Static Analysis of Concurrent Programs

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

Antoine Miné

Year 2017–2018

Course 8
8 November 2017

Concurrent programming

Idea:

Decompose a program into a set of (loosely) interacting processes.

Why concurrent programs?

- exploit parallelism in current computers
  (multi-processors, multi-cores, hyper-threading)

  "Free lunch is over"
  change in Moore's law (×2 transistors every 2 years)

- exploit several computers
  (distributed computing)

- ease of programming
  (GUI, network code, reactive programs)

Models of concurrent programs

Many models:

- process calculi: CSP, $\pi-$calculus, join calculus
- message passing
- shared memory (threads)
- transactional memory
- combination of several models

Example implementations:

- C, C++ with a thread library (POSIX threads, Win32)
- C, C++ with a message library (MPI, OpenMP)
- Java (native threading API)
- Erlang (based on $\pi-$calculus)
- JoCaml (OCaml + join calculus)
- processor-level (interrupts, test-and-set instructions)

Scope

In this course: static thread model

- implicit communication through shared memory
- explicit communication through synchronisation primitives
- fixed number of threads (no dynamic creation of threads)
- numeric programs (real-valued variables)

Goal: static analysis

- infer numeric program invariants
- parametrized by a choice of numeric abstract domains
- discover run-time errors (e.g., division by 0)
- discover data-races (unprotected accesses by concurrent threads)
- discover deadlocks (some threads block each other indefinitely)

Outline

- From sequential to concurrent abstract interpreters
  - alternate sequential semantics
    (denotational semantics with errors)
  - interleaving concurrent semantics
  - (non-relational) interference-based analysis
  - robustness against weakly consistent memory models
  - synchronization: data-races, locks and deadlocks
- Abstract rely-guarantee
  - rely-guarantee proof method
  - complete modular concrete semantics
  - relational interference abstractions

Simple structured numeric language

- finite set of (toplevel) threads: $\texttt{prog}_1$ to $\texttt{prog}_n$
- finite set of numeric program variables $V \in \mathbb{V}$
- finite set of statement locations $\ell \in \mathcal{L}$
- finite set of potential error locations $\omega \in \Omega$

Structured language syntax

$$\begin{array}{lcll}
\texttt{parprog} & ::= & {}^{\ell}\texttt{prog}_1{}^{\ell} \mid\mid \ldots \mid\mid {}^{\ell}\texttt{prog}_n{}^{\ell} & \textit{(parallel composition)} \\
{}^{\ell}\texttt{prog}^{\ell} & ::= & {}^{\ell}V \leftarrow \texttt{exp}^{\ell} & \textit{(assignment)} \\
 & \mid & {}^{\ell}\texttt{if exp} \bowtie 0 \texttt{ then } {}^{\ell}\texttt{prog}^{\ell} \texttt{ fi}^{\ell} & \textit{(conditional)} \\
 & \mid & {}^{\ell}\texttt{while } {}^{\ell}\texttt{exp} \bowtie 0 \texttt{ do } {}^{\ell}\texttt{prog}^{\ell} \texttt{ done}^{\ell} & \textit{(loop)} \\
 & \mid & {}^{\ell}\texttt{prog}; {}^{\ell}\texttt{prog}^{\ell} & \textit{(sequence)} \\
\texttt{exp} & ::= & V \mid [c_1, c_2] \mid -\texttt{exp} \mid \texttt{exp} \diamond_{\omega} \texttt{exp} &
\end{array}$$

$c_1, c_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\diamond \in \{+, -, \times, /\}$, $\bowtie \in \{=, <, \ldots\}$

From sequential to concurrent semantics

Sequential semantics

Reminder: transition systems

Transition system: $(\Sigma, \tau, \mathcal{I})$

- Σ: set of program states
- $\tau \subseteq \Sigma \times \Sigma$: transition relation
  we note $(\sigma, \sigma') \in \tau$ as $\sigma \to_\tau \sigma'$
- $\mathcal{I} \subseteq \Sigma$: set of initial states

Reminder: traces of a transition system

Maximal trace semantics: $\mathcal{M}_\infty \in \mathcal{P}(\Sigma^\infty)$

Set of total executions $\sigma_0, \ldots, \sigma_n, \ldots$

- starting in an initial state $\sigma_0 \in \mathcal{I}$ and either
- ending in a blocking state in $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' \colon \sigma \not\to_\tau \sigma' \,\}$
- or infinite

$$\begin{aligned}\mathcal{M}_\infty \stackrel{\text{def}}{=} & \{\, \sigma_0, \ldots, \sigma_n \mid \sigma_0 \in \mathcal{I} \wedge \sigma_n \in \mathcal{B} \wedge \forall i < n \colon \sigma_i \to_\tau \sigma_{i+1} \,\} \cup \\ & \{\, \sigma_0, \ldots, \sigma_n \ldots \mid \sigma_0 \in \mathcal{I} \wedge \forall i < \omega \colon \sigma_i \to_\tau \sigma_{i+1} \,\}\end{aligned}$$

Reminder: prefix trace abstraction

Finite prefix trace semantics: $\mathcal{T}_p \in \mathcal{P}(\Sigma^*)$

set of finite prefixes of executions:

$$\mathcal{T}_p \stackrel{\text{def}}{=} \{ \sigma_0, \ldots, \sigma_n \mid n \geq 0,\ \sigma_0 \in \mathcal{I},\ \forall i < n: \sigma_i \to_\tau \sigma_{i+1} \}$$

$\mathcal{T}_p$ is an abstraction of the maximal trace semantics

$\mathcal{T}_p = \alpha_{*\preceq}(\mathcal{M}_\infty)$ where $\alpha_{*\preceq}(X) \stackrel{\text{def}}{=} \{ t \in \Sigma^* \mid \exists u \in X: t \preceq u \}$

- can prove safety properties
- cannot prove termination nor inevitability

fixpoint characterisation: $\mathcal{T}_p = \text{lfp}\, F_p$ where

$$F_p(X) = \mathcal{I} \cup \{ \sigma_0, \ldots, \sigma_{n+1} \mid \sigma_0, \ldots, \sigma_n \in X \wedge \sigma_n \to_\tau \sigma_{n+1} \}$$

Reminder: reachable state abstraction

Reachable state semantics: $\mathcal{R} \in \mathcal{P}(\Sigma)$

set of states reachable in any execution:

$\mathcal{R} \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0, \ldots, \sigma_n \colon \sigma_0 \in \mathcal{I}, \forall i < n \colon \sigma_i \to_\tau \sigma_{i+1} \land \sigma = \sigma_n \,\}$

$\mathcal{R}$ is an abstraction of the finite trace semantics: $\mathcal{R} = \alpha_p(\mathcal{T}_p)$

where $\alpha_p(X) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in X \colon \sigma = \sigma_n \,\}$

- $\mathcal{R}$ can prove state safety properties: $\mathcal{R} \subseteq S$
  (executions stay in S)
- $\mathcal{R}$ cannot prove ordering, termination, inevitability properties

fixpoint characterisation: $\mathcal{R} = \text{lfp}\, F_\mathcal{R}$ where

$F_\mathcal{R}(X) = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma' \in X \colon \sigma' \to_\tau \sigma \,\}$

States of a sequential program, with errors

Simple sequential numeric programs: $\texttt{parprog} ::= {}^{\ell^i}\texttt{prog}^{\ell^x}$.

Program states: $\Sigma \stackrel{\text{def}}{=} (\mathcal{L} \times \mathcal{E}) \cup \Omega$

- a control state in $\mathcal{L}$, and
- either a memory state: an environment in $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{R}$
- or an **error state**, in Ω

Initial states:

start at the first control point ℓ^i with variables set to 0:

$\mathcal{I} \stackrel{\text{def}}{=} \{ (\ell^i, \lambda V.0) \}$

Note that $\mathcal{P}(\Sigma) \simeq (\mathcal{L} \to \mathcal{P}(\mathcal{E})) \times \mathcal{P}(\Omega)$:

- a state property in $\mathcal{P}(\mathcal{E})$ at each program point in $\mathcal{L}$
- and a set of errors in $\mathcal{P}(\Omega)$

Expression semantics with errors

Expression semantics: $\mathsf{E}[\![\, \mathrm{exp} \,]\!] : \mathcal{E} \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$

$$\mathsf{E}[\![\, V \,]\!]\,\rho \stackrel{\text{def}}{=} \langle\, \{\, \rho(V) \,\},\, \emptyset \,\rangle$$

$$\mathsf{E}[\![\, [c_1, c_2] \,]\!]\,\rho \stackrel{\text{def}}{=} \langle\, \{\, x \in \mathbb{R} \mid c_1 \leq x \leq c_2 \,\},\, \emptyset \,\rangle$$

$$\mathsf{E}[\![\, -e \,]\!]\,\rho \stackrel{\text{def}}{=} \text{let } \langle\, V,\, O \,\rangle = \mathsf{E}[\![\, e \,]\!]\,\rho \text{ in } \langle\, \{\, -v \mid \in V \,\},\, O \,\rangle$$

$$\mathsf{E}[\![\, e_1 \diamond_\omega e_2 \,]\!]\,\rho \stackrel{\text{def}}{=} \begin{array}{l} \text{let } \langle\, V_1,\, O_1 \,\rangle = \mathsf{E}[\![\, e_1 \,]\!]\,\rho \text{ in} \\ \text{let } \langle\, V_2,\, O_2 \,\rangle = \mathsf{E}[\![\, e_2 \,]\!]\,\rho \text{ in} \\ \langle\, \{\, v_1 \diamond v_2 \mid v_i \in V_i,\, \diamond \neq / \lor v_2 \neq 0 \,\}, \\ \;O_1 \cup O_2 \cup \{\, \omega \text{ if } \diamond = / \land 0 \in V_2 \,\} \,\rangle \end{array}$$

- defined by structural induction on the syntax
- evaluates in an environment ρ to a set of values
- also returns a set of accumulated errors
  (here, only divisions by zero)

Reminders: semantics in equational form

Principle: (without handling errors in Ω)

- see lfp f as the least solution of an equation $x = f(x)$
- partition states by control: $\mathcal{P}(\mathcal{L} \times \mathcal{E}) \simeq \mathcal{L} \to \mathcal{P}(\mathcal{E})$

  $\mathcal{X}_\ell \in \mathcal{P}(\mathcal{E})$: invariant at $\ell \in \mathcal{L}$

  $\forall \ell \in \mathcal{L}$: $\mathcal{X}_\ell \stackrel{\text{def}}{=} \{ m \in \mathcal{E} \mid (\ell, m) \in \mathcal{R} \}$

  $\Longrightarrow$ set of (recursive) equations on $\mathcal{X}_\ell$

Example:

$^{\ell 1}\, i \leftarrow 2;$	$\mathcal{X}_1 = \mathcal{I}$
$^{\ell 2}\, n \leftarrow [-\infty, +\infty];$	$\mathcal{X}_2 = \mathsf{C}[\![\, i \leftarrow 2 \,]\!]\, \mathcal{X}_1$
$^{\ell 3}$ `while` $^{\ell 4}\, i < n$ `do`	$\mathcal{X}_3 = \mathsf{C}[\![\, n \leftarrow [-\infty, +\infty] \,]\!]\, \mathcal{X}_2$
$^{\ell 5}$ `if` $[0,1] = 0$ `then`	$\mathcal{X}_4 = \mathcal{X}_3 \cup \mathcal{X}_7$
$^{\ell 6}\, i \leftarrow i + 1$	$\mathcal{X}_5 = \mathsf{C}[\![\, i < n \,]\!]\, \mathcal{X}_4$
`fi`	$\mathcal{X}_6 = \mathcal{X}_5$
$^{\ell 7}$ `done`	$\mathcal{X}_7 = \mathcal{X}_5 \cup \mathsf{C}[\![\, i \leftarrow i + 1 \,]\!]\, \mathcal{X}_6$
$^{\ell 8}$	$\mathcal{X}_8 = \mathsf{C}[\![\, i \geq n \,]\!]\, \mathcal{X}_4$

Semantics in denotational form

Input-output function $C[\![\, \texttt{prog} \,]\!]$

$C[\![\, \texttt{prog} \,]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega))$

$C[\![\, X \leftarrow e \,]\!]\langle R,\, O \rangle \stackrel{\text{def}}{=} \langle \emptyset,\, O \rangle \sqcup \bigsqcup_{\rho \in R} \langle \{ \rho[X \mapsto v] \mid v \in V_\rho \},\, O_\rho \rangle$

$C[\![\, e \bowtie 0? \,]\!]\langle R,\, O \rangle \stackrel{\text{def}}{=} \langle \emptyset,\, O \rangle \sqcup \bigsqcup_{\rho \in R} \langle \{ \rho \mid \exists v \in V_\rho : v \bowtie 0 \},\, O_\rho \rangle$

where $\langle V_\rho,\, O_\rho \rangle \stackrel{\text{def}}{=} E[\![\, e \,]\!]\rho$

$C[\![\, \texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi} \,]\!]X \stackrel{\text{def}}{=} (C[\![\, s \,]\!] \circ C[\![\, e \bowtie 0? \,]\!])X \sqcup C[\![\, e \not\bowtie 0? \,]\!]X$

$C[\![\, \texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done} \,]\!]X \stackrel{\text{def}}{=}$
$\quad C[\![\, e \not\bowtie 0? \,]\!](\mathrm{lfp}\, \lambda Y. X \sqcup (C[\![\, s \,]\!] \circ C[\![\, e \bowtie 0? \,]\!])Y)$

$C[\![\, s_1;\, s_2 \,]\!] \stackrel{\text{def}}{=} C[\![\, s_2 \,]\!] \circ C[\![\, s_1 \,]\!]$

- mutate memory states in $\mathcal{E}$, accumulate errors in Ω
  ($\sqcup$ is the element-wise $\cup$ in $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)$)
- structured: nested loops yield nested fixpoints
- big-step: forget information on intermediate locations ℓ

Abstract semantics in denotational form

Extend a numeric abstract domain $\mathcal{E}^\sharp$ abstracting $\mathcal{P}(\mathcal{E})$ to $\mathcal{D}^\sharp \stackrel{\text{def}}{=} \mathcal{E}^\sharp \times \mathcal{P}(\Omega)$.

$\mathsf{C}^\sharp[\![\,\texttt{prog}\,]\!] : \mathcal{D}^\sharp \to \mathcal{D}^\sharp$

$\mathsf{C}^\sharp[\![\,X \leftarrow e\,]\!]\langle R^\sharp, O\rangle$ and $\mathsf{C}^\sharp[\![\,e \bowtie 0?\,]\!]\langle R^\sharp, O\rangle$ are given

$\mathsf{C}^\sharp[\![\,\texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi}\,]\!]X^\sharp \stackrel{\text{def}}{=}$
$\quad (\mathsf{C}^\sharp[\![\,s\,]\!] \circ \mathsf{C}^\sharp[\![\,e \bowtie 0?\,]\!])X^\sharp \sqcup^\sharp \mathsf{C}^\sharp[\![\,e \not\bowtie 0?\,]\!]X^\sharp$

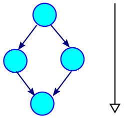
$\mathsf{C}^\sharp[\![\,\texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done}\,]\!]X^\sharp \stackrel{\text{def}}{=}$
$\quad \mathsf{C}^\sharp[\![\,e \not\bowtie 0?\,]\!](\mathrm{lim}\,\lambda Y^\sharp.Y^\sharp \mathbin{\triangledown} (X^\sharp \sqcup (\mathsf{C}^\sharp[\![\,s\,]\!] \circ \mathsf{C}^\sharp[\![\,e \bowtie 0?\,]\!])Y^\sharp))$

$\mathsf{C}^\sharp[\![\,s_1;\ s_2\,]\!] \stackrel{\text{def}}{=} \mathsf{C}^\sharp[\![\,s_2\,]\!] \circ \mathsf{C}^\sharp[\![\,s_1\,]\!]$

- the abstract interpreter mimicks an actual interpreter

Equational vs. denotational form

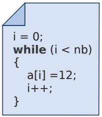
Equational:



$$\begin{cases} \mathcal{X}_1 = \top \\ \mathcal{X}_2 = F_2(\mathcal{X}_1) \\ \mathcal{X}_3 = F_3(\mathcal{X}_1) \\ \mathcal{X}_4 = F_4(\mathcal{X}_3, \mathcal{X}_4) \end{cases}$$

- linear memory in program length
- flexible solving strategy
  flexible context sensitivity
- easy to adapt to concurrency,
  using a product of CFG

Denotational:



$$\mathsf{C}[\![\, \mathtt{while}\ c\ \mathtt{do}\ b\ \mathtt{done} \,]\!]\, X \stackrel{\text{def}}{=}$$
$$\quad \mathsf{C}[\![\, \neg c? \,]\!]\, (\mathrm{lfp}\, \lambda Y. X \cup \mathsf{C}[\![\, b? \,]\!]\, (\mathsf{C}[\![\, c \,]\!]\, Y))$$
$$\mathsf{C}[\![\, \mathtt{if}\ c\ \mathtt{then}\ t\ \mathtt{fi} \,]\!]\, X \stackrel{\text{def}}{=}$$
$$\quad \mathsf{C}[\![\, t \,]\!]\, (\mathsf{C}[\![\, c? \,]\!]\, X) \cup \mathsf{C}[\![\, \neg c? \,]\!]\, X$$
$$\ldots$$

- linear memory in program depth
- fixed iteration strategy
  fixed context sensitivity
  (follows the program structure)
- no inductive definition of the product
  $\Longrightarrow$ thread-modular analysis

Concurrent semantics

Multi-thread execution model

t_1	t_2
$\ell 1$ `while random do`	$\ell 4$ `while random do`
$\ell 2$ `if x < y then`	$\ell 5$ `if y < 100 then`
$\ell 3$ `x ← x + 1`	$\ell 6$ `y ← y + [1,3]`

Execution model:

- finite number of threads
- the memory is shared (x,y)
- each thread has its own program counter
- execution interleaves steps from threads t_1 and t_2
  (assignments and tests are assumed to be atomic)

$\Longrightarrow$ we have the global invariant $0 \leq x \leq y \leq 102$

Labelled transition systems

Labelled transition system: $(\Sigma, \mathcal{A}, \tau, \mathcal{I})$

- Σ: set of program states
- $\mathcal{A}$: set of actions
- $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$: transition relation
  we note $(\sigma, a, \sigma') \in \tau$ as $\sigma \xrightarrow{a}_{\tau} \sigma'$
- $\mathcal{I} \subseteq \Sigma$: set of initial states

Labelled traces: sequences of states interspersed with actions

denoted as $\sigma_0 \stackrel{a_0}{\to} \sigma_1 \stackrel{a_1}{\to} \cdots \sigma_n \stackrel{a_n}{\to} \sigma_{n+1}$

From concurrent programs to labelled transition systems

Notations:

- concurrent program:
  $\texttt{parprog} ::= {}^{\ell_1^i}\texttt{prog}_1{}^{\ell_1^x} \mid\mid \cdots \mid\mid {}^{\ell_n^i}\texttt{prog}_n{}^{\ell_n^x}$
- threads identifiers: $\mathbb{T} \stackrel{\text{def}}{=} \{1, \ldots, n\}$

Program states: $\Sigma \stackrel{\text{def}}{=} ((\mathbb{T} \to \mathcal{L}) \times \mathcal{E}) \cup \Omega$

- a control state $L(t) \in \mathcal{L}$ for each thread $t \in \mathbb{T}$ and
- a single shared memory state $\rho \in \mathcal{E}$
- or an error state $\omega \in \Omega$

Initial states:

threads start at their first control point ℓ_t^i, variables are set to 0:
$\mathcal{I} \stackrel{\text{def}}{=} \{ (\lambda t.\ell_t^i, \lambda V.0) \}$

Actions: thread identifiers: $\mathcal{A} \stackrel{\text{def}}{=} \mathbb{T}$

From concurrent programs to labelled transition systems

Transition relation: $\tau \subseteq \Sigma \times \mathcal{A} \times \Sigma$

$$(L, \rho) \xrightarrow{t}_{\tau} (L', \rho') \overset{\text{def}}{\iff} (L(t), \rho) \rightarrow_{\tau[\texttt{prog}_t]} (L'(t), \rho') \wedge \forall u \neq t\colon L(u) = L'(u)$$

$$(L, \rho) \xrightarrow{t}_{\tau} \omega \overset{\text{def}}{\iff} (L(t), \rho) \rightarrow_{\tau[\texttt{prog}_t]} \omega$$

- based on the transition relation of individual threads seen as sequential processes $\texttt{prog}_t$:
  $\tau[\texttt{prog}] \subseteq (\mathcal{L} \times \mathcal{E}) \times ((\mathcal{L} \times \mathcal{E}) \cup \Omega)$
  - choose a thread t to run
  - update ρ and $L(t)$
  - leave $L(u)$ intact for $u \neq t$

  (See course 3 for the full definition of $\tau[\texttt{prog}]$.)

- each $\sigma \rightarrow \sigma'$ in $\tau[\texttt{prog}_t]$ leads to many transitions in τ!

Interleaved trace semantics

Maximal and finite prefix trace semantics are as before:

Blocking states: $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' : \forall t : \sigma \stackrel{t}{\not\rightarrow}_\tau \sigma' \,\}$

Maximal traces: $\mathcal{M}_\infty$ (finite or infinite)

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \stackrel{t_{n-1}}{\rightarrow} \sigma_n \mid n \geq 0 \wedge \sigma_0 \in \mathcal{I} \wedge \sigma_n \in \mathcal{B} \wedge \forall i < n : \sigma_i \stackrel{t_i}{\rightarrow}_\tau \sigma_{i+1} \,\} \cup \{\, \sigma_0 \stackrel{t_0}{\rightarrow} \sigma_1 \ldots \mid n \geq 0 \wedge \sigma_0 \in \mathcal{I} \wedge \forall i < \omega : \sigma_i \stackrel{t_i}{\rightarrow}_\tau \sigma_{i+1} \,\}$$

Finite prefix traces: $\mathcal{T}_p$

$$\mathcal{T}_p \stackrel{\text{def}}{=} \{\, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \stackrel{t_{n-1}}{\rightarrow} \sigma_n \mid n \geq 0 \wedge \sigma_0 \in \mathcal{I} \wedge \forall i < n : \sigma_i \stackrel{t_i}{\rightarrow}_\tau \sigma_{i+1} \,\}$$

Fixpoint form: $\mathcal{T}_p = \operatorname{lfp} F_p$ where

$$F_p(X) = \mathcal{I} \cup \{\, \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \stackrel{t_n}{\rightarrow} \sigma_{n+1} \mid n \geq 0 \wedge \sigma_0 \stackrel{t_0}{\rightarrow} \cdots \stackrel{t_{n-1}}{\rightarrow} \sigma_n \in X \wedge \sigma_n \stackrel{t_n}{\rightarrow}_\tau \sigma_{n+1} \,\}$$

Abstraction: $\mathcal{T}_p = \alpha_{*\preceq}(\mathcal{M}_\infty)$

Fairness

Fairness conditions: avoid threads being denied to run

Given *enabled*$(\sigma, t) \overset{\text{def}}{\Longleftrightarrow} \exists \sigma' \in \Sigma: \sigma \xrightarrow{t}_\tau \sigma'$,
an infinite trace $\sigma_0 \xrightarrow{t_0} \cdots \sigma_n \xrightarrow{t_n} \cdots$ is:

- weakly fair if $\forall t \in \mathbb{T}$:
  $(\exists i: \forall j \geq i: enabled(\sigma_j, t)) \implies (\forall i: \exists j \geq i: a_j = t)$
  (no thread can be continuously enabled without running)
- strongly fair if $\forall t \in \mathbb{T}$:
  $(\forall i: \exists j \geq i: enabled(\sigma_j, t)) \implies (\forall i: \exists j \geq i: a_j = t)$
  (no thread can be infinitely often enabled without running)

Proofs under fairness conditions given:

- the maximal traces $\mathcal{M}_\infty$ of a program
- a property X to prove (as a set of traces)
- the set F of all (weakly or strongly) fair and of finite traces

$\implies$ prove $\mathcal{M}_\infty \cap F \subseteq X$ instead of $\mathcal{M}_\infty \subseteq X$

Fairness (cont.)

Example: `while` $x \geq 0$ `do` $x \leftarrow x + 1$ `done` $||$ $x \leftarrow -1$

- may not terminate without fairness
- always terminates under weak and strong fairness

Finite prefix trace abstraction

$\mathcal{M}_\infty \cap F \subseteq X$ is abstracted into testing $\alpha_{*\preceq}(\mathcal{M}_\infty \cap F) \subseteq \alpha_{*\preceq}(X)$

for all fairness conditions F, $\alpha_{*\preceq}(\mathcal{M}_\infty \cap F) = \alpha_{*\preceq}(\mathcal{M}_\infty) = \mathcal{T}_p$

$\Longrightarrow$ fairness-dependent properties cannot be proved with finite prefixes only

In the following, we ignore fairness conditions.

(see [Cous85])

Equational state semantics

State abstraction $\mathcal{R}$: as before

- $\mathcal{R} \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0 \stackrel{t_0}{\to} \cdots \sigma_n \colon \sigma_0 \in \mathcal{I}\ \forall i < n \colon \sigma_i \stackrel{t_i}{\to}_\tau \sigma_{i+1} \wedge \sigma = \sigma_n \,\}$
- $\mathcal{R} = \alpha_p(\mathcal{T}_p)$ where $\alpha_p(X) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0 \stackrel{t_0}{\to} \cdots \sigma_n \in X \colon \sigma = \sigma_n \,\}$
- $\mathcal{R} = \text{lfp}\, F_{\mathcal{R}}$ where $F_{\mathcal{R}}(X) = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma' \in X, t \in \mathbb{T} \colon \sigma' \stackrel{t}{\to}_\tau \sigma \,\}$

Equational form: (without handling errors in Ω)

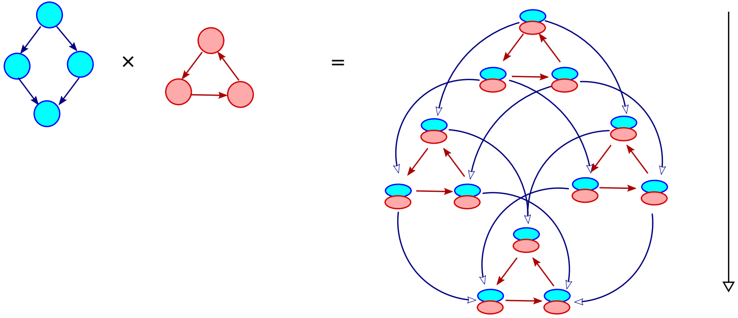
- for each $L \in \mathbb{T} \to \mathcal{L}$, a variable $\mathcal{X}_L$ with value in $\mathcal{E}$
- equations are derived from thread equations $eq(\texttt{prog}_t)$ as:

$$\begin{array}{ll}\mathcal{X}_{L_1} = \bigcup_{t \in \mathbb{T}} \{ & F(\mathcal{X}_{L_2}, \ldots, X_{L_N}) \mid \\ & \exists (\mathcal{X}_{\ell_1} = F(\mathcal{X}_{\ell_2}, \ldots, \mathcal{X}_{\ell_N})) \in eq(\texttt{prog}_t) \colon \\ & \forall i \leq N \colon L_i(t) = \ell_i,\ \forall u \neq t \colon L_i(u) = L_1(u) \,\}\end{array}$$

Join with $\cup$ equations from $eq(\texttt{prog}_t)$ updating a single thread $t \in \mathbb{T}$.

(See course 3 for the full definition of $eq(\texttt{prog})$.)

Equational state semantics (illustration)



Product of control-flow graphs:

- control state = tuple of program points
  $\Longrightarrow$ combinatorial explosion of abstract states
- transfer functions are duplicated

Equational state semantics (example)

Example: inferring $0 \leq x \leq y \leq 102$

t_1	t_2
$\ell 1$ `while random do`	$\ell 4$ `while random do`
$\ell 2$ `if x < y then`	$\ell 5$ `if y < 100 then`
$\ell 3$ `x ← x + 1`	$\ell 6$ `y ← y + [1,3]`

Equation system:

$$\mathcal{X}_{1,4} = \mathcal{I}$$
$$\mathcal{X}_{2,4} = \mathcal{X}_{1,4} \cup \mathsf{C}[\![x \geq y]\!] \mathcal{X}_{2,4} \cup \mathsf{C}[\![x \leftarrow x+1]\!] \mathcal{X}_{3,4}$$
$$\mathcal{X}_{3,4} = \mathsf{C}[\![x < y]\!] \mathcal{X}_{2,4}$$
$$\mathcal{X}_{1,5} = \mathcal{X}_{1,4} \cup \mathsf{C}[\![y \geq 100]\!] \mathcal{X}_{1,5} \cup \mathsf{C}[\![y \leftarrow y+[1,3]]\!] \mathcal{X}_{1,6}$$
$$\mathcal{X}_{2,5} = \mathcal{X}_{1,5} \cup \mathsf{C}[\![x \geq y]\!] \mathcal{X}_{2,5} \cup \mathsf{C}[\![x \leftarrow x+1]\!] \mathcal{X}_{3,5} \cup \mathcal{X}_{2,4} \cup \mathsf{C}[\![y \geq 100]\!] \mathcal{X}_{2,5} \cup \mathsf{C}[\![y \leftarrow y+[1,3]]\!] \mathcal{X}_{2,6}$$
$$\mathcal{X}_{3,5} = \mathsf{C}[\![x < y]\!] \mathcal{X}_{2,5} \cup \mathcal{X}_{3,4} \cup \mathsf{C}[\![y \geq 100]\!] \mathcal{X}_{3,5} \cup \mathsf{C}[\![y \leftarrow y+[1,3]]\!] \mathcal{X}_{3,6}$$
$$\mathcal{X}_{1,6} = \mathsf{C}[\![y < 100]\!] \mathcal{X}_{1,5}$$
$$\mathcal{X}_{2,6} = \mathcal{X}_{1,6} \cup \mathsf{C}[\![x \geq y]\!] \mathcal{X}_{2,6} \cup \mathsf{C}[\![x \leftarrow x+1]\!] \mathcal{X}_{3,6} \cup \mathsf{C}[\![y < 100]\!] \mathcal{X}_{2,5}$$
$$\mathcal{X}_{3,6} = \mathsf{C}[\![x < y]\!] \mathcal{X}_{2,6} \cup \mathsf{C}[\![y < 100]\!] \mathcal{X}_{3,5}$$

Equational state semantics (example)

Example: inferring $0 \le x \le y \le 102$

t_1	t_2
$\ell1$ `while random do` $\ell2$ `if x < y then` $\ell3$ `x ← x + 1`	$\ell4$ `while random do` $\ell5$ `if y < 100 then` $\ell6$ `y ← y + [1,3]`

Pros:

- easy to construct
- easy to further abstract in an abstract domain $\mathcal{E}^\sharp$

Cons:

- explosion of the number of variables and equations
- explosion of the size of equations
  $\Longrightarrow$ efficiency issues
- the equation system does *not* reflect the program structure
  (not defined by induction on the concurrent program)

Wish-list

We would like to:

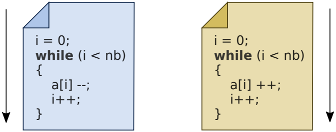
- keep information attached to syntactic program locations
  (control points in $\mathcal{L}$, not control point tuples in $\mathbb{T} \to \mathcal{L}$)
- be able to abstract away control information
  (precision/cost trade-off control)
- avoid duplicating thread instructions
- have a computation structure based on the program syntax
  (denotational style)

Ideally: thread-modular denotational-style semantics

(analyze each thread independently by induction on its syntax)

Simple interference semantics

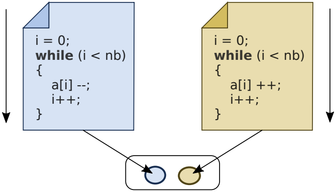
Thread-modular analysis with simple interferences



Principle:

- analyze each thread in isolation

Thread-modular analysis with simple interferences

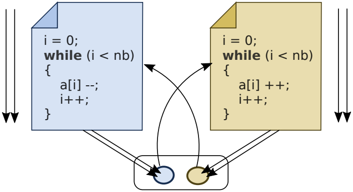


Principle:

- analyze each thread in isolation
- gather the values written into each variable by each thread
  $\Longrightarrow$ so-called interferences

  suitably abstracted in an abstract domain, such as intervals

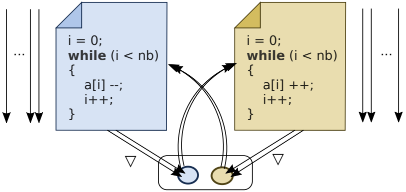
Thread-modular analysis with simple interferences



Principle:

- analyze each thread in isolation
- gather the values written into each variable by each thread
  $\Longrightarrow$ so-called interferences
  suitably abstracted in an abstract domain, such as intervals
- reanalyze threads, injecting these values at each read

Thread-modular analysis with simple interferences



Principle:

- analyze each thread in isolation
- gather the values written into each variable by each thread
  $\Longrightarrow$ so-called interferences
  suitably abstracted in an abstract domain, such as intervals
- reanalyze threads, injecting these values at each read
- iterate until stabilization while widening interferences
  $\Longrightarrow$ one more level of fixpoint iteration

Example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x ← x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y ← y + [1, 3]
```

Example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x ← x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y ← y + [1, 3]
```

Analysis of t_1 in isolation

(1): $x = y = 0$ $\qquad \mathcal{X}_1 = I$

(2): $x = y = 0$ $\qquad \mathcal{X}_2 = \mathcal{X}_1 \cup \mathsf{C}[\![\, x \leftarrow x + 1 \,]\!]\, \mathcal{X}_3 \cup \mathsf{C}[\![\, x \geq y \,]\!]\, \mathcal{X}_2$

(3): $\bot$ $\qquad \mathcal{X}_3 = \mathsf{C}[\![\, x < y \,]\!]\, \mathcal{X}_2$

Example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x ← x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y ← y + [1, 3]
```

Analysis of t_2 in isolation

(4): $x = y = 0$ $\quad \mathcal{X}_4 = I$

(5): $x = 0,\ y \in [0, 102]$ $\quad \mathcal{X}_5 = \mathcal{X}_4 \cup \mathsf{C}[\![\, y \leftarrow y + [1,3] \,]\!]\, \mathcal{X}_6 \cup \mathsf{C}[\![\, y \geq 100 \,]\!]\, \mathcal{X}_5$

(6): $x = 0,\ y \in [0, 99]$ $\quad \mathcal{X}_6 = \mathsf{C}[\![\, y < 100 \,]\!]\, \mathcal{X}_5$

output interferences: $y \leftarrow [1, 102]$

Example

t_1

$\ell 1$ `while random do`
  $\ell 2$ `if x < y then`
    $\ell 3$ $x \leftarrow x + 1$

t_2

$\ell 4$ `while random do`
  $\ell 5$ `if y < 100 then`
    $\ell 6$ $y \leftarrow y + [1, 3]$

Re-analysis of t_1 with interferences from t_2

input interferences: $y \leftarrow [1, 102]$

(1): $x = y = 0$ $\qquad \mathcal{X}_1 = I$
(2): $x \in [0, 102],\ y = 0$ $\qquad \mathcal{X}_2 = \mathcal{X}_{1a} \cup C[\![x \leftarrow x + 1]\!] \mathcal{X}_3 \cup C[\![x \geq (y \mid [1, 102])]\!] \mathcal{X}_2$
(3): $x \in [0, 102],\ y = 0$ $\qquad \mathcal{X}_3 = C[\![x < (y \mid [1, 102])]\!] \mathcal{X}_2$

output interferences: $x \leftarrow [1, 102]$

subsequent re-analyses are identical (fixpoint reached)

Example

t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x ← x + 1
```

t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y ← y + [1, 3]
```

Derived abstract analysis:

- similar to a sequential program analysis, but iterated
  (can be parameterized by arbitrary abstract domains)
- efficient (few reanalyses are required in practice)
- interferences are non-relational and flow-insensitive
  (limit inherited from the concrete semantics)

Limitation:

we get $x, y \in [0, 102]$; we don't get that $x \leq y$

simplistic view of thread interferences (volatile variables)

based on an incomplete concrete semantics!

Denotational semantics with interferences

Interferences in $\mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{V} \times \mathbb{R}$

$\langle\, t,\, X,\, v \,\rangle$ means: t can store the value v into the variable X

We define the analysis of a thread t
with respect to a set of interferences $I \subseteq \mathbb{I}$.

Expressions with interference: for thread t

$$\mathrm{E_t}[\![\, \mathrm{exp} \,]\!] : (\mathcal{E} \times \mathcal{P}(\mathbb{I})) \to (\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\Omega))$$

- Apply interferences to read variables:
  $\mathrm{E_t}[\![\, X \,]\!]\langle\, \rho,\, I \,\rangle \stackrel{\text{def}}{=} \langle\, \{\, \rho(X) \,\} \cup \{\, v \mid \exists u \neq t \colon \langle\, u,\, X,\, v \,\rangle \in I \,\},\, \emptyset \,\rangle$
- Pass recursively I down to sub-expressions:
  $\mathrm{E_t}[\![\, -e \,]\!]\langle\, \rho,\, I \,\rangle \stackrel{\text{def}}{=}$
  $\quad \text{let } \langle\, V,\, O \,\rangle = \mathrm{E_t}[\![\, e \,]\!]\langle\, \rho,\, I \,\rangle \text{ in } \langle\, \{\, -v \mid v \in V \,\},\, O \,\rangle$

  ...

Denotational semantics with interferences (cont.)

Statements with interference: for thread t

$$\mathsf{C_t}⟦\,\texttt{prog}\,⟧ : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})) \rightarrow (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I}))$$

- pass interferences to expressions
- collect new interferences due to assignments
- accumulate interferences from inner statements

$$\mathsf{C_t}⟦\,X \leftarrow e\,⟧\langle\,R,\,O,\,I\,\rangle \stackrel{\text{def}}{=}$$
$$\langle\,\emptyset,\,O,\,I\,\rangle \sqcup \bigsqcup_{\rho\in R}\langle\,\{\,\rho[X\mapsto v]\mid v\in V_\rho\,\},\,O_\rho,\,\{\,\langle\,t,\,X,\,v\,\rangle\mid v\in V_\rho\,\}\,\rangle$$

$$\mathsf{C_t}⟦\,s_1;\,s_2\,⟧ \stackrel{\text{def}}{=} \mathsf{C_t}⟦\,s_2\,⟧ \circ \mathsf{C_t}⟦\,s_1\,⟧$$

...

noting $\langle\,V_\rho,\,O_\rho\,\rangle \stackrel{\text{def}}{=} \mathsf{E_t}⟦\,e\,⟧\langle\,\rho,\,I\,\rangle$
$\sqcup$ is now the element-wise $\cup$ in $\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$

Denotational semantics with interferences (cont.)

Program semantics: $\mathsf{P}[\![\,\texttt{parprog}\,]\!] \subseteq \Omega$

Given $\texttt{parprog} ::= \texttt{prog}_1 \mid\mid \cdots \mid\mid \texttt{prog}_n$, we compute:

$$\mathsf{P}[\![\,\texttt{parprog}\,]\!] \stackrel{\text{def}}{=} \left[\mathsf{lfp}\,\lambda\langle\, O,\, I\,\rangle.\bigsqcup_{t\in\mathbb{T}}\, [\mathsf{C_t}[\![\,\texttt{prog}_t\,]\!]\,\langle\, \mathcal{E}_0,\, \emptyset,\, I\,\rangle]_{\Omega,\mathbb{I}}\right]_{\Omega}$$

- each thread analysis starts in an initial environment set $\mathcal{E}_0 \stackrel{\text{def}}{=} \{\,\lambda V.0\,\}$
- $[X]_{\Omega,\mathbb{I}}$ projects $X \in \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$ on $\mathcal{P}(\Omega) \times \mathcal{P}(\mathbb{I})$ and interferences and errors from all threads are joined

  (the output environments are ignored)
- $\mathsf{P}[\![\,\texttt{parprog}\,]\!]$ only outputs the set of possible run-time errors

Interference abstraction

Abstract interferences $\mathbb{I}^\sharp$

$\mathcal{P}(\mathbb{I}) \stackrel{\text{def}}{=} \mathcal{P}(\mathbb{T} \times \mathbb{V} \times \mathbb{R})$ is abstracted as $\mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{V}) \to \mathcal{R}^\sharp$
where $\mathcal{R}^\sharp$ abstracts $\mathcal{P}(\mathbb{R})$ (e.g. intervals)

Abstract semantics with interferences $\mathsf{C}^\sharp_t[\![s]\!]$

derived from $\mathsf{C}^\sharp[\![s]\!]$ in a generic way:

Example: $\mathsf{C}^\sharp_t[\![X \leftarrow e]\!] \langle R^\sharp, \Omega, I^\sharp \rangle$

- for each Y in e, get its interference $Y^\sharp_\mathcal{R} = \bigsqcup^\sharp_\mathcal{R} \{ I^\sharp\langle u, Y \rangle \mid u \neq t \}$
- if $Y^\sharp_\mathcal{R} \neq \bot^\sharp_\mathcal{R}$, replace Y in e with $get\langle Y, R^\sharp \rangle \sqcup^\sharp_\mathcal{R} Y^\sharp_\mathcal{R}$
  (where $get(Y, R^\sharp)$ extracts the abstract values in $\mathcal{R}^\sharp$ of a variable Y from $R^\sharp \in \mathcal{E}^\sharp$)
- compute $\langle R^{\sharp\prime}, O' \rangle = \mathsf{C}^\sharp[\![e]\!] \langle R^\sharp, O \rangle$
- enrich $I^\sharp\langle t, X \rangle$ with $get(X, R^{\sharp\prime})$

Static analysis with interferences

Abstract analysis

$$\mathsf{P}^\sharp[\![\,\mathtt{parprog}\,]\!] \stackrel{\text{def}}{=} \left[\lim \lambda\langle\, O,\ I^\sharp\,\rangle.\langle\, O,\ I^\sharp\,\rangle \triangledown \bigsqcup\nolimits^\sharp_{t\in\mathbb{T}} \left[\mathsf{C}^\sharp_t[\![\,\mathtt{prog}_t\,]\!]\,\langle\,\mathcal{E}^\sharp_0,\ \emptyset,\ I^\sharp\,\rangle\right]_{\Omega,\mathbb{I}^\sharp}\right]_\Omega$$

- effective analysis by structural induction
- termination ensured by a widening
- parametrized by a choice of abstract domains $\mathcal{R}^\sharp$, $\mathcal{E}^\sharp$

- interferences are flow-insensitive and non-relational in $\mathcal{R}^\sharp$
- thread analysis remains flow-sensitive and relational in $\mathcal{E}^\sharp$

(reminder: $[X]_\Omega$, $[Y]_{\Omega,\mathbb{I}^\sharp}$ keep only X's component in Ω, Y's components in Ω and $\mathbb{I}^\sharp$)

Path-based semantics

Control paths

$atomic ::= X \leftarrow \texttt{exp} \mid \texttt{exp} \bowtie 0?$

Control paths

$\pi : \texttt{prog} \to \mathcal{P}(atomic^*)$

$\pi(X \leftarrow e) \stackrel{\text{def}}{=} \{ X \leftarrow e \}$

$\pi(\texttt{if } e \bowtie 0 \texttt{ then } s \texttt{ fi}) \stackrel{\text{def}}{=} (\{ e \bowtie 0? \} \cdot \pi(s)) \cup \{ e \not\bowtie 0? \}$

$\pi(\texttt{while } e \bowtie 0 \texttt{ do } s \texttt{ done}) \stackrel{\text{def}}{=} \left(\bigcup_{i \geq 0} (\{ e \bowtie 0? \} \cdot \pi(s))^i \right) \cdot \{ e \not\bowtie 0? \}$

$\pi(s_1;\, s_2) \stackrel{\text{def}}{=} \pi(s_1) \cdot \pi(s_2)$

$\pi(\texttt{prog})$ is a (generally infinite) set of finite control paths

Path-based concrete semantics of sequential programs

Join-over-all-path semantics

$\Pi[\![P]\!] : (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \to (\mathcal{P}(\mathcal{E}) \times \mathcal{P}(\Omega)) \qquad P \subseteq atomic^*$

$$\Pi[\![P]\!]\langle R, O \rangle \stackrel{\text{def}}{=} \bigsqcup_{s_1 \cdot \ldots \cdot s_n \in P} (\mathsf{C}[\![s_n]\!] \circ \cdots \circ \mathsf{C}[\![s_1]\!])\langle R, O \rangle$$

Semantic equivalence

$$\mathsf{C}[\![\texttt{prog}]\!] = \Pi[\![\pi(\texttt{prog})]\!]$$

(not true in the abstract)

Advantages:

- easily extended to concurrent programs (path interleavings)
- able to model program transformations (weak memory models)

Path-based concrete semantics of concurrent programs

Concurrent control paths

$$\pi_* \stackrel{\text{def}}{=} \{\text{ interleavings of } \pi(\texttt{prog}_t),\ t \in \mathbb{T} \,\}$$
$$= \{\, p \in \mathit{atomic}^* \mid \forall t \in \mathbb{T},\ \mathit{proj}_t(p) \in \pi(\texttt{prog}_t) \,\}$$

Interleaving program semantics

$$\mathsf{P}_*[\![\texttt{parprog}]\!] \stackrel{\text{def}}{=} [\, \Pi[\![\pi_*]\!] \langle \mathcal{E}_0, \emptyset \rangle \,]_\Omega$$

($\mathit{proj}_t(p)$ keeps only the atomic statement in p coming from thread t)

($\simeq$ sequentially consistent executions [Lamport 79])

Issues:

- too many paths to consider exhaustively
- no induction structure to iterate on

  $\Longrightarrow$ abstract as a denotational semantics
- unrealistic assumptions on granularity and memory consistency

Soundness of the interference semantics

Soundness theorem

$\mathsf{P}_*[\![\, \texttt{parprog} \,]\!] \subseteq \mathsf{P}[\![\, \texttt{parprog} \,]\!]$

Proof sketch:

- define $\sqcap_t[\![\, P \,]\!]\, X \stackrel{\text{def}}{=} \bigsqcup \{\, \mathsf{C}_\mathrm{t}[\![\, s_1; \ldots; s_n \,]\!]\, X \mid s_1 \cdot \ldots \cdot s_n \in P \,\}$,
  then $\sqcap_t[\![\, \pi(s) \,]\!] = \mathsf{C}_\mathrm{t}[\![\, s \,]\!]$;
- given the interference fixpoint $I \subseteq \mathbb{I}$ from $\mathsf{P}[\![\, \texttt{parprog} \,]\!]$,
  prove by recurrence on the length of $p \in \pi_*$ that:
  - $\forall t \in \mathbb{T}, \forall \rho \in [\sqcap[\![\, p \,]\!]\langle\, \mathcal{E}_0,\, \emptyset \,\rangle]_{\mathcal{E}}$,
    $\exists \rho' \in [\sqcap_t[\![\, \mathit{proj}_t(p) \,]\!]\langle\, \mathcal{E}_0,\, \emptyset,\, I \,\rangle]_{\mathcal{E}}$ such that
    $\forall X \in \mathbb{V},\ \rho(X) = \rho'(X)$ or $\langle\, u,\, X,\, \rho(X) \,\rangle \in I$ for some $u \neq t$.
  - $[\sqcap[\![\, p \,]\!]\langle\, \mathcal{E}_0,\, \emptyset \,\rangle]_\Omega \subseteq \bigcup_{t \in \mathbb{T}} [\sqcap_t[\![\, \mathit{proj}_t(p) \,]\!]\langle\, \mathcal{E}_0,\, \emptyset,\, I \,\rangle]_\Omega$

Note: sound but not complete

Weakly consistent memories

Issues with weak consistency

program written

$F_1 \leftarrow 1$;
`if` $F_2 = 0$ `then`
  S_1
`fi`
	$F_2 \leftarrow 1$;
	`if` $F_1 = 0$ `then`
	  S_2
	`fi`

(simplified Dekker mutual exclusion algorithm)

S_1 and S_2 cannot execute simultaneously.

Issues with weak consistency

program written

$F_1 \leftarrow 1$;
`if` $F_2 = 0$ `then`
S_1
`fi`
	$F_2 \leftarrow 1$;
	`if` $F_1 = 0$ `then`
	S_2
	`fi`

$\longrightarrow$

program executed

`if` $F_2 = 0$ `then`
$F_1 \leftarrow 1$;
S_1
`fi`
	`if` $F_1 = 0$ `then`
	$F_2 \leftarrow 1$;
	S_2
	`fi`

(simplified Dekker mutual exclusion algorithm)

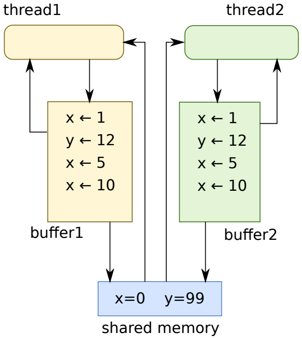
S_1 and S_2 can execute simultaneously.
Not a sequentially consistent behavior!

Caused by:

- write FIFOs, caches, distributed memory
- hardware or compiler optimizations, transformations
- ...

behavior accepted by Java [Mans05]

Hardware memory model example: TSO



Total Store Ordering: model for intel x86

- each thread writes to a FIFO queue
- queues are flushed non-deterministically to the shared memory
- a thread reads back from its queue if possible
  and from shared memory otherwise

Out of thin air principle

original program

R1 ← X; Y ← R1	R ← Y; X ← R2

(example from causality test case #4 for Java by Pugh et al.)

We should not have $R_1 = 42$.

Out of thin air principle

original program

R1 ← X; Y ← R1	R ← Y; X ← R2

⟶

"optimized" program

Y ← 42; R1 ← X; Y ← R1	 R2 ← Y; X ← R2

(example from causality test case #4 for Java by Pugh et al.)

We should not have $R_1 = 42$.

Possible if we allow speculative writes!
$\Longrightarrow$ we disallow this kind of program transformations.

(also forbidden in Java)

Atomicity and granularity

original program

$X \leftarrow X + 1 \mid X \leftarrow X + 1$

We assumed that assignments are atomic. . .

Atomicity and granularity

original program

$X \leftarrow X+1 \mid X \leftarrow X+1$

$\longrightarrow$

executed program

$$\begin{array}{l|l} r_1 \leftarrow X+1 & r_2 \leftarrow X+1 \\ X \leftarrow r_1 & X \leftarrow r_2 \end{array}$$

We assumed that assignments are atomic...
but that may not be the case

The second program admits more behaviors
e.g.: $X = 1$ at the end of the program
[Reyn04]

Path-based definition of weak consistency

Acceptable control path transformations: $p \rightsquigarrow q$

only reduce interferences and errors

- Reordering: $X_1 \leftarrow e_1 \cdot X_2 \leftarrow e_2 \rightsquigarrow X_2 \leftarrow e_2 \cdot X_1 \leftarrow e_1$
  (if $X_1 \notin var(e_2)$, $X_2 \notin var(e_1)$, and e_1 does not stop the program)
- Propagation: $X \leftarrow e \cdot s \rightsquigarrow X \leftarrow e \cdot s[e/X]$
  (if $X \notin var(e)$, $var(e)$ are thread-local, and e is deterministic)
- Factorization: $s_1 \cdot \ldots \cdot s_n \rightsquigarrow X \leftarrow e \cdot s_1[X/e] \cdot \ldots \cdot s_n[X/e]$
  (if X is fresh, $\forall i$, $var(e) \cap lval(s_i) = \emptyset$, and e has no error)
- Decomposition: $X \leftarrow e_1 + e_2 \rightsquigarrow T \leftarrow e_1 \cdot X \leftarrow T + e_2$
  (change of granularity)
- ...

but **NOT:**

- "out-of-thin-air" writes: $X \leftarrow e \rightsquigarrow X \leftarrow 42 \cdot X \leftarrow e$

Soundness of the interference semantics

Interleaving semantics of transformed programs $P'_*[\![\, \texttt{parprog} \,]\!]$

- $\pi'(s) \stackrel{\text{def}}{=} \{\, p \mid \exists p' \in \pi(s) \colon p' \rightsquigarrow^* p \,\}$
- $\pi'_* \stackrel{\text{def}}{=} \{\,\text{interleavings of } \pi'(\texttt{prog}_t),\ t \in \mathbb{T}\,\}$
- $P'_*[\![\, \texttt{parprog} \,]\!] \stackrel{\text{def}}{=} [\, \Pi[\![\, \pi'_* \,]\!] \langle \mathcal{E}_0, \emptyset \rangle \,]_\Omega$

Soundness theorem

$P'_*[\![\, \texttt{parprog} \,]\!] \subseteq P[\![\, \texttt{parprog} \,]\!]$

$\Longrightarrow$ the interference semantics is sound
wrt. weakly consistent memories and changes of granularity

Locks

Scheduling

Synchronization primitives

$$\begin{array}{lcl} \texttt{prog} & ::= & \texttt{lock}(m) \\ & | & \texttt{unlock}(m) \end{array}$$

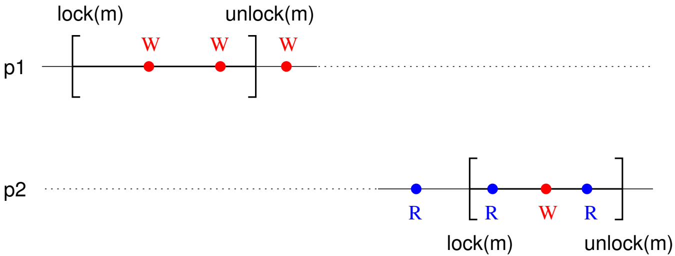
$m \in \mathbb{M}$: finite set of non-recursive mutexes

Scheduling

mutexes ensure mutual exclusion

at each time, each mutex can be locked by a single thread

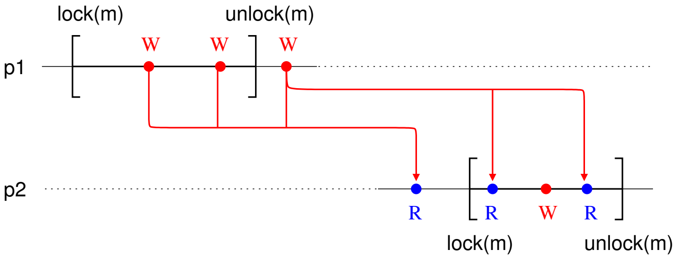
Mutual exclusion



We use a refinement of the simple interference semantics by partitioning wrt. an abstract local view of the scheduler $\mathbb{C}$

- $\mathcal{E} \rightsquigarrow \mathcal{E} \times \mathbb{C}, \quad \mathcal{E}^\sharp \rightsquigarrow \mathbb{C} \to \mathcal{E}^\sharp$
- $\mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{V} \times \mathbb{R} \rightsquigarrow \mathbb{I} \stackrel{\text{def}}{=} \mathbb{T} \times \mathbb{C} \times \mathbb{V} \times \mathbb{R}$,
  $\mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{V}) \to \mathcal{R}^\sharp \rightsquigarrow \mathbb{I}^\sharp \stackrel{\text{def}}{=} (\mathbb{T} \times \mathbb{C} \times \mathbb{V}) \to \mathcal{R}^\sharp$

Mutual exclusion



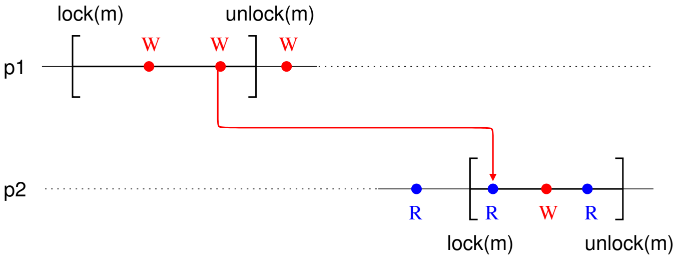
Data-race effects

Across read / write not protected by a mutex.
Partition wrt. mutexes $M \subseteq \mathbb{M}$ held by the current thread t.

- $C_t[\![X \leftarrow e]\!] \langle \rho, M, I \rangle$ adds $\{ \langle t, M, X, v \rangle \mid v \in E_t[\![X]\!] \langle \rho, M, I \rangle \}$ to I
- $E_t[\![X]\!] \langle \rho, M, I \rangle =$
  $\{ \rho(X) \} \cup \{ v \mid \langle t', M', X, v \rangle \in I,\ t \neq t',\ M \cap M' = \emptyset \}$

Bonus: we get a data-race analysis for free!

Mutual exclusion



Well-synchronized effects

- last write before unlock affects first read after lock
- partition interferences wrt. a protecting mutex m (and M)
- $C_t[\![\,\texttt{unlock}(m)\,]\!]\langle\,\rho,\,M,\,I\,\rangle$ stores $\rho(X)$ into I
- $C_t[\![\,\texttt{lock}(m)\,]\!]\langle\,\rho,\,M,\,I\,\rangle$ imports values form I into ρ
- imprecision: non-relational, largely flow-insensitive

Example analysis

abstract consumer/producer

N consumers	N producers

`while 0=0 do`
`  lock(m);`$^{\ell 1}$
`  if X>0 then `$^{\ell 2}$`X←X-1 fi;`
`  unlock(m);`
`  `$^{\ell 3}$`Y←X`
`done`
	`while 0=0 do`
	`  lock(m);`
	`  X←X+1;`
	`  if X>100 then X←100 fi;`
	`  unlock(m)`
	`done`

Assuming we have several (N) producers and consumers:

- no data-race interference (proof of the absence of data-race)
- well-synchronized interferences:
  consumer: $x \leftarrow [0, 99]$
  producer: $x \leftarrow [1, 100]$
- $\implies$ we get that $x \in [0, 100]$

(without locks, if $N > 1$, our concrete semantics cannot bound x!)

Locks and priorities

priority-based critical sections

high thread	low thread
`L ← isLocked(m);`	`lock(m);`
`if L = 0 then`	`Z ← Y;`
`  Y ← Y+1;`	`Y ← 0;`
`  yield()`	`unlock(m)`

Real-time scheduling

- only the highest priority unblocked thread can run
- `lock` and `yield` may block
- `yielding` threads wake up non-deterministically
  preempting lower-priority threads
- explicit synchronisation enforces memory consistency
  prevents data races

Locks and priorities

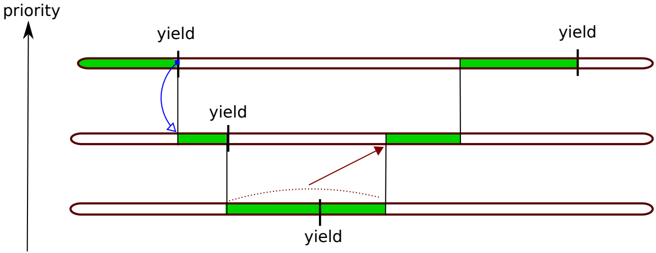
priority-based critical sections

high thread	low thread
`L ← isLocked(m);`	`lock(m);`
`if L = 0 then`	`Z ← Y;`
`  Y ← Y+1;`	`Y ← 0;`
`  yield()`	`unlock(m)`

Partition interferences and environments wrt. scheduling state

- partition wrt. mutexes tested with `isLocked`
- `X ← isLocked(m)` creates two partitions
  - P_0 where $X = 0$ and m is free
  - P_1 where $X = 1$ and m is locked
- P_0 handled as if m where locked
- blocking primitives merge P_0 and P_1 (`lock`, `yield`)

Priority-based scheduling

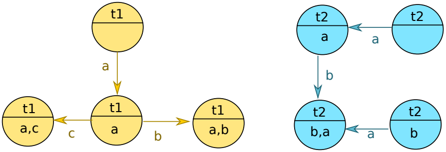


Analysis: refined transfer of interference based on priority

- partition interferences wrt. thread and priority
  support for manual priority change, and for priority ceiling protocol
- higher priority processes inject state from yield into every point
- lower priority processes inject data-race interferences into yield

Deadlock checking

t_1	t_2
lock(a)	lock(a)
lock(c)	lock(b)
unlock(c)	unlock(a)
lock(b)	lock(a)
unlock(b)	unlock(a)
unlock(a)	unlock(b)

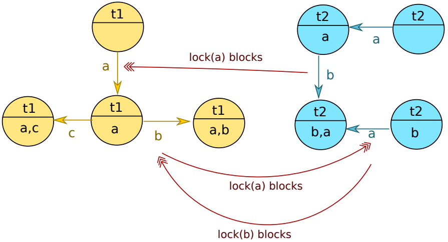


During the analysis, gather:

- all reachable mutex configurations: $R \subseteq \mathbb{T} \times \mathcal{P}(\mathbb{M})$
- lock instructions from these configurations $R \times \mathbb{M}$

Deadlock checking

t_1	t_2
lock(a)	lock(a)
lock(c)	lock(b)
unlock(c)	unlock(a)
lock(b)	lock(a)
unlock(b)	unlock(a)
unlock(a)	unlock(b)



During the analysis, gather:

- all reachable mutex configurations: $R \subseteq \mathbb{T} \times \mathcal{P}(\mathbb{M})$
- lock instructions from these configurations $R \times \mathbb{M}$

Then, construct a blocking graph between lock instructions

- $((t, m), \ell)$ blocks $((t', m'), \ell')$ if

  $t \neq t'$ and $m \cap m' = \emptyset$ (configurations not in mutual exclusion)

  $\ell \in m'$ (blocking lock)

A deadlock is a cycle in the blocking graph.

generalization to larger cycles, with more threads involved in a deadlock, is easy

Abstract rely-guarantee

Rely-guarantee proof method

Reminder: Floyd–Hoare logic

Logic to prove properties about sequential programs [Hoar69].

Hoare triples: $\{P\}\,\texttt{prog}\,\{Q\}$

- annotate programs with logic assertions $\{P\}\,\texttt{prog}\,\{Q\}$

  (if P holds before prog, then Q holds after prog)

- check that $\{P\}\texttt{prog}\{Q\}$ is derivable with the following rules

  (the assertions are program invariants)

$$\frac{}{\{P[e/X]\}\,X \leftarrow e\,\{P\}} \qquad \frac{\{P \wedge e \bowtie 0\}\,s\,\{Q\} \quad P \wedge e \not\bowtie 0 \Rightarrow Q}{\{P\}\,\texttt{if}\ e \bowtie 0\ \texttt{then}\ s\ \texttt{fi}\,\{Q\}}$$

$$\frac{\{P\}\,s_1\,\{Q\} \quad \{Q\}\,s_2\,\{R\}}{\{P\}\,s_1; s_2\,\{R\}} \qquad \frac{\{P \wedge e \bowtie 0\}\,s\,\{P\}}{\{P\}\,\texttt{while}\ e \bowtie 0\ \texttt{do}\ s\ \texttt{done}\,\{P \wedge e \not\bowtie 0\}}$$

$$\frac{\{P'\}\,s\,\{Q'\} \quad P \Rightarrow P' \quad Q' \Rightarrow Q}{\{P\}\,s\,\{Q\}}$$

Floyd–Hoare logic as abstract interpretation

Link with the equational state semantics: $(\mathcal{X}_\ell)_{\ell \in \mathcal{L}}$

Correspondence between ${}^{\ell}\texttt{prog}^{\ell'}$ and $\{P\}\,\texttt{prog}\,\{Q\}$:

- if P (resp. Q) models exactly the points in $\mathcal{X}_\ell$ (resp. $\mathcal{X}_{\ell'}$) then $\{P\}\,\texttt{prog}\,\{Q\}$ is a derivable Hoare triple
- if $\{P\}\,\texttt{prog}\,\{Q\}$ is derivable, then $\mathcal{X}_\ell \models P$ and $\mathcal{X}_{\ell'} \models Q$
  (all the points in $\mathcal{X}_\ell$ (resp. $\mathcal{X}_{\ell'}$) satisfy P (resp. Q))

$\Longrightarrow \mathcal{X}_\ell$ provides the most precise Hoare assertions in a constructive form

- $\gamma(\mathcal{X}^\sharp_\ell)$ provides (less precise) Hoare assertions in a computable form

Owicki–Gries proof method

Extension of Floyd–Hoare to concurrent programs [Owic76].

Principle: add a new rule, for $||$

$$\frac{\{P_1\}\, s_1 \,\{Q_1\} \quad \{P_2\}\, s_2 \,\{Q_2\}}{\{P_1 \wedge P_2\}\, s_1 \,||\, s_2 \,\{Q_1 \wedge Q_2\}}$$

Owicki–Gries proof method

Extension of Floyd–Hoare to concurrent programs [Owic76].

Principle: add a new rule, for $||$

$$\frac{\{P_1\}\, s_1\, \{Q_1\} \quad \{P_2\}\, s_2\, \{Q_2\}}{\{P_1 \wedge P_2\}\, s_1 \,||\, s_2\, \{Q_1 \wedge Q_2\}}$$

This rule is not always sound!

e.g., we have $\{X = 0, Y = 0\}\, X \leftarrow 1\, \{X = 1, Y = 0\}$

and $\{X = 0, Y = 0\}$ $\texttt{if}$ $X = 0$ $\texttt{then}$ $Y \leftarrow 1$ $\texttt{fi}$ $\{X = 0, Y = 1\}$

but not $\{X = 0, Y = 0\}\, X \leftarrow 1 \,||$ $\texttt{if}$ $X = 0$ $\texttt{then}$ $Y \leftarrow 1$ $\texttt{fi}$ $\{\mathit{false}\}$

$\Longrightarrow$ we need a side-condition to the rule:
$\{P_1\}\, s_1\, \{Q_1\}$ and $\{P_2\}\, s_2\, \{Q_2\}$ must not interfere

Owicki–Gries proof method (cont.)

interference freedom

given proofs Δ_1 and Δ_2 of $\{P_1\}\, s_1\, \{Q_1\}$ and $\{P_2\}\, s_2\, \{Q_2\}$

Δ_1 does not interfere with Δ_2 if:
for any Φ appearing before a statement in Δ_1
for any $\{P'_2\}\, s'_2\, \{Q'_2\}$ appearing in Δ_2
$\{\Phi \wedge P'_2\}\, s'_2\, \{\Phi\}$ holds
and moreover $\{Q_1 \wedge P'_2\}\, s'_2\, \{Q_1\}$

i.e.: the assertions used to prove $\{P_1\}\, s_1\, \{Q_1\}$ are stable by s_2

e.g.,
$\{X = 0, Y \in [0,1]\}\; X \leftarrow 1\; \{X = 1, Y \in [0,1]\}$
$\{X \in [0,1], Y = 0\}\; \texttt{if}\ X = 0\ \texttt{then}\ Y \leftarrow 1\ \texttt{fi}\; \{X \in [0,1], Y \in [0,1]\}$
$\Longrightarrow$ $\{X = 0, Y = 0\}\; X \leftarrow 1 \,||\, \texttt{if}\ X = 0\ \texttt{then}\ Y \leftarrow 1\ \texttt{fi}\; \{X = 1, Y \in [0,1]\}$

Summary:

- pros: the invariants are local to threads
- cons: the proof is not compositional

  (proving one thread requires delving into the proof of other threads)

$\Longrightarrow$ not satisfactory

Jones' rely-guarantee proof method

Idea: explicit interferences with (more) annotations [Jone81].

Rely-guarantee "quintuples": $R, G \vdash \{P\}\ \texttt{prog}\ \{Q\}$

- if P is true before `prog` is executed
- and the effect of other threads is included in R (rely)
- then Q is true after `prog`
- and the effect of `prog` is included in G (guarantee)

where:

- P and Q are assertions on states (in $\mathcal{P}(\Sigma)$)
- R and G are assertions on transitions (in $\mathcal{P}(\Sigma \times \mathcal{A} \times \Sigma)$)

The parallel composition rule becomes:

$$\frac{R \vee G_2, G_1 \vdash \{P_1\}\ s_1\ \{Q_1\} \qquad R \vee G_1, G_2 \vdash \{P_2\}\ s_2\ \{Q_2\}}{R, G_1 \vee G_2 \vdash \{P_1 \wedge P_2\}\ s_1 \mid\mid s_2\ \{Q_1 \wedge Q_2\}}$$

Rely-guarantee example

checking t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x ← x+1
  fi
done
```

$\ell 1: \; x = y = 0$
$\ell 2: \; x, y \in [0, 102], \; x \leq y$
$\ell 3: \; x \in [0, 101], \; y \in [1, 102], \; x < y$

checking t_2

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y ← y + [1,3]
  fi
done
```

at $\ell 4: \; x = y = 0$
at $\ell 5: \; x, y \in [0, 102], \; x \leq y$
at $\ell 6: \; x \in [0, 99], \; y \in [0, 99], \; x \leq y$

Rely-guarantee example

checking t_1

```
ℓ1 while random do
  ℓ2 if x < y then
    ℓ3 x ← x+1
  fi
done
```

x unchanged
y incremented
$0 \leq y \leq 102$

$\ell 1: \; x = y = 0$
$\ell 2: \; x, y \in [0, 102], \; x \leq y$
$\ell 3: \; x \in [0, 101], \; y \in [1, 102], \; x < y$

checking t_2

y unchanged
$0 \leq x \leq y$

```
ℓ4 while random do
  ℓ5 if y < 100 then
    ℓ6 y ← y + [1,3]
  fi
done
```

at $\ell 4: \; x = y = 0$
at $\ell 5: \; x, y \in [0, 102], \; x \leq y$
at $\ell 6: \; x \in [0, 99], \; y \in [0, 99], \; x \leq y$

In this example:

- guarantee exactly what is relied on ($R_1 = G_1$ and $R_2 = G_2$)
- rely and guarantee are global assertions

Benefits of rely-guarantee:

- invariants are still local to threads
- checking a thread does not require looking at other threads, only at an abstraction of their semantics

Auxiliary variables

Example

t_1	t_2
$^{\ell 1}\ x \leftarrow x + 1\ ^{\ell 2}$	$^{\ell 3}\ x \leftarrow x + 1\ ^{\ell 4}$

Goal: prove $\{x = 0\}\ t_1 \mid\mid t_2\ \{x = 2\}$.

Auxiliary variables

Example

t_1	t_2
$\ell 1$ $x \leftarrow x + 1$ $\ell 2$	$\ell 3$ $x \leftarrow x + 1$ $\ell 4$

Goal: prove $\{x = 0\}\ t_1 \mid\mid t_2\ \{x = 2\}$.

we must rely on and guarantee that
each thread increments x exactly once!

Solution: auxiliary variables
do not change the semantics but store extra information:

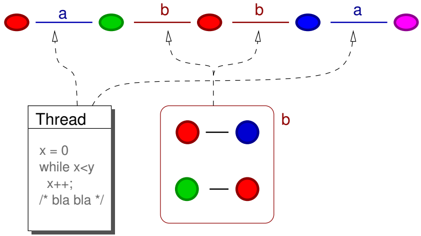
- past values of variables (history of the computation)
- program counter of other threads (pc_t)

Example: for t_1:

$$\{(pc_2 = \ell 3 \wedge x = 0) \vee (pc_2 = \ell 4 \wedge x = 1)\}$$
$$x \leftarrow x + 1$$
$$\{(pc_2 = \ell 3 \wedge x = 1) \vee (pc_2 = \ell 4 \wedge x = 2)\}$$

Rely-guarantee as abstract interpretation

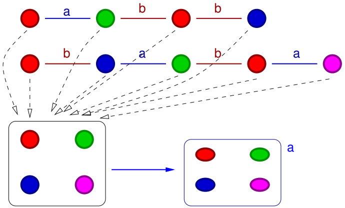
Modularity: main idea



Main idea: separate execution steps

- from the current thread *a*
  - found by analysis by induction on the syntax of *a*
- from other threads *b*
  - given as parameter in the analysis of *a*
  - inferred during the analysis of *b*

Trace decomposition



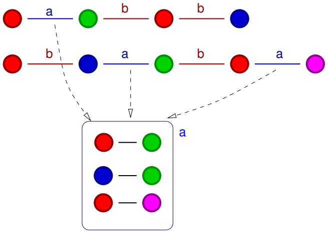
Reachable states projected on thread t: $\mathcal{RI}(t)$

- attached to thread control point in $\mathcal{L}$, not control state in $\mathbb{T} \to \mathcal{L}$
- remember other thread's control point as "auxiliary variables"
  (required for completeness)

$$\mathcal{RI}(t) \stackrel{\text{def}}{=} \pi_t(\mathcal{R}) \;\subseteq \mathcal{L} \times (\mathbb{V} \cup \{\, pc_{t'} \mid t \neq t' \in \mathbb{T} \,\}) \to \mathbb{R}$$

where $\pi_t(R) \stackrel{\text{def}}{=} \{\, \langle\, L(t),\, \rho\,[\forall t' \neq t\colon pc_{t'} \mapsto L(t')] \,\rangle \mid \langle\, L,\, \rho \,\rangle \in R \,\}$

Trace decomposition



Interferences generated by t: $A(t)$ ($\simeq$ guarantees on transitions)

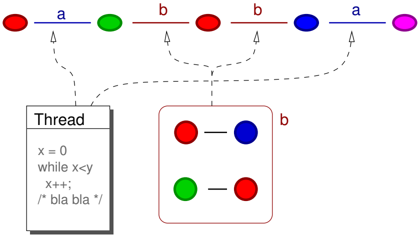
Extract the transitions with action t observed in $\mathcal{T}_p$

(subset of the transition system, containing only transitions actually used in reachability)

$A(t) \stackrel{\text{def}}{=} \alpha^{\mathbb{I}}(\mathcal{T}_p)(t)$

where $\alpha^{\mathbb{I}}(X)(t) \stackrel{\text{def}}{=} \{\, \langle\, \sigma_i,\, \sigma_{i+1} \,\rangle \,|\, \exists \sigma_0 \stackrel{a_1}{\to} \sigma_1 \cdots \stackrel{a_n}{\to} \sigma_n \in X \colon a_{i+1} = t \,\}$

Thread-modular concrete semantics



Principle: express $\mathcal{RI}(t)$ and $A(t)$ directly, without computing $\mathcal{T}_p$

States: $\mathcal{RI}$

Interleave:

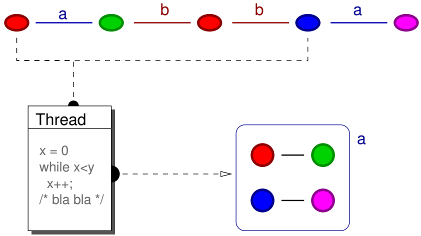
- transitions from the current thread t
- transitions from interferences A by other threads

$\mathcal{RI}(t) = \text{lfp } R_t(A)$, where

$$R_t(Y)(X) \stackrel{\text{def}}{=} \pi_t(I) \cup \{\, \pi_t(\sigma') \mid \exists \pi_t(\sigma) \in X \colon \sigma \xrightarrow{t}_\tau \sigma' \,\} \cup \{\, \pi_t(\sigma') \mid \exists \pi_t(\sigma) \in X \colon \exists t' \neq t \colon \langle \sigma, \sigma' \rangle \in Y(t') \,\}$$

$\Longrightarrow$ similar to reachability for a sequential program, up to A

Thread-modular concrete semantics



Principle: express $\mathcal{R}I(t)$ and $A(t)$ directly, without computing $\mathcal{T}_p$

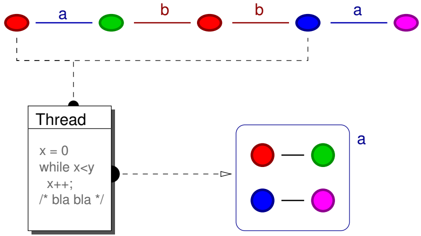
Interferences: A

Collect transitions from a thread t and reachable states $\mathcal{R}$:

$A(t) = B(\mathcal{R}I)(t)$, where

$B(Z)(t) \stackrel{\text{def}}{=} \{ \langle \sigma, \sigma' \rangle \mid \pi_t(\sigma) \in Z(t) \wedge \sigma \xrightarrow{t}_{\tau} \sigma' \}$

Thread-modular concrete semantics



Principle: express $\mathcal{RI}(t)$ and $A(t)$ directly, without computing $\mathcal{T}_p$

Recursive definition:

- $\mathcal{RI}(t) = \mathrm{lfp}\, R_t(A)$
- $A(t) = B(\mathcal{RI})(t)$

$\Longrightarrow$ express the most precise solution as nested fixpoints:

$$\mathcal{RI} = \mathrm{lfp}\, \lambda Z.\lambda t.\, \mathrm{lfp}\, R_t(B(Z))$$

Completeness: $\forall t\colon \mathcal{RI}(t) \simeq \mathcal{R}$ (π_t is bijective thanks to auxiliary variables)

Fixpoint form

Constructive fixpoint form:

Use Kleene's iteration to construct fixpoints:

- $\mathcal{R}I = \text{lfp } H = \bigsqcup_{n\in\mathbb{N}} H^n(\lambda t.\emptyset)$
  in the pointwise powerset lattice $\prod_{t\in\mathbb{T}} \{t\} \to \mathcal{P}(\Sigma_t)$
- $H(Z)(t) = \text{lfp } R_t(B(Z)) = \bigcup_{n\in\mathbb{N}}(R_t(B(Z)))^n(\emptyset)$
  in the powerset lattice $\mathcal{P}(\Sigma_t)$

  (similar to the sequential semantics of thread t in isolation)

$\Longrightarrow$ nested iterations

Abstract rely-guarantee

Suggested algorithm: nested iterations with acceleration

once abstract domains for states and interferences are chosen

- start from $\mathcal{RI}^\sharp_0 \stackrel{\text{def}}{=} A^\sharp_0 \stackrel{\text{def}}{=} \lambda t.\bot^\sharp$
- while $A^\sharp_n$ is not stable
  - compute $\forall t \in \mathbb{T}: \mathcal{RI}^\sharp_{n+1}(t) \stackrel{\text{def}}{=} \text{lfp } R^\sharp_t(A^\sharp_n)$
    by iteration with widening $\triangledown$

    ($\simeq$ separate analysis of each thread)
  - compute $A^\sharp_{n+1} \stackrel{\text{def}}{=} A^\sharp_n \triangledown B^\sharp(\mathcal{RI}^\sharp_{n+1})$
- when $A^\sharp_n = A^\sharp_{n+1}$, return $\mathcal{RI}^\sharp_n$

$\Longrightarrow$ thread-modular analysis
parameterized by abstract domains
able to easily reuse existing sequential analyses

Thread-modular abstractions

Flow-insensitive abstraction

Flow-insensitive abstraction:

- reduce as much control information as possible
- but keep flow-sensitivity on each thread's control location

Local state abstraction: remove auxiliary variables

$$\alpha_{\mathcal{R}}^{nf}(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho_{|\mathbb{V}}) \,|\, (\ell, \rho) \in X \,\} \cup (X \cap \Omega)$$

Interference abstraction: remove all control state

$$\alpha_{A}^{nf}(Y) \stackrel{\text{def}}{=} \{\, (\rho, \rho') \,|\, \exists L, L' \in \mathbb{T} \to \mathcal{L}: ((L, \rho), (L', \rho')) \in Y \,\}$$

Flow-insensitive abstraction (cont.)

Flow-insensitive fixpoint semantics: (omitting errors Ω)

We apply $\alpha^{nf}_{\mathcal{R}}$ and α^{nf}_{A} to the nested fixpoint semantics.

$\mathcal{RI}^{nf} \stackrel{\text{def}}{=} \text{lfp}\, \lambda Z.\lambda t.\, \text{lfp}\, R^{nf}{}_t(B^{nf}(Z))$, where

$B^{nf}(Z)(t) \stackrel{\text{def}}{=} \{\, (\rho,\rho') \mid \exists \ell,\ell' \in \mathcal{L}\colon (\ell,\rho) \in Z(t) \wedge (\ell,\rho) \to_t (\ell',\rho') \,\}$
(extract interferences from reachable states)

$R^{nf}_t(Y)(X) \stackrel{\text{def}}{=} R^{loc}_t(X) \cup A^{nf}_t(Y)(X)$ (interleave steps)

$R^{loc}_t(X) \stackrel{\text{def}}{=} \{(\ell^i_t, \lambda V.0)\} \cup \{\, (\ell',\rho') \mid \exists (\ell,\rho) \in X\colon (\ell,\rho) \to_t (\ell',\rho') \,\}$ (thread step)

$A^{nf}_t(Y)(X) \stackrel{\text{def}}{=} \{\, (\ell,\rho') \mid \exists \rho,\, u \neq t\colon (\ell,\rho) \in X \wedge (\rho,\rho') \in Y(u) \,\}$ (interference step)

where $\to_t$ is the transition relation for thread t alone: $\tau[\texttt{prog}_t]$

Cost/precision trade-off:

- less variables
  $\Longrightarrow$ subsequent numeric abstractions are more efficient
- sufficient to analyze our first example (slide 26)
- insufficient to analyze $x \leftarrow x+1 \,||\, x \leftarrow x+1$ (slide 35)

Retrieving the simple interference-based analysis

Cartesian abstraction: on interferences

- forget the relations between variables
- forget the relations between values before and after transitions
  (input-output relationship)
- only remember which variables are modified, and their value:

$$\alpha_A^{nr}(Y) \stackrel{\text{def}}{=} \lambda V.\{\, x \in \mathbb{V} \mid \exists (\rho, \rho') \in Y: \rho(V) \neq x \wedge \rho'(V) = x \,\}$$

- to apply interferences, we get, in the nested fixpoint form:

$$A_t^{nr}(Y)(X) \stackrel{\text{def}}{=} \{\, (\ell, \rho[V \mapsto v]) \mid (\ell, \rho) \in X, V \in \mathbb{V}, \exists u \neq t: v \in Y(u)(V) \,\}$$

- no modification on the state
  (the analysis of each thread can still be relational)

$\Longrightarrow$ we get back our simple interference analysis!

Finally, use a numeric abstract domain $\alpha : \mathcal{P}(\mathbb{V} \to \mathbb{R}) \to \mathcal{D}^\sharp$
(for interferences, $\mathbb{V} \to \mathcal{P}(\mathbb{R})$ is abstracted as $\mathbb{V} \to \mathcal{D}^\sharp$)

A note on unbounded threads

Extension: relax the finiteness constraint on $\mathbb{T}$

- there is still a finite syntactic set of threads $\mathbb{T}_s$
- some threads $\mathbb{T}_\infty \subseteq \mathbb{T}_s$ can have several instances

  (possibly an unbounded number)

Flow-insensitive analysis:

- local state and interference domains have finite dimensions

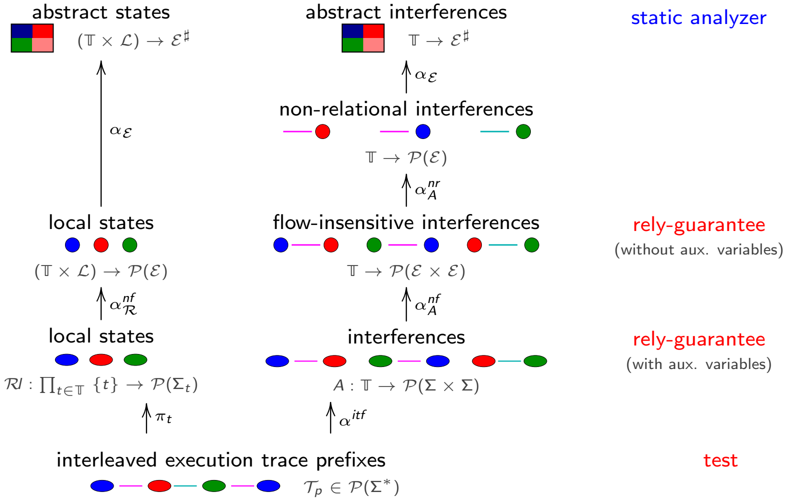
  ($\mathcal{E}_t$ and $(\mathcal{L} \times \mathcal{E}) \times (\mathcal{L} \times \mathcal{E})$, as opposed to $\mathcal{E}$ and $\mathcal{E} \times \mathcal{E}$)
- all instances of a thread $t \in \mathbb{T}_s$ are isomorphic

  $\Longrightarrow$ iterate the analysis on the finite set $\mathbb{T}_s$ (instead of $\mathbb{T}$)
- we must handle self-interferences for threads in $\mathbb{T}_\infty$:

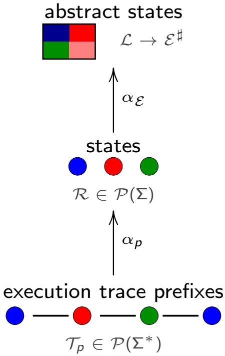
  $A_t^{nf}(Y)(X) \stackrel{\text{def}}{=}$

  $\{\, (\ell, \rho') \mid \exists \rho,\, u\colon (u \neq t \vee t \in \mathbb{T}_\infty) \wedge (\ell, \rho) \in X \wedge (\rho, \rho') \in Y(u) \,\}$

From traces to thread-modular analyses



Compare with sequential analyses



static analyzer

reachability

test

Beyond simple interferences

Weakly relational interferences

Clock thread

```
while Clock < 10^6 do
  Clock ← Clock + 1;
  ...
done
```

Accumulator thread

```
while random do
  Prec ← Clock;
  ...
  delta ← Clock - Prec;
  if random then x ← x + delta endif;
  ...
done
```

- `clock` is a global, increasing clock
- `x` accumulates periods of time
- no overflow on `Clock - Prec`, nor `x ← x + delta`

To prove this we need relational abstractions of interferences
(keep input-output relationships)

Monotonicity abstraction

Abstraction:

map variables to $\nearrow$ monotonic or $\top$ don't know

$$\alpha_A^{\text{mono}}(Y) \stackrel{\text{def}}{=} \lambda V.\text{if } \forall \langle \rho, \rho' \rangle \in Y: \rho(V) \leq \rho'(V) \text{ then } \nearrow \text{ else } \top$$

- keep some input-output relationships
- forgets all relations between variables
- flow-insensitive

Inference and use

- **gather:**
  $A^{\text{mono}}(t)(V) = \nearrow \iff$
  all assignments to V in t have the form $V \leftarrow V + e$, with $e \geq 0$
- **use:** combined with non-relational interferences
  if $\forall t: A^{\text{mono}}(t)(V) = \nearrow$
  then any test with non-relational interference $C[\![X \leq (V \mid [a, b])]\!]$
  can be strengthened into $C[\![X \leq V]\!]$

Relational invariant interferences

Abstraction: keep relations maintained by interferences

- remove control state in interferences (α_A^{nf})
- keep mutex state M (set of mutexes held)
- forget input-output relationships
- keep relationships between variables

$$\alpha_A^{\text{inv}}(Y) \stackrel{\text{def}}{=} \{ \langle M, \rho \rangle \mid \exists \rho' : \langle \langle M, \rho \rangle, \langle M, \rho' \rangle \rangle \in Y \vee \langle \langle M, \rho' \rangle, \langle M, \rho \rangle \rangle \in Y \}$$

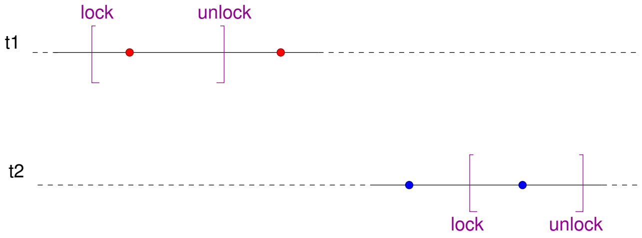
$\langle M, \rho \rangle \in \alpha_A^{\text{inv}}(Y) \implies \langle M, \rho \rangle \in \alpha_A^{\text{inv}}(Y)$ after any sequence of interferences from Y

Lock invariant:

$$\{ \rho \mid \exists t \in \mathcal{T}, M : \langle M, \rho \rangle \in \alpha_A^{\text{inv}}(\mathbb{I}(t)), m \notin M \}$$

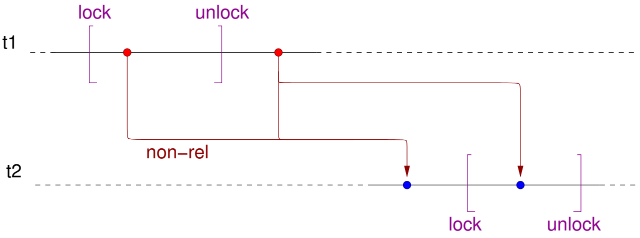
- property maintained outside code protected by m
- possibly broken while m is locked
- restored before unlocking m

Relational lock invariants



Improved interferences: mixing simple interferences and lock invariants

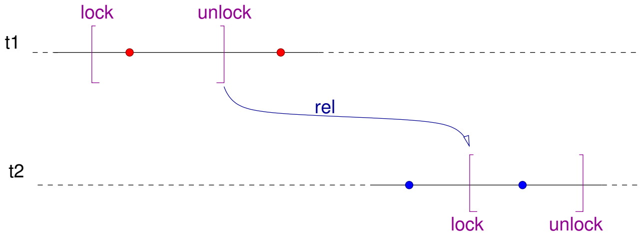
Relational lock invariants



Improved interferences: mixing simple interferences and lock invariants

- apply non-relational data-race interferences
  unless threads hold a common lock (mutual exclusion)

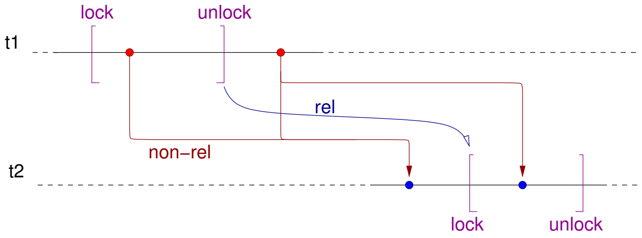
Relational lock invariants



Improved interferences: mixing simple interferences and lock invariants

- apply non-relational data-race interferences
  unless threads hold a common lock (mutual exclusion)
- apply non-relational well-synchronized interferences at `lock` points
  then intersect with the lock invariant
- gather lock invariants for `lock` / `unlock` pairs

Relational lock invariants



Improved interferences: mixing simple interferences and lock invariants

- apply non-relational data-race interferences unless threads hold a common lock (mutual exclusion)
- apply non-relational well-synchronized interferences at `lock` points then intersect with the lock invariant
- gather lock invariants for `lock` / `unlock` pairs

Weakly relational interference example

analyzing t_1

t_1	t_2

`while random do`
`lock(m);`
`if x < y then`
`x ← x + 1;`
`unlock(m)`
	x unchanged
	y incremented
	$0 \leq y \leq 102$

analyzing t_2

t_1	t_2

y unchanged
$0 \leq x$, $x \leq y$
	`while random do`
	`lock(m);`
	`if y < 100 then`
	`y ← y + [1,3];`
	`unlock(m)`

Using all three interference abstractions:

- non-relational interferences $(0 \leq y \leq 102, 0 \leq x)$
- lock invariants, with the octagon domain $(x \leq y)$
- monotonic interferences (y monotonic)

we can prove automatically that $x \leq y$ holds

Subsequence interference

t_1: clock in H	t_2: sample H into C	t_3: accumulate time in T

`while random do`
`if H < 10,000 then`
`H ← H+1`
	`while random do`
	`C ← H`
		`while random do`
		`if random then T ← 0`
		`else T ← T + (C-L)`
		`L ← C`

Problem: we wish to prove that $T \leq L \leq C \leq H$

it is sufficient to prove the monotony of H, C, and L

but monotony is not transitive

X is only assigned monotonic variables $\not\Longleftarrow$ X is monotonic

$\Longrightarrow$ we infer an additional property implying monotony

Abstraction: subsequence

- $A^{\text{sseq}}(t)(V) = \{\, W \in \mathbb{V} \mid V\text{'s values are a subsequence of } W\text{'s values} \,\}$
- $\alpha_{\mathcal{R}}^{\text{sseq}}(X)(V) \stackrel{\text{def}}{=} \{\, W \mid$
  $\forall \langle\langle\, \ell_0,\, \rho_0 \,\rangle, \ldots, \langle\, \ell_n,\, \rho_n \,\rangle\rangle \in X: \exists i_0, \ldots, i_n:$
  $\forall k: i_k \leq k \wedge i_k \leq i_{k+1} \wedge \forall j: \rho_j(V) = \rho_{i_j}(W) \,\}$

based on a **trace version** of the modular semantics

Summary

Conclusion

We presented static analysis methods that are:

- inspired from thread-modular proof methods
- abstractions of complete concrete semantics
  (for safety properties)
- sound for all interleavings
- sound for weakly consistent memory semantics
  (when using non-relational, flow-insensitive interference abstraction)
- aware of scheduling, priorities and synchronization
- parametrized by abstract domains
  (independent domains for state abstraction and interference abstraction)

Bibliography

Bibliography

[Bour93] **F. Bourdoncle**. *Efficient chaotic iteration strategies with widenings.* In Proc. FMPA'93, LNCS vol. 735, pp. 128–141, Springer, 1993.

[Carr09] **J.-L. Carré & C. Hymans**. *From single-thread to multithreaded: An efficient static analysis algorithm.* In arXiv:0910.5833v1, EADS, 2009.

[Cous84] **P. Cousot & R. Cousot**. *Invariance proof methods and analysis techniques for parallel programs.* In Automatic Program Construction Techniques, chap. 12, pp. 243–271, Macmillan, 1984.

[Cous85] **R. Cousot**. *Fondements des méthodes de preuve d'invariance et de fatalité de programmes parallèles.* In Thèse d'Etat es sc. math., INP Lorraine, Nancy, 1985.

[Hoar69] **C. A. R. Hoare**. *An axiomatic basis for computer programming.* In Com. ACM, 12(10):576–580, 1969.

Bibliography (cont.)

[Jone81] **C. B. Jones**. *Development methods for computer programs including a notion of interference.* In PhD thesis, Oxford University, 1981.

[Lamp77] **L. Lamport**. *Proving the correctness of multiprocess programs.* In IEEE Trans. on Software Engineering, 3(2):125–143, 1977.

[Lamp78] **L. Lamport**. *Time, clocks, and the ordering of events in a distributed system.* In Comm. ACM, 21(7):558–565, 1978.

[Mans05] **J. Manson, B. Pugh & S. V. Adve**. *The Java memory model.* In Proc. POPL'05, pp. 378–391, ACM, 2005.

[Miné12] **A. Miné**. *Static analysis of run-time errors in embedded real-time parallel C programs.* In LMCS 8(1:26), 63 p., arXiv, 2012.

[Owic76] **S. Owicki & D. Gries**. *An axiomatic proof technique for parallel programs I.* In Acta Informatica, 6(4):319–340, 1976.

Bibliography (cont.)

[Reyn04] **J. C. Reynolds**. *Toward a grainless semantics for shared-variable concurrency.* In Proc. FSTTCS'04, LNCS vol. 3328, pp. 35–48, Springer, 2004.

[Sara07] **V. A. Saraswat, R. Jagadeesan, M. M. Michael & C. von Praun**. *A theory of memory models.* In Proc. PPoPP'07, pp. 161–172, ACM, 2007.