of* $p=\mu(b)$ *and* $q=\mu(b^2)$*, from two different angles.*

$$\begin{aligned} r(a) &= \frac{p^2+q^2-2pq-p-q+4-\Delta_1}{2\Delta_2} \\ r(b) &= \frac{q^3-3q^2+p^2q-5pq+2p+6q-(2-q)\Delta_1}{2(q-p)\Delta_2} \\ r(b^2) &= \frac{p^3-3p^2+pq^2-5pq+6p+2q-(2-p)\Delta_1}{2(p-q)\Delta_2}\,, \end{aligned}$$

with

$$\begin{aligned} \Delta_1 &= \sqrt{p^4+q^4-2p^3-2q^3+2p^2q^2-6p^2q-6pq^2+5p^2+5q^2+6pq} \\ \Delta_2 &= p^2+q^2-pq-2p-2q+4\,. \end{aligned}$$

Set $r=\mu(a)=1-p-q$. By Corollary 3.6, the drift is then:

$$\gamma = \frac{2r\Big(pq-p-q+\sqrt{(p^2+q^2)(3+(r+p)^2+(r+q)^2)+2pq(2r+1)}\Big)}{(r+p)^2+(r+q)^2-pq+2}\,. \qquad (26)$$

In Figure 2, we have represented the drift as a function of the probability μ. We see that the surface is smooth but rather complex with a saddle point. As a curiosity, the drift is maximized for $r=z_0, p=1-z_0, q=0$ (or $r=z_0, p=0, q=1-z_0$)), where z_0 is the root of $[z^6+12z^4-4z^3+47z^2-48z+12]$ whose numerical value is $0.490275\cdots$. The corresponding numerical value of the drift is $\gamma_{\max}=0.163379\cdots$. This was not a priori obvious!

We are in the domain of application of Proposition 3.9. On the other hand, the harmonic measure is never shift-invariant. This can be proved by showing directly that the Stationary Traffic Equations have no solution.

4.3. *Random walks on* $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$

Consider the probability μ such that $\mu(a) = \mu(b) = p, \mu(a^2) = \mu(b^2) = q = 1/2 - p$. According to Proposition 3.8, the harmonic measure is ergodic. The Stationary Traffic Equations (12) can be solved explicitly. We get

$$r(a) = r(b) = \frac{4p - 3 + \sqrt{16p^2 - 8p + 5}}{4(4p-1)}, \; r(a^2) = r(b^2) = \frac{4p + 1 - \sqrt{16p^2 - 8p + 5}}{4(4p-1)}. \tag{27}$$

In particular, the drift is

$$\gamma \; = \; \frac{1}{2} - p + \big(2p - \frac{1}{2}\big) r(1) \; = \; -\frac{1}{4} + \frac{1}{4}\sqrt{16p^2 - 8p + 5}\,. \tag{28}$$

At last, consider the case $\mu(a) = p, \mu(a^2) = q$, and $\mu(b) = \mu(b^2) = (1 - p - q)/2$. Here, it is not difficult to check that there is no solution to the Stationary Traffic Equations. However, solving explicitly the Traffic Equations is feasible. The formulas for r are too lengthy to be reproduced here. But, for the drift, several simplifications occur and we obtain the following formula:

$$\gamma = 2(1 - p - q)\sqrt{\frac{p^2 + q^2 + pq}{p^2 + q^2 - 2pq + 3}}\,.$$

In the subcase $p = q$, we get $\gamma = 2p(1 - 2p)$, a formula that can also be obtained using (22). For the general nearest neighbor random walk on $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/3\mathbb{Z}$, we did not succeed in completely solving the Traffic Equations.

4.4. *The simple random walks on* $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$

We now consider the whole family of groups $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}, k \geq 3$. However, we have to compromise by considering only simple random walks with respect to a minimal set of generators. In Figure 3 (left), we show this simple random walk in the case $\mathbb{Z}/4\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$.

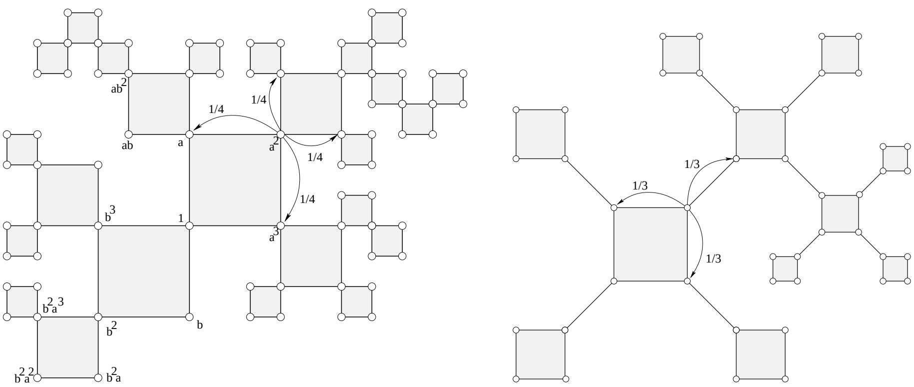

FIGURE 3. *The simple random walk on* $\mathbb{Z}/4\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$ *(left) and* $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$ *(right).*

We obtain "semi-explicit" formulas: we define recursively a family of polynoms of one variable $(F_k)_k$, and the harmonic measure is expressed as a function of the unique solution in $(0,1)$ of $F_k = 1$. For $k \geq 6$, we have no closed form formula for this unique root.

Consider the free product $G_1 \star G_2 = \mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$. Let a and b be the generators of the two cyclic groups. Consider the simple random walk $(\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}, \mu)$ with $\mu(a) = \mu(b) = \mu(a^{-1}) = \mu(b^{-1}) = 1/4$.

According to Proposition 3.8, the harmonic measure is ergodic. To determine it, we need to solve the Stationary Traffic Equations (12). Set $r(i) = r(a^i) + r(b^i) = 2r(a^i) = 2r(b^i)$. The Stationary Traffic Equations are:

$$r = \frac{r}{4}\Big[\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} + \begin{bmatrix} r(1) & \cdots & r(k-1) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ r(1) & \cdots & r(k-1) \end{bmatrix} \Big]$$

If $[r(1), r(2), \ldots, r(k-1)]$ is a solution, then it is clear that $[r(k-1), \ldots, r(2), r(1)]$ is also a solution. By uniqueness, we deduce that $r(i) = r(k-i)$ for all i. In particular, we have $r(1) = r(k-1)$. For convenience, set $r(0) = r(k) = 1$. Then the Stationary Traffic Equations rewrite as:

$$\forall i \in \{1, k-1\}, \qquad r(i+1) + r(i-1) = 2(2 - r(1))r(i)\,. \tag{29}$$

Consider the applications $F_n : [0,1] \to \mathbb{R}, n \in \mathbb{N}$, defined by

$$F_0(x) = 1, \quad F_1(x) = x, \quad \forall n \geq 2,\ F_n(x) = 2(2-x)F_{n-1}(x) - F_{n-2}(x)\,. \tag{30}$$

To illustrate, here are the first values of F_i:

$$F_2 = -2x^2 + 4x - 1, \quad F_3 = 4x^3 - 16x^2 + 17x - 4, \quad F_4 = -8x^4 + 48x^3 - 96x^2 + 72x - 15.$$

By construction, the unique solution $r \in \mathring{\mathcal{B}}$ of the Traffic Equations (29) satisfies $r(i) = F_i(r(1))$ for all $i \in \{1, \ldots, k-1\}$. Hence, it is enough to determine $r(1)$. Set $x_k = r(1)$. We have $F_i(x_k) = F_{k-i}(x_k)$ for all $i \in \{1, \ldots, k-1\}$. Now, using (30) twice, we get

$$\begin{aligned} F_k(x_k) = 2(2-x_k)F_{k-1}(x_k) - F_{k-2}(x_k) &= 2(2-x_k)F_1(x_k) - F_2(x_k) \\ = 2(2-x_k)F_1(x_k) - \big[2(2-x_k)F_1(x_k) - F_0(x_k)\big] &= F_0(x_k) = 1\,. \end{aligned}$$

Next lemma shows that $F_k(x_k) = 1$ is actually a characterization of x_k.

Lemma 4.1. *For $k \geq 3$, the equation $F_k(x) = 1$ has a unique solution in $(0,1)$.*

The results are gathered in the theorem below.

Theorem 4.2. *Consider the group $\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}$, $k \geq 3$, the generators of the two cyclic groups being respectively a and b. Consider the simple random walk $(\mathbb{Z}/k\mathbb{Z} \star \mathbb{Z}/k\mathbb{Z}, \mu)$ with $\mu(a) = \mu(a^{-1}) = \mu(b) = \mu(b^{-1}) = 1/4$. The harmonic measure is the ergodic Markovian multiplicative measure associated with*

$$r = [x_k, F_2(x_k), F_3(x_k), \ldots, F_2(x_k), x_k]$$

where x_k is the unique solution in $(0,1)$ of the equation $F_k(x) = 1$. The drift is $\gamma_k = (1 - x_k)/2$. It is a strictly increasing function of k and $\lim_k \gamma_k = 1/3$.

To illustrate, here is the vector r for $\mathbb{Z}/4\mathbb{Z}\star\mathbb{Z}/4\mathbb{Z}$:

$$\big[r(a),r(a^2),r(a^3)\big]=\big[r(b),r(b^2),r(b^3)\big]=\Big[\frac{3-\sqrt{5}}{8},\frac{\sqrt{5}}{4}-\frac{1}{2},\frac{3-\sqrt{5}}{8}\Big]\,. \qquad (31)$$

Now, here is a table of the first values of γ_k, given either in closed form or numerically when no closed form could be found. Set $\mathbb{Z}_k=\mathbb{Z}/k\mathbb{Z}$.

	$\mathbb{Z}_3\star\mathbb{Z}_3$	$\mathbb{Z}_4\star\mathbb{Z}_4$	$\mathbb{Z}_5\star\mathbb{Z}_5$	$\mathbb{Z}_6\star\mathbb{Z}_6$	$\mathbb{Z}_7\star\mathbb{Z}_7$
γ	$\frac{1}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{13}-1}{8}$	0.330851...	0.332515...

4.5. *The simple random walks on* $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$

We study simple random walks on $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$ for a minimal set of generators. The model considered is illustrated in Figure 3 (right) in the case $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/4\mathbb{Z}$. The approach and results are similar to the ones in §4.4. The groups $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$ are known as the Hecke groups.

Consider the applications $G_n:[0,1]\to\mathbb{R}, n\in\mathbb{N}$, defined by

$$G_0(x)=\frac{1}{4}+\frac{x}{2},\quad G_1(x)=x,\quad \forall n\geq 2,\ G_n(x)=\frac{8(1-x)}{3-2x}G_{n-1}(x)-G_{n-2}(x)\,. \qquad (32)$$

Theorem 4.3. *Consider the group $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z}$, $k\geq 3$, the generators of the two cyclic groups being respectively a and b. Consider the simple random walk $(\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/k\mathbb{Z},\mu)$ with $\mu(a)=\mu(b)=\mu(b^{-1})=1/3$. Let the functions G_i be defined as in (32) and let y_k be the unique solution in $(0,1/2)$ of $G_{k-1}(y_k)=y_k$. The harmonic measure is the Markovian multiplicative measure associated with r: $r(a)=G_0(y_k)$, $\forall i\in\{1,\dots,k-1\}$, $r(b^i)=G_i(y_k)$. The drift is $\gamma_k=(1-2y_k)/3$. It is a strictly increasing function of k and* $\lim_k\gamma_k=2/9$.

Here is the vector r for $\mathbb{Z}/2\mathbb{Z}\star\mathbb{Z}/4\mathbb{Z}$:

$$\big[r(a),r(b),r(b^2),r(b^3)\big]=\Big[\frac{7-\sqrt{7}}{12},\frac{2}{3}-\frac{\sqrt{7}}{6},\frac{-11+5\sqrt{7}}{12},\frac{2}{3}-\frac{\sqrt{7}}{6}\Big]\,. \qquad (33)$$

Here is the drift, either in closed form or numerically, for the small values of k.

	$\mathbb{Z}_2\star\mathbb{Z}_3$	$\mathbb{Z}_2\star\mathbb{Z}_4$	$\mathbb{Z}_2\star\mathbb{Z}_5$	$\mathbb{Z}_2\star\mathbb{Z}_6$	$\mathbb{Z}_2\star\mathbb{Z}_7$
γ	$\frac{2}{15}$	$\frac{\sqrt{7}-1}{9}$	$\frac{2\sqrt{61}-4}{57}$	0.213412...	0.217921...

5. *Entropy and Extremal Generators*

We study Vershik's notion of extremal generators for free products of finite groups. The results are of two kinds, structural results derived from the shape of the harmonic measure (Prop. 5.2 and 5.3), and explicit computations obtained by solving the Traffic Equations (Section 5.1).

The *entropy* of a probability measure μ with finite support S is defined by

$$H(\mu) = -\sum_{x\in S} \mu(x)\log[\mu(x)] \,. \tag{34}$$

The entropy is maximized for the uniform measure U on S.

Let G be a free product of finite groups and let (G,μ) be a random walk defined as in §2. Let $(X_n)_n$ be a realization of the random walk. The *entropy of* (G,μ), introduced by Avez [**1**], is

$$h = h(\mu) = \lim_n \frac{H(\mu^{*n})}{n} = \lim_n -\frac{1}{n}\log \mu^{*n}(X_n) \,, \tag{35}$$

$a.s.$ and in L^p, for all $1 \leq p < \infty$. The existence of the limits as well as their equality follow from Kingman's Subadditive Ergodic Theorem (see [**1**] and Derriennic [**4**]). In general, the entropy is difficult to compute, see [**4**]. But not in our case.

Proposition 5.1. *Let r be the unique solution in $\mathring{\mathcal{B}}$ of the Traffic Equations. Set $q(a) = r(a)/r(\Sigma\backslash\Sigma_a)$ for all $a \in \Sigma$. The entropy is given by:*

$$h = -\sum_{a\in\Sigma} \mu(a) \int \log\Big[\frac{da^{-1}\mu^\infty}{d\mu^\infty}(\xi)\Big]\, d\mu^\infty(\xi) \tag{36}$$

$$= -\sum_{a\in\Sigma}\mu(a)\Big[\log\Big[\frac{1}{q(a^{-1})}\Big]r(a^{-1}) + \sum_{b\in\Sigma_a\backslash a^{-1}} \log\Big[\frac{q(ab)}{q(b)}\Big]r(b) + \log[q(a)]\sum_{b\in\Sigma\backslash\Sigma_a} r(b)\Big] \tag{37}$$

where $da^{-1}\mu^\infty/d\mu^\infty$ is the Radon-Nikodym derivative of $a^{-1}\mu^\infty$ with respect to μ^∞.

Formula (36) is proved for instance in [**15**, Corollary 4.5] in the context of the free group. The proof adapts easily to the present setting. The formulation in (37) follows directly from the one in (36), using that the harmonic measure is Markovian multiplicative.

The *growth* of the group G with respect to the finite set of generators Σ is

$$v = v(\Sigma) = \lim_n \frac{1}{n}\log \#\{g \in G, |g| = n\} \,. \tag{38}$$

The limit exists by subadditivity. The following fundamental inequality was proved and highlighted in [**26**] (see also [**9**] and [**10**, **14**]):

$$h/\gamma \leq v \,, \tag{39}$$

where we recall that γ is the drift of the random walk. The interpretation is that the proportion of typical elements visited by the walk is less than or equal to the total number of elements.

This inequality is reminiscent of the one between metric entropy and topological entropy in symbolic dynamics [**16**]. Also, for a free product of finite groups, h/γ

and v can be interpreted respectively as the Hausdorff dimension of the harmonic measure μ^∞ and the Hausdorff dimension of its support. See Prop. 5.3 for a precise statement.

Observe that v and γ depend on Σ but not h, and that h and γ depend on μ but not v. It might be enlightning to view the inequality as $h(\mu)/v(\Sigma) \leq \gamma(\Sigma, \mu)$. Define

$$Q(\Sigma) = \sup_{\mu \in \mathcal{S}} \frac{h(\mu)}{\gamma(\Sigma,\mu)v(\Sigma)} ,$$

where $\mathcal{S}$ is the set of probability measures on Σ such that: (i) $\forall a, \mu(a) = \mu(a^{-1})$ (μ is symmetric), and (ii) $\bigcup_n \mathrm{supp}(\mu^{*n}) = G$. The set $\mathcal{S}$ is not closed due to condition (ii). The closure $\overline{\mathcal{S}}$ is the set of probability measures on Σ satisfying only (i). It might be the case that the supremum in $Q(\Sigma)$ is attained only for a measure $\mu \in \overline{\mathcal{S}} \backslash \mathcal{S}$, for which the random walk is not transient, and for which $\gamma = 0$. We will see an occurence of this situation in Section 5.1.

It is suggested by Vershik [**26**] to view $Q(\Sigma)$ as a measure of the 'quality' of the set of generators Σ. If $Q(\Sigma) = 1$ then Σ is said to be *extremal.*

This notion is investigated in [**27**] for braid groups and partially commutative free groups. In particular, it is proved that the natural generators are not extremal for partially commutative free groups of the type $\langle\, a_1, \ldots, a_n \mid \forall i = 1, \ldots, n-1,\ a_i a_{i+1} = a_{i+1} a_i \,\rangle$, for n large enough. Below, we continue the study of this notion using free products of finite groups.

Consider a free product $G_1 \star \cdots \star G_n$ with $|G_i| = |G_j|$ for all i,j. Set $\Sigma = \sqcup_i G_i \backslash \{1_{G_i}\}$. It is elementary to prove that Σ is extremal. Indeed consider the uniform distribution μ on Σ. It is clear by symmetry that the harmonic measure should be the uniform measure on L^∞, that is, $\forall u, v \in L \cap \Sigma^k$, $\mu^\infty(u\Sigma^{\mathbb{N}}) = \mu^\infty(v\Sigma^{\mathbb{N}})$. The exact values of h, γ, and v follow readily. Set $k = |G_i \backslash \{1_{G_i}\}|$, and $K = nk = |\Sigma|$. We have

$$v = \log\bigl[K-k\bigr], \quad \gamma = -\frac{1}{K} + (1 - \frac{k}{K}), \quad h = \log\bigl[K-k\bigr]\bigl(-\frac{1}{K} + (1-\frac{k}{K})\bigr) .$$

In particular, $h/\gamma = v$. This is mentioned in [**26**, Section 2.5].

The argument collapses when the different groups G_i do not have the same cardinality. In particular, there is no reason to expect the uniform ditribution on Σ to maximize $h/(\gamma v)$. Indeed, the result $Q(\Sigma) = 1$ is still true but non-elementary. We adress this point in Proposition 5.2. For definitions and details regarding subshifts and measures of maximal entropy, see [**16**].

PROPOSITION 5.2. *Consider a free product of finite groups $G_1 \star \cdots \star G_n$. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$ and set $k_i = |\Sigma_i|$. Let ρ be the unique positive solution of the equation: $\sum_{i=1}^n k_i/(x+k_i) = 1$. Consider the probability measure μ on $\Sigma = \sqcup_i \Sigma_i$ defined by $\mu = \sum_i k_i/(\rho + k_i) U_{\Sigma_i}$, where U_{Σ_i} is the uniform distribution on Σ_i. The following properties hold.*

(i) We have $h/(\gamma v) = 1$ for (G, μ). The generators Σ are extremal.

(ii) The harmonic measure μ^∞ associated with (G, μ) is defined by

$$\forall u_1 \cdots u_k \in L, u_k \in \Sigma_j, \quad \mu^\infty(u_1 \cdots u_k \Sigma^{\mathbb{N}}) = \frac{1}{\rho^{k-1}} \frac{1}{(\rho + k_j)} . \qquad (40)$$

(iii) Consider the shift: $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}},\ (x_n)_n \mapsto (x_{n+1})_n$. *Then the symbolic dynamical system* (L^∞, τ) *is a subshift of finite type whose measure of maximal entropy* ν_{max} *is defined by*

$$\forall u_1 \cdots u_k \in L, u_1 \in \Sigma_i, u_k \in \Sigma_j, \quad \nu_{max}(u_1 \cdots u_k \Sigma^{\mathbb{N}}) = \frac{1}{(\rho + k_i)} \frac{1}{\rho^{k-2}} \frac{1}{(\rho + k_j)} \,. \tag{41}$$

Proof. Let us compute the growth v of $G = (G_1 \star \cdots \star G_n)$. Consider the matrix

$$A = \begin{bmatrix} 0 & k_2 & k_3 & \cdots & k_n \\ k_1 & 0 & k_3 & \cdots & k_n \\ k_1 & k_2 & 0 & \cdots & k_n \\ \vdots & \vdots & & & \vdots \\ k_1 & k_2 & \cdots & 0 & k_n \\ k_1 & k_2 & \cdots & k_{n-1} & 0 \end{bmatrix} . \tag{42}$$

Clearly $\sum_{ij} k_i A_{ij}^{n-1}$ is equal to the number of elements of G of length n (with respect to Σ). It follows that $v = \log(\rho)$, where ρ is the spectral radius of A. According to the Perron-Frobenius Theorem, there exists a unique $Y = [Y_1, \ldots, Y_n], Y_i > 0, \sum_i Y_i = 1$, such that $YA = \rho Y$, i.e. $\forall i$,

$$k_i (\sum_{j \neq i} Y_j) = \rho Y_i \iff k_i(1 - Y_i) = \rho Y_i \iff Y_i = \frac{k_i}{\rho + k_i} \,.$$

Therefore, the spectral radius ρ satisfies:

$$\frac{k_1}{\rho + k_1} + \cdots + \frac{k_n}{\rho + k_n} = 1 \,. \tag{43}$$

Since the homographic maps $k_i/(x + k_i)$ are strictly decreasing functions of x in $\mathbb{R}_+$, the Equation (43) characterizes ρ.

Now consider the probability measure μ on Σ defined by $\mu = \sum_i k_i/(\rho + k_i) U_{\Sigma_i}$, where U_{Σ_i} is the uniform distribution on Σ_i. For simplicity, set $\mu_i = \mu(\Sigma_i)/k_i = 1/(\rho + k_i) = \mu(a)$ for all $a \in \Sigma_i$. Let r be the solution to the Traffic Equations and let q be the probabilities of ever hitting the generators. Set $r_i = r(\Sigma_i)/k_i = r(a), \forall a \in \Sigma_i$, and $q_i = q(\Sigma_i)/k_i = q(a), \forall a \in \Sigma_i$. Observe that we have

$$\sum_i k_i \mu_i = 1, \qquad \sum_i k_i r_i = 1, \qquad \forall i,\ q_i = \frac{r_i}{\sum_{j \neq i} k_j r_j} = \frac{r_i}{1 - k_i r_i} \,. \tag{44}$$

Using Formulas (7) and (37), we get the drift:

$$\gamma = \sum_i k_i r_i \left[-\mu_i + \sum_{j \neq i} k_j \mu_j \right] = 1 - \sum_i k_i (k_i + 1) \mu_i r_i \,, \tag{45}$$

and the entropy:

$$h = -\sum_i \mu_i \log(q_i) \left[-r_i + \sum_{j \neq i} k_j r_j \right] = \sum_i \mu_i \log(1/q_i) - \sum_i \mu_i \log(1/q_i) r_i \left[k_i + 1 \right] . \tag{46}$$

To obtain the formula for h, we have used that for all i and all $a, b \in \Sigma_i$, such that $a * b \in \Sigma_i$, we have $q(a) = q(a * b)$, so $\log[q(a * b)/q(a)] = 0$.

Assume that we have: $\forall i, \log(1/q_i) = v$. Then it follows from (45) and (46) that

$h/\gamma = v$. Let us prove that $\log(1/q_i) = v$ for all i. Recall that q_i satisfies the Equations (13):

$$q_i = \mu_i + (k_i\mu_i - \mu_i)q_i + q_i \sum_{j\neq i} k_j\mu_j q_j \,. \tag{47}$$

Assume that $q_i = q_j = q$ for all i, j. Then we get

$$(1 - k_i\mu_i)q^2 - (1 - k_i\mu_i + \mu_i)q + \mu_i = 0 \,.$$

The two solutions of this second order equation are: $q = 1$ and $q = \mu_i/(1 - k_i\mu_i)$. Since we must have $0 < q < 1$, the right solution is the second one, and we deduce that:

$$\mu_i = \frac{1}{1/q + k_i} \quad \Longrightarrow \quad \sum_i \frac{k_i}{1/q + k_i} = 1 \,.$$

Comparing with (43), we conclude that $1/q = \rho$. Backtracking in the argument, we conclude that $\forall i, q_i = 1/\rho$, is a solution to the Equations (47). Now the Equations (47) have at most one solution in $(0,1)^\Sigma$, otherwise, following the proof of Theorem 3.5, one would get several solutions in $\mathring{\mathcal{B}}$ for the Traffic Equations, which is impossible. We conclude that: $\forall i, \log(1/q_i) = \log(\rho) = v$. It completes the proof of $h/(\gamma v) = 1$.

Observe that: $r_i = \mu_i = Y_i/k_i = 1/(\rho + k_i)$. Applying Theorem 3.5, we get that the harmonic measure μ^∞ is given by (40).

Consider the translation shift: $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, $(x_n)_n \mapsto (x_{n+1})_n$. The symbolic dynamical system (L^∞, τ) is clearly a subshift of finite type. For such a system, the measure of *maximal entropy* is well-known, and it turns out to be precisely the measure $\nu_{\max}$ given in (41). □

We now turn our attention to Hausdorff dimensions. The Hausdorff dimensions of a metric space (X, d) and a Borel measure ν on X are denoted respectively by $\mathrm{HD}(X)$ and $\mathrm{HD}(\nu)$, see for instance [**20**] for the definitions.

PROPOSITION 5.3. *Let $G = G_1 \star \cdots \star G_n$ be a free product of finite groups. Set $\Sigma_i = G_i \backslash \{1_{G_i}\}$ and $\Sigma = \sqcup_i \Sigma_i$. Let v be the growth of G with respect to Σ. Denote by d the metric defined on the set L^∞ of right-infinite normal form words by $d(\xi_1, \xi_2) = e^{-|\xi_1 \wedge \xi_2|}$ where $\xi_1 \wedge \xi_2$ is the greatest common prefix of ξ_1 and ξ_2. Let ν_{max} be the measure of maximal entropy of the dynamical system (L^∞, τ), see (41). The following properties hold.*

(i) We have $\mathrm{HD}(\nu_{max}) = \mathrm{HD}(L^\infty) = v$.

(ii) Let μ be a probability on Σ whose support generates G. We have $\mathrm{HD}(\mu^\infty) = h/\gamma$.

Proof. The equality $\mathrm{HD}(L^\infty) = v$ is analogous to [**15**, Proposition 1.9]. By taking some precautions, the proof can be adapted. Let μ^∞ be the harmonic measure associated with the probability μ defined in Prop. 5.2. With μ^∞ in the role of the measure m, the proof of [**15**, Proposition 1.9] goes through. Here, it is central to know that μ^∞ is closely related to $\nu_{\max}$ to be able to adapt the argument from [**15**, Proposition 1.9]. The equality $\mathrm{HD}(\mu^\infty) = h/\gamma$ is proved in [**15**, Theorem 4.15] for a finitely generated free group. The proof adapts easily to the present setting. □

5.1. *Explicit computations*

Let us go back to extremal generators. After having settled Prop. 5.2, other natural questions become in order. Is there a unique measure on Σ maximizing $h/(\gamma v)$? Are other sets of generators extremal?

We believe that some new light is shed on these questions by the three results below. These results are obtained by solving explicitly the Traffic Equations in the cases considered. The proofs and the details are to be found in the Math ArXiv appendix [**19**].

• A. Consider a free product of two finite groups, $G_1 \star G_2$. Consider the probability μ_p on Σ such that $\mu_p = pU_{\Sigma_1} + (1-p)U_{\Sigma_2}$, where U_{Σ_i} is the uniform distribution on Σ_i, and where $p \in (0,1)$. Then we have $h/(\gamma v) = 1$, for the whole family of probabilities μ_p. This does not extend to the free product of more than 2 groups, see B, nor to minimal sets of generators $S \subset \Sigma$, see C. According to Prop. 5.3, we have $\mathrm{HD}(\mu_p^\infty) = \mathrm{HD}(L^\infty)$ for $p \in (0,1)$. This contrasts sharply with [**15**, Theorem 2.1].

• B. Consider the group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, and let a, b, and c be the non-identity elements of the three cyclic groups. Consider the family of probability measures $\mu_p, p \in (0, 1/2)$, defined by: $\mu_p(a) = \mu_p(b) = p, \mu_p(c) = 1-2p$. Among them, the only probability such that $h/(\gamma v) = 1$ is $\mu_{1/3}$.

• C. We now give three examples of various behaviours for a generating set S of a free product $G_1 \star G_2$ such that $S \subsetneq \Sigma$, the natural generators.

First, consider the group $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$. Let a and b be respective generators of the two cyclic groups. Consider the minimal set of generators $S = \{a, b, b^{-1}\}$. Here we have $h/(\gamma v) < 1$ for all μ in $\mathcal{S}$. Nevertheless, the set S is extremal, the supremum of $h/(\gamma v)$ being equal to 1.

Consider now the group $\mathbb{Z}/3\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$. The minimal set of generators $S = \{a, a^{-1}, b, b^{-1}\}$ is extremal. Actually, the only symmetric probability measure on S for which $h = \gamma v$ is $\mu = p\,(\delta_a + \delta_{a^{-1}}) + (1/2 - p)(\delta_b + \delta_{b^{-1}})$ where $p = 0.43269\cdots$ is the middle root of the polynomial $5x^3 - 13x^2 + 7x - 1$. Observe that $\mu \in \mathcal{S}$.

Last, consider the group $\mathbb{Z}/4\mathbb{Z} \star \mathbb{Z}/4\mathbb{Z}$. The minimal set of generators $S = \{a, a^{-1}, b, b^{-1}\}$ is not extremal. Indeed,

$$Q(S) = \frac{5+\sqrt{5}}{4} \frac{\log\bigl(1/2 + \sqrt{5}/2\bigr)}{\log(1+\sqrt{2})} = 0.987686\cdots .$$

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