Rule-based modeling and application to biomolecular networks Abstract interpretation of protein-protein interactions networks Solution of the Questions Set

Jérôme Feret

LIENS (INRIA,ÉNS,CNRS)

1 Abstract Interpretation

Definition 1 (partial order). *A partial order $(D, \leq)$ is given by a set D and a binary relation $\leq \in D \times D$ such that:*

1. *(reflexivity)* $\forall a \in D,\ a \leq a$;
2. *(antisymmetry)* $\forall a, a' \in D,\ [a \leq a' \wedge a' \leq a] \implies a = a'$;
3. *(transitivity) and* $\forall a, a', a" \in D,\ [a \leq a' \wedge a' \leq a''] \implies a \leq a"$.

Definition 2 (closure). *Given a partial order $(D, \leq)$ and a mapping $\rho : D \to D$.*

1. *We say that ρ is a upper closure operator, if and only if:*
   (a) *(idempotence)* $\forall d \in D,\ \rho(\rho(d)) = \rho(d)$;
   (b) *(extensivity)* $\forall d \in D,\ d \leq \rho(d)$;
   (c) *(monotonicity)* $\forall d, d' \in D,\ d \leq d' \implies \rho(d) \leq \rho(d')$.

2. *We say that ρ is a lower closure operator, if and only if:*
   (a) *(idempotence)* $\forall d \in D,\ \rho(\rho(d)) = \rho(d)$;
   (b) *(antiextensivity)* $\forall d \in D,\ \rho(d) \leq d$;
   (c) *(monotonicity)* $\forall d, d' \in D,\ d \leq d' \implies \rho(d) \leq \rho(d')$.

Definition 3 (least upper bound). *Given a partial order $(D, \leq)$ and a subset $X \subseteq A$, we say that $m \in D$ is a least upper bound for X, if and only if:*

1. *(bound)* $\forall a \in X,\ a \leq m$;
2. *(least one) and* $\forall a \in D,\ [\forall a' \in X, a' \leq a] \implies m \leq a$.

By antisymmetry, if it exists a least upper bound is unique, thus we call it the least upper bound.

Definition 4 (greatest lower bound). *Given a partial order $(D, \leq)$ and a subset $X \subseteq A$, we say that $m \in D$ is a greatest lower bound for X, if and only if:*

1. *(bound)* $\forall a \in X,\ m \leq a$;
2. *(least one) and* $\forall a \in D,\ [\forall a' \in X, a \leq a'] \implies a \leq m$.

By antisymmetry, if it exists a greatest lower bound is unique, thus we call it the greatest lower bound.

Definition 5 (complete lattice). *Given a partial order $(D, \leq)$, we say that D is a complete lattice if any subset X has a least upper bound $\sqcup X$.*

In a complete lattice, any subset X has a greatest lower bound $\sqcap X$. Moreover,

$$\sqcap(X) = \sqcup\{d \in X \mid \forall x \in X, d \leq x\}.$$

The element $\top = \sqcup(D)$ is the greatest element of D, and the element $\bot = \sqcup(\emptyset)$ is the least element. A complete lattice is usually denoted by $(D, \leq, \bot, \top, \sqcup, \sqcap)$.

Proof. Let us show that the hypothesis of Def. 4 are satisfied.

- Let x be an element of X.
  By Def. 1.(1), we have $x \leq x$.
  Thus by Def. 3.(1), we have $x \leq \sqcup\{d \in X \mid \forall x \in X, d \leq x\}$.

- Let m be an element of D such that for any element $x \in X$, $m \leq x$.
  By Def. 3.(2), we have $\sqcup\{d \in X \mid \forall x \in X, d \leq x\} \leq m$.

Thus by Def. 4, $\sqcup\{d \mid \forall x \in X, d \leq x\}$ is the greatest least bound of X.
□

Definition 6 (chain-complete partial order). *Given a partial order $(D, \leq)$, we say that $(D, \leq)$ is a chain-complete partial order if and only if any chain $X \subseteq D$ has a least upper bound $\sqcup X$.*

A chain-complete partial order is denoted by a triple $(D, \leq, \sqcup)$.

Definition 7 (inductive function). *Given a chain-complete partial order $(D, \subseteq, \cup)$, we say that a function $\mathbb{F} : D \to D$ is inductive if and only if the two following properties are satisfied:*

1. *$\forall x \in D$, $x \subseteq \mathbb{F}(x) \implies \mathbb{F}(x) \subseteq \mathbb{F}(\mathbb{F}(x))$;*
2. *for any chain C of elements in D such that $x \subseteq \mathbb{F}(x)$, for any $x \in C$, we have: $\cup C \subseteq \mathbb{F}(\cup C)$.*

Proposition 1. *Let $(D, \subseteq, \cup)$ be a chain-complete partial order and $\mathbb{F} : D \to D$ be a function such that: $\forall x, y \in D, x \subseteq y \implies \mathbb{F}(x) \subseteq \mathbb{F}(y)$.*

Then $\mathbb{F}$ is an inductive function.

Proof. Let us prove that the hypotheses of Def. 7 are satisfies:

1. Let $x_0 \in D$ be an element such that $x_0 \subseteq \mathbb{F}(x_0)$.
   Since $\mathbb{F}$ is monotonic, it follows that $\mathbb{F}(x_0) \subseteq \mathbb{F}(\mathbb{F}(x_0))$.

2. Let C be a chain of elements in D such that, for any element $x \in C$, $x \subseteq \mathbb{F}(x)$.
   Let $x \in C$ be an element.
   By Def. 3.(1), $x \subseteq \cup C$.
   Since $\mathbb{F}$ is monotonic, we have: $\mathbb{F}(x) \subseteq \mathbb{F}(\cup X)$;
   Since, by hypothesis, $x \subseteq \mathbb{F}(x)$ and by Prop. 1.(3), it follows that $x \subseteq \mathbb{F}(\cup X)$;

   Thus, by Def. 3.(2), $\cup C \subseteq \mathbb{F}(\cup C)$.

□

Definition 8 (inductive definition). *Let $(D, \subseteq, \cup)$ be a chain-complete partial order, $x_0 \in D$ be an element such that $x_0 \subseteq \mathbb{F}(x_0)$, and $\mathbb{F} : D \to D$ be an inductive function.*

There exists a unique collection of elements (X_o) such that for any ordinal o:

$$\begin{cases} X_o = x_0 & \text{whenever } o = 0 \\ X_o = \mathbb{F}(X_{o-1}) & \text{whenever } o \text{ is a succesor ordinal} \\ X_o = \cup\{X_\beta \mid \beta < o\} & \text{otherwise.} \end{cases}$$

The collection (X_o) is called the transfinite iteration of $\mathbb{F}$ starting from x_0. For each ordinal o, the element X_o is usually denoted by $\mathbb{F}^o(x_0)$.

Proof. We show by induction over the ordinals, that for any ordinal o_0, there exists a unique family of elements $(X_o)_{o<o_0}$ such that the three following properties are satisfied:

- (a)

$$\begin{cases} X_o = x_0 & \text{whenever } o = 0, \\ X_o = \mathbb{F}(X_{o-1}) & \text{whenever } o \text{ is a succesor ordinal,} \\ X_o = \cup\{X_\beta \mid \beta < o\} & \text{otherwise.} \end{cases}$$

- (b) $(X_o)_{o<o_0}$ is increasing,
- (c) and for any ordinal $o < o_0$, $X_o \subseteq \mathbb{F}(X_o)$.

1. (a) There exists a unique element X_0 such that $X_0 = x_0$.
   (b) (x_0) is an increasing family (of one element).
   (c) By hypothesis, $x_0 \subseteq \mathbb{F}(x_0)$.

2. Let o_0 be an ordinal.
   We assume that there exists a unique family $(X_o)_{o\leq o_0}$ such that the equations (a) are satisfied.
   We also assume that $(X_o)_{o\leq o_0}$ is increasing and that for any ordinal $o \leq o_0$, $X_o \subseteq \mathbb{F}(X_o)$.
   We define $Y_o = X_o$ whenever $o \leq o_0$ and $Y_{o_0+1} = \mathbb{F}(X_{o_0})$.

   (a) The family $(Y_o)_{o\leq o_0+1}$ satisfies the equations (a).

   (b) Now we consider a family $(Z_o)_{o\leq o_0+1}$ of elements in D which satisfies the equations (a).
   Then by induction hypotheses (uniqueness), we have $Z_o = Y_o$ for any ordinal $o \leq o_0$.
   Moreover, since $(Z_o)_{o\leq o_0+1}$ satisfies the equations (a), we have $Z_{o_0+1} = \mathbb{F}(Z_{o_0})$.
   Since $Z_{o_0} = Y_{o_0}$, it follows by extensionality that $\mathbb{F}(Z_{o_0}) = \mathbb{F}(Y_{o_0})$.
   Moreover, we have: $\mathbb{F}(Y_{o_0}) = Y_{o_0+1}$.
   So $Z_{o_0+1} = Y_{o_0+1}$.
   Thus $(Z_o)_{o\leq o_0+1} = (Y_o)_{o\leq o_0+1}$.

   (c) By induction hypotheses, $(Y_o)_{o\leq o_0}$ is increasing.
   By induction hypotheses again $Y_{o_0} \leq \mathbb{F}(Y_{o_0})$.
   Since $Y_{o_0+1} = \mathbb{F}(Y_{o_0})$, it follows that $Y_{o_0} \subseteq Y_{o_0+1}$.
   Thus $(Y_o)_{o\leq o_0+1}$ is increasing.

   (d) By induction hypotheses, for any $o \leq o_0$, $Y_o \subseteq \mathbb{F}(Y_o)$.
   Since $\mathbb{F}$ is inductive, by Def. 7.(1), it follows that $\mathbb{F}(Y_{o_0}) \subseteq \mathbb{F}(\mathbb{F}(Y_{o_0}))$.
   Since $Y_{o_0+1} = \mathbb{F}(Y_{o_0})$, we get $Y_{o_0+1} \subseteq \mathbb{F}(Y_{o_0+1})$.

3. Let o_0 be a limit ordinal.
   We assume that there exists a unique family $(X_o)_{o<o_0}$ such that the equations (a) are satisfied.
   We define $Y_o = X_o$ whenever $o < o_0$ and $Y_{o_0} = \cup\{X_\beta \mid \beta < o_0\}$.

   (a) The family $(Y_o)_{o\leq o_0}$ satisfies the equations (a).

(b) Now we consider a family $(Z_o)_{o\leq o_0}$ of elements in D which satisfies the equations (a).
Then by induction hypotheses (uniqueness), we have $Z_o = Y_o$ for any ordinal $o < o_0$.
Moreover, since $(Z_o)_{o\leq o_0}$ satisfies the equations (a), we have $Z_{o_0} = \cup\{Z_\beta \mid \beta < o_0\}$.
Since $Z_\beta = Y_\beta$, for any $\beta < o_0$, it follows that: $\cup\{Z_\beta \mid \beta < o_0\} = \cup\{Y_\beta \mid \beta < o_0\}$.
Moreover, we have: $\cup\{Y_\beta \mid \beta < o_0\} = Y_{o_0}$.
So $Z_{o_0} = Y_{o_0}$.
Thus $(Z_o)_{o\leq o_0} = (Y_o)_{o\leq o_0}$.

(c) By induction hypotheses, $(Y_o)_{o<o_0}$ is increasing.
By Def. 3.(1), for any ordinal $o < o_0$, we have: $Y_o \leq \cup\{Y_{o'} \mid o' < o_0\}$.
Since $Y_{o_0} = \cup\{Y_{o'} \mid o' < o_0\}$, it follows that $Y_o \subseteq Y_{o_0}$, for any ordinal $o \leq o_0$.

(d) By induction hypotheses, for any $o < o_0$, $Y_o \subseteq \mathbb{F}(Y_o)$.
Since $\mathbb{F}$ is inductive, by Def. 7.(2), it follows that $\cup\{Y_o \mid o < o_0\} \subseteq \mathbb{F}(\cup\{Y_o \mid o < o_0\})$.
Since $Y_{o_0} = \cup\{Y_o \mid o < o_0\}$, we get $Y_{o_0} \subseteq \mathbb{F}(Y_{o_0})$.

□

Proposition 2. *Let $(D, \subseteq, \cup)$ be a chain-complete partial order, $x_0 \in D$ be an element such that $x_0 \subseteq \mathbb{F}(x_0)$, and $\mathbb{F} : D \to D$ an inductive function.*
Then:

1. *for any pair of ordinals (o, o'), $[o < o'] \implies \mathbb{F}^o(x_0) \subseteq \mathbb{F}^{o'}(x_0)$;*
2. *for any ordinal o, $x_0 \subseteq \mathbb{F}^o(x_0)$.*

Proof. The assertion 1 is implied by the hypotheses induction of the proof that Def. 8 is well-defined.
The assertion 2 follows from the fact that for any ordinal, $0 \leq o$, and by the assertion 1.
□

Lemma 1 (least fix-point). *Let:*

1. *$(D, \subseteq, \cup)$ be a chain-complete partial order;*
2. *$\mathbb{F} \in D \to D$ be a monotonic map;*
3. *$x_0 \in D$ be an element such that: $x_0 \subseteq \mathbb{F}(x_0)$.*

Then: there exists $y \in D$ such that:

- $x_0 \subseteq y$,
- $\mathbb{F}(y) = y$,
- $\forall z \in D,\ [[\mathbb{F}(z) = z \wedge x_0 \subseteq z] \implies y \subseteq z]$.

This element is called the least fix-point of $\mathbb{F}$ which is greater than x_0, and is written $lfp_{x_0}\mathbb{F}$.

Proof. Let $x_0 \in D$, such that $x_0 \subseteq \mathbb{F}(x_0)$.

By hypothesis, $\mathbb{F}$ is monotonic.
By Prop. 1, $\mathbb{F}$ is inductive.
By Def. 8, it follows that the collection $(\mathbb{F}^o(x_0))_o$ indexed over the ordinals is well-defined.

By Prop. 2.(1), the collection $(\mathbb{F}^o(x_0))_o$ is increasing.
Since D is a set, the collection $(\mathbb{F}^o(x_0))_o$ is ultimately stationary.

Thus there exists an ordinal o such that $\mathbb{F}^o(x_0) = \mathbb{F}^{o+1}(x_0)$.

Thus, $\mathbb{F}(\mathbb{F}^o(x_0)) = \mathbb{F}^o(x_0)$.

By Prop. 2.(2), for any ordinal o, we have: $x_0 \subseteq \mathbb{F}^o(x_0)$.

Consider another fix-point $y \in D$ such that $x_0 \subseteq y$.
We have $y = \mathbb{F}(y)$.

Let us show by transfinite induction that $\mathbb{F}^o(x_0) \subseteq y$.

- We have, by hypothesis, $x_0 \subseteq y$.
  Since, $\mathbb{F}^0(x_0) = x_0$, it follows that $\mathbb{F}^0(x_0) \subseteq y$.

- Let us consider an ordinal o such that $\mathbb{F}^o(x_0) \subseteq y$.
  Since, $\mathbb{F}$ is monotonic, we have $\mathbb{F}(\mathbb{F}^o(x_0)) \subseteq \mathbb{F}(y)$.
  Then by Def. 8, $\mathbb{F}^{o+1}(x_0) = \mathbb{F}(\mathbb{F}^o(x_0))$.
  And by hypothesis $\mathbb{F}(y) = y$.
  Thus $\mathbb{F}^{o+1}(x_0) \subseteq y$.

- Let us consider an ordinal o such that for any $\beta < o$, we have $\mathbb{F}^\beta(x_0) \subseteq y$.
  By Def. 3.(2), we get that $\cup\{\mathbb{F}^\beta(x_0) \mid \beta < o\} \subseteq y$.
  By Def. 8, $\mathbb{F}^o(x_0) = \cup\{\mathbb{F}^\beta(x_0) \mid \beta < o\}$.
  Thus, $\mathbb{F}^o(x_0) \subseteq y$.

Thus $\mathbb{F}^o(x_0)$ is the least fix-point of $\mathbb{F}$.
□

Remark 1. We have seen in this proof that, under the hypotheses of Lemma 1, $\mathit{lfp}_{x_0}\mathbb{F} = \mathbb{F}^o(x_0)$ for a given ordinal o.

Definition 9 (Galois connexion). *Given two partial orders $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$, we say that the pair of maps (α, γ) forms a Galois connection between D and $D^\sharp$ if and only if:*

1. $\alpha \,:\, D \to D^\sharp$;
2. $\gamma \,:\, D^\sharp \to D$;
3. *and* $\forall d \in D,\ \forall d^\sharp \in D^\sharp, [\alpha(d) \sqsubseteq d^\sharp \Leftrightarrow d \subseteq \gamma(d^\sharp)]$.

In such a case, we write:

$$D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp.$$

Proposition 3. *Let $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$ be partial orders, and $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ be a Galois connexion. The following properties are satisfied:*

1. $\forall d \in D,\ d \subseteq \gamma(\alpha(d))$;
2. $\forall d^\sharp \in D^\sharp,\ \alpha(\gamma(d^\sharp)) \sqsubseteq d^\sharp$;
3. *(α is monotonic)* $\forall d, d' \in D,\ d \subseteq d' \implies \alpha(d) \sqsubseteq \alpha(d')$;
4. *(γ is monotonic)* $\forall d^\sharp, d'^\sharp \in D^\sharp,\ d^\sharp \sqsubseteq d'^\sharp \implies \gamma(d^\sharp) \subseteq \gamma(d'^\sharp)$;
5. $\forall d \in D,\ \alpha(d) = \alpha(\gamma(\alpha(d)))$;
6. $\forall d^\sharp \in D^\sharp,\ \gamma(d^\sharp) = \gamma(\alpha(\gamma(d)))$;
7. *$\gamma \circ \alpha$ is an upper closure operator;*
8. *$\alpha \circ \gamma$ is a lower closure operator.*

Proof. Let $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$ be partial orders, and $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ be a Galois connexion.

1. Let $d \in D$ be an element.

   By Def. 1.(1), we have: $\alpha(d) \sqsubseteq \alpha(d)$.
   By Def. 9.(3), it follows that: $d \subseteq \gamma(\alpha(d))$.

2. Let $d^\sharp \in D^\sharp$ be an element.

   By Def. 1.(1), we have: $\gamma(d^\sharp) \subseteq \gamma(d^\sharp)$.
   By Def. 9.(3), it follows that: $\alpha(\gamma(d^\sharp)) \subseteq d^\sharp$.

3. Let $d, d' \in D$ be two elements such that $d \subseteq d'$.

   By hypothesis, we have $d \subseteq d'$.
   Moreover, by Prop. 3.(1), we have $d' \subseteq \gamma(\alpha(d'))$.
   Thus by Def. 1.(3), we get: $d \subseteq \gamma(\alpha(d'))$.
   By Def. 9.(3), it follows that: $\alpha(d) \sqsubseteq \alpha(d')$.

4. Let $d^\sharp, d'^\sharp \in D^\sharp$ be two elements such that $d^\sharp \sqsubseteq d'^\sharp$.

   By Prop. 3.(2), we have $\alpha(\gamma(d^\sharp)) \subseteq d^\sharp$.
   Moreover, by hypothesis, we have $d^\sharp \sqsubseteq d'^\sharp$.
   Thus by Def. 1.(3), we get: $\alpha(\gamma(d^\sharp)) \sqsubseteq d'^\sharp$.
   By Def. 9.(3), it follows that: $\gamma(d^\sharp) \sqsubseteq \gamma(d'^\sharp)$.

5. Let $d \in D$ be an element.

   By Prop. 3.(1), we have: $d \subseteq \gamma(\alpha(d))$.
   By Prop. 3.(3), it follows that $\alpha(d) \sqsubseteq \alpha(\gamma(\alpha(d)))$.

   By Def. 1.(1), we have: $\gamma(\alpha(d)) \subseteq \gamma(\alpha(d))$
   By Def. 9.(3), it follows that: $\alpha(\gamma(\alpha(d)) \sqsubseteq \alpha(d)$.

   By Def. 1.(2), it follows that $\alpha(d) = \alpha(\gamma(\alpha(d)))$.

6. Let $d^\sharp \in D^\sharp$ be an element.

   By Prop. 3.(2), we have: $\alpha(\gamma(d^\sharp) \sqsubseteq d^\sharp$.
   By Prop. 3.(4), it follows that $\gamma(\alpha(\gamma(d^\sharp))) \subseteq \gamma(d^\sharp)$.

   By Def. 1.(1), we have: $\alpha(\gamma(d^\sharp)) \sqsubseteq \alpha(\gamma(d^\sharp))$
   By Def. 9.(3), it follows that: $\gamma(d^\sharp) \subseteq \gamma(\alpha(\gamma(d^\sharp)))$.

   By Def. 1.(2), it follows that $\gamma(d^\sharp) = \gamma(\alpha(\gamma(d^\sharp)))$.

7. Let $d, d' \in D$ such that $d \subseteq d'$.

   (a) By Prop. 3.(6), we have $\gamma(\alpha(\gamma(\alpha(d)))) = \gamma(\alpha(d))$.
   (b) By Prop. 3.(1), we have $d \subseteq \gamma(\alpha(d))$.

(c) By Prop. 3.(3), we have $\alpha(d) \sqsubseteq \alpha(d')$.
Then by prop. 3.(4), it follows that $\gamma(\alpha(d)) \subseteq \gamma(\alpha(d'))$.

8. Let $d^\sharp, d'^\sharp \in D^\sharp$ such that $d^\sharp \sqsubseteq d'^\sharp$.

   (a) By Prop. 3.(5), we have $\alpha(\gamma(\alpha(\gamma(d^\sharp)))) = \alpha(\gamma(d^\sharp))$.
   (b) By Prop. 3.(2), we have $\alpha(\gamma(d^\sharp)) \sqsubseteq d^\sharp$.
   (c) By Prop. 3.(4), we have $\gamma(d^\sharp) \subseteq \gamma(d'^\sharp)$.
   Then by prop. 3.(3), it follows that $\alpha(\gamma(d^\sharp)) \sqsubseteq \alpha(\gamma(d'))$.

□

Proposition 4. *Let $(D, \subseteq, \bot, \top, \cup, \cap)$ and $(D^\sharp, \sqsubseteq, \bot^\sharp, \top^\sharp, \sqcup, \sqcap)$ be two complete lattices. Let α be a mapping between D and $D^\sharp$ such that for any subset $X \subseteq D$, we have $\alpha(\cup X) = \sqcup\{\alpha(d) \mid d \in X\}$.*

Then there exists a unique mapping γ between $D^\sharp$ and D such that:

$$D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$$

is a Galois connexion.

Moreover, for any element $d^\sharp \in D^\sharp$, we have:

$$\gamma(d^\sharp) = \cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}.$$

Proof. Let $(D, \subseteq, \bot, \top, \cup, \cap)$ and $(D^\sharp, \sqsubseteq, \bot^\sharp, \top^\sharp, \sqcup, \sqcap)$ be two complete lattices. Let α be a mapping between D and $D^\sharp$ such that for any subset $X \subseteq D$, we have $\alpha(\cup X) = \sqcup\{\alpha(d) \mid d \in X\}$.

1. (α is monotonic)
   Let $d, d' \in D$, such that $d \subseteq d'$.
   By Def. 3, we have $\cup\{d, d'\} = d'$.

   Thus, we have: $\alpha(d') = \alpha(\cup\{d, d'\})$.
   By the hypothesis on α, we have $\alpha(\cup\{d, d'\}) = \sqcup\{\alpha(d), \alpha(d')\}$.
   Thus, $\alpha(d') = \sqcup\{\alpha(d), \alpha(d')\}$.
   And by Def. 3.(1), it follow that $\alpha(d) \sqsubseteq \alpha(d')$.

2. (existence)
   Let γ' be the mapping between $D^\sharp$ and D such that:

   $$\gamma'(d^\sharp) = \cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}.$$

   Let $d \in D$ and $d^\sharp \in D^\sharp$.
   - We assume that $\alpha(d) \sqsubseteq d^\sharp$.
     We have: $\gamma'(d^\sharp) = \cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}$.
     Thus, by Def. 3.(1), we have $d \subseteq \gamma'(d^\sharp)$.

   - We assume that $d \subseteq \gamma'(d^\sharp)$.
     By hypothesis, we have: $\gamma'(d^\sharp) = \cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}$.
     Thus, $d \subseteq \cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}$.
     Since α is monotonic, we have: $\alpha(d) \sqsubseteq \alpha(\cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\})$.
     By hypothesis on α, we have $\alpha(\cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}) = \sqcup\{\alpha(d) \mid \alpha(d) \sqsubseteq d^\sharp\}$.
     Thus, $\alpha(d) \sqsubseteq \sqcup\{\alpha(d) \mid \alpha(d) \sqsubseteq d^\sharp\}$.
     For any $d \in D$, such that $\alpha(d) \sqsubseteq d^\sharp$, we have $\alpha(d) \sqsubseteq d^\sharp$.
     Thus, by Def. 3.(1), we have $\sqcup\{\alpha(d) \mid \alpha(d) \sqsubseteq d^\sharp\} \sqsubseteq d^\sharp$.
     By Def. 1.(3), we get: $\alpha(d) \sqsubseteq d^\sharp$.

Thus:

$$D \underset{\alpha}{\overset{\gamma'}{\leftrightarrows}} D^\sharp.$$

3. (uniqueness) Let γ such that:

$$D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp.$$

Let $d^\sharp \in D^\sharp$ be an abstract element.

For any $d \in D$ such that $\alpha(d) \sqsubseteq d^\sharp$, we have by Def. 9.(3), $d \subseteq \gamma(d^\sharp)$.
By hypothesis, $\gamma'(d^\sharp) = \cup\{d \mid \alpha(d) \sqsubseteq d^\sharp\}$.
Thus, Def. 3.(2), we get that $\gamma'(d^\sharp) \subseteq \gamma(d^\sharp)$.

By prop.3.(2), we have $\alpha(\gamma(d^\sharp)) \subseteq d^\sharp$.
We have already proved that:

$$D \underset{\alpha}{\overset{\gamma'}{\leftrightarrows}} D^\sharp.$$

is a Galois connexion.
Thus, by Def. 9.(3), we have $\gamma(d^\sharp) \subseteq \gamma'(d^\sharp)$.

By Def. 1.(2), we get that $\gamma(d^\sharp) = \gamma'(d^\sharp)$.
Thus $\gamma = \gamma'$.

□

Proposition 5. *Given $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$ two partial orders, $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ a Galois connexion, and $X \subseteq D$ a subset of D, if, X has a least upper bound $\cup X$ and $\{\alpha(d) \mid d \in X\}$ has a least upper bound $\sqcup\{\alpha(d) \mid d \in X\}$, then we have:*

$$\alpha(\cup X) = \sqcup\{\alpha(d) \mid d \in X\}.$$

Proof. Let $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$ be two partial orders, $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ be a Galois connexion, and $X \subseteq D$ be a subset of D, such that X has a least upper bound $\cup X$ and $\{\alpha(d) \mid d \in X\}$ has a least upper bound $\sqcup\{\alpha(d) \mid d \in X\}$.

- Let d be an element in X.

  Since X has a least upper bound, we have by Def. 3.(1), $d \subseteq \cup X$.
  By Prop. 3.(3), we have $\alpha(d) \sqsubseteq \alpha(\cup X)$.

  Since $\{\alpha(d) \mid d \in X\}$ has a least upper bound, and by Def. 3.(2), it follows that $\sqcup\{\alpha(d) \mid d \in X\} \sqsubseteq \alpha(\cup X)$.

- Let d be an element in X.

  By Prop. 3.(1), we have $d \subseteq \gamma(\alpha(d))$.
  Since $\{\alpha(d) \mid d \in X\}$ has a least upper bound, and by Def. 3.(1), we have $\alpha(d) \sqsubseteq \sqcup\{\alpha(d) \mid d \in X\}$.
  Thus by Prop. 3.(4), it follows that $\gamma(\alpha(d)) \subseteq \gamma(\sqcup\{\alpha(d) \mid d \in X\})$.
  By Def. 1.(3), it follows that $d \subseteq \gamma(\sqcup\{\alpha(d) \mid d \in X\})$.

  Since X has a least upper bound, and by Def. 3.(2), it follows that $\cup X \subseteq \gamma(\sqcup\{\alpha(d) \mid d \in X\})$.

  By Def. 9.(3), we get that $\alpha(\cup X) \sqsubseteq \sqcup\{\alpha(d) \mid d \in X\}$.

By Def. 1.(2), we conclude that $\alpha(\cup X) = \sqcup\{\alpha(d) \mid d \in X\}$.

□

Proposition 6. *Given $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$ two partial orders, $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ a Galois connexion, and $X^\sharp \subseteq D^\sharp$ a subset of $D^\sharp$, if, $X^\sharp$ has a least upper bound $\sqcup X^\sharp$ and $\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$ has a least upper bound $\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$, then we have:*

$$\gamma(\sqcup X^\sharp) = \gamma(\alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\})).$$

Proof. Let $(D, \subseteq)$ and $(D^\sharp, \sqsubseteq)$ be two partial orders, $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ be a Galois connexion, and $X^\sharp \subseteq D^\sharp$ be a subset of $D^\sharp$, such that: $X^\sharp$ has a least upper bound $\sqcup X^\sharp$ and $\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$ has a least upper bound $\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$.

– Let $d^\sharp$ be an element in $X^\sharp$.
  Since $X^\sharp$ has a least upper bound, we have by Def. 3.(1), $d^\sharp \sqsubseteq \sqcup X^\sharp$.
  By Prop. 3.(4), we have $\gamma(d) \subseteq \gamma(\sqcup X^\sharp)$.

  Since $\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$ has a least upper bound, and by Def. 3.(2), it follows that $\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\} \subseteq \gamma(\sqcup X^\sharp)$.

  Then, by Prop. 3.(4) and Prop. 3.(3), we have $\gamma(\alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\})) \subseteq \gamma(\alpha(\gamma(\sqcup X^\sharp)))$.
  But, by Prop. 3.(6), we have $\gamma(\alpha(\gamma(\sqcup X^\sharp))) = \gamma(\sqcup X^\sharp)$.
  Thus, it follows that: $\gamma(\alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\} \subseteq \gamma(\sqcup X^\sharp)$.

– Let $d^\sharp$ be an element in $X^\sharp$.

  By Prop. 3.(2), we have $d^\sharp \sqsubseteq \alpha(\gamma(d^\sharp))$.

  Since $\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$ has a least upper bound, and by Def. 3.(1), we have $\gamma(d^\sharp) \subseteq \cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}$.
  Thus by Prop. 3.(3), it follows that $\alpha(\gamma(d^\sharp)) \subseteq \alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\})$.
  By Def. 1.(3), it follows that $d^\sharp \subseteq \alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\})$.
  Since $X^\sharp$ has a least upper bound, and by Def. 3.(2), it follows that $\sqcup X^\sharp \sqsubseteq \alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\})$.
  By Prop. 3.(4), we get that $\gamma(\sqcup X^\sharp) \subseteq \gamma(\alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}))$.

By Def. 1.(2), we conclude that $\gamma(\sqcup X^\sharp) = \gamma(\alpha(\cup\{\gamma(d^\sharp) \mid d^\sharp \in X^\sharp\}))$.

□

Lemma 2. *Let:*

1. *$(D, \subseteq, \cup)$ and $(D^\sharp, \sqsubseteq, \sqcup)$ be chain-complete partial orders;*
2. *$D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ be a Galois connexion;*
3. *$\mathbb{F} \in D \to D$ be a monotonic mapping;*
4. *$\mathbb{F}^\sharp \in D^\sharp \to D^\sharp$ be mapping such that: $[\forall d^\sharp \in D^\sharp,\ \mathbb{F}(\gamma(d^\sharp)) \subseteq \gamma(\mathbb{F}^\sharp(d^\sharp))]$;*
5. *$x_0 \in D$ such that $x_0 \subseteq \mathbb{F}(x_0)$.*

   Then:

$$\alpha(x_0) \sqsubseteq \mathbb{F}^\sharp(\alpha(x_0)).$$

Proof. Let us show that $\alpha(x_0) \sqsubseteq \mathbb{F}^\sharp(\alpha(x_0))$.

We have: $x_0 \subseteq \mathbb{F}(x_0)$.
By Prop. 3.(1), we have: $x_0 \subseteq \gamma(\alpha(x_0))$.
Then, since $\mathbb{F}$ is monotonic, it follows that $\mathbb{F}(x_0) \subseteq \mathbb{F}(\gamma(\alpha(x_0)))$.
By hypothesis, $\mathbb{F}(\gamma(\alpha(x_0))) \subseteq \gamma(\mathbb{F}^\sharp(\alpha(x_0)))$.
Thus, $x_0 \subseteq \gamma(\mathbb{F}^\sharp(\alpha(x_0)))$. By Def. 9.(3), it follows that $\alpha(x_0) \subseteq \mathbb{F}^\sharp(\alpha(x_0))$.

□

Theorem 1 (soundness). *Let:*

1. $(D, \subseteq, \cup)$ *and* $(D^\sharp, \sqsubseteq, \sqcup)$ *be chain-complete partial orders;*
2. $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^\sharp$ *be a Galois connexion;*
3. $\mathbb{F} \in D \to D$ *and* $\mathbb{F}^\sharp \in D^\sharp \to D^\sharp$ *be monotonic mappings such that:* $[\forall d^\sharp \in D^\sharp,\ \mathbb{F}(\gamma(d^\sharp)) \subseteq \gamma(\mathbb{F}^\sharp(d^\sharp))]$*;*
4. $x_0 \in D$ *be an element such that:* $x_0 \subseteq \mathbb{F}(x_0)$*.*

Then, both $lfp_{x_0}\mathbb{F}$ *and* $lfp_{\alpha(x_0)}\mathbb{F}^\sharp$ *exist, and moreover:*

$$lfp_{x_0}\mathbb{F} \subseteq \gamma(lfp_{\alpha(x_0)}\mathbb{F}^\sharp).$$

Proof. We assume that the hypotheses of The. 1 are satisfied.

1. We have $x_0 \subseteq \mathbb{F}(x_0)$ and $\mathbb{F}$ is monotonic.
   Thus, by Lem. 1, $\mathbb{F}$ has a least fix-point greater than x_0.
   Moreover, by Rem. 1, there exists an ordinal o such that $lfp_{x_0}\mathbb{F} = \mathbb{F}^o(x_0)$.

2. By Lem. 2, $\alpha(x_0) \subseteq \mathbb{F}^\sharp(\alpha(x_0))$.

   Thus, by Lem. 1, $\mathbb{F}^\sharp$ has a least fix-point greater than x_0.
   Moreover, by Rem. 1, there exists an ordinal $o^\sharp$ such that $lfp_{\alpha(x_0)}\mathbb{F}^\sharp = \mathbb{F}^{\sharp o^\sharp}(\alpha(x_0))$.

3. We consider an ordinal β such that $o \leq \beta$ and $o^\sharp \leq \beta$.
   We have: $lfp_{x_0}\mathbb{F} = \mathbb{F}^\beta(x_0)$ and $lfp_{\alpha(x_0)}\mathbb{F}^\sharp = \mathbb{F}^{\sharp\beta}(\alpha(x_0))$.
   We show by transfinite induction that for any ordinal o, $\mathbb{F}^o(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.

   - By Def. 8, we have $\mathbb{F}^0(x_0) = x_0$ and $\mathbb{F}^{\sharp 0}(\alpha(x_0)) = \alpha(x_0)$.
     By Prop. 3.(1), we have $x_0 \subseteq \gamma(\alpha(x_0))$.
     Thus, $\mathbb{F}^0(x_0) \subseteq \gamma(\mathbb{F}^{\sharp 0}(\alpha(x_0)))$.

   - We consider an ordinal o such that $\mathbb{F}^o(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
     By Def. 8, we have: $\mathbb{F}^{o+1}(x_0) = \mathbb{F}(\mathbb{F}^o(x_0))$.
     Since $\mathbb{F}$ is monotonic, we have: $\mathbb{F}(\mathbb{F}^o(x_0)) \subseteq \mathbb{F}(\gamma(\mathbb{F}^{\sharp o}(\alpha(x_0))))$.
     By hypothesis, $\mathbb{F}(\gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))) \subseteq \gamma(\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))))$.
     Then, by Def. 8, we have: $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) = \mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
     And by extensionality, $\gamma(\mathbb{F}^{\sharp o+1}(\alpha(x_0))) = \gamma(\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))))$.
     Thus: $\mathbb{F}^{o+1}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o+1}(\alpha(x_0)))$.

– We consider an ordinal o such that for any ordinal $\beta < o$ we have: $\mathbb{F}^{\beta}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0)))$.
By Def. 8, we have: $\mathbb{F}^{o}(x_0) = \cup\{\mathbb{F}^{\beta}(x_0) \mid \beta < o\}$.
Thus, by Def. 3.(1), we get that, for any ordinal β such that $\beta < o$, $\mathbb{F}^{o}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0)))$.
Thus, since $\{\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}$ is a chain, by Def. 6, and by Def. 3.(2), it follows that: $\mathbb{F}^{o}(x_0) \subseteq \cup\{\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}$.

For any ordinal β such that $\beta < o$,
by Def. 3.(1), we have: $\mathbb{F}^{\sharp\beta}(\alpha(x_0)) \subseteq \sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}$;
then by Prop. 3.(4), we get that: $\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \subseteq \gamma(\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\})$.
Then by Def. 3.(2), it follows that $\cup\{\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\} \subseteq \gamma(\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\})$;

By Def. 8, $\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\} = \mathbb{F}^{\sharp o}(\alpha(x_0))$.
Thus, by extensionality, $\gamma(\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}) = \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
It follows that: $\mathbb{F}^{o}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.

□

Theorem 2. *We suppose that:*

1. $(D, \subseteq)$ *be a partial order;*
2. $(D^{\sharp}, \sqsubseteq, \sqcup)$ *be chain-complete partial order;*
3. $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} D^{\sharp}$ *be a Galois connexion;*
4. $\mathbb{F} \in D \to D$ *and* $\mathbb{F}^{\sharp} \in D^{\sharp} \to D^{\sharp}$ *are monotonic;*
5. $\forall d^{\sharp} \in D^{\sharp}$, $\mathbb{F}(\gamma(d^{\sharp})) \subseteq \gamma(\mathbb{F}^{\sharp}(d^{\sharp}))$*;*
6. $x_0, inv \in D$ *such that:*
   – $x_0 \subseteq \mathbb{F}(x_0) \subseteq \mathbb{F}(inv) \subseteq inv$,
   – $inv = \gamma(\alpha(inv))$,
   – *and* $\alpha(\mathbb{F}(\gamma(\alpha(inv)))) = \mathbb{F}^{\sharp}(\alpha(inv))$*;*

Then, $lfp_{\alpha(x_0)}\mathbb{F}^{\sharp}$ *exists and* $\gamma(lfp_{\alpha(x_0)}\mathbb{F}^{\sharp}) \subseteq inv$*.*

Proof. Let us show this result.

– By Lem. 2, $\alpha(x_0) \subseteq \mathbb{F}^{\sharp}(\alpha(x_0))$.

Thus, by Lem. 1, $\mathbb{F}^{\sharp}$ has a least fix-point greater than x_0.
Moreover, by Rem. 1, there exists an ordinal $o^{\sharp}$ such that $lfp_{\alpha(x_0)}\mathbb{F}^{\sharp} = \mathbb{F}^{\sharp o^{\sharp}}(\alpha(x_0))$.

– Let us show by induction over $o^{\sharp}$ that $\mathbb{F}^{\sharp o^{\sharp}}(\alpha(x_0)) \sqsubseteq \alpha(inv)$.

- By Def. 8, we have $\mathbb{F}^{\sharp 0}(\alpha(x_0)) = \alpha(x_0)$.
Thus, by Def. 1.(1), $\alpha(x_0) \sqsubseteq \alpha(x_0)$.
So, $\mathbb{F}^{\sharp 0}(\alpha(x_0)) \sqsubseteq \alpha(x_0)$.

By hypothesis, $x_0 \subseteq inv$.
By Prop. 3.(3), we get that $\alpha(x_0) \sqsubseteq \alpha(inv)$.

Thus, by Def. 1.(3), it follows that $\mathbb{F}^{\sharp 0}(\alpha(x_0)) \sqsubseteq \alpha(inv)$.

- Let o be an ordinal such that $\mathbb{F}^{\sharp o}(\alpha(x_0)) \sqsubseteq \alpha(inv)$.

  Since $\mathbb{F}^\sharp$ is monotonic, we have $\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))) \sqsubseteq \mathbb{F}^\sharp(\alpha(inv))$.
  By Def. 8, $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) = \mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
  By hypothesis, $\alpha(\mathbb{F}(\gamma(\alpha(inv)))) = \mathbb{F}^\sharp(\alpha(inv))$.
  Thus, $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq \alpha(\mathbb{F}(\gamma(\alpha(inv))))$.

  By hypothesis, $\gamma(\alpha(inv)) = inv$.
  Thus, by extensionality, $\mathbb{F}(\gamma(\alpha(inv))) = \mathbb{F}(inv)$.
  By hypothesis, $\mathbb{F}(inv) \subseteq inv$.
  Thus, $\mathbb{F}(\gamma(\alpha(inv))) \subseteq inv$.
  By Prop. 3.(3), $\alpha(\mathbb{F}(\gamma(\alpha(inv)))) \subseteq \alpha(inv)$.

  Thus, by Def. 1.(3), $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq \alpha(inv)$.

- Let o be an ordinal such that for any ordinal $\beta < o$, we have $\mathbb{F}^{\sharp \beta}(\alpha(x_0)) \sqsubseteq \alpha(inv)$.

  By Def. 3.(2), $\sqcup\{\mathbb{F}^{\sharp \beta}(\alpha(x_0)) \mid \beta < o\} \sqsubseteq \alpha(inv)$.
  By Def. 8, $\mathbb{F}^{\sharp o}(\alpha(x_0)) = \sqcup\{\mathbb{F}^{\sharp \beta}(\alpha(x_0)) \mid \beta < o\}$.
  Thus, $\mathbb{F}^{\sharp o}(\alpha(x_0)) \sqsubseteq \alpha(inv)$.

  Thus, $lfp_{\alpha(x_0)}\mathbb{F}^\sharp \sqsubseteq \alpha(inv)$.

– We have seen that $lfp_{\alpha(x_0)}\mathbb{F}^\sharp \sqsubseteq \alpha(inv)$.
  By Prop. 3.(4), we have: $\gamma(lfp_{\alpha(x_0)}\mathbb{F}^\sharp) \subseteq \gamma(\alpha(inv))$.
  By hypothesis, $\gamma(\alpha(inv)) = inv$.
  Thus, $\gamma(lfp_{\alpha(x_0)}\mathbb{F}^\sharp) \subseteq inv$.

□

Theorem 3. *We suppose that:*

1. $(D, \subseteq, \cup)$ *and* $(D^\sharp, \sqsubseteq, \sqcup)$ *are chain-complete partial orders;*
2. $(D, \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (D^\sharp, \sqsubseteq)$ *is a Galois connexion;*
3. $\mathbb{F} : D \to D$ *is a monotonic map;*
4. x_0 *is a concrete element such that* $x_0 \subseteq \mathbb{F}(x_0)$*;*
5. $\mathbb{F} \circ \gamma \mathrel{\dot{\subseteq}} \gamma \circ \mathbb{F}^\sharp$*;*
6. $\mathbb{F}^\sharp \circ \alpha = \alpha \circ \mathbb{F} \circ \gamma \circ \alpha$.

Then:

– $lfp_{x_0}\mathbb{F}$ *and* $lfp_{\alpha(x_0)}\mathbb{F}^\sharp$ *exist;*
– $lfp_{x_0}\mathbb{F} \in \gamma(D^\sharp) \iff lfp_{x_0}\mathbb{F} = \gamma(lfp_{\alpha(x_0)}\mathbb{F}^\sharp)$.

Proof. We assume that the hypotheses of The. 3 are satisfied.

1. We have $x_0 \subseteq \mathbb{F}(x_0)$ and $\mathbb{F}$ is monotonic.
   Thus, by Lem. 1, $\mathbb{F}$ has a least fix-point greater than x_0.
   Moreover, by Rem. 1, there exists an ordinal $o^\bullet$ such that $lfp_{x_0}\mathbb{F} = \mathbb{F}^{o^\bullet}(x_0)$.

2. Let us show, by induction over the ordinal o_0, that there exists a unique collection of elements $(X^\sharp_o)_{o<o_0}$ such that for any ordinal $o < o_0$:
   - i.
$$\begin{cases} X^\sharp_o = \alpha(x_0) & \text{whenever } o = 0 \\ X^\sharp_o = \mathbb{F}^\sharp(X^\sharp_{o-1}) & \text{whenever } o \text{ is a succesor ordinal} \\ X^\sharp_o = \sqcup\{X^\sharp_\beta \mid \beta < o\} & \text{otherwise.} \end{cases}$$
   - ii. for any ordinal $o < o_0$, there exists an element $d \in D$ such that $X^\sharp_o = \alpha(d)$,
   - iii. $(X^\sharp_o)_{o<o_0}$ is increasing,
   - iv. and for any ordinal $o < o_0$, $X^\sharp_o \sqsubseteq \mathbb{F}^\sharp(X^\sharp_o)$.

   (a) i. There exists a unique element $X^\sharp_0$ such that $X^\sharp_0 = \alpha(x_0)$.

   ii. $\alpha(x_0) = \alpha(x_0)$.

   iii. $(\alpha(x_0))$ is an increasing family (of one element).

   iv. By hypothesis, $x_0 \subseteq \mathbb{F}(x_0)$.

   By Prop. 3.(1), $x_0 \subseteq \gamma(\alpha(x_0))$.
   Since $\mathbb{F}$ is monotonic, $\mathbb{F}(x_0) \subseteq \mathbb{F}(\gamma(\alpha(x_0)))$.

   Thus, by Def. 1.(3), it follows that $x_0 \subseteq \mathbb{F}(\gamma(\alpha(x_0)))$.

   By Prop. 3.(3), we get that: $\alpha(x_0) \sqsubseteq \alpha(\mathbb{F}(\gamma(\alpha(x_0))))$.

   By hypothesis, $\mathbb{F}^\sharp(\alpha(x_0)) = \alpha(\mathbb{F}(\gamma(\alpha(x_0))))$.
   Thus, $\alpha(x_0) \sqsubseteq \mathbb{F}^\sharp(\alpha(x_0))$.

   (b) Let o_0 be an ordinal.
   We assume that there exists a unique family $(X^\sharp_o)_{o\le o_0}$ such that the equations (a) are satisfied.
   We also assume that there exists a family of elements $(X_o)_{o\le o_0}$ such that for any ordinal, $\alpha(X_o) = X^\sharp_o$, that $(X^\sharp_o)_{o\le o_0}$ is increasing and that for any ordinal $o \le o_0$, $X^\sharp_o \sqsubseteq \mathbb{F}^\sharp(X_o)$.
   We define $Y^\sharp_o = X^\sharp_o$ whenever $o \le o_0$ and $Y^\sharp_{o_0+1} = \mathbb{F}^\sharp(X^\sharp_{o_0})$.

   i. The family $(Y^\sharp_o)_{o\le o_0+1}$ satisfies the equations (a).

   ii. Now we consider a family $(Z^\sharp_o)_{o\le o_0+1}$ of elements in $D^\sharp$ which satisfies the equations (a).

   By induction hypotheses (uniqueness), we have $Z^\sharp_o = Y^\sharp_o$ for any ordinal $o \le o_0$.

   Moreover, since $(Z^\sharp_o)_{o\le o_0+1}$ satisfies the equations (a), we have $Z^\sharp_{o_0+1} = \mathbb{F}^\sharp(Z^\sharp_{o_0})$.
   Since $Z^\sharp_{o_0} = Y^\sharp_{o_0}$, it follows by extensionality that $\mathbb{F}^\sharp(Z^\sharp_{o_0}) = \mathbb{F}^\sharp(Y^\sharp_{o_0})$.
   Moreover, we have: $\mathbb{F}^\sharp(Y^\sharp_{o_0}) = Y^\sharp_{o_0+1}$.
   Thus $Z^\sharp_{o+1} = Y^\sharp_{o+1}$.

   It follows that $(Z^\sharp_o)_{o\le o_0+1} = (Y^\sharp_o)_{o\le o_0+1}$.

iii. By induction hypotheses, there exists a family $(X_o)_{o \leq o_0}$ such that $(Y_o^\sharp)_{o \leq o_0} = (\alpha(X_o))_{o \leq o_0}$.

It follows that $Y_{o_0}^\sharp = \alpha(X_o)$.
By extensionality, $\mathbb{F}^\sharp(Y_{o_0}^\sharp) = \mathbb{F}^\sharp(\alpha(X_o))$.
By hypothesis, $Y_{o_0+1}^\sharp = \mathbb{F}^\sharp(Y_{o_0}^\sharp)$.
By hypothesis, $\mathbb{F}^\sharp(\alpha(X_o)) = \alpha(\mathbb{F}(\gamma(\alpha(X_0))))$.
Thus, $Y_{o_0+1}^\sharp = \alpha(\mathbb{F}(\gamma(\alpha(X_0))))$.

We define $X_{o_0+1} = \mathbb{F}(\gamma(\alpha(X_0)))$.
We have $Y_{o_0+1}^\sharp = \alpha(X_{o_0+1})$.

Since $(Y_o^\sharp)_{o \leq o_0} = (\alpha(X_o))_{o \leq o_0}$, it follows that $(Y_o^\sharp)_{o \leq o_0+1} = (\alpha(X_o))_{o \leq o_0+1}$.

iv. By induction hypotheses, $(Y_o^\sharp)_{o \leq o_0}$ is increasing.
By induction hypotheses again $Y_{o_0}^\sharp \sqsubseteq \mathbb{F}^\sharp(Y_{o_0}^\sharp)$.
Since $Y_{o_0+1}^\sharp = \mathbb{F}^\sharp(Y_{o_0}^\sharp)$, it follows that $Y_{o_0}^\sharp \sqsubseteq Y_{o_0+1}^\sharp$.
Thus $(Y_o^\sharp)_{o \leq o_0+1}$ is increasing.

v. By induction hypotheses, for any $o \leq o_0$, $Y_o^\sharp \sqsubseteq \mathbb{F}^\sharp(Y_o^\sharp)$.

Moreover, $Y_{o_0}^\sharp \sqsubseteq Y_{o_0+1}^\sharp$.
Since $Y_{o_0}^\sharp = \alpha(X_{o_0})$ and $Y_{o_0+1}^\sharp = \alpha(X_{o_0+1})$, it follows that $\alpha(X_{o_0}) \sqsubseteq \alpha(X_{o_0+1})$.
By Prop. 3.(4), since $\mathbb{F}$ is monotonic, and by Prop. 3.(3), $\alpha(\mathbb{F}(\gamma(\alpha(X_{o_0})))) \sqsubseteq \alpha(\mathbb{F}(\gamma(\alpha(X_{o_0+1}))))$.
By hypothesis, $\alpha \circ \mathbb{F} \circ \gamma \circ \alpha = \mathbb{F}^\sharp \circ \alpha$, thus $\mathbb{F}^\sharp(\alpha(X_{o_0})) \sqsubseteq \mathbb{F}^\sharp(\alpha(X_{o_0+1}^\sharp))$.
Since, $Y_{o_0}^\sharp = \alpha(X_{o_0})$ and $Y_{o_0+1}^\sharp = \alpha(X_{o_0+1})$, it follows that $\mathbb{F}^\sharp(Y_{o_0}^\sharp) \sqsubseteq \mathbb{F}^\sharp(\mathbb{F}^\sharp(Y_{o_0}^\sharp))$.
By induction hypothesis, $Y_{o_0+1}^\sharp = \mathbb{F}^\sharp(Y_{o_0}^\sharp)$.
Thus, $Y_{o_0+1}^\sharp \sqsubseteq \mathbb{F}^\sharp(Y_{o_0+1}^\sharp)$.

Thus, we denote by $\mathbb{F}^{\sharp o}(\alpha(x_0))$ the unique collection which satisfies the equations (2).

(c) Let us show that $\mathbb{F}^\sharp$ has a fix-point.

The collection $(\mathbb{F}^{\sharp o}(\alpha(x_0)))$ which is indexed over the ordinals is increasing.
Since $D^\sharp$ is a set, it follows that there exists an ordinal $o^\sharp$, such that $\mathbb{F}^{\sharp o^\sharp}(\alpha(x_0)) = \mathbb{F}^{\sharp o^\sharp+1}(\alpha(x_0))$.
Since $(\mathbb{F}^{\sharp o}(\alpha(x_0)))$ satisfied equation (2), it follows that $\mathbb{F}^\sharp(\mathbb{F}^{\sharp o^\sharp}(\alpha(x_0))) = \mathbb{F}^{\sharp o^\sharp}(\alpha(x_0))$.
Moreover, we have already proven that $\alpha(x_0) \sqsubseteq \mathbb{F}^{\sharp o^\sharp}(\alpha(x_0))$.

(d) Let is show that $\mathbb{F}^{\sharp o}(\alpha(x_0))$ is the least fix-point of $\mathbb{F}^\sharp$.

Consider another fix-point $y^\sharp \in D^\sharp$ such that $\alpha(x_0) \sqsubseteq y^\sharp$.
We have $y^\sharp = \mathbb{F}^\sharp(y^\sharp)$.

Let us show by transfinite induction that $\mathbb{F}^{\sharp o^\sharp}(\alpha(x_0)) \sqsubseteq y^\sharp$.
- We have, by hypothesis, $\alpha(x_0) \sqsubseteq y^\sharp$.
  Since, $\mathbb{F}^{\sharp 0}(\alpha(x_0)) = \alpha(x_0)$, it follows that $\mathbb{F}^{\sharp 0}(\alpha(x_0)) \sqsubseteq \alpha(y)$.

– Let us consider an ordinal o such that $\mathbb{F}^{\sharp o}(\alpha(x_0)) \sqsubseteq y^\sharp$.
We know that $\mathbb{F}^{\sharp o}(\alpha(x_0)) \in \alpha(D)$.
Thus there exists an element $x \in D$ such that $\mathbb{F}^{\sharp o}(\alpha(x_0)) = \alpha(x)$.
Then, $\alpha(x) \sqsubseteq y^\sharp$.
By Prop. 3.(4), since $\mathbb{F}$ is monotonic, and by Prop. 3.(3), $\alpha(\mathbb{F}(\gamma(\alpha(x)))) \sqsubseteq \alpha(\mathbb{F}(\gamma(y^\sharp)))$.

By hypothesis, $\mathbb{F}(\gamma(y^\sharp)) \subseteq \gamma(\mathbb{F}^\sharp(y^\sharp))$.
By Prop. 3.(3), we get that $\alpha(\mathbb{F}(\gamma(y^\sharp))) \sqsubseteq \alpha(\gamma(\mathbb{F}^\sharp(y^\sharp)))$.

By Prop. 3.(2), $\alpha(\gamma(\mathbb{F}^\sharp(y^\sharp))) \sqsubseteq \mathbb{F}^\sharp(y^\sharp)$.

Thus, by Def. 1.(3), $\alpha(\mathbb{F}(\gamma(\alpha(x)))) \sqsubseteq \mathbb{F}^\sharp(y^\sharp)$.

By hypothesis, $\alpha(\mathbb{F}(\gamma(\alpha(x)))) = \mathbb{F}^\sharp(\alpha(x))$.
Moreover, $\alpha(x) = \mathbb{F}^{\sharp o}(\alpha(x_0))$.
Thus, by extensionality, $\alpha(\mathbb{F}(\gamma(\mathbb{F}^{\sharp o}(\alpha(x_0))))) = \mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
But by definition, $\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))) = \mathbb{F}^{\sharp o+1}(\alpha(x_0))$.
Thus, $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq \mathbb{F}^\sharp(y^\sharp)$.

By hypothesis, $\mathbb{F}^\sharp(y^\sharp) = y^\sharp$.
Thus $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq y^\sharp$.

– Let us consider an ordinal o such that for any $\beta < o$, we have $\mathbb{F}^{\sharp\beta}(x_0) \sqsubseteq y$.
By Def. 3.(2), we get that $\sqcup\{\mathbb{F}^{\sharp\beta}(x_0) \mid \beta < o\} \sqsubseteq y$.
By hypothesis, $\mathbb{F}^{\sharp o}(x_0) = \sqcup\{\mathbb{F}^{\sharp\beta}(x_0) \mid \beta < o\}$.
Thus, $\mathbb{F}^{\sharp o}(x_0) \subseteq y$.

Thus, $\mathbb{F}^{\sharp o^\sharp}$ is the least fix-point of $\mathbb{F}^\sharp$ which is bigger than $\alpha(x_0)$.

(e) Let us prove that $lfp_{x_0}\mathbb{F} \subseteq \gamma(lfp_{\alpha(x_0)}\mathbb{F}^\sharp)$.

We consider an ordinal β such that $o^\bullet \leq \beta$ and $o^\sharp \leq \beta$.

We have: $lfp_{x_0}\mathbb{F} = \mathbb{F}^\beta(x_0)$ and $lfp_{\alpha(x_0)}\mathbb{F}^\sharp = \mathbb{F}^{\sharp\beta}(\alpha(x_0))$.
We show by transfinite induction that for any ordinal o, $\mathbb{F}^o(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.

– By hypotheses, we have $\mathbb{F}^0(x_0) = x_0$ and $\mathbb{F}^{\sharp 0}(\alpha(x_0)) = \alpha(x_0)$.
By Prop. 3.(1), we have $x_0 \subseteq \gamma(\alpha(x_0))$.
Thus, $\mathbb{F}^0(x_0) \subseteq \gamma(\mathbb{F}^{\sharp 0}(\alpha(x_0)))$.

– We consider an ordinal o such that $\mathbb{F}^o(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
By Def. 8, we have: $\mathbb{F}^{o+1}(x_0) = \mathbb{F}(\mathbb{F}^o(x_0))$.
Since $\mathbb{F}$ is monotonic, we have: $\mathbb{F}(\mathbb{F}^o(x_0)) \subseteq \mathbb{F}(\gamma(\mathbb{F}^{\sharp o}(\alpha(x_0))))$.
By hypothesis, $\mathbb{F}(\gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))) \subseteq \gamma(\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))))$.
Then, by hypothesis, we have: $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) = \mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
And by extensionality, $\gamma(\mathbb{F}^{\sharp o+1}(\alpha(x_0))) = \gamma(\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))))$.
Thus: $\mathbb{F}^{o+1}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o+1}(\alpha(x_0)))$.

– We consider an ordinal o such that for any ordinal $\beta < o$ we have: $\mathbb{F}^{\beta}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0)))$.
By Def. 8, we have: $\mathbb{F}^{o}(x_0) = \cup\{\mathbb{F}^{\beta}(x_0) \mid \beta < o\}$.
Thus, by Def. 3.(1), we get that, for any ordinal β such that $\beta < o$, $\mathbb{F}^{o}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0)))$.
Thus, since $\{\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}$ is a chain, by Def. 6, and by Def. 3.(2), it follows that: $\mathbb{F}^{o}(x_0) \subseteq \cup\{\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}$.

For any ordinal β such that $\beta < o$,
by Def. 3.(1), we have: $\mathbb{F}^{\sharp\beta}(\alpha(x_0)) \sqsubseteq \sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}$;
then by Prop. 3.(4), we get that: $\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \subseteq \gamma(\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\})$.
Then by Def. 3.(2), it follows that $\cup\{\gamma(\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\} \subseteq \gamma(\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\})$;

By hypothesis, $\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\} = \mathbb{F}^{\sharp o}(\alpha(x_0))$.
Thus, by extensionality, $\gamma(\sqcup\{\mathbb{F}^{\sharp\beta}(\alpha(x_0))) \mid \beta < o\}) = \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
It follows that: $\mathbb{F}^{o}(x_0) \subseteq \gamma(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.

Thus, $\mathit{lfp}_{x_0}\mathbb{F} \subseteq \gamma(\mathit{lfp}_{\mathbb{F}^\sharp}\alpha(x_0))$.

(f) Let us prove that: $\mathit{lfp}_{x_0}\mathbb{F} \in \gamma(D^\sharp) \iff \mathit{lfp}_{x_0}\mathbb{F} = \gamma(\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp)$.

i. We assume that $\mathit{lfp}_{x_0}\mathbb{F} = \gamma(\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp)$.

Then, by definition of $\gamma(D^\sharp)$, $\mathit{lfp}_{x_0}\mathbb{F} \in \gamma(D^\sharp)$.

ii. Now we assume that $\mathit{lfp}_{x_0}\mathbb{F} \in \gamma(D^\sharp)$.

A. We know that: $\mathit{lfp}_{x_0}\mathbb{F} \subseteq \gamma(\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp)$.

B. Let us prove that: $\gamma(\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp) \subseteq \mathit{lfp}_{x_0}\mathbb{F}$.

We propose to prove by induction over the ordinals that $\mathbb{F}^{\sharp\beta}(\alpha(x_0)) \sqsubseteq \alpha(\mathit{lfp}_{x_0}\mathbb{F})$.

$\star$ We have $x_0 \subseteq \mathit{lfp}_{x_0}\mathbb{F}$.
By Prop. 3.(3), $\alpha(x_0) \sqsubseteq \alpha(\mathit{lfp}_{x_0}\mathbb{F})$.

$\star$ Let us assume that there exists an ordinal o, such that $\mathbb{F}^{\sharp o}(\alpha(x_0)) \sqsubseteq \alpha(\mathit{lfp}_{x_0}\mathbb{F})$.
There exists $x \in D$, such that $\mathbb{F}^{\sharp o}(\alpha(x_0)) = \alpha(x)$.
Thus $\alpha(x) \sqsubseteq \alpha(\mathit{lfp}_{x_0}(x_0))$.
By Prop. 4, since $\mathbb{F}$ is monotonic, and by Prop. 3, $\alpha(\mathbb{F}(\gamma(\alpha(x)))) \sqsubseteq \alpha(\mathbb{F}(\gamma(\alpha(\mathit{lfp}_{x_0}(\mathbb{F})))))$.

By hypothesis, $\alpha(\mathbb{F}(\gamma(\alpha(x)))) = \mathbb{F}^\sharp(\alpha(x))$.
Since $\mathbb{F}^{\sharp o}(\alpha(x_0)) = \alpha(x)$, by extensionality, we get that: $\mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0))) = \mathbb{F}^\sharp(\alpha(x))$.
Since by equations (2), it follows that $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) = \mathbb{F}^\sharp(\mathbb{F}^{\sharp o}(\alpha(x_0)))$.
Thus, $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq \alpha(\mathbb{F}(\gamma(\alpha(\mathit{lfp}_{x_0}\mathbb{F}))))$.

By Prop. 3.(1), $\gamma(\alpha(\mathit{lfp}_{x_0}\mathbb{F})) \sqsubseteq \mathit{lfp}_{x_0}\mathbb{F}$.
Since $\mathbb{F}$ is monotonic, $\mathbb{F}(\gamma(\alpha(\mathit{lfp}_{x_0}\mathbb{F}))) \sqsubseteq \mathbb{F}(\mathit{lfp}_{x_0}\mathbb{F})$.
But $\mathbb{F}(\mathit{lfp}_{x_0}\mathbb{F}) = \mathit{lfp}_{x_0}\mathbb{F}$.

Thus, $\mathbb{F}(\gamma(\alpha(lfp_{x_0}\mathbb{F}))) \sqsubseteq lfp_{x_0}\mathbb{F}$.
By Prop. 3.(3), $\alpha(\mathbb{F}(\gamma(\alpha(lfp_{x_0}\mathbb{F})))) \sqsubseteq \alpha(lfp_{x_0}\mathbb{F})$.

By Def. 1.(3), it follows that: $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq \alpha(lfp_{x_0}\mathbb{F})$.

$\star$ Let us assume that there exists an ordinal o_0, such that for any ordinal $o < o_0$, $\mathbb{F}^{\sharp o}(\alpha(x_0)) \sqsubseteq \alpha(lfp_{x_0}\mathbb{F})$.

Since $(\mathbb{F}^{\sharp o}(\alpha(x_0)))$ is a chain, $\sqcup\{\mathbb{F}^{\sharp o}(\alpha(x_0)) \mid o < o_0\}$ exists.
By Def. 3.(2), $\sqcup\{\mathbb{F}^{\sharp o}(\alpha(x_0)) \mid o < o_0\} \sqsubseteq \alpha(lfp_{x_0}\mathbb{F})$.
By equations (2), we have $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) = \sqcup\{\mathbb{F}^{\sharp o}(\alpha(x_0)) \mid o < o_0\}$.
Thus, $\mathbb{F}^{\sharp o+1}(\alpha(x_0)) \sqsubseteq \alpha(lfp_{x_0}\mathbb{F})$.

We have proved that $lfp_{\alpha(x_0)}\mathbb{F}^{\sharp} \sqsubseteq \alpha(lfp_{x_0}\mathbb{F})$.
By Prop. 3.(4), $\gamma(lfp_{\alpha(x_0)}\mathbb{F}^{\sharp}) \subseteq \gamma(\alpha(lfp_{x_0}\mathbb{F}))$.
But since, $lfp_{x_0}\mathbb{F} \in \gamma(D^{\sharp})$, there exists $x \in D$, such that $\gamma(x) = lfp_{x_0}\mathbb{F}$.
By extensionality, $\gamma(\alpha(\gamma(x))) = \gamma(\alpha(lfp_{x_0}\mathbb{F}))$.
By Prop. 3.(6), $\gamma(x) = \gamma(\alpha(\gamma(x)))$.
Thus $\gamma(\alpha(lfp_{x_0}\mathbb{F})) = lfp_{x_0}\mathbb{F}$.
It follows that: $\gamma(lfp_{\alpha(x_0)}\mathbb{F}^{\sharp}) \subseteq lfp_{x_0}\mathbb{F}$.

Thus $lfp_{x_0}\mathbb{F} = \gamma(lfp_{\alpha(x_0)}\mathbb{F}^{\sharp})$.

□

Corollary 1 (relative completeness). *We suppose that:*

1. $(D, \subseteq, \cup)$ *and* $(D^{\sharp}, \sqsubseteq, \sqcup)$ *are chain-complete partial orders;*
2. $(D, \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (D^{\sharp}, \sqsubseteq)$ *is a Galois connexion;*
3. *for any chain* $X^{\sharp} \subseteq D^{\sharp}$, $\cup(\gamma(X^{\sharp})) \in \gamma(D^{\sharp})$*;*
4. $\mathbb{F} : D \to D$ *is a monotonic map;*
5. x_0 *is a concrete element such that* $x_0 \subseteq \mathbb{F}(x_0)$*;*
6. $\alpha \circ \mathbb{F} \circ \gamma = \mathbb{F}^{\sharp}$*;*
7. $x_0 \in \gamma(D^{\sharp})$*;*
8. $\mathbb{F}(\gamma(D^{\sharp})) \subseteq \gamma(D^{\sharp})$.

Then, both $lfp_{x_0}\mathbb{F}$ *and* $lfp_{\alpha(x_0)}\mathbb{F}^{\sharp}$ *exist, and moreover:*

$$lfp_{x_0}\mathbb{F} = \gamma(lfp_{\alpha(x_0)}\mathbb{F}^{\sharp}).$$

Proof. We assume that the hypotheses of The. 1 are satisfied.

- By hypothesis 4, $\mathbb{F}$ is monotonic.
  By hypothesis 5, $x_0 \subseteq \mathbb{F}(x_0)$.
  Thus, by Lem. 1, $\mathbb{F}$ has a least fix-point greater than x_0.
  Moreover, by Rem. 1, there exists an ordinal o such that $lfp_{x_0}\mathbb{F} = \mathbb{F}^{o}(x_0)$.

- Let us show by induction over the ordinal o that $\mathbb{F}^{o}(x_0) \in \gamma(D^{\sharp})$.

- We have $\mathbb{F}^0(x_0) = x_0$.

  By hypothesis 7, $x_0 \in \gamma(D^\sharp)$.

  Thus $\mathbb{F}^0(x_0) \in \gamma(D^\sharp)$.

- We assume that there exists an ordinal β such that $\mathbb{F}^\beta(x_0) \in \gamma(D^\sharp)$.

  By induction hypothesis, $\mathbb{F}^\beta(x_0) \in \gamma(D^\sharp)$.
  By hypothesis 8, $\mathbb{F}(\mathbb{F}^\beta(x_0)) \in \gamma(D^\sharp)$.
  Since $\mathbb{F}^{\beta+1}(x_0) = \mathbb{F}(\mathbb{F}^\beta(x_0))$.
  It follows that $\mathbb{F}^{\beta+1}(x_0) \in \gamma(D^\sharp)$.

- We assume that there exists an ordinal β such that for any ordinal $\beta' < \beta$, $\mathbb{F}^{\beta'}(x_0) \in \gamma(D^\sharp)$.

  We have $\mathbb{F}^\beta(x_0) = \cup\{\mathbb{F}^{\beta'} \mid \beta' < \beta\}$.
  By hypothesis 3, $\mathbb{F}^\beta(x_0) \in \gamma(D^\sharp)$.

Thus, since $\mathit{lfp}_{x_0}\mathbb{F} = \mathbb{F}^o(x_0)$, it follows that $\mathit{lfp}_{x_0}\mathbb{F} \in \gamma(D^\sharp)$.
All the hypotheses of The. 3 are satisfied.
Thus, $\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp$ exists.
Moreover, since $\mathit{lfp}_{x_0}\mathbb{F} \in \gamma(D^\sharp)$, it follows that: $\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp = \gamma(\mathit{lfp}_{\alpha(x_0)}\mathbb{F}^\sharp)$.

□

2 Site-graphs

Let $\mathbb{N}$ be a countable set of agent identifiers.
Let $\mathcal{A}$ be a finite set of agent types.
Let $\mathcal{S}$ be a finite set of site types.

Definition 10 (site-graphs). *A site-graph is a triple $(Ag, Site, Link)$ where:*

- *$Ag : \mathbb{N} \rightharpoonup \mathcal{A}$ is a partial map between $\mathbb{N}$ and $\mathcal{A}$ such that the subset of $\mathbb{N}$ of the elements i such that $Ag(i)$ is defined is finite;*
- *$Site \subseteq \mathbb{N} \times \mathcal{S}$ is a subset of $\mathbb{N} \times \mathcal{S}$ such that for any pair $(i, s) \in Site$, $Ag(i)$ is defined;*
- *$Link \subseteq Site^2$ is a relation over Site such that:*
  1. *for any site $a \in Site$, $(a, a) \notin Link$;*
  2. *for any pair $(a, b) \in Link$, we have $(b, a) \in Link$;*
  3. *for any sites $a, b, b' \in Site$, if both $(a, b) \in Link$ and $(a, b') \in Link$, then $b = b'$.*

  Whenever $(a, b) \in Link$, we say that there is a link between the site a and the site b.
  Whenever $a \in Site$, but there exists no $b \in Site$ such that $(a, b) \in Link$, we say that a is free.

Definition 11 (embeddings). *An embedding between two site-graphs $(Ag, Site, Link)$ and $(Ag', Site', Link')$ is given by a partial mapping $\phi : \mathbb{N} \rightharpoonup \mathbb{N}$, such that:*

1. *(agent mapping) For any $i \in \mathbb{N}$, $Ag(i)$ is defined if and only if $\phi(i)$ is defined;*
2. *(well-formedness) For any $i \in \mathbb{N}$, if $Ag(i)$ is defined, then $Ag'(\phi(i))$ is defined;*
3. *(into mapping) For any $i, i' \in \mathbb{N}$, if $\phi(i)$ and $\phi'(i)$ are defined, then $\phi(i) = \phi(i') \implies i = i'$;*
4. *(agent types) For any $i \in \mathbb{N}$, if $Ag(i)$ is defined, then $Ag(i) = Ag'(\phi(i))$;*

5. *(site types) For any site $(i,s) \in Site$, $(\phi(i),s) \in Site'$;*
6. *(free sites) For any pair $(i,s) \in Site$ such that for any $(i',s') \in Site$, $((i,s),(i',s')) \notin Link$, then for any $(i'',s'') \in Site'$, $((\phi(i),s),(i'',s'')) \notin Link$;*
7. *(links) For any link $((i,s),(i',s')) \in Link$, $((\phi(i),s),(\phi(i'),s')) \in Link'$.*

Definition 12 (automorphism). *An embedding between a site-graph and itself is called an automorphism.*

Definition 13 (paths). *Let $\mathcal{G} = (Ag, Site, Link)$ be a site-graph. We define a path of length $n > 0$ in the site-graph $\mathcal{G}$ a sequence $(i_k, s_k)_{0 \leq k \leq 2\times n-1}$ of $2 \times n$ pairs of sites in Site such that:*

1. *For any j such that $0 \leq j < n$, $((i_{2\times j}, s_{2\times j}),(i_{2\times j+1}, s_{2\times j+1})) \in Link$.*
2. *For any j such that $1 \leq j < n$, $i_{2\times j} = i_{2\times j-1}$ and $s_{2\times j} \neq s_{2\times j-1}$.*

Proposition 7 (sub-paths). *Let $\mathcal{G} = (Ag, Site, Link)$ be a site-graph and $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ be a path of length $n > 0$ in the site-graph $\mathcal{G}$. Let m, m' be two integers such that $0 \leq m < m' \leq n$, then, $(i_k, s_k)_{2\times m\leq k\leq 2\times m'-1}$ is a path in the site-graph $\mathcal{G}$.*

Proof. We have $m' - m > 0$.
For any integer k such that $2 \times m \leq k \leq 2 \times m' - 1$, we have by Def. 13, $(i_k, s_k) \in Site$.
Moreover,

1. for any integer k such that $m \leq k < m'$, by Def. 13.(1), $((i_{2\times k}, s_{2\times k}),(i_{2\times k+1}, s_{2\times k+1})) \in Link$;
2. for any integer k such that $m < k < m'$, by Def. 13.(2), $i_{2\times k} = i_{2\times k-1}$ and $s_{2\times k} \neq s_{2\times k-1}$.

By Def. 13, it follows that $(i_k, s_k)_{2\times m\leq k\leq 2\times m'-1}$ is a path in the site-graph $\mathcal{G}$.
□

Proposition 8 (path composition). *Let $\mathcal{G} = (Ag, Site, Link)$ be a site-graph and $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ and $(i'_k, s'_k)_{0\leq k\leq 2\times n'-1}$ be two paths of length $n > 0$ and $n' > 0$ in the site-graph $\mathcal{G}$ such that $i_{2\times n-1} = i'_0$ and $s_{2\times n-1} \neq s'_0$.*
Then, the sequence $(i''_k, s''_k)_{0\leq k\leq 2\times(n+n')-1}$ where:

$$\begin{cases} (i''_k, s''_k) = (i_k, s_k) & \textit{whenever } 0 \leq k \leq 2 \times n - 1 \\ (i''_k, s''_k) = (i'_{k-2\times n}, s'_{k-2\times n}) & \textit{whenever } 2 \times n \leq k \leq 2 \times (n + n') - 1 \end{cases}$$

is a path of length $n + n'$ in $\mathcal{G}$.

Proof. Let $\mathcal{G} = (Ag, Site, Link)$ be a site-graph and $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ and $(i'_k, s'_k)_{0\leq k\leq 2\times n'-1}$ be two paths of size $n > 0$ and $n' > 0$ in the site-graph $\mathcal{G}$ such that $i_{2\times n-1} = i'_0$ and $s_{2\times n-1} \neq s'_0$.
We have $2 \times (n + n') > 0$.
We consider the sequence $(i''_k, s''_k)_{0\leq k\leq 2\times(n+n')-1}$ which is defined as follows:

$$\begin{cases} (i''_k, s''_k) = (i_k, s_k) & \text{whenever } 0 \leq k \leq 2 \times n - 1 \\ (i''_k, s''_k) = (i'_{k-2\times n}, s'_{k-2\times n}) & \text{whenever } 2 \times n \leq k \leq 2 \times (n + n') - 1 \end{cases}$$

Let k be an integer such that $0 \leq k \leq 2 \times (n + n') - 1$.

- We assume that $k \leq 2 \times n - 1$.

  We have: $(i''_k, s''_k) = (i_k, s_k)$.
  Thus, by Def. 13, $(i_k, s_k) \in Site$.
  Thus $(i''_k, s''_k) \in Site$.

– We assume that $k > 2 \times n - 1$.

We have: $(i''_k, s''_k) = (i'_{k-2\times n}, s'_{k-2\times n})$.
Thus, by Def. 13, $(i'_{k-2\times n}, s'_{k-2\times n}) \in \mathit{Site}$.
Thus $(i''_k, s''_k) \in \mathit{Site}$.

– Let k be an integer such that $0 \leq k < n + n'$.

- We assume that $k < n$.

  We have $(i''_{2\times k}, s''_{2\times k}) = (i_{2\times k}, s_{2\times k})$ and $(i''_{2\times k+1}, s''_{2\times k+1}) = (i_{2\times k+1}, s_{2\times k+1})$.
  Since $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ is a path, by Def. 13.(1), $((i_{2\times k}, s_{2\times k}), (i_{2\times k+1}, s_{2\times k+1})) \in \mathit{Link}$.
  Thus, $((i''_{2\times k}, s''_{2\times k}), (i''_{2\times k+1}, s''_{2\times k+1})) \in \mathit{Link}$.

- We assume that $k \geq n$.

  We have $(i''_{2\times k}, s''_{2\times k}) = (i'_{2\times(k-n)}, s'_{2\times(k-n)})$ and $(i''_{2\times k+1}, s''_{2\times k+1}) = (i'_{2\times(k-n)+1}, s'_{2\times(k-n)+1})$.
  We know that the sequence $(i'_k, s'_k)_{0\leq k\leq 2\times n'-1}$ is a path.
  By Def. 13.(1), $((i'_{2\times(k-n)}, s'_{2\times(k-n)}), (i'_{2\times(k-n)+1}, s'_{2\times(k-n)+1})) \in \mathit{Link}$.
  Thus, $((i''_{2\times k}, s''_{2\times k}), (i''_{2\times k+1}, s''_{2\times k+1})) \in \mathit{Link}$.

– Let k be an integer such that $1 \leq k < n + n'$.

- We assume that $k < n$.

  We have $(i''_{2\times k}, s''_{2\times k}) = (i_{2\times k}, s_{2\times k})$ and $(i''_{2\times k-1}, s''_{2\times k-1}) = (i_{2\times k-1}, s_{2\times k-1})$.
  Since $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ is a path, by Def. 13.(2), $i_{2\times k} = i_{2\times k-1}$ and $s_{2\times k} \neq s_{2\times k-1}$.
  Thus, $i''_{2\times k} = i''_{2\times k-1}$ and $s''_{2\times k} \neq s''_{2\times k-1}$.

- We assume that $k = n$.

  We have $i''_{2\times k} = i'_0$, $i''_{2\times k-1} = i_{2\times n-1}$, $s''_{2\times k} = s'_0$, $s''_{2\times k-1} = s_{2\times n-1}$.
  By hypothesis, $i'_0 = i_{2\times n-1}$ and $s'_0 \neq s_{2\times n-1}$.
  Thus, $i''_{2\times k} = i''_{2\times k-1}$ and $s''_{2\times k} \neq s''_{2\times k-1}$.

- We assume that $k > n$.

  We have $(i''_{2\times k}, s''_{2\times k}) = (i'_{2\times(k-n)}, s'_{2\times(k-n)})$ and $(i''_{2\times k-1}, s''_{2\times k-1}) = (i'_{2\times(k-n)-1}, s'_{2\times(k-n)-1})$.
  Since $(i'_k, s'_k)_{0\leq k\leq 2\times n'-1}$ is a path, by Def. 13.(2), $i'_{2\times(k-n)} = i'_{2\times(k-n)-1}$ and $s'_{2\times(k-n)} \neq s'_{2\times(k-n)-1}$.
  Thus, $i''_{2\times k} = i''_{2\times k-1}$ and $s''_{2\times k} \neq s''_{2\times k-1}$.

Thus, by Def. 13, $(i''_k, s''_k)_{0\leq k\leq 2\times(n+n')-1}$ is a path in $\mathcal{G}$.
□

Proposition 9 (path image). *Let $\mathcal{G} = (\mathit{Ag}, \mathit{Site}, \mathit{Link})$ be a site-graph, ϕ be an automorphism of $\mathcal{G}$, and $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ be a path of length $n > 0$ in $\mathcal{G}$, then $(\phi(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is a path of length n in $\mathcal{G}$.*

Proof. Let $\mathcal{G} = (Ag, Site, Link)$ be a site-graph, ϕ be an automorphism of $\mathcal{G}$, and $(i_k, s_k)_{0 \leq k \leq 2\times n-1}$ be a path in $\mathcal{G}$, then $(\phi(i_k), s_k)_{0 \leq k \leq 2\times n-1}$ is a path in $\mathcal{G}$.

– Let k be an integer such that $0 \leq k \leq 2 \times n - 1$.

  By Def. 13, $(i_k, s_k) \in Site$.
  By Def. 10, $Ag(i_k)$ is defined.
  By Def. 11.(1), $\phi(i_k)$ is defined.
  By Def. 11.(2), $Ag(\phi(i_k))$ is defined.
  By Def. 11.(5), $(\phi(i_k), s_k) \in Site$.

– Let k be an integer such that $0 \leq k < n$.

  By Def. 13.(1), $((i_{2\times k}, s_{2\times k}), (i_{2\times k+1}, s_{2\times k+1})) \in Link$.
  By Def. 11.(7), $((\phi(i_{2\times k}), s_{2\times k}), (\phi(i_{2\times k+1}), s_{2\times k+1})) \in Link$.

– Let k be an integer such that $1 \leq k < n$.

  By Def. 13.(2), $i_{2\times k} = i_{2\times k-1}$ and $s_{2\times k} \neq s_{2\times k-1}$.
  By extensionality, $\phi(i_{2\times k}) = \phi(i_{2\times k-1})$.

Thus, by Def. 13, $(\phi(i_k), s_k)_{0 \leq k \leq 2\times n-1}$ is a path in $\mathcal{G}$.
□

Definition 14 (connected components). *A site-graph $(Ag, Site, Link)$ is a connected component, if and only if, for any pair $(i, i') \in \mathbb{N}^2$ of agent identifiers such that $Ag(i)$ and $Ag(i')$ are defined and $i \neq i'$, there exists a pair $(s, s') \in \mathcal{S}^2$ of site types, such that $(i, s) \in Site$, $(i', s') \in Site$, and there is a path in $\mathcal{G}$ between the site (i, s) and the site (i', s').*

Definition 15 (cycle). *Let $\mathcal{G}$ be a site-graph. A cycle of length $n > 0$ is a path $(i_k, s_k)_{0 \leq k \leq 2\times n-1}$ in the site-graph $\mathcal{G}$ such that $i_0 = i_{2\times n-1}$ and $s_0 \neq s_{2\times n-1}$.*

Lemma 1 (rigidity) *An embedding between two connected components is fully characterized by the image of one agent.*

Proof. Let $\mathcal{G} = (Ag, Site, Link)$ and $\mathcal{G}' = (Ag', Site', Link')$ be two connected components and ϕ, ϕ' be two embeddings between $\mathcal{G}$ and $\mathcal{G}'$.

Let $i \in \mathbb{N}$ be an agent identifier such that $Ag(i)$ is defined.
We assume that $\phi(i) = \phi'(i)$.

For any agent identifier $i' \in \mathbb{N}$,

– We assume that $Ag(i')$ is not defined.

  Then by Def. 11.(1), neither $\phi(i')$ nor $\phi'(i')$ are defined.

– We assume that $Ag(i')$ is defined and that $i' = i$.

By hypothesis, $\phi(i) = \phi'(i)$.
Thus, $\phi(i') = \phi'(i')$.

– We assume that $Ag(i')$ is defined and that $i' \neq i$.

By Def. 14 and since $i \neq i'$, there exist two sites s and s' and a path $(i_k, s_k)_{0 \leq k \leq 2\times n-1}$ of length $n > 0$ between (i, s) and (i', s').
Moreover, by Def. 11.(1), both $\phi(i)$ and $\phi'(i)$ are defined.

By absurd, let us assume that $\phi(i') \neq \phi'(i')$ and that n is minimal for this property.
We have $n > 0$.

- For any $j \in \mathbb{N}$, such that $0 \leq j < n$, we have by Def. 13.(1), $((i_{2\times j}, s_{2\times j}), (i_{2\times j+1}, s_{2\times j+1})) \in \mathit{Link}$;
- For any j such that $1 \leq j < n$, we have by Def. 13.(2), $i_{2\times j} = i_{2\times j-1}$ and $s_{2\times j} = s_{2\times j-1}$.

We consider two cases:

1. We assume that $n = 1$.

   We have $\phi(i_{2\times n}) = \phi'(i_{2\times n})$.

2. We assume that $n \geq 2$.

   Thus, by Def. 13, $(i_k, s_k)_{0\leq k\leq 2\times(n-1)+1}$ is a path between $i_0 = i$ and $i_{2\times(n-1)+1}$.
   Since n is minimal, we get that $\phi(i_{2\times(n-1)+1}) = \phi'(i_{2\times(n-1)+1})$.
   By Def. 13.(2), we have $i_{2\times(n-1)+1} = i_{2\times(n-1)+2}$ and $s_{2\times(n-1)+1} \neq s_{2\times(n-1)+2}$.
   Thus, by extensionality, $\phi(i_{2\times(n-1)+1}) = \phi(i_{2\times(n-1)+2})$ and $\phi'(i_{2\times(n-1)+1}) = \phi'(i_{2\times(n-1)+2})$.
   Thus, $\phi(i_{2\times n}) = \phi'(i_{2\times n})$.
   By Def. 13.(1), we have $((i_{2\times n}, s_{2\times n}), (i_{2\times n+1}, s_{2\times n+1})) \in \mathit{Link}$.
   Thus, by Def. 11.(7), $((\phi(i_{2\times n}), s_{2\times n}), (\phi(i_{2\times n+1}), s_{2\times n+1})) \in \mathit{Link}$
   and $((\phi'(i_{2\times n}), s_{2\times n}), (\phi'(i_{2\times n+1}), s_{2\times n+1})) \in \mathit{Link}$.
   Since $\phi(i_{2\times n}) = \phi'(i_{2\times n})$, it follows that $((\phi(i_{2\times n}), s_{2\times n}), (\phi(i_{2\times n+1}), s_{2\times n+1})) \in \mathit{Link}$
   and $((\phi(i_{2\times n}), s_{2\times n}), (\phi'(i_{2\times n+1}), s_{2\times n+1})) \in \mathit{Link}$.
   Then, by Def. 10.(3), it follows that $\phi(i_{2\times n+1}) = \phi'(i_{2\times n+1})$.
   Thus, since $i' = i_{2\times n+1}$, $\phi(i') = \phi'(i')$ which is absurd.

So whenever $Ag(i')$ is defined, $\phi(i') = \phi'(i')$.

Thus ϕ and ϕ' are equal.
□

Proposition 10. *Let $\mathcal{G} = (Ag, \mathit{Site}, \mathit{Link})$ be a connected component without any cycle. Let ϕ be an automorphism of $\mathcal{G}$. Let i be an agent identifier such that $Ag(i)$ is defined. Let $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ be a path between i and $\phi(i)$.*
Then $s_0 = s_{2\times n-1}$.

Proof. Let $\mathcal{G} = (Ag, \mathit{Site}, \mathit{Link})$ be a connected component without any cycle.
Let ϕ be an automorphism of $\mathcal{G}$.
Let i be an agent identifier such that $Ag(i)$ is defined.

Let $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ be a path between i and $\phi(i)$ such that $s_0 \neq s_{2\times n-1}$.

Let us prove by induction over m, that for any $m \in \mathbb{N}$, $(\phi^m(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.

- We assume that $m = 0$.

  The sequence $(\phi^m(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is equal to the sequence $(i_k, s_k)_{0\leq k\leq 2\times n-1}$.
  By hypothesis, $(i_k, s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.
  Thus, $(\phi^m(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.

- We consider $m \in \mathbb{N}$ such that $(\phi^m(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.

  By Prop. 9, $(\phi(\phi^m(i_k)), s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.
  Since the sequence, $(\phi(\phi^m(i_k)), s_k)_{0\leq k\leq 2\times n-1}$ is equal to the sequence $(\phi^{m+1}(i_k), s_k)_{0\leq k\leq 2\times n-1}$.
  $(\phi^{m+1}(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.

Let us prove by induction over m', that for any $m, m' \in \mathbb{N}$, such that $m < m'$, there exists a path $(i'_k, s'_k)_{0\leq k\leq 2\times n'-1}$ in $\mathcal{G}$ such that $i'_0 = \phi^m(i_0)$, $i'_{2\times n'-1} = \phi^{m'}(i_0)$, $s'_0 = s_0$, and $s'_{2\times n'-1} = s_{2\times n-1}$.

- We assume that $m' = m + 1$.

  We have $\phi^{m'}(i_0) = \phi^m(\phi(i_0))$.
  We have proved that $(\phi^m(i_k), s_k)_{0\leq k\leq 2\times n-1}$ is a path in $\mathcal{G}$.
  Moreover, $\phi^m(i_0) = \phi^m(i_0)$.

  Since $i_{2\times n-1} = \phi(i_0)$, by extensionaly, $\phi(\phi^m(i_0)) = \phi(\phi^m(i_0))$.
  So $\phi^m(i_{2\times n-1}) = \phi^{m'}(i_{2\times n-1})$.

  Lastly, $s_0 = s_0$ and $s_{2\times n-1} = s_{2\times n-1}$.

- We assume that there exist $m, m' \in \mathbb{N}$, such that $m < m'$ and a path $(i'_k, s'_k)_{0\leq k\leq 2\times n'-1}$ in $\mathcal{G}$ such that $i'_0 = \phi^m(i_0)$ and $i'_{2\times n'-1} = \phi^{m'}(i_0)$ such that $s'_0 = s_0$ and $s'_{2\times n'-1} = s_{2\times n-1}$.

  We have already proved that there exists a path $(i''_k, s''_k)_{0\leq k\leq 2\times n''-1}$ in $\mathcal{G}$ such that $i''_0 = \phi^{m'}(i_0)$, $i''_{2\times n''-1} = \phi^{m'+1}(i_0)$, $s''_0 = s_0$ and $s''_{2\times n''-1} = s_{2\times n-1}$.

  Since $s_0 \neq s_{2\times n-1}$, by Prop. 8, there exists a path between the site $(\phi^m(i_0), s_0)$ and the site $(\phi^{m'+1}(i_0), s_{2\times n-1})$ in $\mathcal{G}$.

By Def. 10, Def. 11.(1), and Def. 11.(2), the set $\{\phi^{m''}(i_0) \mid m'' \in \mathbb{N}\}$ is finite.
Thus there exists $m < m'$ such that $\phi^m(i_0) = \phi^{m'}(i_0)$.
By Def. 15, there exists a cycle in $(Ag, Site, Link)$, which is absurd.
□

Lemma 2 (automorphism) *Let $\mathcal{G} = (Ag, Site, Link)$ be a connected component without any cycle.*

– *$\mathcal{G}$ has at most two automorphisms.*
– *If ϕ is a automorphism over $\mathcal{G}$, such that there exists $i \in \mathbb{N}$, such that $Ag(i)$ is defined and $\phi(i) \neq i$, then there exist two agent identifiers $i, i' \in \mathbb{N}$ and a site type $s \in \mathcal{S}$, such that $Ag(i) = Ag(i')$, $(i,s),(i',s) \in$ Site, and $((i,s),(i',s)) \in$ Link.*

Proof. Let $(Ag, Site, Link)$ be a connected component without any cycle.

– By Def. 11, the identify function restricted to the elements $i \in \mathbb{N}$ such that $Ag(i)$ is defined, is an automorphism.

– Let us assume that there exists another automorphism ϕ of $(Ag, Site, Link)$.

  - Let us show that for any agent identifier $i \in \mathbb{N}$ such that $Ag(i)$ is defined, then $\phi(i) \neq i$.

    We assume that there exists $i \in \mathbb{N}$ such that $Ag(i)$ is defined and $\phi(i) = i$.
    Then, ϕ and the restriction of the identify function to the elements $i \in \mathbb{N}$ such that $Ag(i)$ is defined are two embeddings between $(Ag, Site, Link)$ and $(Ag, Site, Link)$.
    Since $(Ag, Site, Link)$ is connected, by Lem. 1, ϕ is equal to the restriction of the identify function to the elements $i \in \mathbb{N}$ such that $Ag(i)$ is defined are two embeddings between $(Ag, Site, Link)$ and $(Ag, Site, Link)$, which is absurd.

  - Let $i \in \mathbb{N}$ be an agent identifier such that $Ag(i)$ is defined.

    Since $(Ag, Site, Link)$ is connected and $i \neq \phi(i)$, we can consider a path $(i_k, s_k)_{0 \leq k \leq 2\times n-1}$ between i and $\phi(i)$.

    By Prop. 10, $s_0 = s_{2\times n-1}$.

    Let us prove by induction, that for any $k \in \mathbb{N}$, such that $0 \leq k \leq n$, $Ag(i_k) = Ag(i_{2\times n-1-k})$, $s_k = s_{2\times n-1-k}$, $\phi(i_k) = i_{2\times n-1-k}$.

    ∗ We assume that $k = 0$.

      By Def. 13, we have $i_0 = i$ and $i_{2\times n-1} = \phi(i)$.
      By Def. 11.(4), $Ag(\phi(i)) = Ag(\imath)$.
      Thus, $Ag(i_0) = Ag(i_{2\times n-1})$.

      By hypothesis, we have $s_0 = s_{2\times n-1}$.

      By hypothesis, we have $\phi(i_0) = i_{2\times n-1}$.

    ∗ We assume that there exists $k \in \mathbb{N}$ such that $0 \leq k < n$, $Ag(i_k) = Ag(i_{2\times n-k-1})$, $s_k = s_{2\times n-k-1}$ and $\phi(i_k) = i_{2\times n-1-k}$.

      · We assume that k is even.

        We have by Def. 13.(1), $((i_k, s_k),(i_{k+1}, s_{k+1})) \in$ *Link*
        and $((i_{2\times n-k}, s_{2\times n-k}),(i_{2\times n-k+1}, s_{2\times n-k+1})) \in$ *Link*.
        By Def. 10, $((i_{2\times n-k+1}, s_{2\times n-k+1}),(i_{2\times n-k}, s_{2\times n-k})) \in$ *Link*.

By Def. 11.(1), $\phi(i_k)$ and $\phi(i_{k+1})$ are defined.
By Def. 11.(2), $Ag(\phi(i_k))$ and $Ag(\phi(i_{k+1}))$ are defined.
By Def. 11.(5), $(\phi(i_k), s_k) \in \mathit{Site}$ and $(\phi(i_{k+1}), s_{k+1}) \in \mathit{Site}$.
By Def. 11.(7), $((\phi(i_k), s_k), (\phi(i_{k+1}), s_{k+1})) \in \mathit{Link}$.
By induction hypothesis, $\phi(i_k) = i_{2\times n+1-k}$ and $s_k = s_{2\times n+1-k}$.
Thus, $((i_{2\times n-k+1}, s_{2\times n-k+1}), (\phi(i_{k+1}), s_{k+1})) \in \mathit{Link}$.
We already proved that $((i_{2\times n-k+1}, s_{2\times n-k+1}), (i_{2\times n-k}, s_{2\times n-k})) \in \mathit{Link}$.
By Def. 10.(3), it follows that $\phi(i_{k+1}) = i_{2\times n-k}$ and $s_{k+1} = s_{2\times n-k}$.

· We assume that k is odd and $k < n$

We have by Def. 13.(2), $i_k = i_{k+1}$ and $i_{2\times n-k} = i_{2\times n-k+1}$.
By induction hypothesis, $\phi(i_k) = i_{2\times n-k+1}$.
By extensionality, $\phi(i_{k+1}) = i_{2\times n-k+1}$.
Thus, $\phi(i_{k+1}) = i_{2\times n-k}$.
We can deduce that $i_{k+1} \neq i_{2\times n-k}$.
Since, moreover, $(i_l, s_l)_{0\leq l\leq 2\times n+1}$ is a path and $k+1$ is even, $2\times n-k-1$ is even, and $k+1 < 2\times n-k+1$, and by Prop. 7, $(i_l, s_l)_{k+1\leq l\leq 2\times n-k}$ is a path between (i_{l+1}, s_{l+1}) and $(\phi(i_{l+1}), s_{2\times n-k})$.
Thus, by Lem. 10, $s_{k+1} = s_{2\times n-k}$.
By Def. 10, $((i_{2\times n-k+1}, s_{2\times n-k+1}), (i_{2\times n-k}, s_{2\times n-k})) \in \mathit{Link}$.
By Def. 11.(1), $\phi(i_k)$ and $\phi(i_{k+1})$ are defined.
By Def. 11.(2), $Ag(\phi(i_k))$ and $Ag(\phi(i_{k+1}))$ are defined.
By Def. 11.(5), $(\phi(i_k), s_k) \in \mathit{Site}$ and $(\phi(i_{k+1}), s_{k+1}) \in \mathit{Site}$.
By Def. 11.(7), $((\phi(i_k), s_k), (\phi(i_{k+1}), s_{k+1})) \in \mathit{Link}$.
By induction hypothesis, $\phi(i_k) = i_{2\times n+1-k}$ and $s_k = s_{2\times n+1-k}$.
Thus, $((i_{2\times n-k+1}, s_{2\times n-k+1}), (\phi(i_{k+1}), s_{k+1})) \in \mathit{Link}$.
We already proved that $((i_{2\times n-k+1}, s_{2\times n-k+1}), (i_{2\times n-k}, s_{2\times n-k})) \in \mathit{Link}$.
By Def. 10.(3), it follows that $\phi(i_{k+1}) = i_{2\times n-k}$ and $s_{k+1} = s_{2\times n-k}$.

Thus, we have $(Ag(i_n), s_n) = (Ag(i_{n+1}), s_{n+1})$. and $\phi(i_n) = i_{n+1}$.
□

Lemma 3 (Euler) *If a site-graph has no cycle, then it has an agent with at most one bound site.*

Proof. Let $\mathcal{G} = (Ag, \mathit{Site}, \mathit{Link})$ be a site-graph such that for any agent identifier $i \in \mathbb{N}$ such that $Ag(i)$ is defined, there exists two links $((i_1, s_1), (i_2, s_2)), ((i'_1, s'_1), (i'_2, s'_2)) \in \mathit{Link}$ such that $i_1 = i'_1 = i$ and $s_1 \neq s'_1$.

We can assume, without any loss of generality, that the set $\mathbb{N}$ and $\mathcal{S}$ are totally ordered.
We define the following sequence $(x_n)_{n\in\mathbb{N}}$ of sites:

$$\begin{cases} x_0 = (\textsc{min}\{i \in \mathbb{N} \mid Ag(i) \text{ is defined }\}, \textsc{min}\{s \mid (\textsc{min}\{i \in \mathbb{N} \mid Ag(i) \text{ is defined }\}, s) \text{ is bound in } \mathcal{G}\}) \\ x_{2\times n+1} = (x', s') \mid ((x_{2\times n}, s_{2\times n}), (x', s')) \in \mathit{Link} \\ x_{2\times n+2} = (x_{2\times n+1}, \textsc{min}\{s \mid s \neq s_{2\times n+1} \wedge (x_{2\times n+1}, s) \text{ is bound in } \mathcal{G}\}). \end{cases}$$

Let us prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is well-defined and for any $n \in \mathbb{N}$, $Ag(n)$ is defined, and (x_n) is bound in $(Ag, \mathit{Site}, \mathit{Link})$.

– x_0 is well-defined, since any site has at least two bound sites.
  Let us denote $x_0 = (i_0, s_0)$.
  By definition, $Ag(i_0)$ is defined, and x_0 is bound in $\mathcal{G}$.

- Let us assume that $x_{2\times n}$ is well-defined, that $Ag(\text{FST}(x_{2\times n}))$ is defined, and $x_{2\times n}$ is bound in $\mathcal{G}$.

  Let us denote $x_{2\times n} = (i_{2\times_n}, s_{2\times n})$.
  Since $x_{2\times n}$ is bound in $\mathcal{G}$, by Def. 10, there exists a unique pair (i', s') such that $(x_{2\times n}, (i', s')) \in \mathit{Link}$.
  Moreover, by Def. 10, $Ag(i')$ is defined and (i', s') is bound in $\mathcal{G}$.

- Let us assume that $x_{2\times m+1}$ is well-defined, that $Ag(\text{FST}(x_{2\times n+1}))$ is defined.

  Let us denote $x_{2\times n+1} = (i_{2\times_n+1}, s_{2\times n+1})$.
  By hypothesis, $i_{2\times n+1}$ has at least two bound sites.
  Thus the set $\{s \mid s \neq s_{2\times n+1} \wedge (x_{2\times n+1}, s)$ is bound in $\mathcal{G}\}$ is not empty, and $x_{2\times n}$ is well defined.
  Moreover, $i_{2\times n+1} = i_{2\times n}$ and $Ag(i_{2\times n})$ is defined, thus $Ag(i_{2\times n+1})$ is defined.
  Lastly, $x_{2\times n+1}$ is bound in $\mathcal{G}$.

By Def. 10, the set of the elements $i \in \mathbb{N}$ such that $Ag(i)$ is defined is finite.
Moreover $\mathcal{S}$ is finite.
Thus the Cartesian product between the set of the elements $i \in \mathbb{N}$ such that $Ag(i)$ is defined and $\mathcal{S}$ is finite.
Thus the set $\{x_{2\times k} \mid k \in \mathbb{N}\}$ is finite.
Thus, there exists k and k' such that $k < k'$ and $x_{2\times k} = x_{2\times k'}$.
Let us prove that the sequence $(x_l)_{2\times k \leq l \leq 2\times k'+1}$ is a path between $\text{FST}(x_{2\times k})$ and $\text{FST}(x_{2\times k'})$.

- We have $k' > k$.

- For any integer l such that $k \leq l \leq k'$, we have, by definition of $(x_n)_{n\in\mathbb{N}}$, $(x_{2\times l}, x_{2\times l+1}) \in \mathit{Link}$;

- For any integer l such that $k \leq l \leq k'$, we have, by definition of $(x_n)_{n\in\mathbb{N}}$, $\text{FST}(x_{2\times l+1}) = \text{FST}(x_{2\times l+2)}))$ and $\text{SND}(x_{2\times l+1}) \neq \text{SND}(x_{2\times l+2)}))$.

This is absurd, thus there exists an agent identifier $i \in \mathbb{N}$ such that $Ag(i)$ is defined and such that there exists at most one site $s \in \mathcal{S}$ such that $(i, s) \in \mathit{Site}$ and (i, s) is bound in $(Ag, \mathit{Site}, \mathit{Link})$.
□