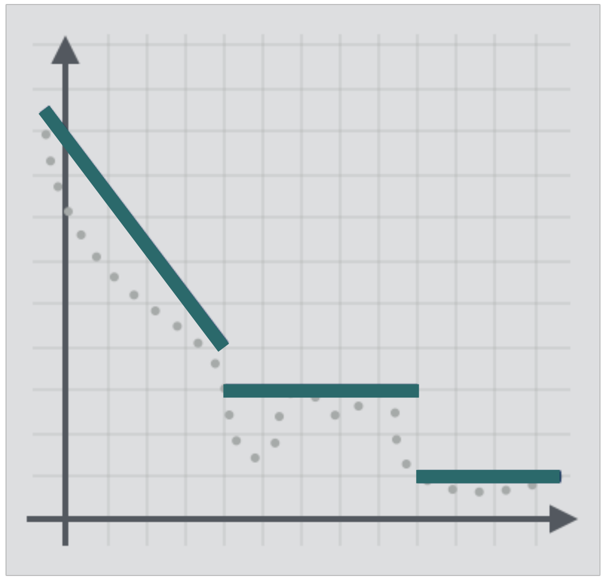
Liveness Analysis

MPRI 2-6: Abstract Interpretation, Application to Verification and Static Analysis



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Year 2021-2022

Liveness Properties

- **Guarantee Properties**
  "something good eventually happens at least once"
  - Example: Program Termination
- **Recurrence Properties**
  "something good eventually happens infinitely often"
  - Example: Starvation Freedom

Guarantee Properties

Guarantee Properties

"something good eventually happens at least once"

$$\text{AF}\,\varphi$$

$$\varphi ::= e \bowtie 0 \;\mid\; \ell : e \bowtie 0 \;\mid\; \varphi \wedge \varphi \mid \varphi \vee \varphi \qquad\qquad \ell \in \mathscr{L}$$

Example:

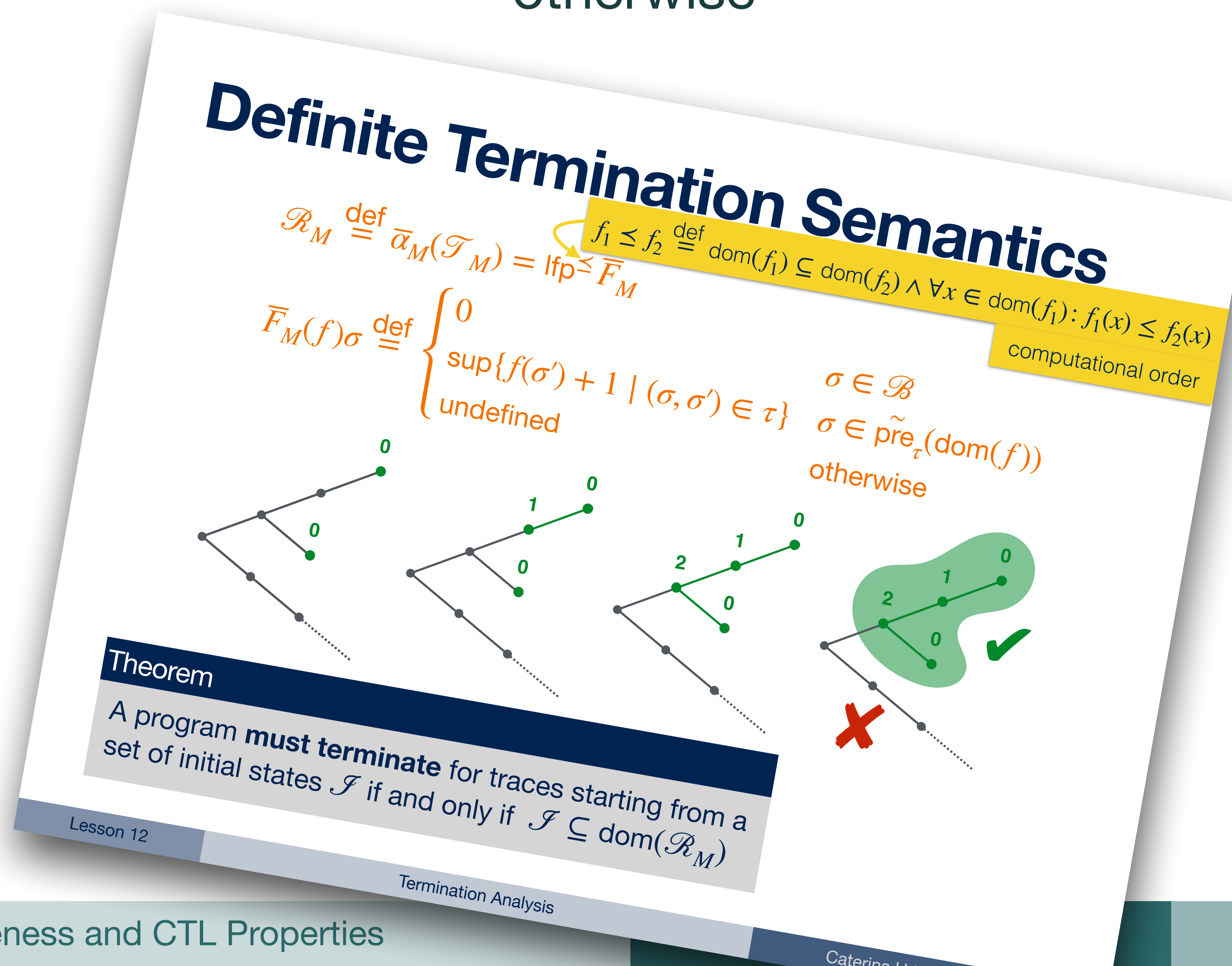
```
1x ← [-∞, +∞]
 while 2(x ≥ 0) do
      3x ← x + 1
od4
 while 5(0 ≥ 0) do
      if 6(x ≤ 10) do
            7x ← x + 1
      else
            8x ← -x
od9
```

AF $(x = 3)$ is satisfied for $\mathscr{I} \overset{\text{def}}{=} \{(1,\rho) \in \Sigma \mid \rho(x) \leq 3\}$

Guarantee Semantics

$$\mathcal{R}_G^{\varphi} \overset{\text{def}}{=} \text{lfp}^{\preceq}\, \overline{F}_G[\{\sigma \in \Sigma \mid \sigma \models \varphi\}]$$

$$\overline{F}_G[S]f \overset{\text{def}}{=} \lambda\sigma\,.\begin{cases} 0 & \sigma \in S \\ \sup\{f(\sigma') + 1 \mid (\sigma, \sigma') \in \tau\} & \sigma \notin S \wedge \sigma \in \widetilde{\text{pre}}_\tau(\text{dom}(f)) \\ \text{undefined} & \text{otherwise} \end{cases}$$



Guarantee Semantics

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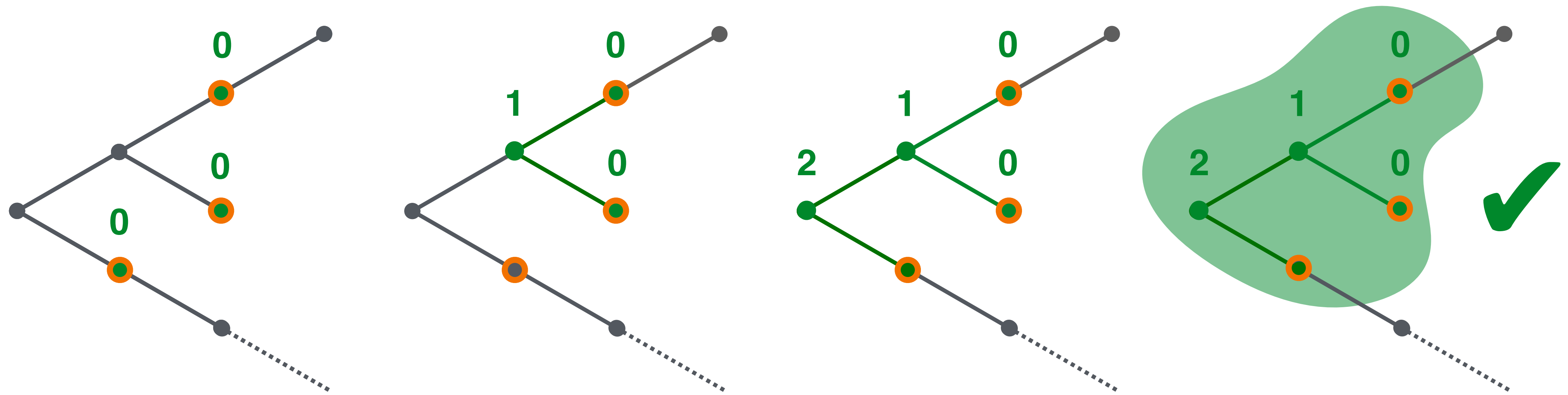


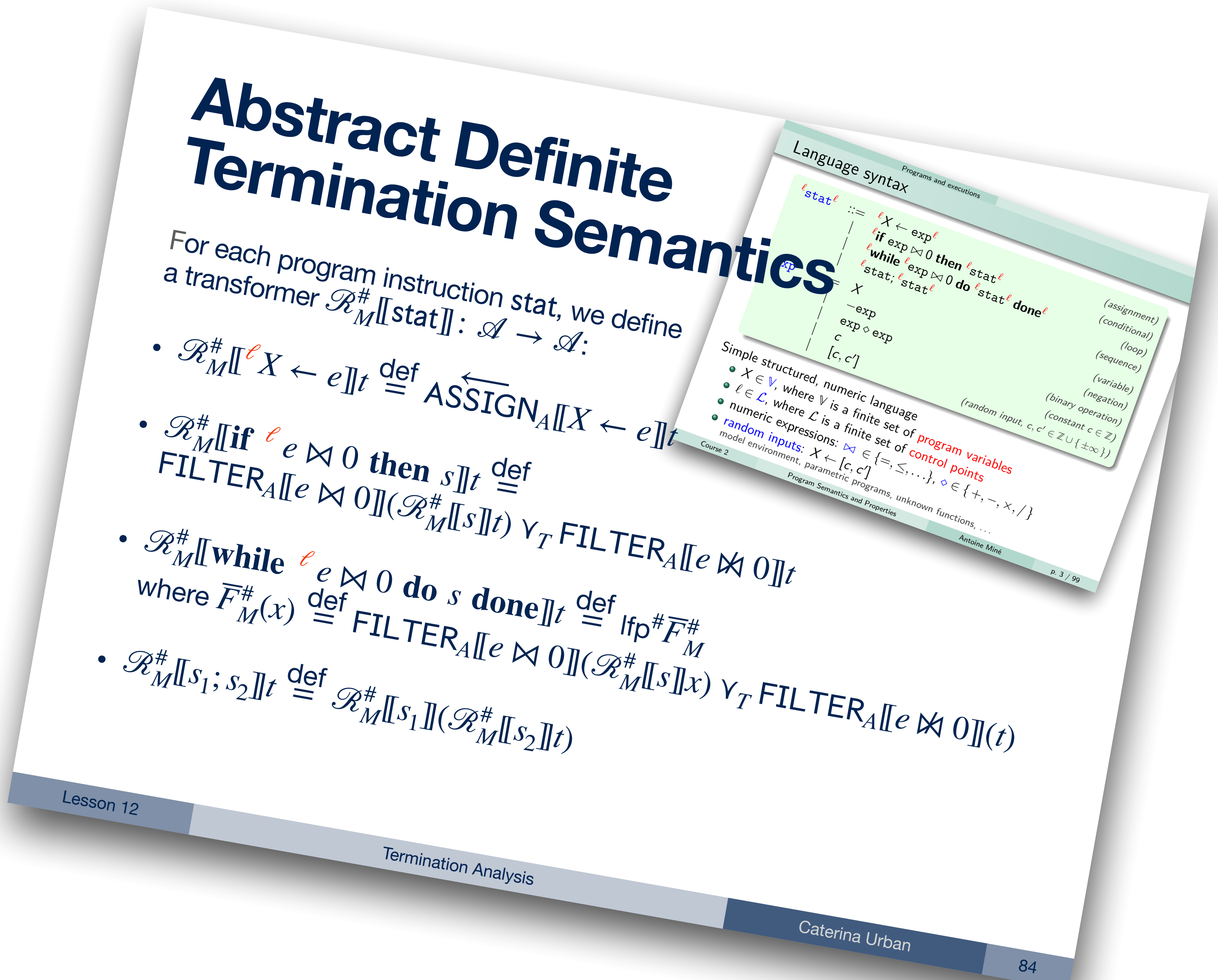
Theorem

A program satisfies a **guarantee property** AF φ for traces starting from a set of initial states $\mathcal{I}$ if and only if $\mathcal{I} \subseteq \text{dom}(\mathcal{R}_G^{\varphi})$

Abstract Guarantee Semantics

For each program instruction stat, we define $\mathcal{R}_G^{\varphi\#}[\![\texttt{stat}]\!] : \mathcal{A} \rightarrow \mathcal{A}$:

- $\mathcal{R}_G^{\varphi\#}[\![{}^{\ell}X \leftarrow e]\!]t \stackrel{\text{def}}{=} \text{RESET}_A^G[\![\varphi]\!](\overleftarrow{\text{ASSIGN}}_A[\![X \leftarrow e]\!]t)$



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  where $X \stackrel{\text{def}}{=} \text{FILTER}_A[\![e \bowtie 0]\!](\mathcal{R}_G^{\varphi\#}[\![s]\!]t) \curlyvee_T \text{FILTER}_A[\![e \not\bowtie 0]\!]t)$

- $\mathcal{R}_G^{\varphi\#}[\![\textbf{while}\ ^{\ell}\ e \bowtie 0 \ \textbf{do}\ s\ \textbf{done}]\!]t \stackrel{\text{def}}{=} \text{lfp}^{\#} \overline{F}_G^{\varphi\#}$
  where $\overline{F}_G^{\varphi\#}(x) \stackrel{\text{def}}{=} \text{RESET}_A^G[\![\varphi]\!](X)$
  $X \stackrel{\text{def}}{=} \text{FILTER}_A[\![e \bowtie 0]\!](\mathcal{R}_G^{\varphi\#}[\![s]\!]x) \curlyvee_T \text{FILTER}_A[\![e \not\bowtie 0]\!](t))$

- $\mathcal{R}_G^{\varphi\#}[\![s_1; s_2]\!]t \stackrel{\text{def}}{=} \mathcal{R}_G^{\varphi\#}[\![s_1]\!](\mathcal{R}_G^{\varphi\#}[\![s_2]\!]t)$

Abstract Guarantee Semantics

Example

int : x, y

while [1]$(x \geq 0)$ do

[2]$x := x + 1$

od

while [3](true) do

if [4]$(\ x \leq 10\ )$

[5]$x := x + 1$

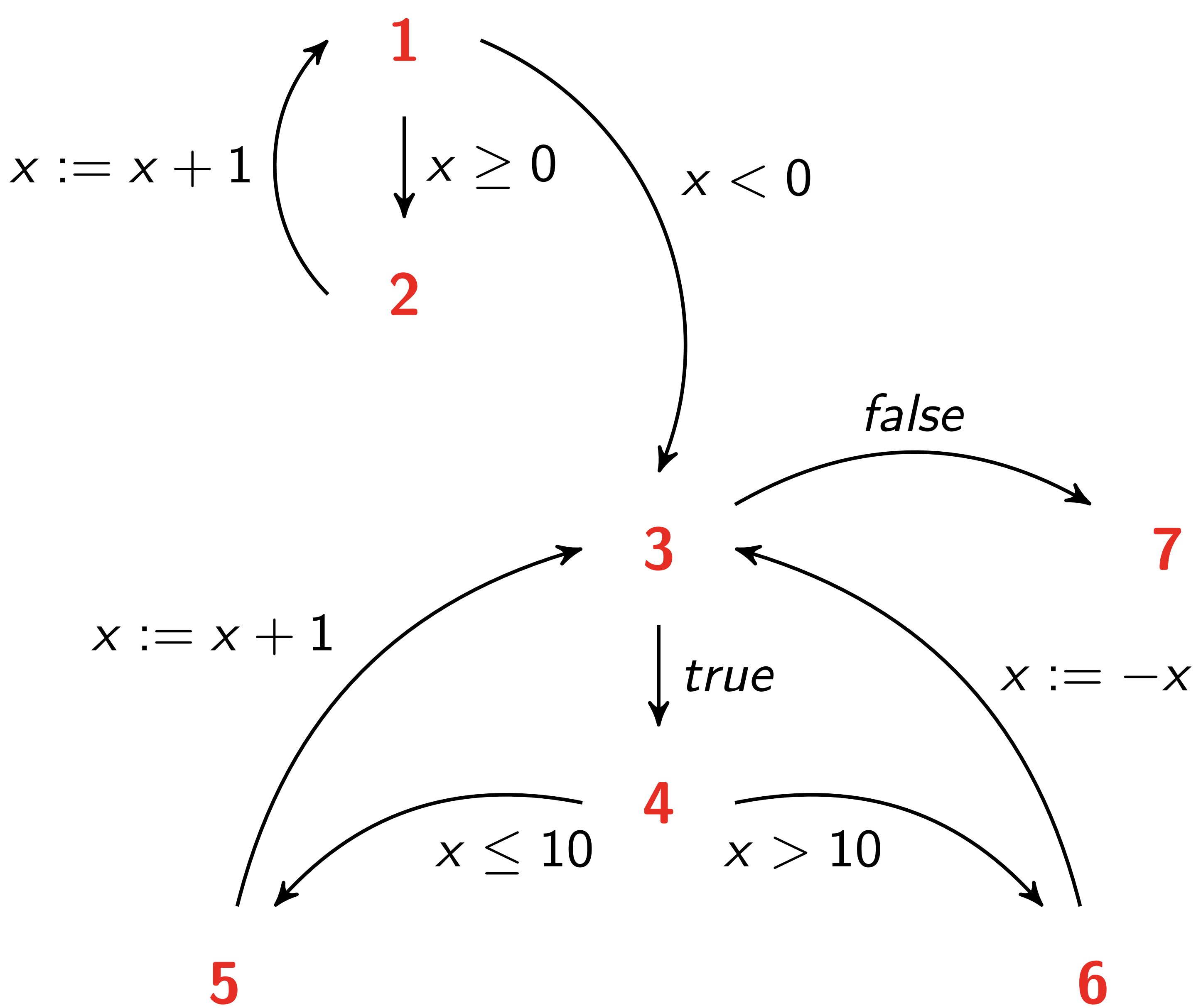
else

[6]$x := -x$

od[7]

Property

$$AF\,(x = 3)$$



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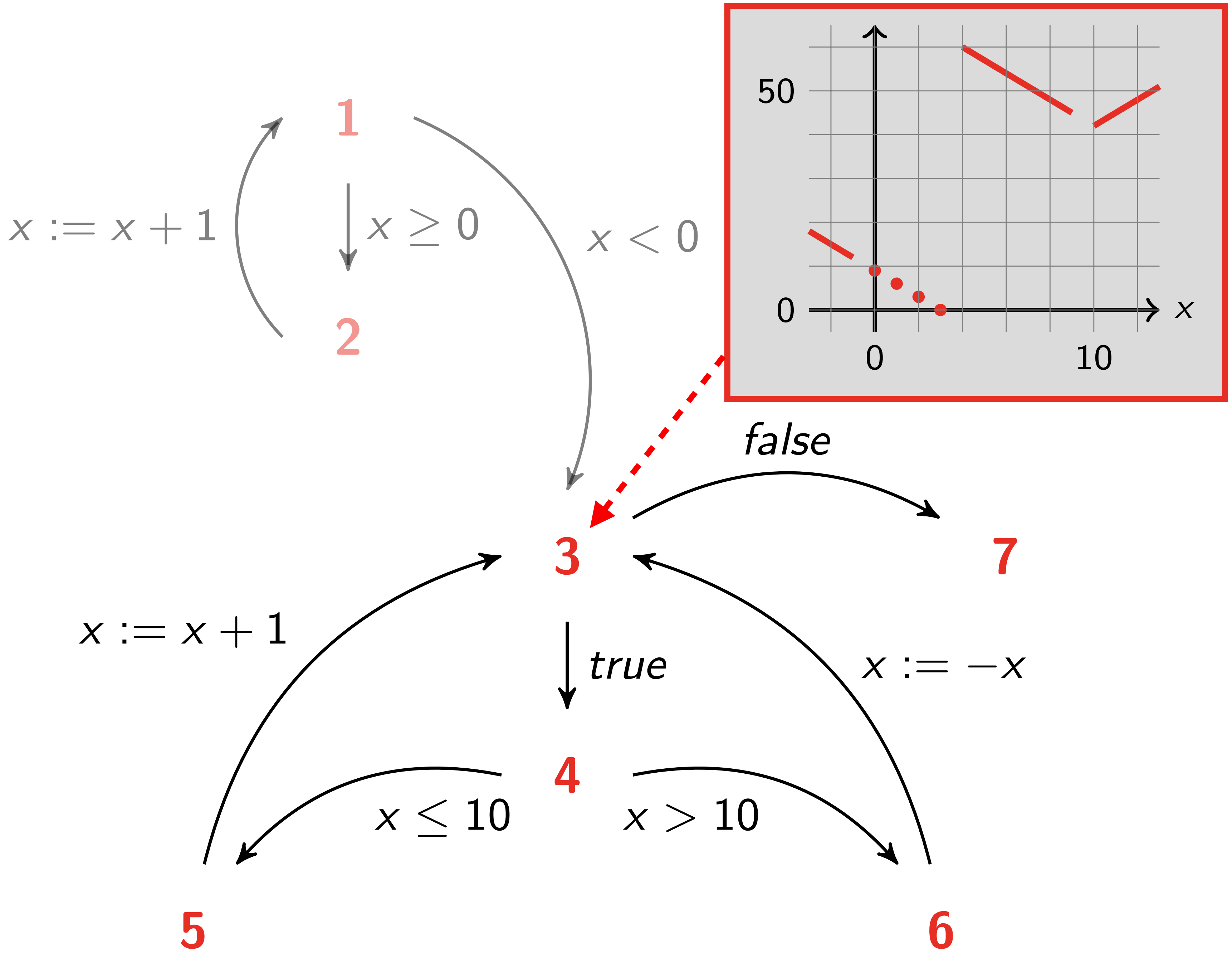

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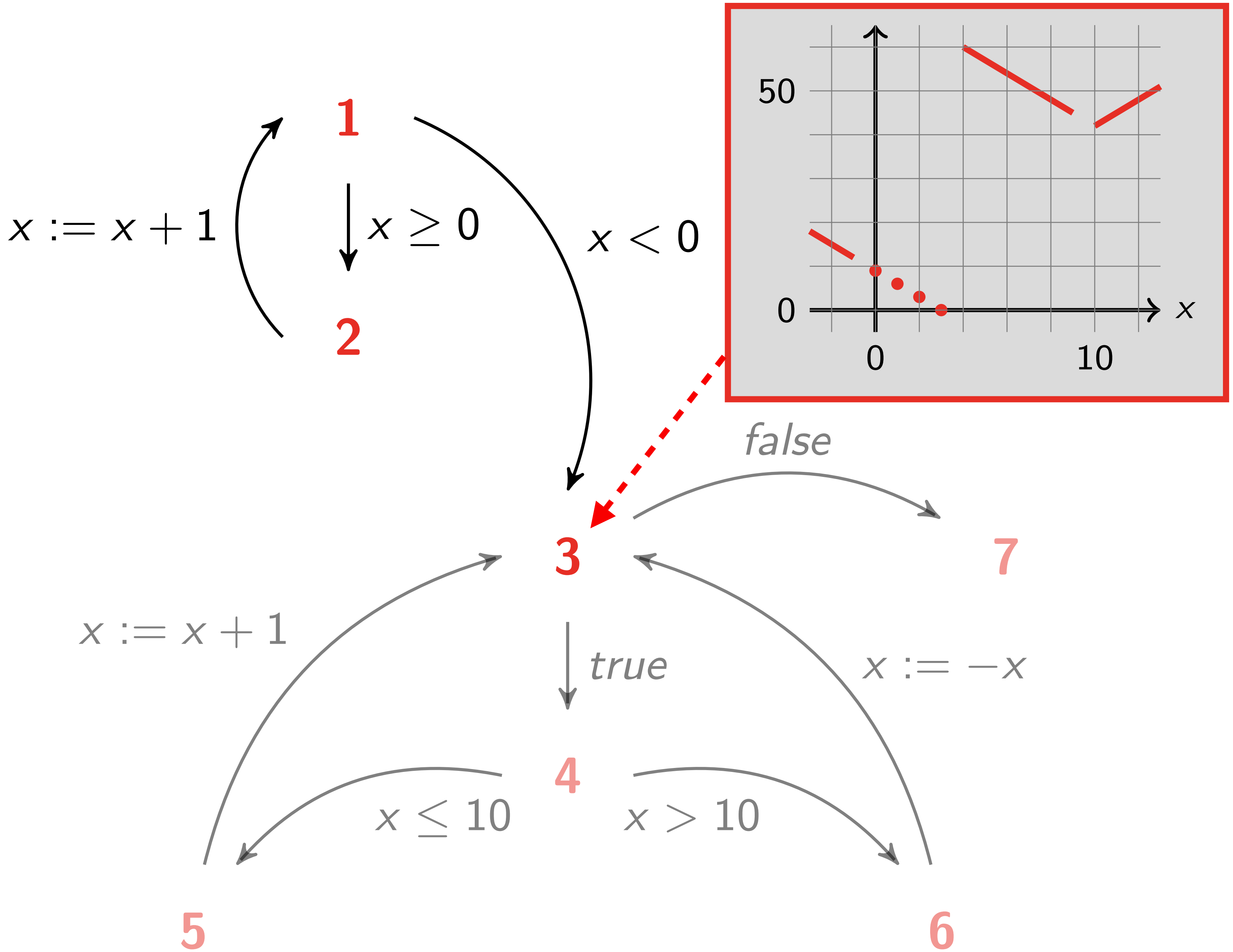

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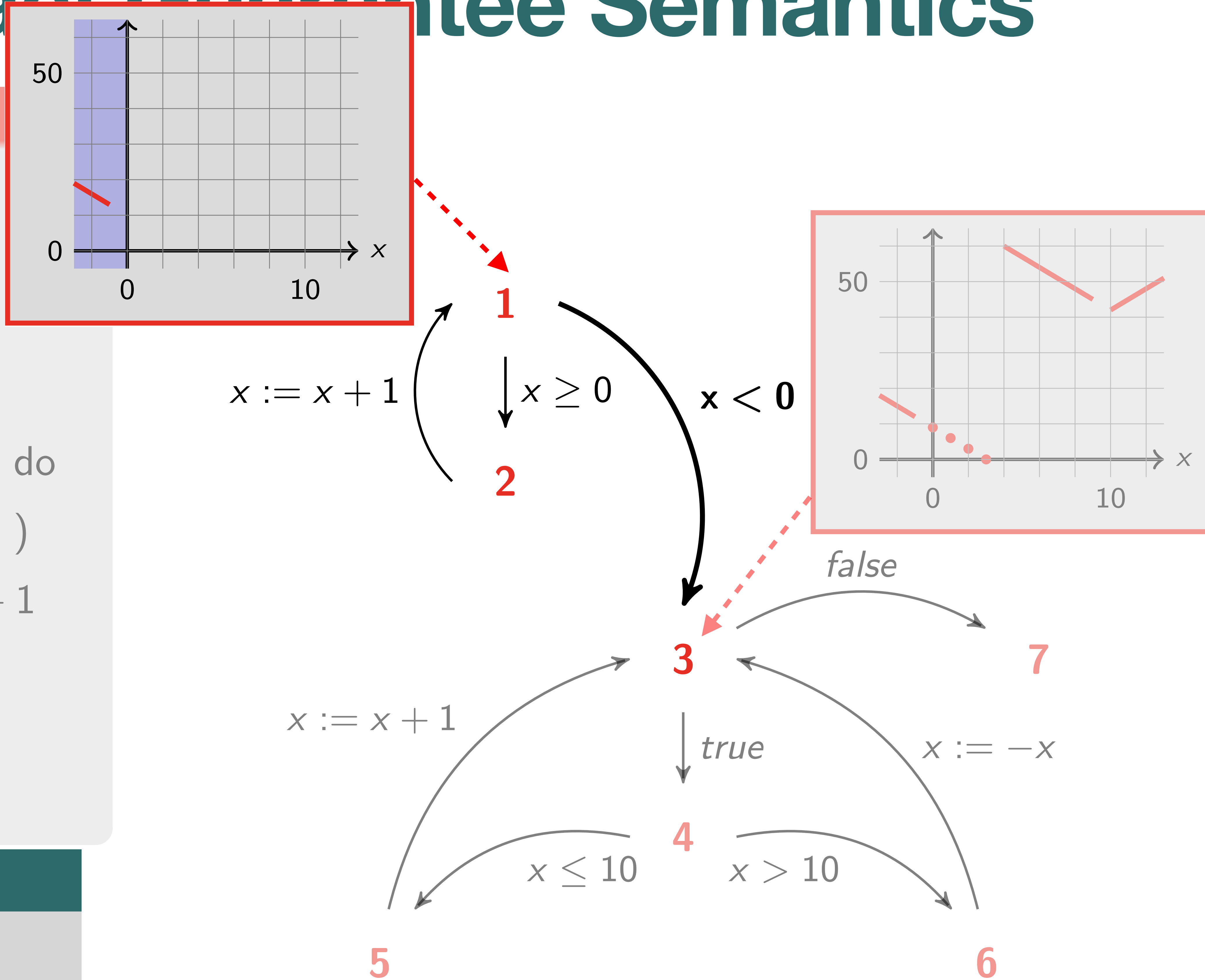

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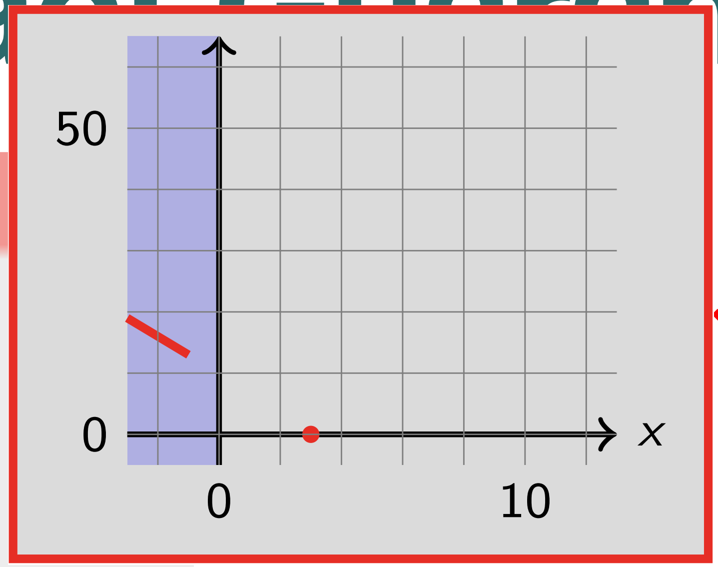

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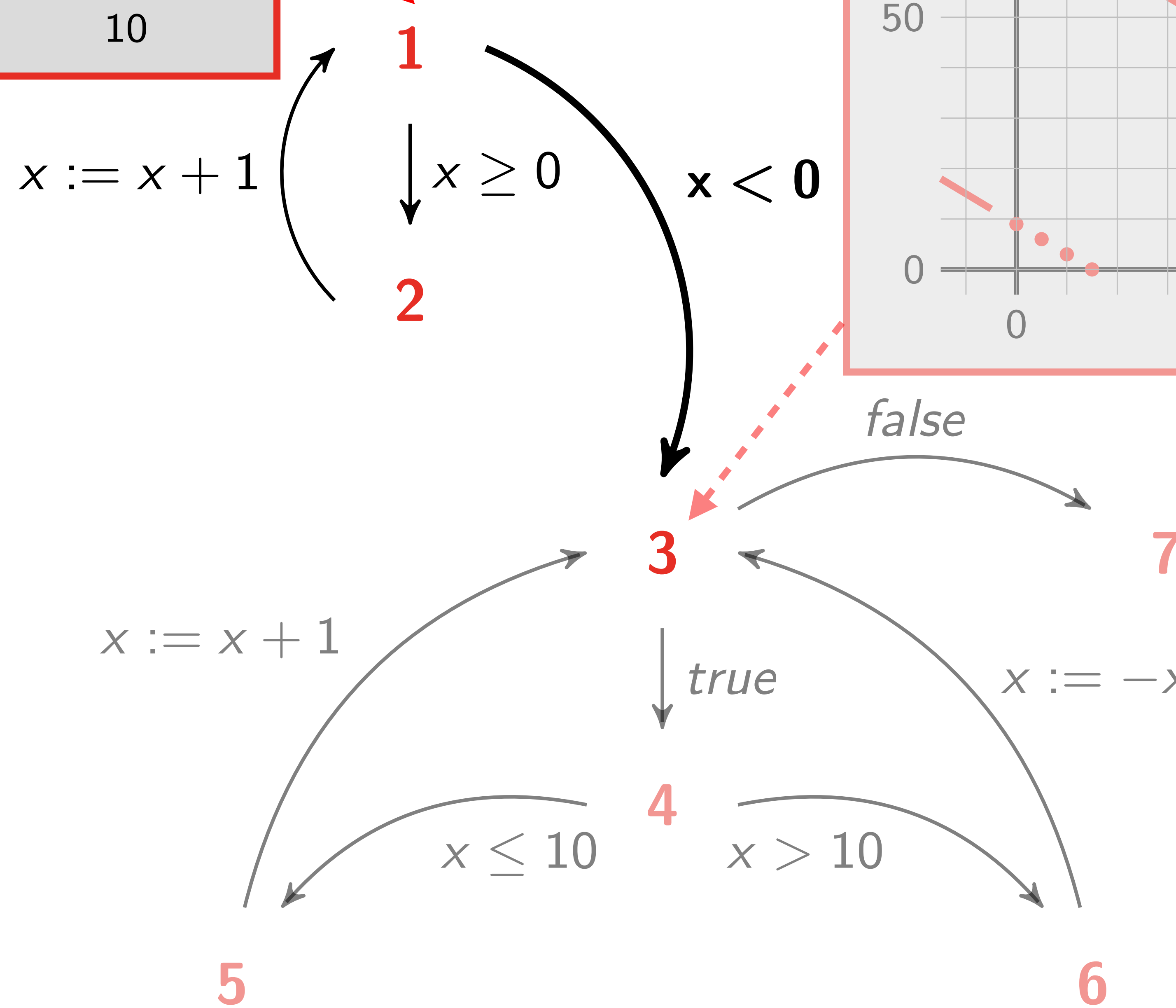

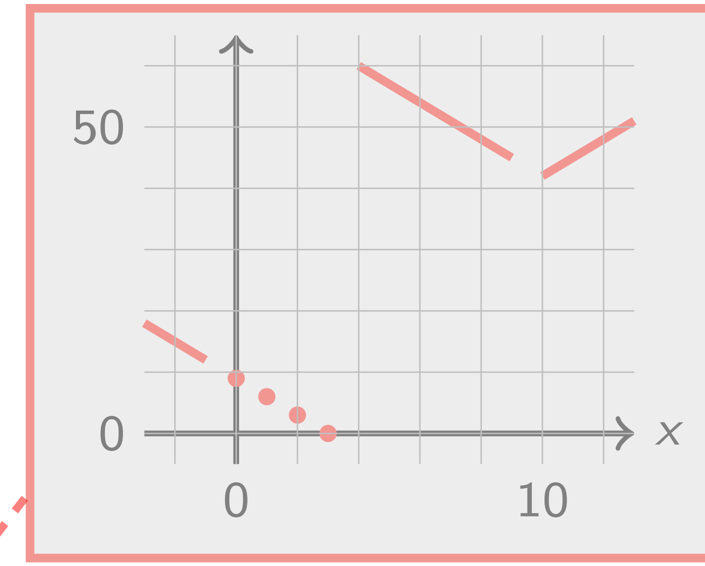

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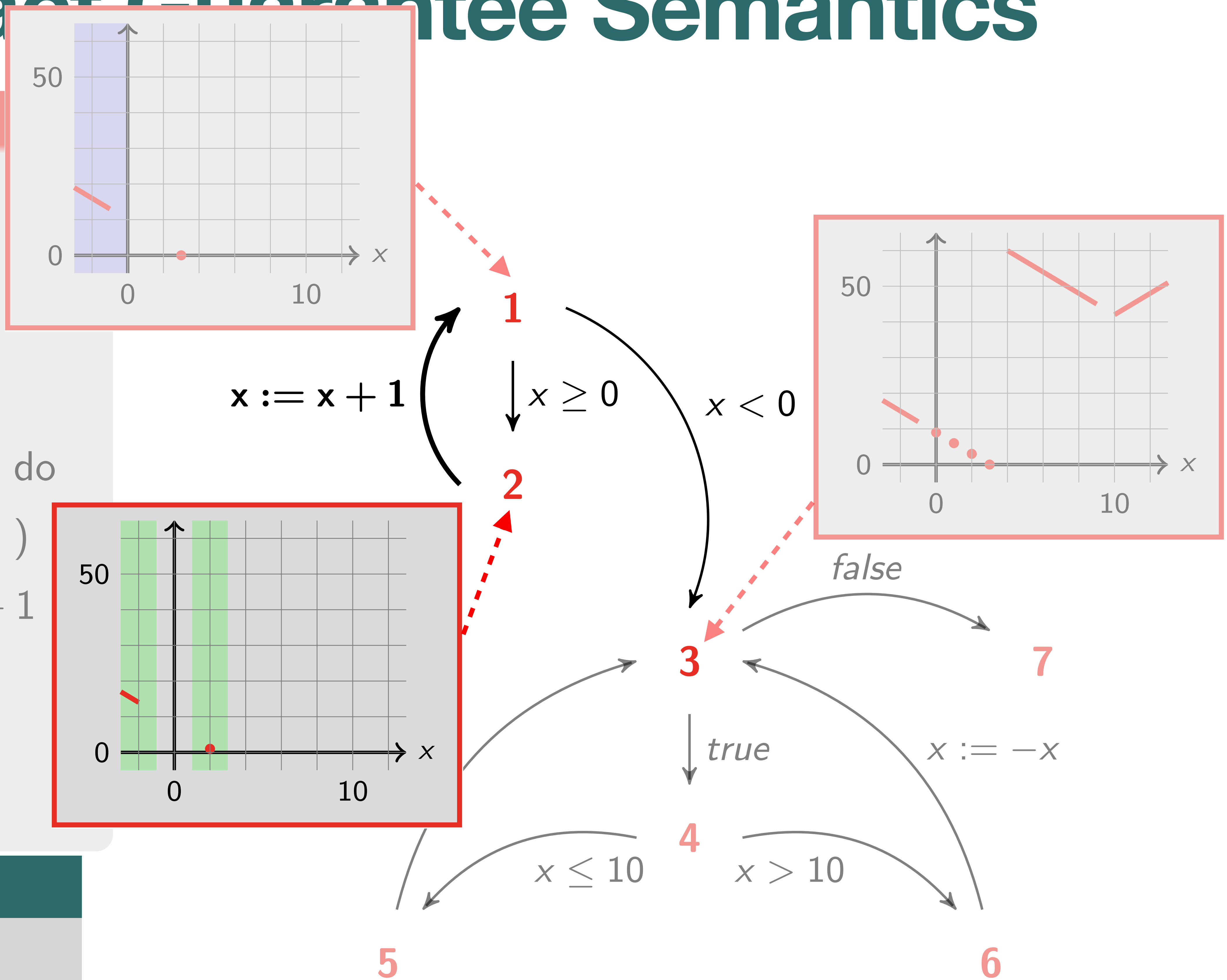

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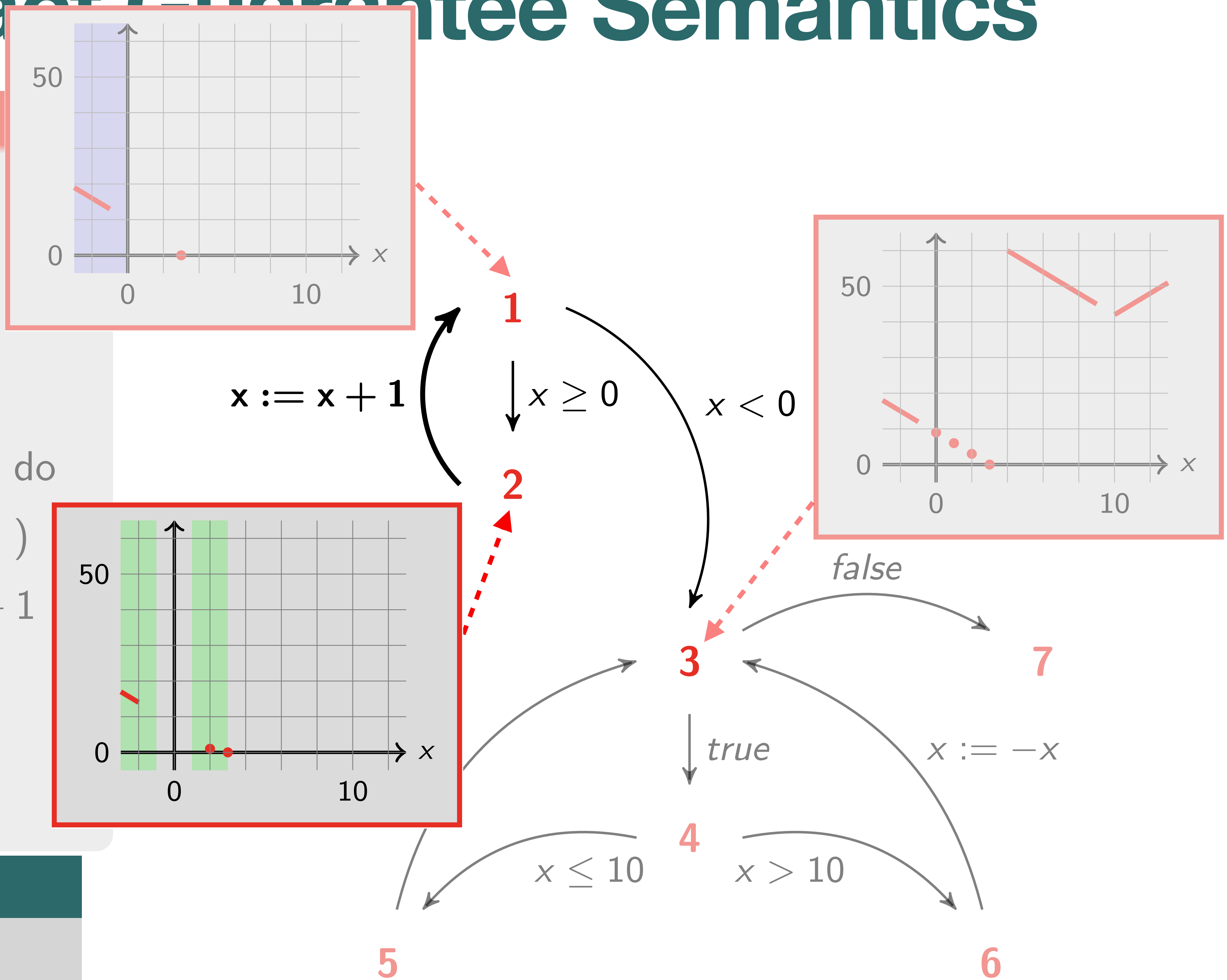

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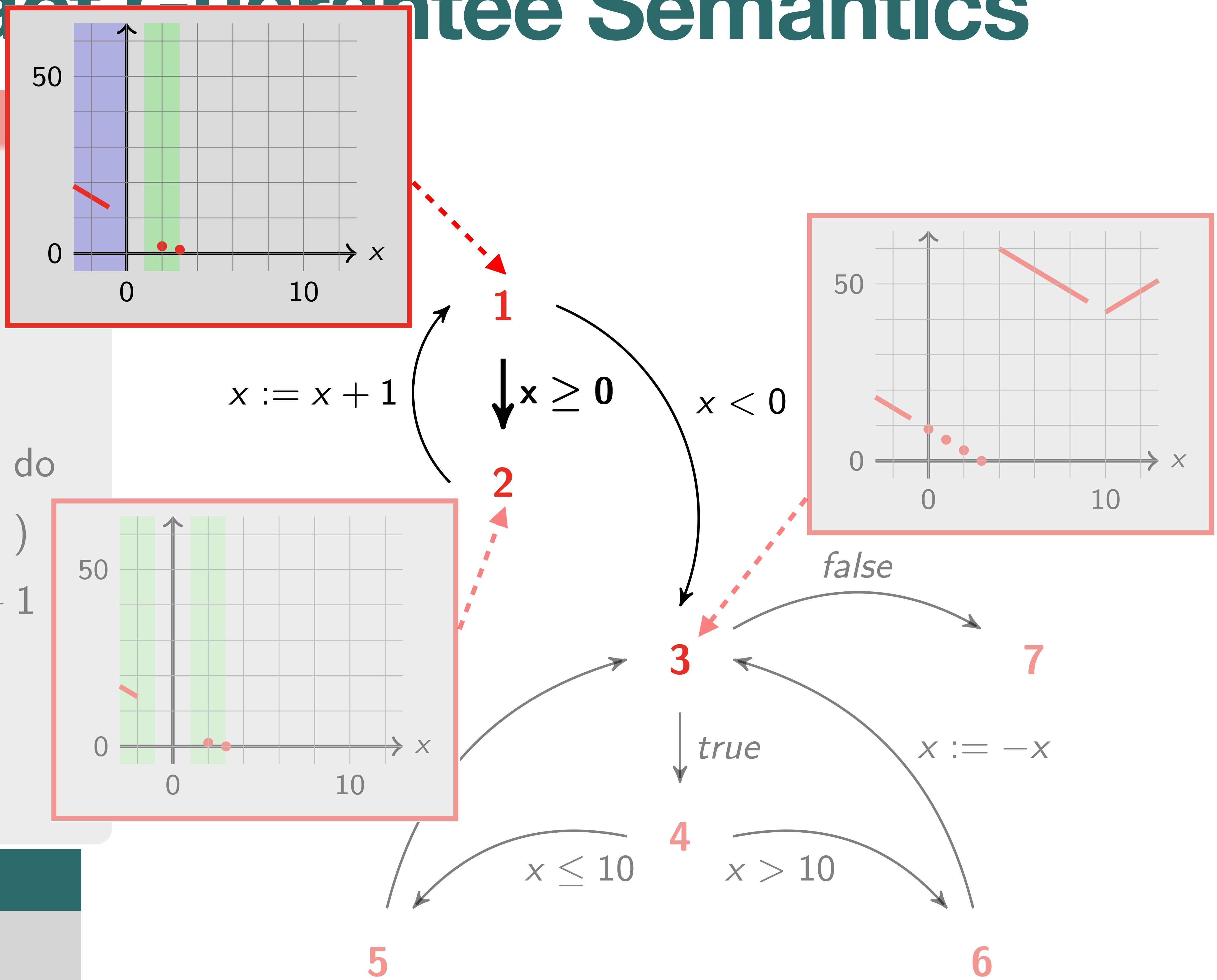

Abstract Guarantee Semantics

Example

```
int : x, y
while ¹(x ≥ 0)
    ²x := x + 1
od
while ³( true ) do
    if ⁴( x ≤ 10 )
        ⁵x := x + 1
    else
        ⁶x := −x
od⁷
```

Property

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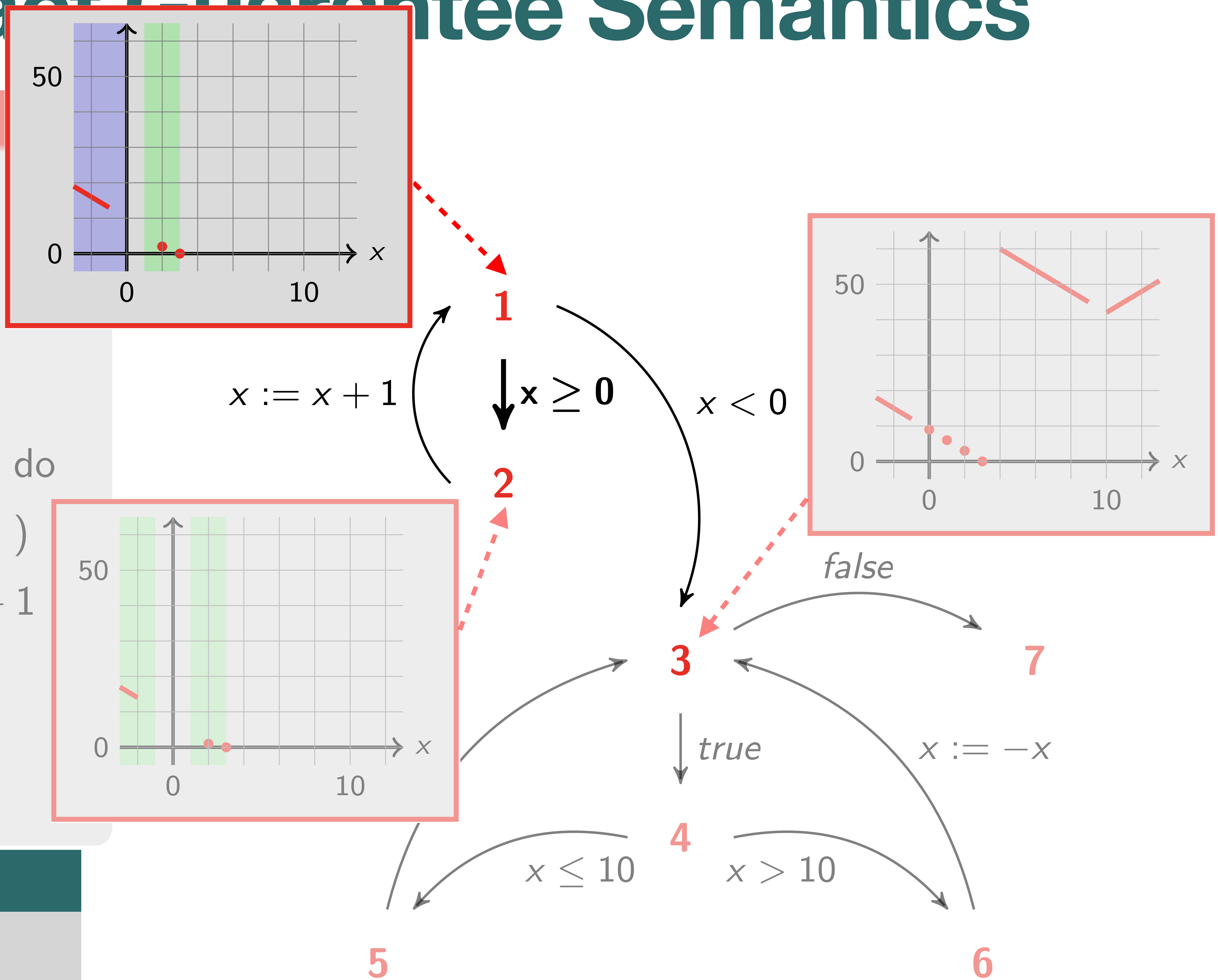

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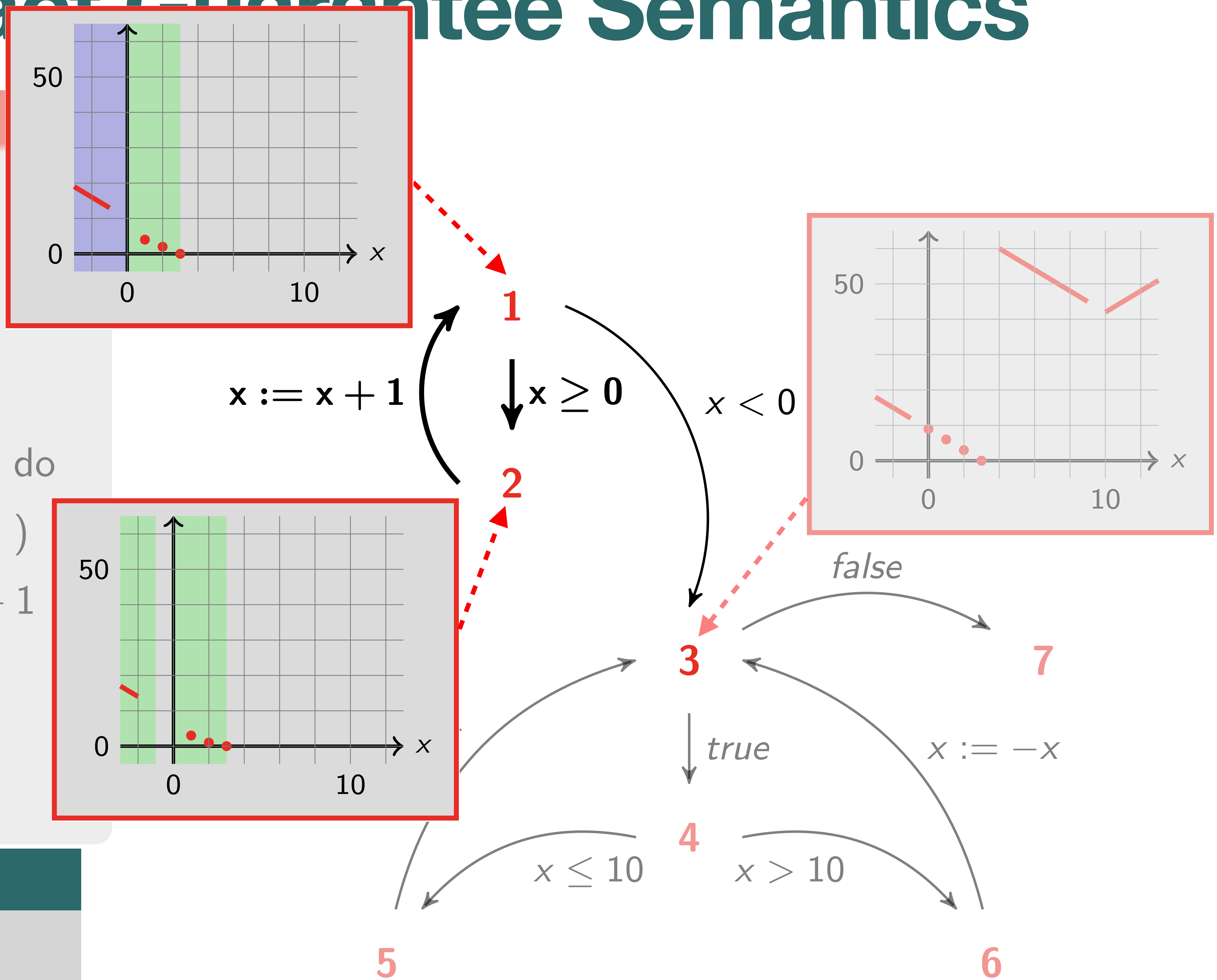

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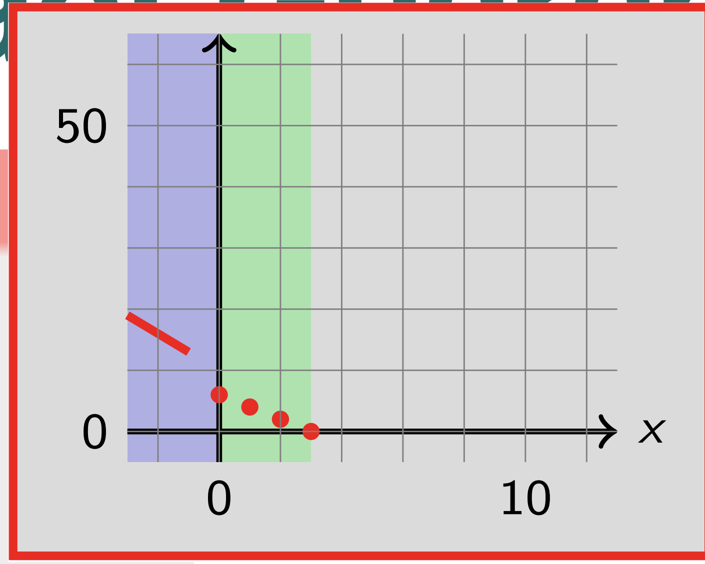

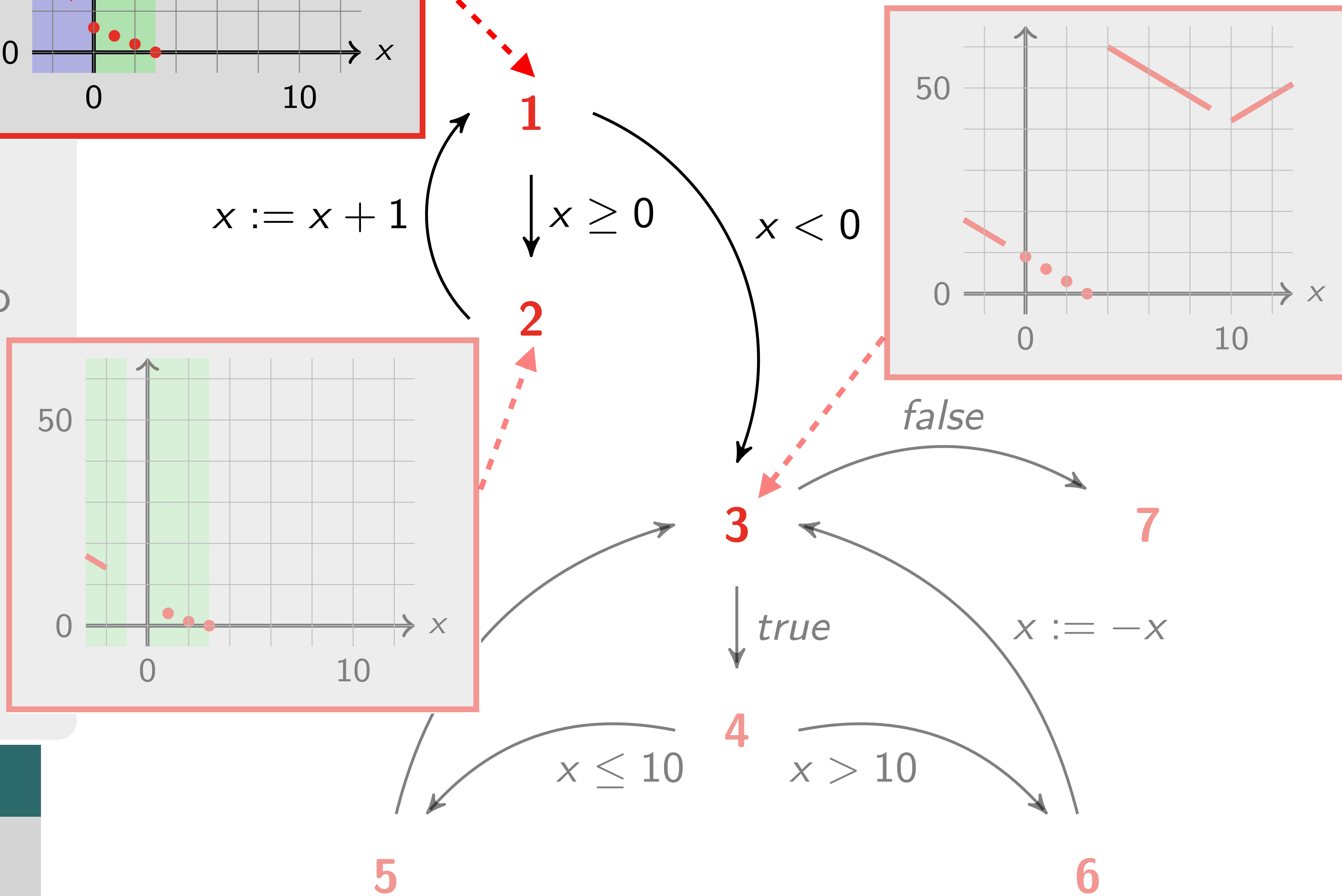

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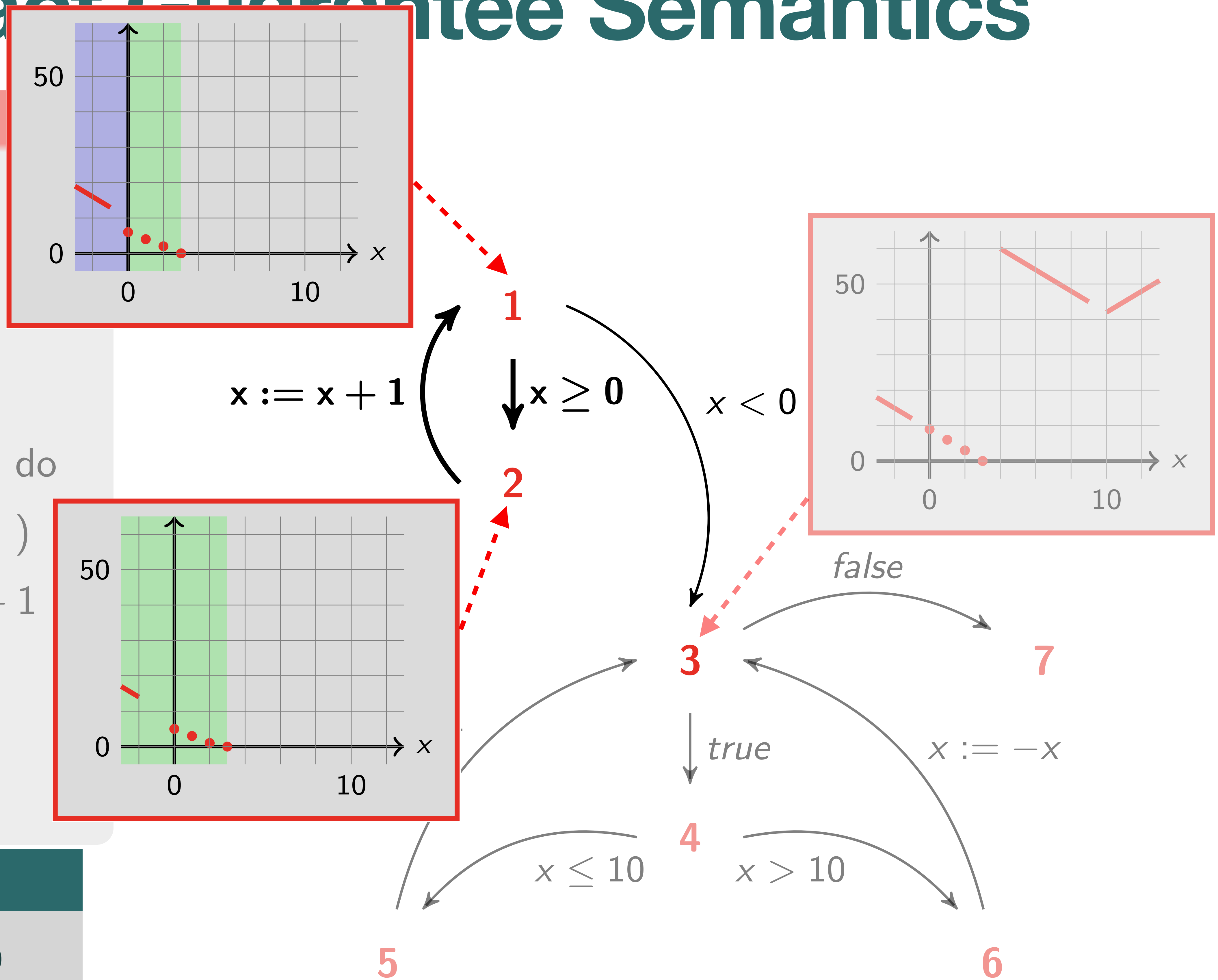

Abstract Guarantee Semantics

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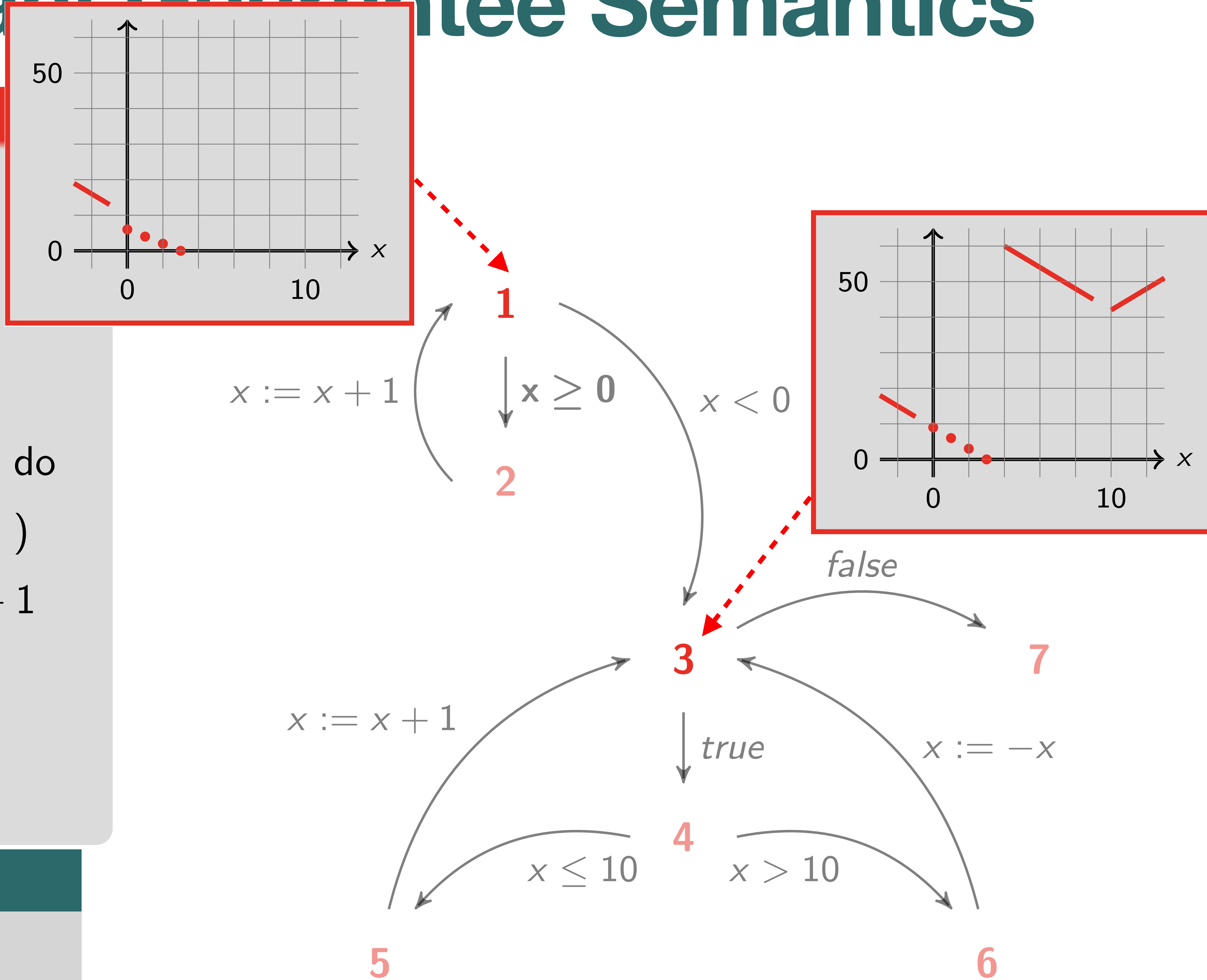


Abstract Guarantee Semantics

Example

$\text{int} : x,\ y$

$\text{while}\ ^{1}(x \geq 0)$

$\quad ^{2}x := x + 1$

od

$\text{while}\ ^{3}(\ \text{true}\ )\ \text{do}$

$\quad \text{if}\ ^{4}(\ x \leq 10\ )$

$\qquad ^{5}x := x + 1$

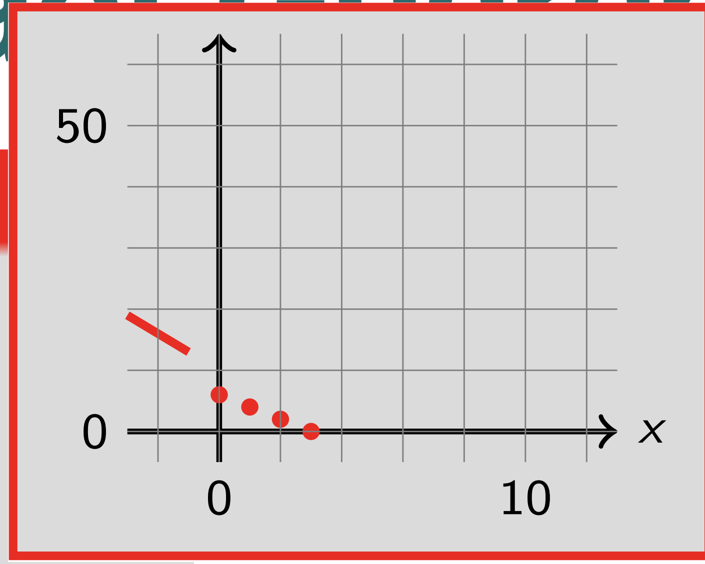
$\quad \text{else}$

$\qquad ^{6}x := -x$

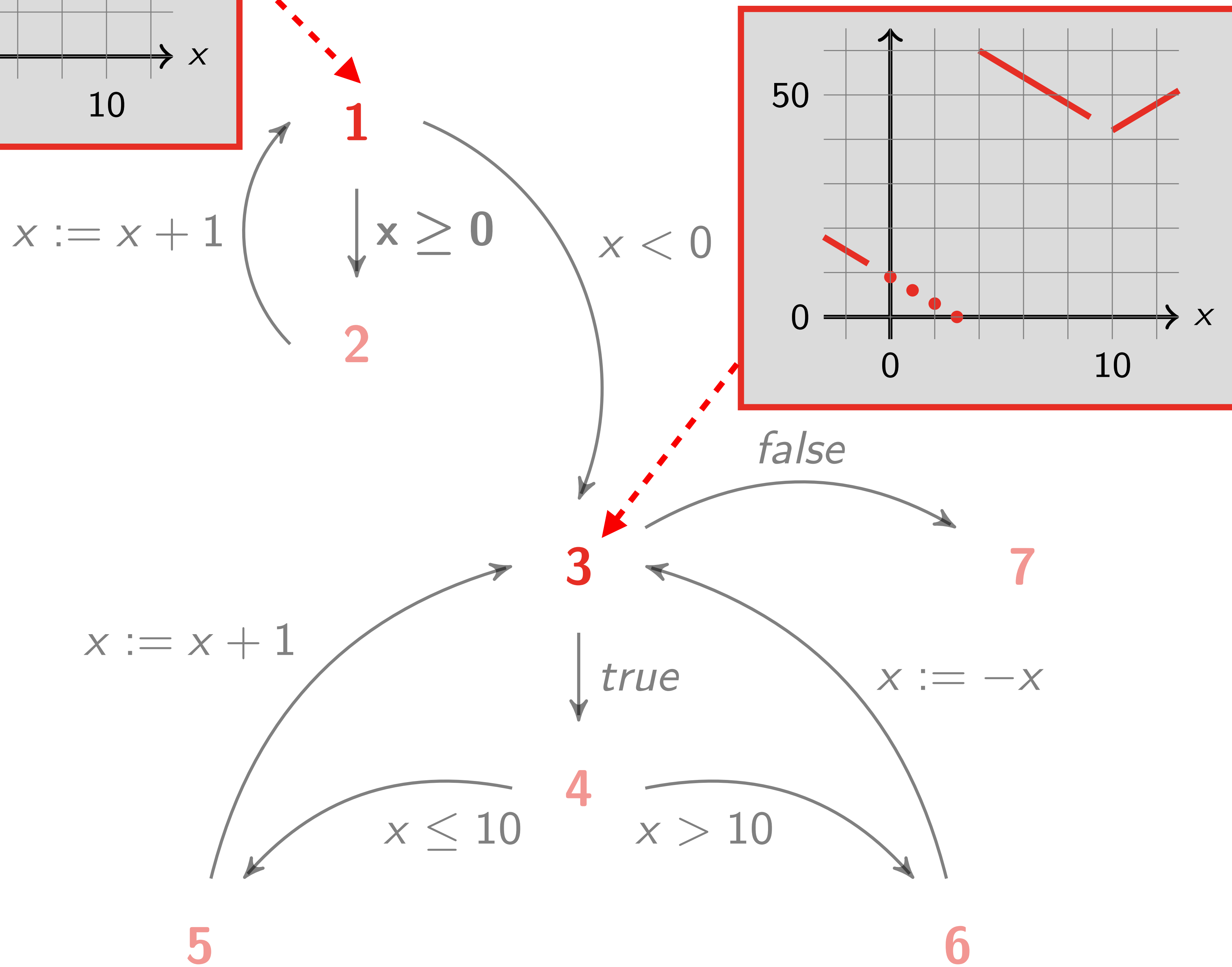
od^{7}

Property

$AF\,(x = 3)$



the analysis gives $x \leq 3$ as **sufficient precondition**



Abstract Guarantee Semantics

Definition

The **abstract guaranteee semantics** $\mathcal{R}_G^{\varphi\#}[\![\text{stat}^{\ell}]\!] \in \mathcal{A}$ of a program stat^{ℓ} is:

$$\mathcal{R}_G^{\varphi\#}[\![\text{stat}^{\ell}]\!] \stackrel{\text{def}}{=} \mathcal{R}_G^{\varphi\#}[\![\text{stat}]\!](\text{RESET}_A^G[\![\varphi]\!](\text{LEAF}: \bot_F))$$

where $\mathcal{R}_G^{\varphi\#}[\![\text{stat}]\!] : \mathcal{A} \to \mathcal{A}$ is the abstract guarantee semantics of each program instruction stat

Corollary (Soundness)

A program stat^{ℓ} satisfies a **guarantee property** AF φ for traces starting from a set of initial states $\mathcal{I}$ if $\mathcal{I} \subseteq \text{dom}(\gamma_A(\mathcal{R}_G^{\varphi\#}[\![\text{stat}^{\ell}]\!]))$

Recurrence Properties

Recurrence Properties

"something good eventually happens infinitely often"

$$\text{AG AF } \varphi$$

$\varphi ::= e \bowtie 0 \mid \ell : e \bowtie 0 \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$ $\qquad \ell \in \mathscr{L}$

Example:

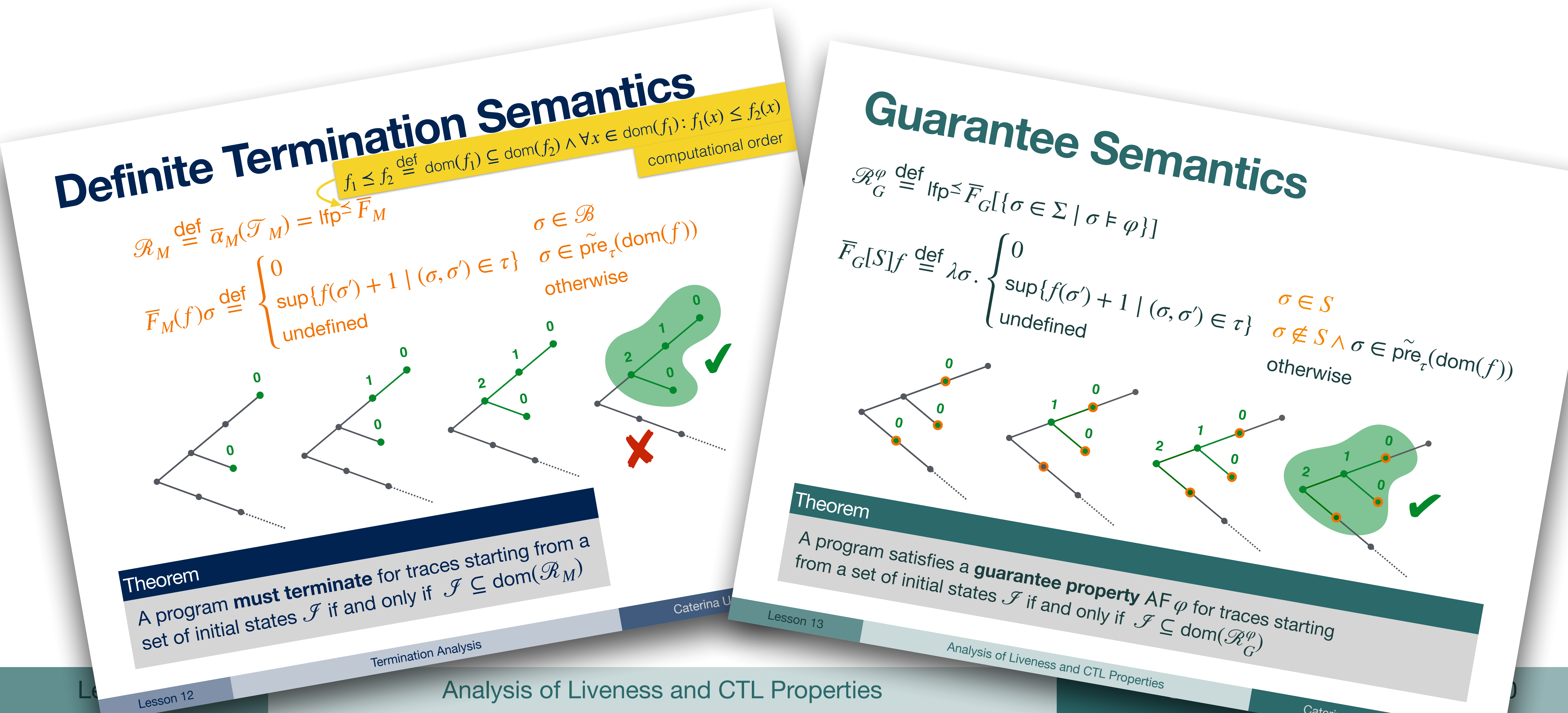
AG AF $(x = 3)$ is satisfied for $\mathscr{I} \stackrel{\text{def}}{=} \{(1,\rho) \in \Sigma \mid \rho(x) < 0\}$

```
¹x ← [-∞, +∞]
 while ²(x ≥ 0) do
      ³x ← x + 1
od⁴
 while ⁵(0 ≥ 0) do
      if ⁶(x ≤ 10) do
            ⁷x ← x + 1
      else
            ⁸x ← -x
od⁹
```

Recurrence Semantics

$$\mathcal{R}_R^{\varphi} \overset{\text{def}}{=} \text{gfp}_{\mathcal{R}_G^{\varphi}}^{\preceq} \overline{F}_R$$

$$\overline{F}_R(f)\sigma \overset{\text{def}}{=} \begin{cases} f(s) & \sigma \in \widetilde{\text{pre}}_{\tau}(\text{dom}(f)) \\ \text{undefined} & \text{otherwise} \end{cases}$$



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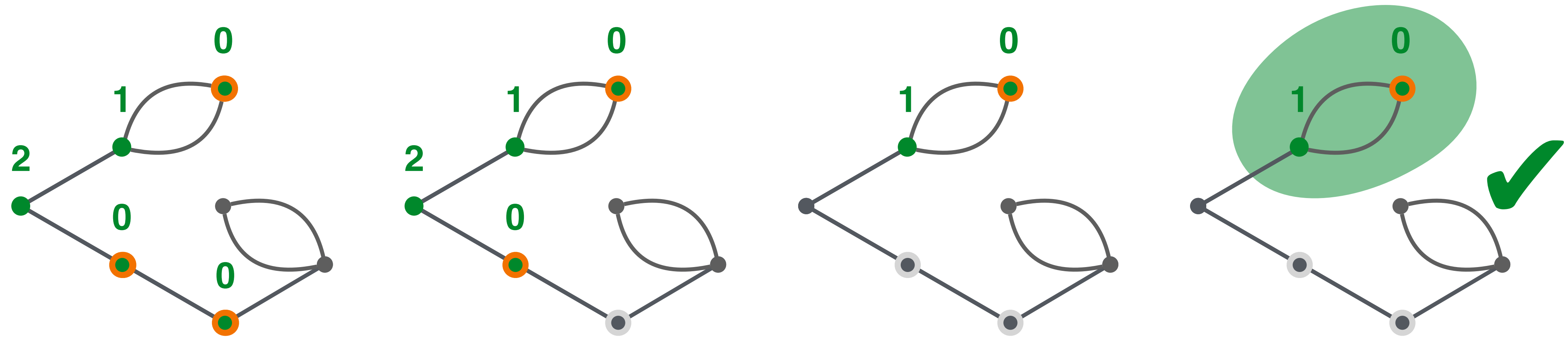

Theorem

A program satisfies a **recurrence property** AG AF φ for traces starting from a set of initial states $\mathcal{I}$ if and only if $\mathcal{I} \subseteq \text{dom}(\mathcal{R}_R^{\varphi})$

Abstract Recurrence Semantics

For each program instruction stat, we define $\mathscr{R}_G^{\varphi\#}[\![\text{stat}]\!] : \mathscr{A} \to \mathscr{A}$:

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- $\mathscr{R}_R^{\varphi\#}[\![\textbf{while } {}^{\ell}\, e \bowtie 0 \textbf{ do } s \textbf{ done}]\!]t \stackrel{\text{def}}{=} \text{gfp}_{G(t)}^{\#}\overline{F}_R^{\varphi\#}$
  where $G \stackrel{\text{def}}{=} \mathscr{R}_G^{\varphi\#}[\![\textbf{while } {}^{\ell}\, e \bowtie 0 \textbf{ do } s \textbf{ done}]\!]$
  $\overline{F}_R^{\varphi\#}(x) \stackrel{\text{def}}{=} \text{RESET}_A^R[\![\varphi]\!](X)$
  $X \stackrel{\text{def}}{=} \text{FILTER}_A[\![e \bowtie 0]\!](\mathscr{R}_R^{\varphi\#}[\![s]\!]x) \curlyvee_T \text{FILTER}_A[\![e \not\bowtie 0]\!](t))$

- $\mathscr{R}_R^{\varphi\#}[\![s_1; s_2]\!]t \stackrel{\text{def}}{=} \mathscr{R}_R^{\varphi\#}[\![s_1]\!](\mathscr{R}_R^{\varphi\#}[\![s_2]\!]t)$

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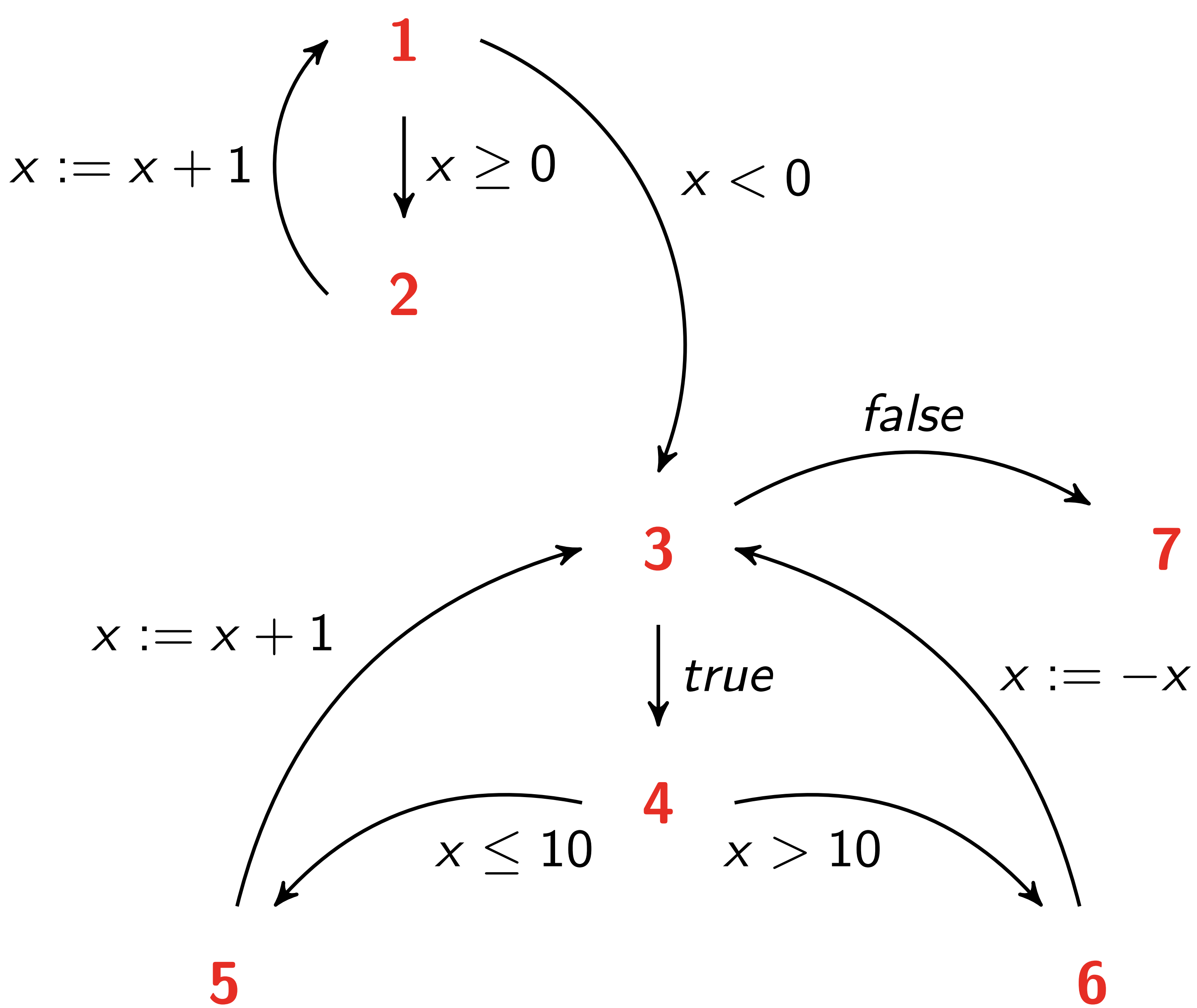
Property

$\text{AGAF}\,(x = 3)$



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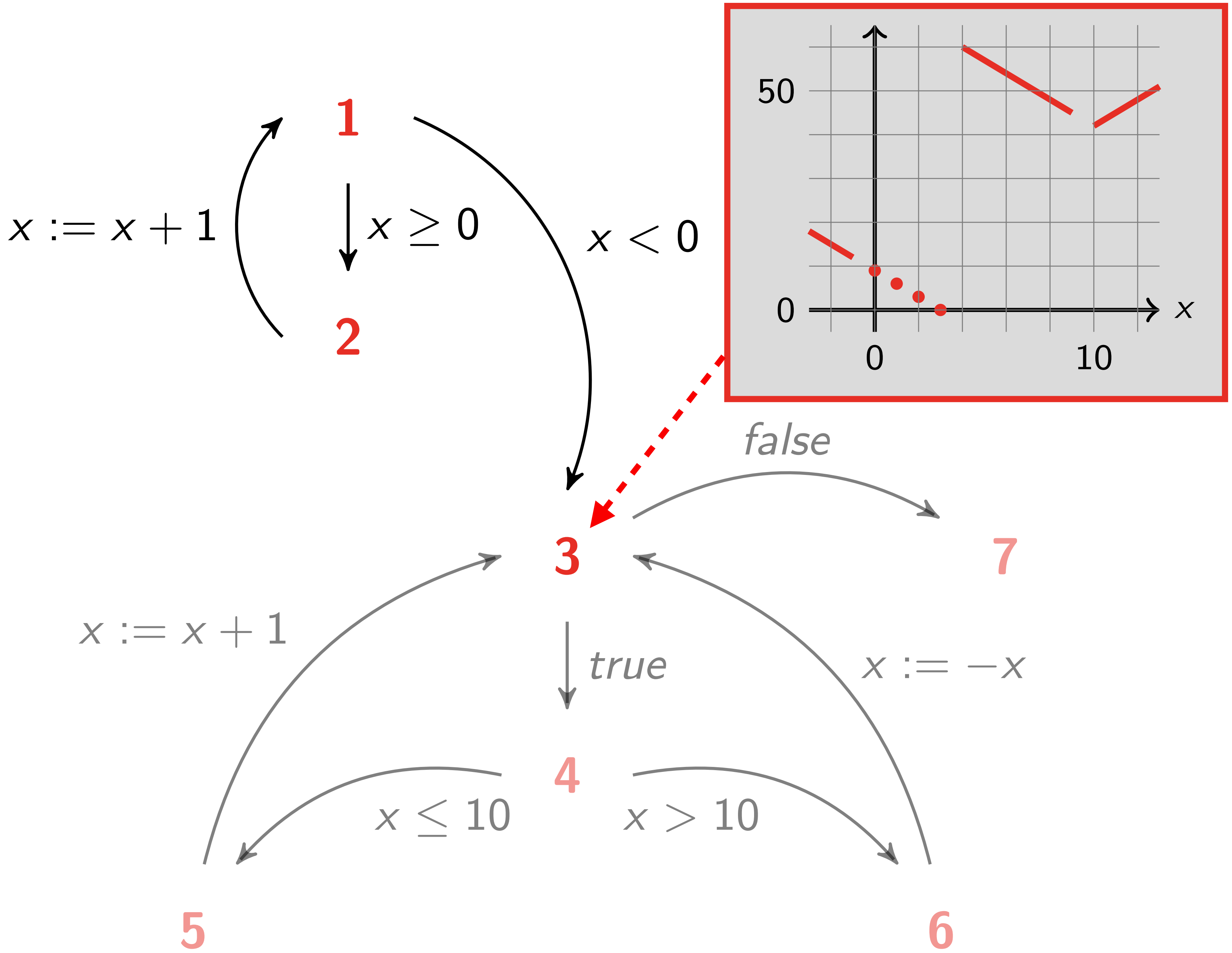

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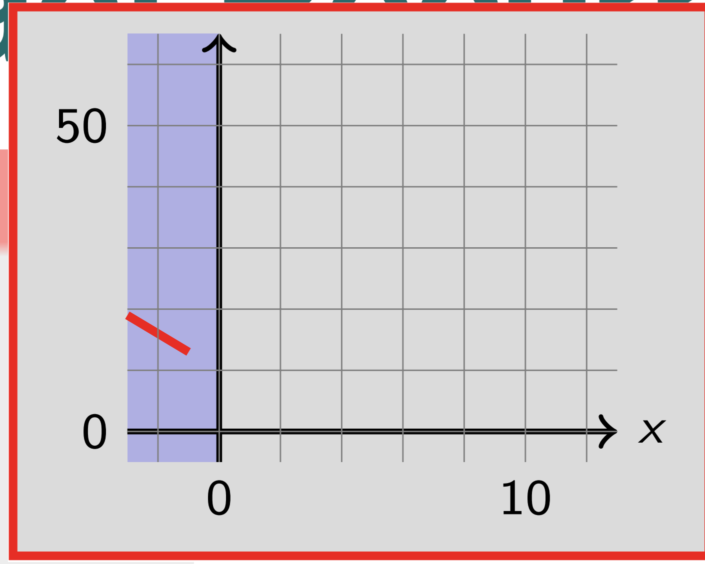

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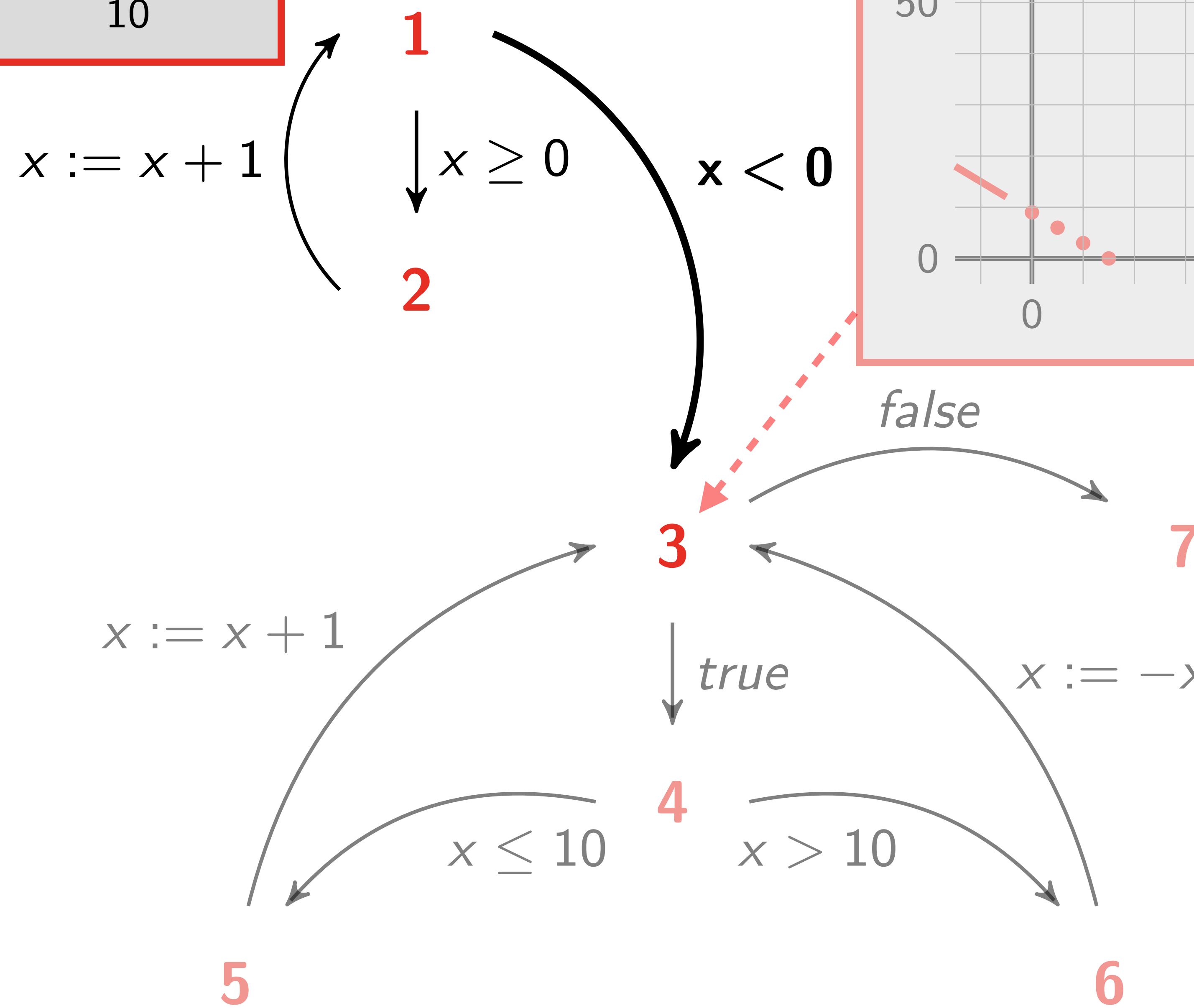

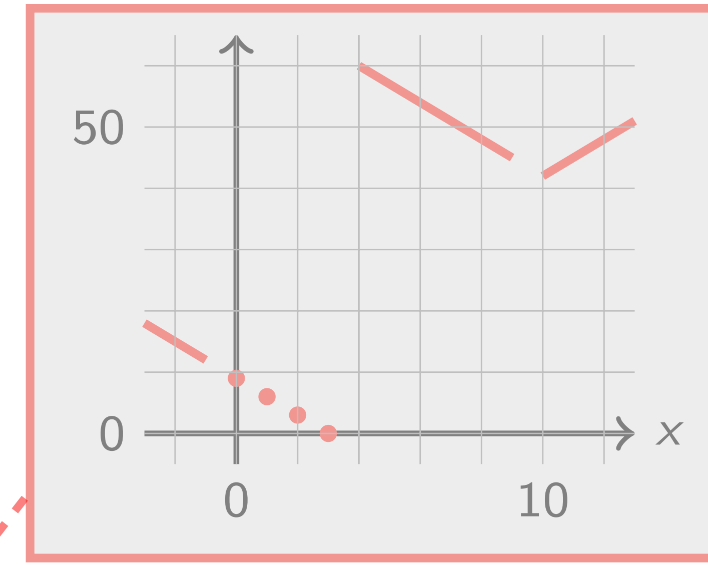

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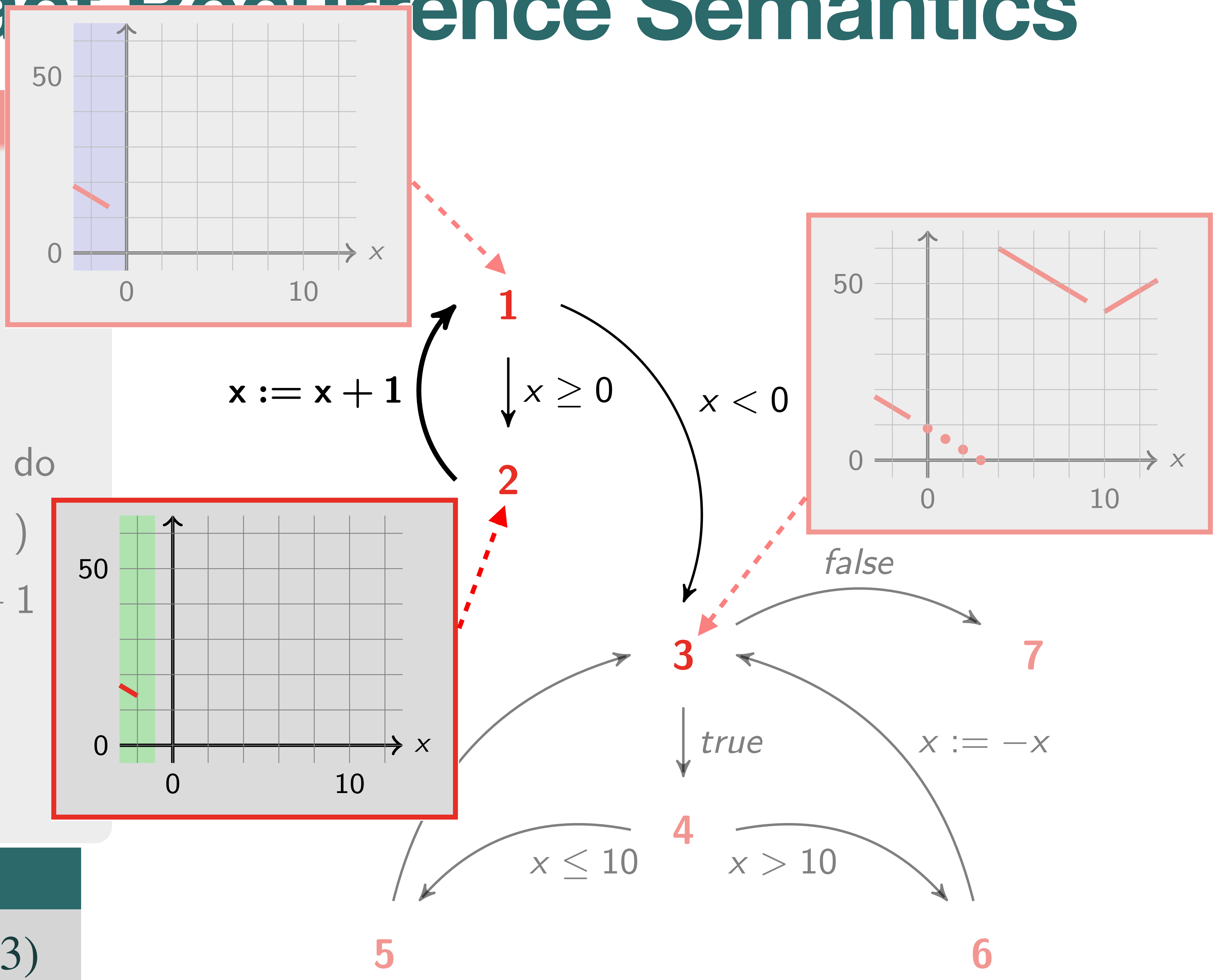

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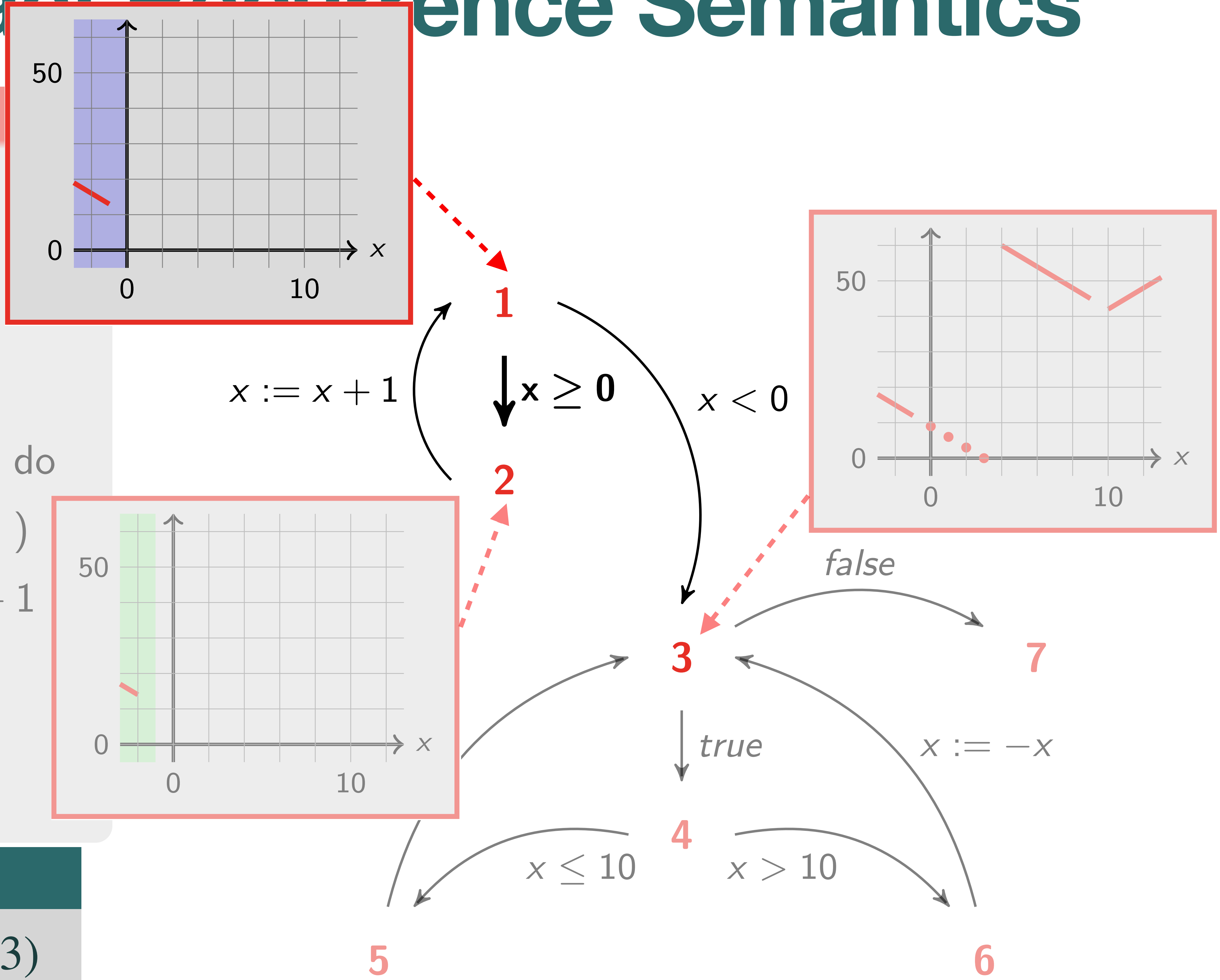

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while $^{1}(x \geq 0)$

$^{2}x := x + 1$

od

while $^{3}($ true $)$ do

if $^{4}($ $x \leq 10$ $)$

$^{5}x := x + 1$

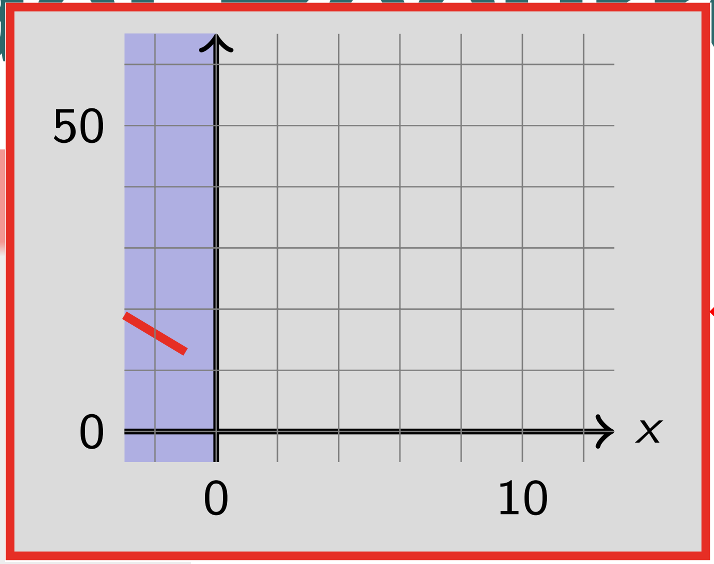
else

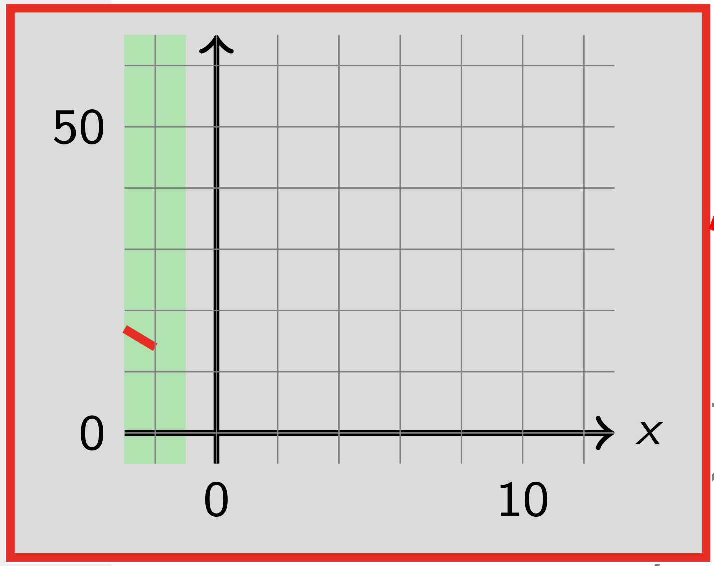
$^{6}x := -x$

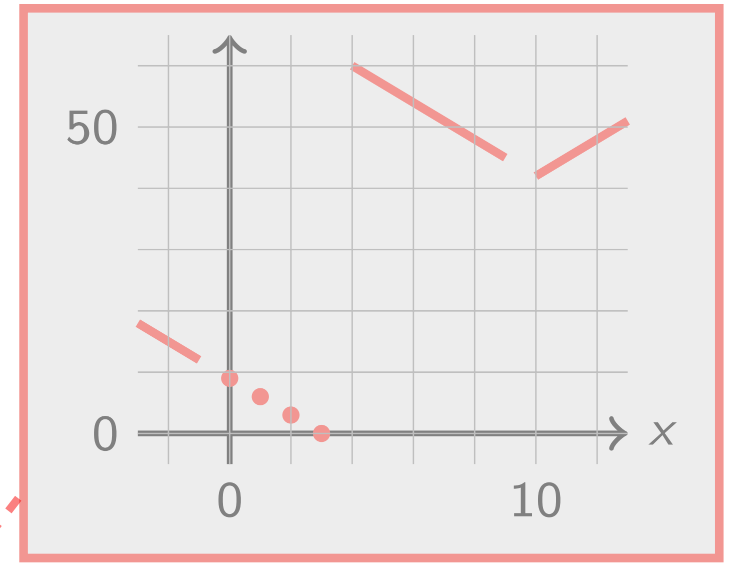
od^{7}

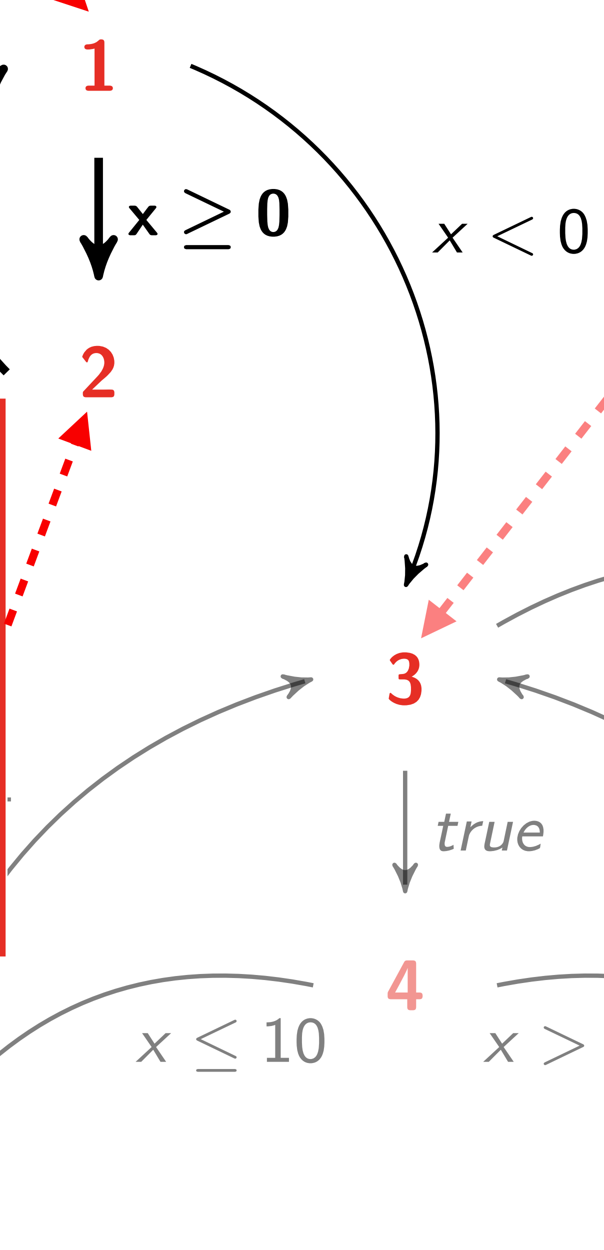
Property

$\text{AGAF}\,(x = 3)$

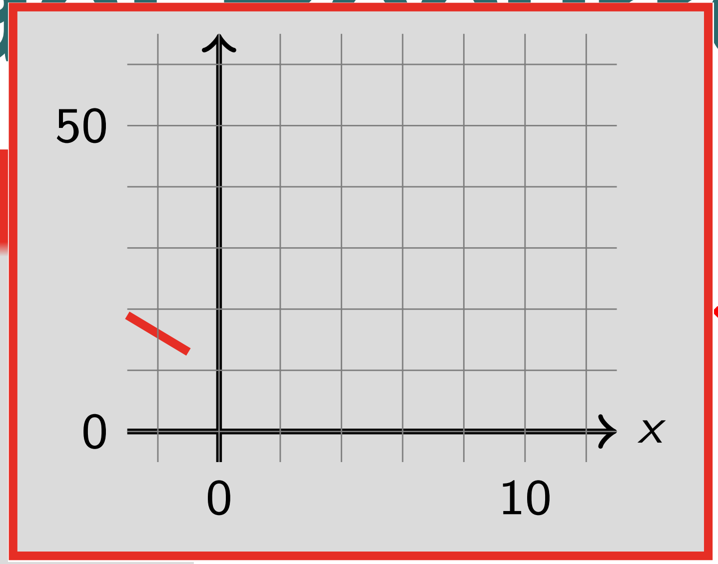








Abstract Recurrence Semantics



the analysis gives $x < 0$ as **sufficient precondition**

Example

$\text{int} : x,\ y$

$\text{while } {}^{1}(x \geq 0)$

$\quad {}^{2}x := x + 1$

od

$\text{while } {}^{3}(\ \text{true}\ ) \text{ do}$

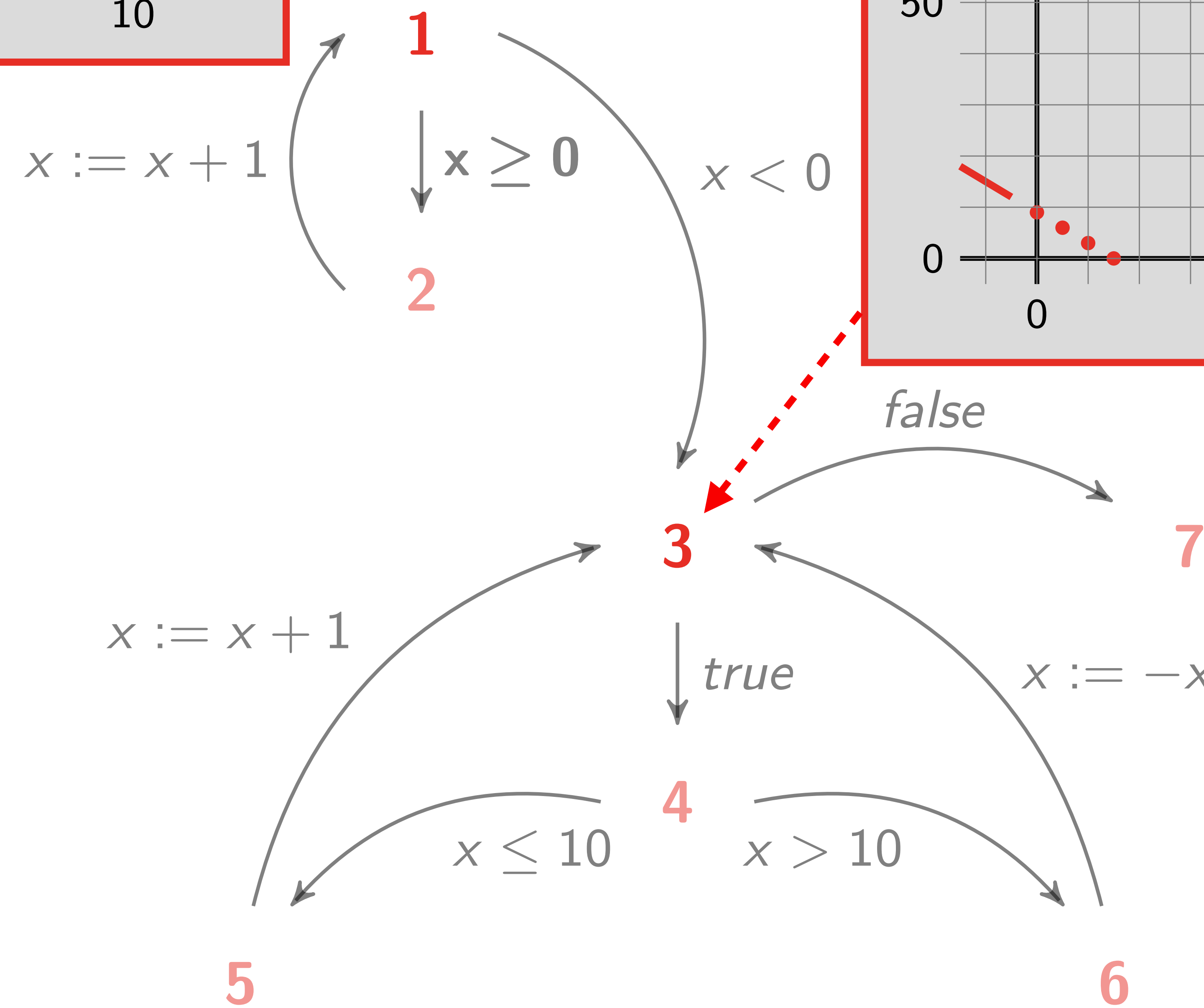
$\quad \text{if } {}^{4}(\ x \leq 10\ )$

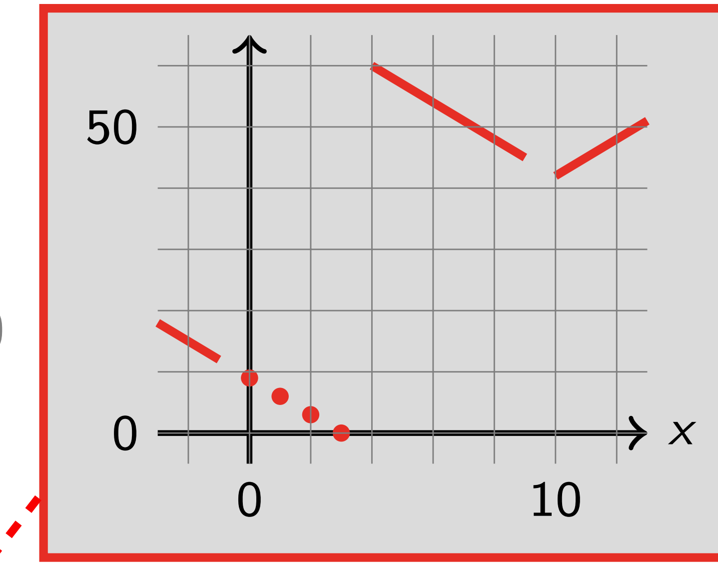
$\quad\quad {}^{5}x := x + 1$

$\quad \text{else}$

$\quad\quad {}^{6}x := -x$

od^{7}





Property

$\text{AGAF}\,(x = 3)$

Abstract Recurrence Semantics

Definition

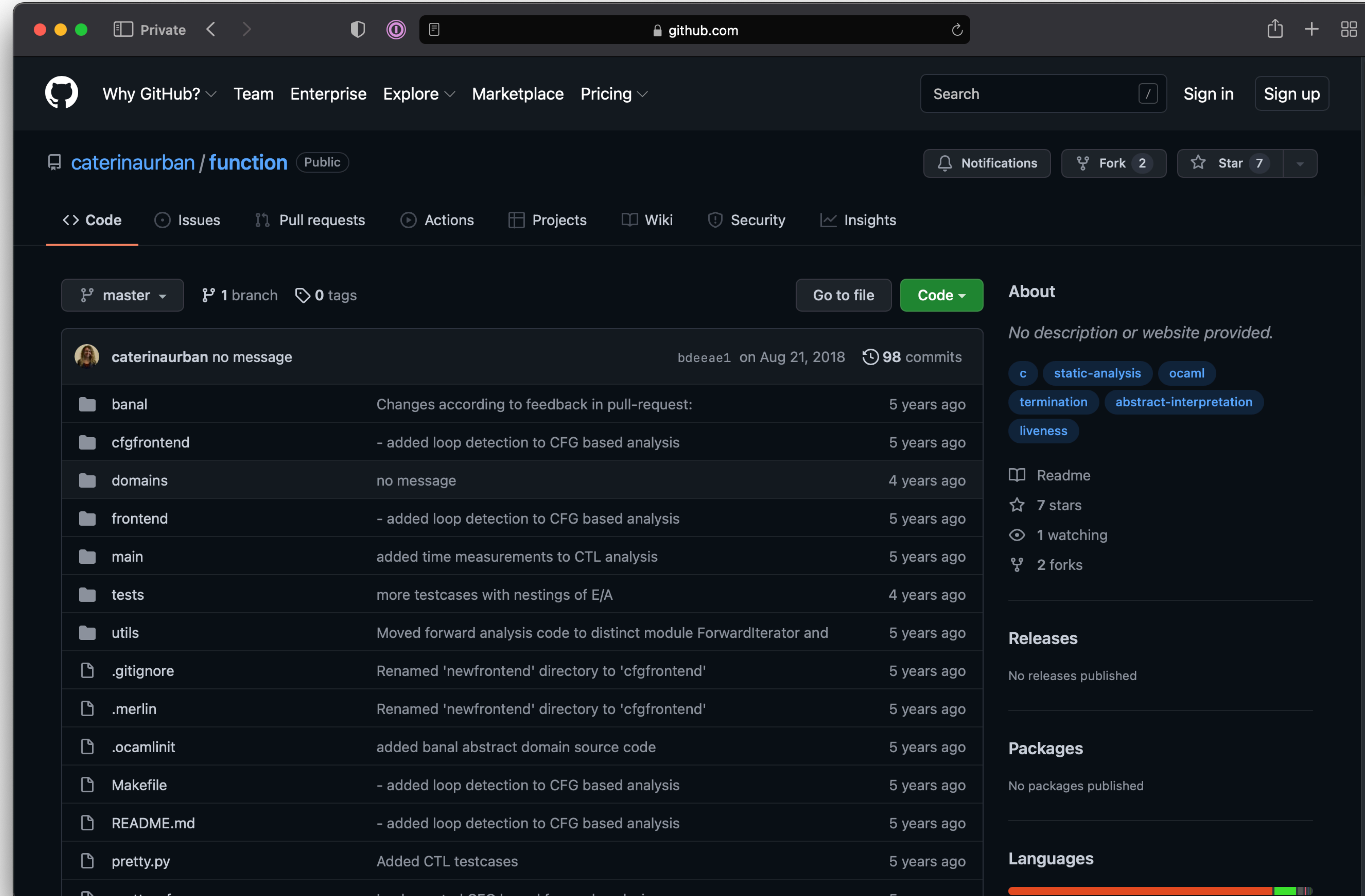
The **abstract recurrence semantics** $\mathcal{R}_R^{\varphi\#}[\![\text{stat}^{\ell}]\!] \in \mathcal{A}$ of a program stat^{ℓ} is:

$$\mathcal{R}_R^{\varphi\#}[\![\text{stat}^{\ell}]\!] \stackrel{\text{def}}{=} \mathcal{R}_R^{\varphi\#}[\![\text{stat}]\!](\text{LEAF}: \bot_F)$$

where $\mathcal{R}_R^{\varphi\#}[\![\text{stat}]\!] : \mathcal{A} \rightarrow \mathcal{A}$ is the abstract recurrence semantics of each program instruction stat

Corollary (Soundness)

A program stat^{ℓ} satisfies a **recurrence property** AG AF φ for traces starting from a set of initial states $\mathcal{I}$ if $\mathcal{I} \subseteq \text{dom}(\gamma_A(\mathcal{R}_R^{\varphi\#}[\![\text{stat}^{\ell}]\!]))$


Private
github.com
Why GitHub? Team Enterprise Explore Marketplace Pricing
Search
Sign in
Sign up
caterinaurban / function Public
Notifications
Fork 2
Star 7
Code Issues Pull requests Actions Projects Wiki Security Insights
master
1 branch
0 tags
Go to file
Code
caterinaurban no message
bdeeae1 on Aug 21, 2018
98 commits
banal Changes according to feedback in pull-request: 5 years ago
cfgfrontend - added loop detection to CFG based analysis 5 years ago
domains no message 4 years ago
frontend - added loop detection to CFG based analysis 5 years ago
main added time measurements to CTL analysis 5 years ago
tests more testcases with nestings of E/A 4 years ago
utils Moved forward analysis code to distinct module ForwardIterator and 5 years ago
.gitignore Renamed 'newfrontend' directory to 'cfgfrontend' 5 years ago
.merlin Renamed 'newfrontend' directory to 'cfgfrontend' 5 years ago
.ocamlinit added banal abstract domain source code 5 years ago
Makefile - added loop detection to CFG based analysis 5 years ago
README.md - added loop detection to CFG based analysis 5 years ago
pretty.py Added CTL testcases 5 years ago
About
No description or website provided.
c static-analysis ocaml
termination abstract-interpretation
liveness
Readme
7 stars
1 watching
2 forks
Releases
No releases published
Packages
No packages published
Languages

Bibliography

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[Urban17] **Caterina Urban**, Antoine Miné. Inference of Ranking Functions for Proving Temporal Properties by Abstract Interpretation. In COMLAN, 2017.