Introduction

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

Antoine Miné

CNRS, École normale supérieure

course i, 2012–2013

Ariane 5, Flight 501

Maiden flight of the Ariane 5 Launcher, 4 June 1996.

Ariane 5, Flight 501

40s after launch. . .

Ariane 5, Flight 501

- **Cause:** software error[1]
  - arithmetic overflow in unprotected data conversion from 64-bit float to 16-bit integer types[2]

```
P_M_DERIVE(T_ALG.E_BH) :=
  UC_16S_EN_16NS (TDB.T_ENTIER_16S
    ((1.0/C_M_LSB_BH) * G_M_INFO_DERIVE(T_ALG.E_BH)));
```

  - software exception not caught $\Longrightarrow$ computer switched off
  - all backup computers run the same software
    all computers switched off, no guidance
    $\Longrightarrow$ rocket self-destructs

- **Cost:** estimated at more than 370 000 000 US$[3]

---

1 J.-L. Lions et al., Ariane 501 Inquiry Board report.

2 J.-J. Levy. Un petit bogue, un grand boum. Séminaire du Département d'informatique de l'ENS, 2010.

3 M. Dowson. "The Ariane 5 Software Failure". Software Engineering Notes 22 (2): 84, March 1997.

How can we avoid such failures?

- Choose a safe programming language.
  C (low level) / Ada, Java (high level)

- Carefully design the software.
  many software development methods exist

- Program well.
  is it art or science?

- Test the software extensively.

How can we avoid such failures?

- Choose a safe programming language.
  C (low level) / Ada, Java (high level)
  yet, Ariane 5 software is written in Ada
- Carefully design the software.
  many software development methods exist
  yet, critical embedded software follow strict development processes
- Program well.
  is it art or science?
- Test the software extensively.
  yet, the erroneous code was well tested... on Ariane 4!

$\Longrightarrow$ **not sufficient!**

How can we avoid such failures?

- Choose a safe programming language.
  C (low level) / Ada, Java (high level)
  yet, Ariane 5 software is written in Ada
- Carefully design the software.
  many software development methods exist
  yet, critical embedded software follow strict development processes
- Program well.
  is it art or science?
- Test the software extensively.
  yet, the erroneous code was well tested... on Ariane 4!

  $\Longrightarrow$ **not sufficient!**

We should use **formal methods.**

provide rigorous, mathematical insurance

Invariants and programs

```
assume X in [0,1000];

I := 0;

while I < X do

    I := I + 2;


assert I in [0,???]
```

[4] R. W. Floyd. "Assigning meanings to programs". In Proc. Amer. Math. Soc. Symposia in Applied Mathematics, vol. 19, pp. 19–31, 1967.

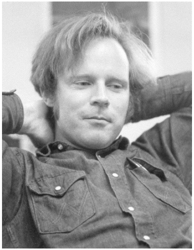
Invariants and programs

```
assume X in [0,1000];

I := 0;

while I < X do

    I := I + 2;


assert I in [0,1000]
```

[4] R. W. Floyd. "Assigning meanings to programs". In Proc. Amer. Math. Soc. Symposia in Applied Mathematics, vol. 19, pp. 19–31, 1967.

```
assume X in [0,1000];
```
$\{X \in [0,1000]\}$
```
I := 0;
```
$\{X \in [0,1000], I = 0\}$
```
while I < X do
```
$\{X \in [0,1000], I \in [0,998]\}$
```
    I := I + 2;
```
$\{X \in [0,1000], I \in [2,1000]\}$

$\{X \in [0,1000], I \in [0,1000]\}$
```
assert I in [0,1000]
```



Robert Floyd[4]

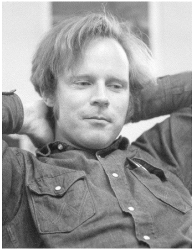
invariant: property true of all the executions of the program

[4] R. W. Floyd. "Assigning meanings to programs". In Proc. Amer. Math. Soc. Symposia in Applied Mathematics, vol. 19, pp. 19–31, 1967.

Invariants and programs

```
assume X in [0,1000];
```
$\{X \in [0,1000]\}$
```
I := 0;
```
$\{X \in [0,1000], I = 0\}$
```
while I < X do
```
$\{X \in [0,1000], I \in \{0,2,\ldots,996,998\}\}$
```
    I := I + 2;
```
$\{X \in [0,1000], I \in \{2,4,\ldots,998,1000\}\}$

$\{X \in [0,1000], I \in \{0,2,\ldots,998,1000\}\}$
```
assert I in [0,1000]
```



Robert Floyd[4]

inductive invariant: invariant that can be proved to hold by induction on loop iterates

[4] R. W. Floyd. "Assigning meanings to programs". In Proc. Amer. Math. Soc. Symposia in Applied Mathematics, vol. 19, pp. 19–31, 1967.

Logics and programs

$$\frac{}{\{P[e/X]\}\ \texttt{X := e}\ \{P\}} \qquad \frac{\{P\}\ \texttt{C}_1\ \{R\} \quad \{R\}\ \texttt{C}_2\ \{Q\}}{\{P\}\ \texttt{C}_1;\texttt{C}_2\ \{Q\}}$$

$$\frac{\{P \,\&\, b\}\ \texttt{C}\ \{P\}}{\{P\}\ \texttt{while b do C}\ \{P \,\&\, \neg b\}}$$

...

Tony Hoare[5]

- sound logic to prove program properties, (rel.) complete
- proofs can be checked automatically
  (e.g., using proof assistants: Coq, PVS, Isabelle, HOL, etc.)

[5] C. A. R. Hoare. "An Axiomatic Basis for Computer Programming". Commun. ACM 12(10): 576-580 (1969).
[6] How Many Lines of Code in Windows?". Knowing.NET. December 6, 2005.

Logics and programs

$$\frac{}{\{P[e/X]\}\ \texttt{X := e}\ \{P\}} \qquad \frac{\{P\}\ \texttt{C}_1\ \{R\} \quad \{R\}\ \texttt{C}_2\ \{Q\}}{\{P\}\ \texttt{C}_1;\texttt{C}_2\ \{Q\}}$$

$$\frac{\{P \,\&\, b\}\ \texttt{C}\ \{P\}}{\{P\}\ \texttt{while b do C}\ \{P \,\&\, \neg b\}}$$

...

Tony Hoare[5]

- sound logic to prove program properties, (rel.) complete
- proofs can be checked automatically
  (e.g., using proof assistants: Coq, PVS, Isabelle, HOL, etc.)
- requires annotations
  but manual annotation is not practical for large programs!
  (e.g., Windows XP: 45 Mlines[6])

[5] C. A. R. Hoare. "An Axiomatic Basis for Computer Programming". Commun. ACM 12(10): 576-580 (1969).

[6] How Many Lines of Code in Windows?". Knowing.NET. December 6, 2005.

$$O(P, D) \in \{yes, no, \perp\}$$

O

P

D

The computer O runs the program P on the data D
and answers (yes,no)... or does not answer ($\perp$).

$$O(P, D) \in \{yes, no, \bot\}$$

O

P

P'

Note that programs are also a kind of data!
They can be fed to other programs. (e.g., to compilers)

Static analyzer A.
Given a program P:

- $O(A, P) = yes \iff \forall D,\ O(P, D)$ is safe
- $O(A, P) \neq \bot$ (the static analysis always terminates)

Static analysis

Static analyzer A.
Given a program P:

- $O(A, P) = yes \iff \forall D,\ O(P, D)$ is safe
- $O(A, P) \neq \perp$ (the static analysis always terminates)

$\implies$ P is proved safe even before it is run!

Fundamental undecidability

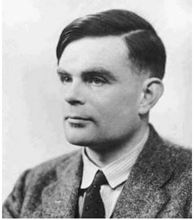
There cannot exist a static analyzer A proving the termination of every terminating program P.

Proof sketch:

$$A(P \cdot D) : O(A, P \cdot D) = \left| \begin{array}{l} \textit{yes} \text{ if } O(P, D) \neq \bot \\ \textit{no} \text{ otherwise} \end{array} \right.$$

$A'(X)$: `while A(X·X) do` *nothing* ; *no*

do we have $O(A', A') = \bot$ or $\neq \bot$? neither!
$\implies A$ cannot exist



Alan Turing[7]

[7] A. M. Turing. "Computability and definability". The Journal of Symbolic Logic, vol. 2, pp. 153–163, (1937).

[8] H. G. Rice. "Classes of Recursively Enumerable Sets and Their Decision Problems." Trans. Amer. Math. Soc. 74, 358-366, 1953.

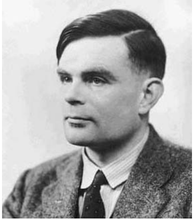
There cannot exist a static analyzer A proving the termination of every terminating program P.

Proof sketch:

$$A(P \cdot D) : O(A, P \cdot D) = \left| \begin{array}{l} \textit{yes} \text{ if } O(P, D) \neq \bot \\ \textit{no} \text{ otherwise} \end{array} \right.$$

$A'(X)$: `while A(X·X) do` *nothing* ; *no*

do we have $O(A', A') = \bot$ or $\neq \bot$? neither!
$\implies A$ cannot exist



Alan Turing[7]

All "interesting" properties are undecidable![8]

[7] A. M. Turing. "Computability and definability". The Journal of Symbolic Logic, vol. 2, pp. 153–163, (1937).

[8] H. G. Rice. "Classes of Recursively Enumerable Sets and Their Decision Problems." Trans. Amer. Math. Soc. 74, 358-366, 1953.

Approximate static analysis

An approximate static analyzer A always answers in finite time ($\neq \bot$):

- either *yes*: the program P is definitely safe (soundness)
- either *no*: I don't know (incompleteness)

Sufficient to prove the safety of (some) programs.
Fails on infinitely many programs...

Approximate static analysis

An approximate static analyzer A always answers in finite time ($\neq \bot$):

- either *yes*: the program P is definitely safe (soundness)
- either *no*: I don't know (incompleteness)

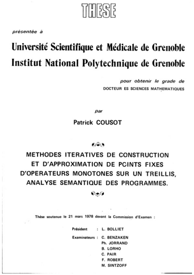
Sufficient to prove the safety of (some) programs.
Fails on infinitely many programs...

$\Longrightarrow$ We should adapt the analyzer A to

- a class of programs to verify, and
- a class of safety properties to check.

Abstract interpretation

Patrick Cousot[9]



THESE

présentée à

Université Scientifique et Médicale de Grenoble
Institut National Polytechnique de Grenoble

pour obtenir le grade de
DOCTEUR ES SCIENCES MATHEMATIQUES

par

Patrick COUSOT

METHODES ITERATIVES DE CONSTRUCTION
ET D'APPROXIMATION DE POINTS FIXES
D'OPERATEURS MONOTONES SUR UN TREILLIS,
ANALYSE SEMANTIQUE DES PROGRAMMES.

Thèse soutenue le 21 mars 1978 devant la Commission d'Examen :

Président : L. BOLLIET
Examinateurs : C. BENZAKEN
Ph. JORRAND
B. LORHO
C. PAIR
F. ROBERT
M. SINTZOFF

General theory of the approximation and comparison of program semantics:

- unifies many semantics
- allows the definition of static analyses
  that are correct by construction

[9] P. Cousot. "Méthodes itératives de construction et d'approximation de points fixes d'opérateurs monotones sur un treillis, analyse sémantique des programmes." Thèse És Sciences Mathématiques, 1978.

Abstract interpretation

$(\mathcal{S}_0)$
```
assume X in [0,1000];
```
$(\mathcal{S}_1)$
```
I := 0;
```
$(\mathcal{S}_2)$
```
while
```
$(\mathcal{S}_3)$
```
I < X do
```
$(\mathcal{S}_4)$
```
    I := I + 2;
```
$(\mathcal{S}_5)$

$(\mathcal{S}_6)$

program

program	semantics
$(\mathcal{S}_0)$	
`assume X in [0,1000];`	$\mathcal{S}_i \in \mathcal{D} = \mathcal{P}(\{\mathtt{I},\mathtt{X}\} \to \mathbb{Z})$
$(\mathcal{S}_1)$	$\mathcal{S}_0 = \{\,(i,x) \mid i,x \in \mathbb{Z}\,\} \quad = \top$
`I := 0;`	$\mathcal{S}_1 = \{\,(i,x) \in \mathcal{S}_0 \mid x \in [0,1000]\,\} \quad = F_1(\mathcal{S}_0)$
$(\mathcal{S}_2)$	$\mathcal{S}_2 = \{\,(0,x) \mid \exists i,\ (i,x) \in \mathcal{S}_1\,\} \quad = F_2(\mathcal{S}_1)$
`while` $(\mathcal{S}_3)$ `I < X do`	$\mathcal{S}_3 = \mathcal{S}_2 \cup \mathcal{S}_5$
$(\mathcal{S}_4)$	$\mathcal{S}_4 = \{\,(i,x) \in \mathcal{S}_3 \mid i < x\,\} \quad = F_4(\mathcal{S}_3)$
`I := I + 2;`	$\mathcal{S}_5 = \{\,(i+2,x) \mid (i,x) \in \mathcal{S}_4\,\} \quad = F_5(\mathcal{S}_4)$
$(\mathcal{S}_5)$	$\mathcal{S}_6 = \{\,(i,x) \in \mathcal{S}_3 \mid i \geq x\,\} \quad = F_6(\mathcal{S}_3)$
$(\mathcal{S}_6)$	

Concrete semantics $\mathcal{S}_i \in \mathcal{D} = \mathcal{P}(\{\mathtt{I},\mathtt{X}\} \to \mathbb{Z})$:

- smallest solution of a system of equations
- strongest invariant (and an inductive invariant)
- not computable in general

```
(S0)
assume X in [0,1000];
(S1)
I := 0;
(S2)
while (S3) I < X do
    (S4)
    I := I + 2;
    (S5)
(S6)
```

program

$\mathcal{S}_i^\sharp \in \mathcal{D}^\sharp$

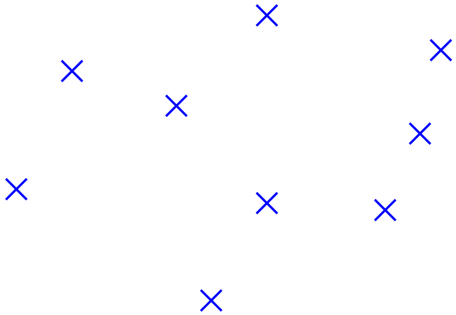
$$\mathcal{S}_0^\sharp = \top^\sharp$$
$$\mathcal{S}_1^\sharp = F_1^\sharp(\mathcal{S}_0^\sharp)$$
$$\mathcal{S}_2^\sharp = F_2^\sharp(\mathcal{S}_1^\sharp)$$
$$\mathcal{S}_3^\sharp = S_2^\sharp \cup^\sharp \mathcal{S}_5^\sharp$$
$$\mathcal{S}_4^\sharp = F_4^\sharp(\mathcal{S}_3^\sharp)$$
$$\mathcal{S}_5^\sharp = F_5^\sharp(\mathcal{S}_4^\sharp)$$
$$\mathcal{S}_6^\sharp = F_6^\sharp(\mathcal{S}_3^\sharp)$$

semantics

Abstract semantics $\mathcal{S}_i^\sharp \in \mathcal{D}^\sharp$:

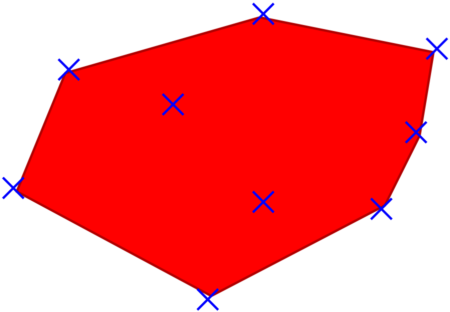
- $\mathcal{D}^\sharp$ subset of properties of interest
  (with a machine representation)
- $F^\sharp : \mathcal{D}^\sharp \to \mathcal{D}^\sharp$ over-approximates the effect of $F : \mathcal{D} \to \mathcal{D}$ in $\mathcal{D}^\sharp$
  (with effective algorithms)

Numeric abstract domain examples



concrete sets $\mathcal{D}$: $\{(0,3),(5.5,0),(12,7),\ldots\}$

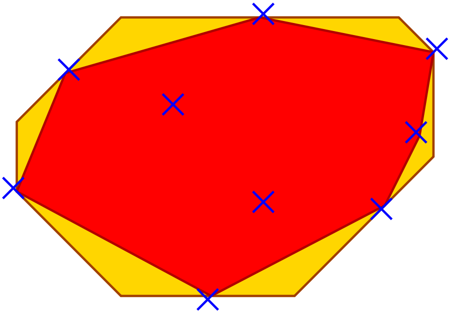
Numeric abstract domain examples



concrete sets $\mathcal{D}$: $\{(0,3),(5.5,0),(12,7),\ldots\}$

abstract polyhedra $\mathcal{D}^\sharp_p$: $6X + 11Y \geq 33 \wedge \cdots$

Numeric abstract domain examples

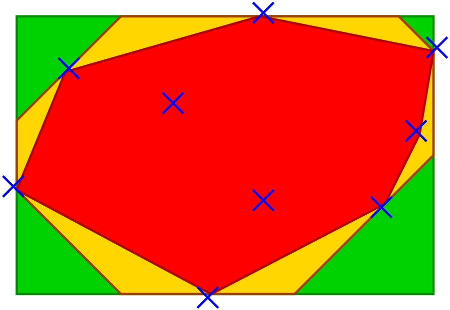


concrete sets $\mathcal{D}$: $\{(0,3),(5.5,0),(12,7),\ldots\}$

abstract polyhedra $\mathcal{D}^\sharp_p$: $6X + 11Y \geq 33 \wedge \cdots$

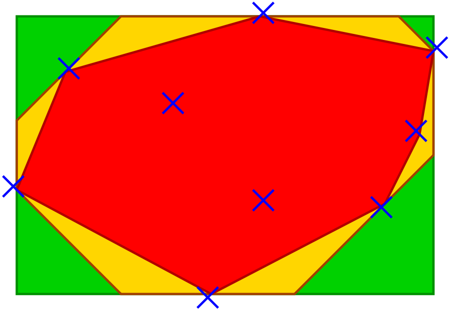
abstract octagons $\mathcal{D}^\sharp_o$: $X + Y \geq 3 \wedge Y \geq 0 \wedge \cdots$

Numeric abstract domain examples



concrete sets $\mathcal{D}$:	$\{(0,3),(5.5,0),(12,7),\ldots\}$
abstract polyhedra $\mathcal{D}^\sharp_p$:	$6X + 11Y \geq 33 \wedge \cdots$
abstract octagons $\mathcal{D}^\sharp_o$:	$X + Y \geq 3 \wedge Y \geq 0 \wedge \cdots$
abstract intervals $\mathcal{D}^\sharp_i$:	$X \in [0,12] \wedge Y \in [0,8]$

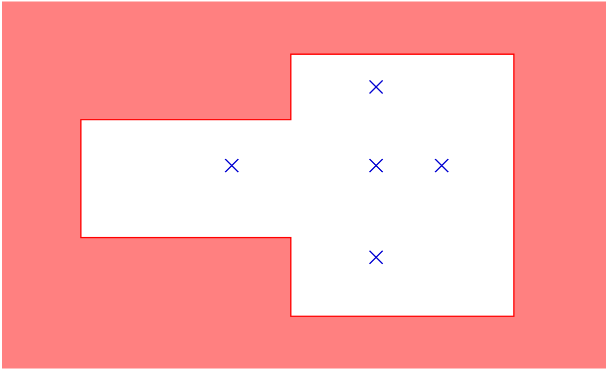
Numeric abstract domain examples



concrete sets $\mathcal{D}$:	$\{(0,3),(5.5,0),(12,7),\ldots\}$	not computable
abstract polyhedra $\mathcal{D}^\sharp_p$:	$6X + 11Y \geq 33 \wedge \cdots$	exponential cost
abstract octagons $\mathcal{D}^\sharp_o$:	$X + Y \geq 3 \wedge Y \geq 0 \wedge \cdots$	cubic cost
abstract intervals $\mathcal{D}^\sharp_i$:	$X \in [0,12] \wedge Y \in [0,8]$	linear cost

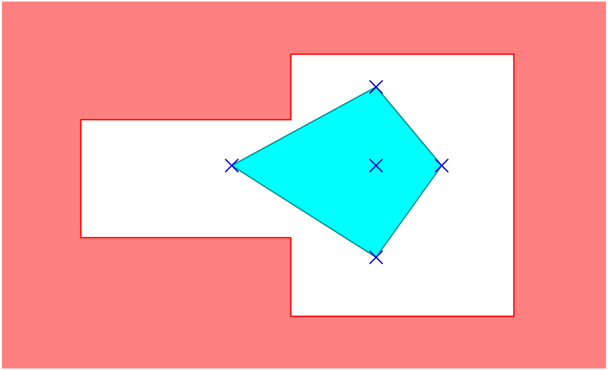
Trade-off between cost and expressiveness / precision

Correctness proof and false alarms



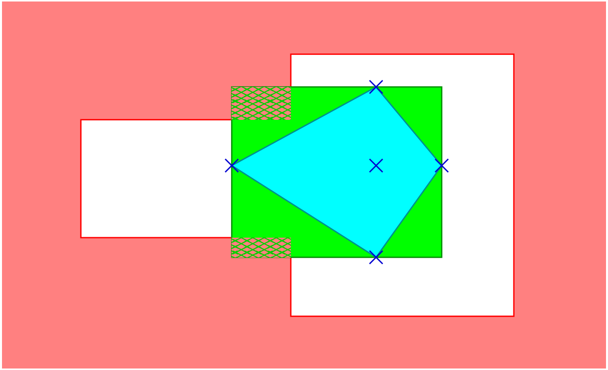
The program is correct (blue $\cap$ red $= \emptyset$).

Correctness proof and false alarms



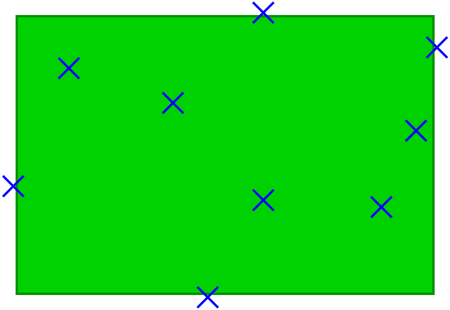
The program is correct (blue $\cap$ red $= \emptyset$).
The polyhedra domain can prove the correctness (cyan $\cap$ red $= \emptyset$).

Correctness proof and false alarms



The program is correct (blue $\cap$ red $= \emptyset$).
The polyhedra domain can prove the correctness (cyan $\cap$ red $= \emptyset$).
The interval domain cannot (green $\cap$ red $\neq \emptyset$, false alarm).

Numeric abstract domain examples (cont.)



abstract semantics $F^\sharp$ in the interval domain $\mathcal{D}^\sharp_i$

- `I:=I+2`: $(\mathtt{I} \in [\ell, h]) \mapsto (\mathtt{I} \in [\ell+2, h+2])$
- $\cup^\sharp$: $(\mathtt{I} \in [\ell_1, h_1]) \cup^\sharp (\mathtt{I} \in [\ell_2, h_2])$
  $= (\mathtt{I} \in [\min(\ell_1, \ell_2), \max(h_1, h_2)])$
- ...

Galois connection

$$(\mathcal{D}, \subseteq) \xleftarrow[\alpha]{\gamma} (\mathcal{D}^\sharp, \subseteq^\sharp)$$

$$\alpha(X) \subseteq^\sharp Y^\sharp \iff X \subseteq \gamma(Y^\sharp)$$

Évariste Galois

Use:

- $\alpha(X)$ is the best abstraction of X in $\mathcal{D}^\sharp$
- $F^\sharp = \alpha \circ F \circ \gamma$ is the best abstraction of F in $\mathcal{D}^\sharp \to \mathcal{D}^\sharp$

Galois connection

$$(\mathcal{D}, \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\mathcal{D}^\sharp, \subseteq^\sharp)$$

$$\alpha(X) \subseteq^\sharp Y^\sharp \iff X \subseteq \gamma(Y^\sharp)$$

Évariste Galois

Use:

- $\alpha(X)$ is the best abstraction of X in $\mathcal{D}^\sharp$
- $F^\sharp = \alpha \circ F \circ \gamma$ is the best abstraction of F in $\mathcal{D}^\sharp \to \mathcal{D}^\sharp$

Example: in the interval domain $\mathcal{D}_i^\sharp$

- $[\ell_1, h_1] \subseteq_i^\sharp [\ell_2, h_2] \iff \ell_1 \geq \ell_2 \wedge h_1 \leq h_2$
- $\gamma_i([\ell, h]) = \{\, x \in \mathbb{Z} \mid \ell \leq x \leq h \,\}$
- $\alpha_i(X) = [\min X, \max X]$

Resolution by iteration and extrapolation

Challenge: the equation system is recursive: $\vec{\mathcal{S}}^\sharp = \vec{F}^\sharp(\vec{\mathcal{S}}^\sharp)$.

Solution: resolution by iteration: $\vec{\mathcal{S}}^{\sharp\,0} = \emptyset^\sharp$, $\vec{\mathcal{S}}^{\sharp\,i+1} = \vec{F}^\sharp(\vec{\mathcal{S}}^{\sharp\,i})$.

e.g., $\mathcal{S}_3^\sharp$: $\mathtt{I} \in \emptyset$, $\mathtt{I} = 0$, $\mathtt{I} \in [0,2]$, $\mathtt{I} \in [0,4]$, $\ldots$, $\mathtt{I} \in [0,1000]$

Resolution by iteration and extrapolation

Challenge: the equation system is recursive: $\vec{\mathcal{S}}^\sharp = \vec{F}^\sharp(\vec{\mathcal{S}}^\sharp)$.

Solution: resolution by iteration: $\vec{\mathcal{S}}^{\sharp\,0} = \emptyset^\sharp$, $\vec{\mathcal{S}}^{\sharp\,i+1} = \vec{F}^\sharp(\vec{\mathcal{S}}^{\sharp\,i})$.

e.g., $\mathcal{S}_3^\sharp$: $\mathtt{I} \in \emptyset$, $\mathtt{I} = 0$, $\mathtt{I} \in [0,2]$, $\mathtt{I} \in [0,4]$, ..., $\mathtt{I} \in [0,1000]$

Challenge: infinite or very long sequence of iterates in $\mathcal{D}^\sharp$

Solution: extrapolation operator $\triangledown$

e.g., $[0,2] \triangledown [0,4] = [0,+\infty[$

- remove unstable bounds and constraints
- ensures the convergence in finite time
- inductive reasoning (through generalisation)

Resolution by iteration and extrapolation

Challenge: the equation system is recursive: $\vec{\mathcal{S}}^\sharp = \vec{F}^\sharp(\vec{\mathcal{S}}^\sharp)$.

Solution: resolution by iteration: $\vec{\mathcal{S}}^{\sharp\,0} = \emptyset^\sharp$, $\vec{\mathcal{S}}^{\sharp\,i+1} = \vec{F}^\sharp(\vec{\mathcal{S}}^{\sharp\,i})$.

e.g., $\mathcal{S}_3^\sharp$: $\texttt{I} \in \emptyset$, $\texttt{I} = 0$, $\texttt{I} \in [0,2]$, $\texttt{I} \in [0,4]$, $\ldots$, $\texttt{I} \in [0,1000]$

Challenge: infinite or very long sequence of iterates in $\mathcal{D}^\sharp$

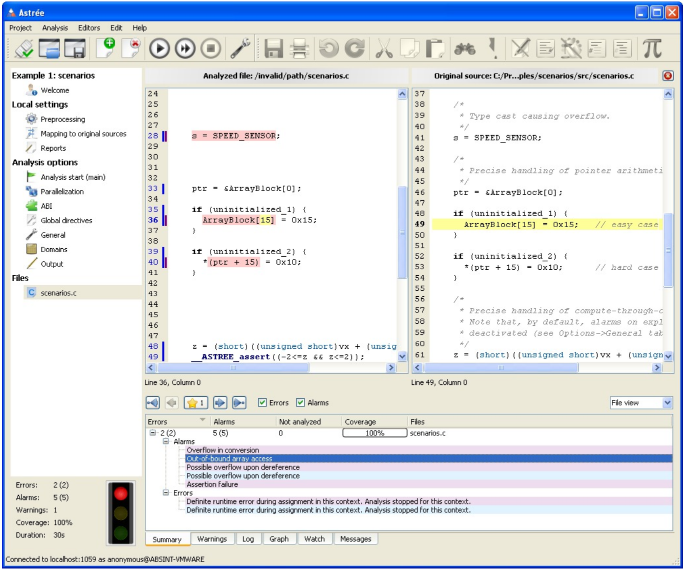
Solution: extrapolation operator $\triangledown$

e.g., $[0,2] \triangledown [0,4] = [0,+\infty[$

- remove unstable bounds and constraints
- ensures the convergence in finite time
- inductive reasoning (through generalisation)

$\Longrightarrow$ effective solving method $\longrightarrow$ static analyzer!

The Astrée static analyzer



The Astrée static analyzer

Analyseur statique de programmes temps-réels embarqués

(static analyzer for real-time embedded software)

- developed at ENS (since 2001)

  B. Blanchet, P. Cousot, R. Cousot, J. Feret,
  L. Mauborgne, D. Monniaux, A. Miné, X. Rival

- industrialized and made commercially available by AbsInt (since 2009)

Astrée

`www.astree.ens.fr`

AbsInt

`www.absint.com`

The Astrée static analyzer

Specialized:

- for the analysis of run-time errors
  (arithmetic overflows, array overflows, divisions by 0, etc.)
- on embedded critical C software
  (no dynamic memory allocation, no recursivity)
- in particular on control / command software
  (reactive programs, intensive floating-point computations)
- intended for validation
  (analysis does not miss any error and tries to minimise false alarms)

The Astrée static analyzer

Specialized:

- for the analysis of run-time errors
  (arithmetic overflows, array overflows, divisions by 0, etc.)
- on embedded critical C software
  (no dynamic memory allocation, no recursivity)
- in particular on control / command software
  (reactive programs, intensive floating-point computations)
- intended for validation
  (analysis does not miss any error and tries to minimise false alarms)

Approximately 40 abstract domains are used at the same time:

- numeric domains (intervals, octagons, ellipsoids, etc.)
- boolean domains
- domains expressing properties on the history of computations

Astrée applications (at ENS)

Airbus A340-300 (2003)

Airbus A380 (2004)

(model of) ESA ATV (2008)

- size: from 70 000 to 860 000 lines of C
- analysis time: from 45mn to $\simeq$40h
- alarm(s): 0 (proof of absence of run-time error)

Other applications of abstract interpretation

- Analysis of dynamic memory data-structures (*shape analysis*).
- Analysis of parallel, distributed, and multi-thread programs.
- Analysis of probabilistic programs.
- Analysis of biological systems.
- Security analysis (*information flow*).
- Termination analysis.
- Cost analysis.
- Analyses to enable compiler optimisations.
- ...