Program Semantics

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

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Goal

Discuss several flavors of **concrete** semantics:

- independently from programming languages (transition systems)
- defined in a constructive way (as fixpoints)
- compare their expressive power (link by abstractions)

Plan:

- transition systems
- state semantics (forward and backward)
- trace semantics (finite and infinite)
- relational semantics
- state and trace properties

Transition systems

Transition systems: definition

Language-neutral formalism to discuss program semantics.

Transition system: (Σ, τ)

- set of states Σ,

  (memory states, $\lambda-$terms, configurations, etc., generally infinite)

- transition relation $\tau \subseteq \Sigma \times \Sigma$.

(Σ, τ) is a general form of small-step operational semantics.

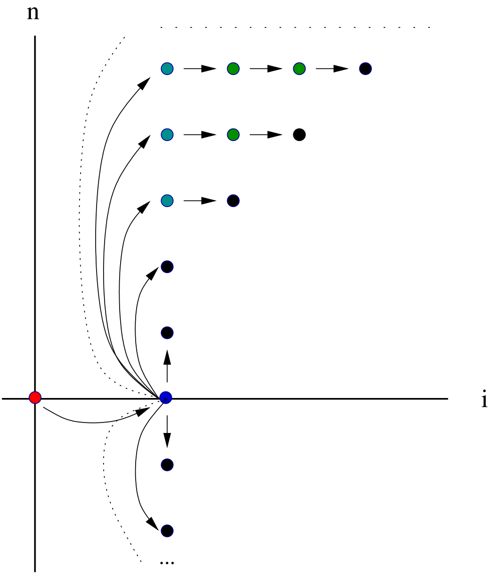
$(\sigma, \sigma') \in \tau$ is noted $\sigma \to \sigma'$:

starting in state σ, after one execution step, we can go to state σ'.

Transition system: example

$i \leftarrow 2;$
$n \leftarrow [-\infty, +\infty];$
while $i < n$ **do**
  if ? **then**
    $i \leftarrow i + 1$

$\Sigma \stackrel{\text{def}}{=} \{i, n\} \to \mathbb{Z}$



From programs to transition systems

Example: on a simple imperative language.

Language syntax

${}^{\ell}stat^{\ell}$	$::=$	${}^{\ell}X \leftarrow expr^{\ell}$	*(assignment)*
	$\mid$	${}^{\ell}$**if** $expr \bowtie 0$ **then** ${}^{\ell}stat^{\ell}$	*(conditional)*
	$\mid$	${}^{\ell}$**while** ${}^{\ell}expr \bowtie 0$ **do** ${}^{\ell}stat^{\ell}$	*(loop)*
	$\mid$	${}^{\ell}stat; {}^{\ell}stat^{\ell}$	*(sequence)*

- $X \in \mathbb{V}$, where $\mathbb{V}$ is a finite set of program variables,
- $\ell \in \mathcal{L}$ is a finite set of control labels,
- $\bowtie \in \{=, \leq, \ldots\}$, the syntax of *expr* is left undefined.
  (see next course)

Program states: $\Sigma \stackrel{\text{def}}{=} \mathcal{L} \times \mathcal{E}$ are composed of:

- a control state in $\mathcal{L}$,
- a memory state in $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{R}$.

From programs to transition systems

Transitions: $\tau[{}^{\ell}stat^{\ell'}] \subseteq \Sigma \times \Sigma$ is defined by induction on the syntax.

Assuming that expression semantics is given as $\mathsf{E}[\![\, e \,]\!] : \mathcal{E} \to \mathcal{P}(\mathbb{R})$.
(see next course)

$$\tau[{}^{\ell1}X \leftarrow e^{\ell2}] \stackrel{\text{def}}{=} \{\, (\ell1, \rho) \to (\ell2, \rho[X \mapsto v]) \mid \rho \in \mathcal{E},\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\}$$

$$\tau[{}^{\ell1}\textbf{if } e \bowtie 0 \textbf{ then } {}^{\ell2}s^{\ell3}] \stackrel{\text{def}}{=}$$
$$\{\, (\ell1, \rho) \to (\ell2, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \bowtie 0 \,\} \cup$$
$$\{\, (\ell1, \rho) \to (\ell3, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \not\bowtie 0 \,\} \cup \tau[{}^{\ell2}s^{\ell3}]$$

$$\tau[{}^{\ell1}\textbf{while } {}^{\ell2}e \bowtie 0 \textbf{ do } {}^{\ell3}s^{\ell4}] \stackrel{\text{def}}{=}$$
$$\{\, (\ell1, \rho) \to (\ell2, \rho) \mid \rho \in \mathcal{E} \,\} \cup$$
$$\{\, (\ell2, \rho) \to (\ell3, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \bowtie 0 \,\} \cup$$
$$\{\, (\ell2, \rho) \to (\ell4, \rho) \mid \rho \in \mathcal{E},\ \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon v \not\bowtie 0 \,\} \cup \tau[{}^{\ell3}s^{\ell2}]$$

$$\tau[{}^{\ell1}s_1;\, {}^{\ell2}s_2{}^{\ell3}] \stackrel{\text{def}}{=} \tau[{}^{\ell1}s_1{}^{\ell2}] \cup \tau[{}^{\ell2}s_2{}^{\ell3}]$$

Initial, final, blocking states

Initial and final states:

Transition systems (Σ, τ) are often enriched with:

- $\mathcal{I} \subseteq \Sigma$ a set of distinguished initial states,
- $\mathcal{F} \subseteq \Sigma$ a set of distinguished final states.

(e.g., limit observation to executions starting in an initial state
and ending in a final state)

Blocking states $\mathcal{B}$:

- states with no successor $\mathcal{B} \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' \in \Sigma \colon \sigma \not\to \sigma' \,\}$,
- model both correct program termination and program errors,
  (correct exit, program stuck, unhandled exception, etc.)
- often include (or equal) final states $\mathcal{F}$.

Note: we can always remove blocking states by completing τ:
$\tau' \stackrel{\text{def}}{=} \tau \cup \{\, (\sigma, \sigma) \mid \sigma \in \mathcal{B} \,\}$. (add self-loops)

Transition systems: additional examples

A large variety of semantics can be expressed as a transition system, with different notions of states and transitions.

Example: the $\lambda-$calculus

- Syntax: $t ::= x \mid \lambda x.t \mid t\ u$
- Small-step operational semantics:

$$\frac{}{(\lambda x.M)N \rightsquigarrow M[x/N]} \qquad \frac{M \rightsquigarrow M'}{M\ N \rightsquigarrow M'\ N} \qquad \frac{N \rightsquigarrow N'}{M\ N \rightsquigarrow M\ N'}$$

  (states are terms, transitions are rewriting rule applications)

- Big-step operational semantics: *(higher level)*

$$\frac{}{\lambda x.M \Downarrow \lambda x.M} \qquad \frac{M \Downarrow \lambda x.L \quad N \Downarrow V_2 \quad L[x/V_2] \Downarrow V_1}{M\ N \Downarrow V_1}$$

- Krivine abstract machine: *(lower level)*
  sequence of instructions: $\mathcal{I} \stackrel{\text{def}}{=} Grab \mid Access(\mathbb{Z}) \mid Push(\mathcal{I}) \mid \mathcal{I};\mathcal{I}$
  executed over configurations *(i.e., state)* $(inst, env, state)$
  transitions, e.g.: $\langle Access(0) \cdot C, (C_0, e_0) \cdot e, s\rangle \rightarrow \langle C_0, e_0, s\rangle, \ldots$

State semantics

State operators

Post-image, pre-image

Forward and backward images, in $\mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$:

- successors: (forward, post-image)

  $$\text{post}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma' \mid \exists \sigma \in S \colon \sigma \to \sigma' \,\}$$

- predecessors: (backward, pre-image)

  $$\text{pre}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma' \in S \colon \sigma \to \sigma' \,\}$$

post_τ and pre_τ are complete $\cup-$morphisms in $(\mathcal{P}(\Sigma), \subseteq, \cup, \cap, \emptyset, \Sigma)$.
$(\text{post}_\tau(\cup_{i \in I} S_i) = \cup_{i \in I} \text{post}_\tau(S_i),\ \text{pre}_\tau(\cup_{i \in I} S_i) = \cup_{i \in I} \text{pre}_\tau(S_i))$

post_τ and pre_τ are strict. $(\text{post}_\tau(\emptyset) = \text{pre}_\tau(\emptyset) = \emptyset)$

We have: $\text{pre}_\tau(S) = \cup \{\, \text{pre}_\tau(\{s\}) \mid s \in S \,\}$ and $\text{post}_\tau(S) = \cup \{\, \text{post}_\tau(\{s\}) \mid s \in S \,\}$.

Dual images

Dual post-images and pre-images:

- $\widetilde{\text{pre}}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma \mid \forall \sigma' : \sigma \to \sigma' \implies \sigma' \in S \,\}$
  (states such that all successors satisfy S)
- $\widetilde{\text{post}}_\tau(S) \stackrel{\text{def}}{=} \{\, \sigma' \mid \forall \sigma : \sigma \to \sigma' \implies \sigma \in S \,\}$
  (states such that all predecessors satisfy S)

$\widetilde{\text{pre}}_\tau$ and $\widetilde{\text{post}}_\tau$ are complete $\cap-$morphisms and not strict.

$\widetilde{\text{post}}$ is not much used. . .

Correspondences between images and dual images

We have the following correspondences:

- **inverse:** $\mathsf{pre}_\tau = \mathsf{post}_{(\tau^{-1})}$ $\quad \mathsf{post}_\tau = \mathsf{pre}_{(\tau^{-1})}$

  $\widetilde{\mathsf{pre}}_\tau = \widetilde{\mathsf{post}}_{(\tau^{-1})}$ $\quad \widetilde{\mathsf{post}}_\tau = \widetilde{\mathsf{pre}}_{(\tau^{-1})}$

  (where $\tau^{-1} \stackrel{\text{def}}{=} \{\, (\sigma, \sigma') \mid (\sigma', \sigma) \in \tau \,\}$)

- **Galois connections:**

$$(\mathcal{P}(\Sigma), \subseteq) \xrightleftharpoons[\mathsf{post}_\tau]{\widetilde{\mathsf{pre}}_\tau} (\mathcal{P}(\Sigma), \subseteq) \text{ and}$$

$$(\mathcal{P}(\Sigma), \subseteq) \xrightleftharpoons[\mathsf{pre}_\tau]{\widetilde{\mathsf{post}}_\tau} (\mathcal{P}(\Sigma), \subseteq).$$

<u>proof:</u>

$\mathsf{post}_\tau(A) \subseteq B \iff \{\, \sigma' \mid \exists \sigma \in A : \sigma \to \sigma' \,\} \subseteq B \iff (\forall \sigma \in A : \sigma \to \sigma' \implies \sigma' \in B) \iff (A \subseteq \{\, \sigma \mid \forall \sigma' : \sigma \to \sigma' \implies \sigma' \in B \,\}) \iff A \subseteq \widetilde{\mathsf{pre}}_\tau(B)$;
other directions are similar.

Deterministic systems

Determinism:

- (Σ, τ) is deterministic if $\forall \sigma \in \Sigma$: $|\,\mathrm{post}_\tau(\{\sigma\})| = 1$,
  (every state has a single successor, no blocking state)
- most transition systems are non-deterministic.
  (e.g., effect of input $X \leftarrow [0, 10]$, program termination)

We have the following correspondences:

- $\forall S$: $\mathcal{B} \subseteq \widetilde{\mathrm{pre}}_\tau(S) \subseteq \mathrm{pre}_\tau(S) \cup \mathcal{B}$.
  When $\mathcal{B} = \emptyset$, then $\widetilde{\mathrm{pre}}_\tau(S) \subseteq \mathrm{pre}_\tau(S)$.
- If τ is deterministic, then $\mathcal{B} = \emptyset$,
  $\mathrm{pre}_\tau = \widetilde{\mathrm{pre}}_\tau$ and $\mathrm{post}_\tau = \widetilde{\mathrm{post}}_\tau$.

post: reachability state semantics

Forward reachability

$\mathcal{R}(\mathcal{I})$: states reachable from $\mathcal{I}$ in the transition system

$$\mathcal{R}(\mathcal{I}) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0, \ldots, \sigma_n \colon \sigma_0 \in \mathcal{I}, \sigma = \sigma_n, \forall i \colon \sigma_i \to \sigma_{i+1} \,\}$$
$$= \bigcup_{n \geq 0} \mathrm{post}_\tau^n(\mathcal{I})$$

(reachable $\iff$ reachable from $\mathcal{I}$ in n steps of τ for some $n \geq 0$)

$\mathcal{R}(\mathcal{I})$ can be expressed in fixpoint form:

$$\mathcal{R}(\mathcal{I}) = \mathrm{lfp}\; F_\mathcal{R} \text{ where } F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \mathrm{post}_\tau(S)$$

($F_\mathcal{R}$ shifts S and adds back $\mathcal{I}$)

<u>Alternate characterization:</u> $\mathcal{R} = \mathrm{lfp}_\mathcal{I}\; G_\mathcal{R}$ where $G_\mathcal{R}(S) \stackrel{\text{def}}{=} S \cup \mathrm{post}_\tau(S)$.
($G_\mathcal{R}$ shifts S by τ and accumulates the result with S)

(proofs on next slide)

Forward reachability: proof

proof: of $\mathcal{R}(\mathcal{I}) = \text{lfp}\, F_\mathcal{R}$ where $F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$

$(\mathcal{P}(\Sigma), \subseteq)$ is a CPO and post_τ is continuous, hence $F_\mathcal{R}$ is continuous:
$F_\mathcal{R}(\cup_{i\in I} A_i) = \cup_{i \in I} F_\mathcal{R}(A_i)$.

By Kleene's theorem, $\text{lfp}\, F_\mathcal{R} = \cup_{n\in\mathbb{N}} F^n_\mathcal{R}(\emptyset)$.

We prove by recurrence on n that: $\forall n$: $F^n_\mathcal{R}(\emptyset) = \cup_{i<n} \text{post}^i_\tau(\mathcal{I})$.
(states reachable in less than n steps)

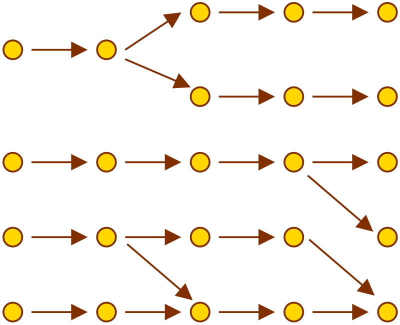
- $F^0_\mathcal{R}(\emptyset) = \emptyset$
- assuming the property at n,

$$\begin{array}{rcl} F^{n+1}_\mathcal{R}(\emptyset) & = & F_\mathcal{R}(\bigcup_{i<n} \text{post}^i_\tau(\mathcal{I})) \\ & = & \mathcal{I} \cup \text{post}_\tau(\bigcup_{i<n} \text{post}^i_\tau(\mathcal{I})) \\ & = & \mathcal{I} \cup \bigcup_{i<n} \text{post}_\tau(\text{post}^i_\tau(\mathcal{I})) \\ & = & \mathcal{I} \cup \bigcup_{1\leq i<n+1} \text{post}^i_\tau(\mathcal{I}) \\ & = & \bigcup_{i<n+1} \text{post}^i_\tau(\mathcal{I}) \end{array}$$

Hence: $\text{lfp}\, F_\mathcal{R} = \cup_{n\in\mathbb{N}} F^n_\mathcal{R}(\emptyset) = \cup_{i\in\mathbb{N}} \text{post}^i_\tau(\mathcal{I}) = \mathcal{R}(\mathcal{I})$.

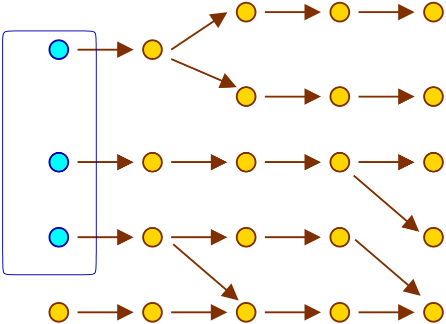
The proof is similar for the alternate form, given that $\text{lfp}_\mathcal{I}\, G_\mathcal{R} = \cup_{n\in\mathbb{N}} G^n_\mathcal{R}(\mathcal{I})$ and $G^n_\mathcal{R}(\mathcal{I}) = F^{n+1}_\mathcal{R}(\emptyset) = \cup_{i\leq n} \text{post}^i_\tau(\mathcal{I})$.

Forward reachability: graphical illustration



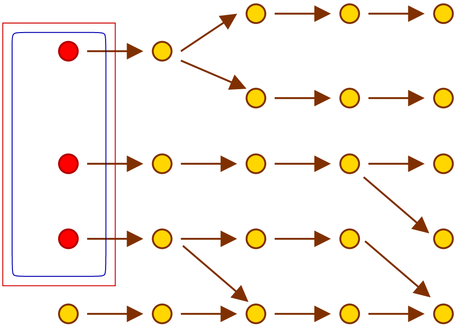
Transition system.

Forward reachability: graphical illustration



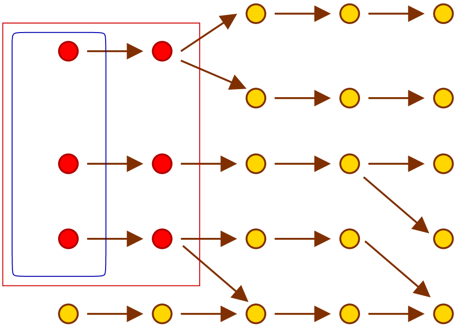
Initial states $\mathcal{I}$.

Forward reachability: graphical illustration



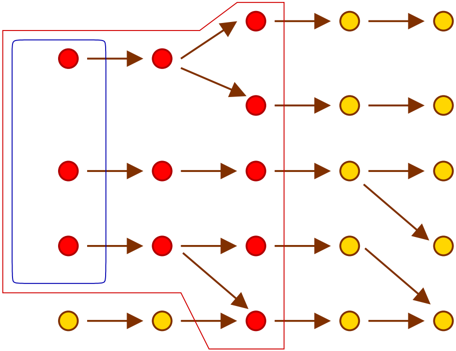
Iterate $F^1_{\mathcal{R}}(\mathcal{I})$.

Forward reachability: graphical illustration



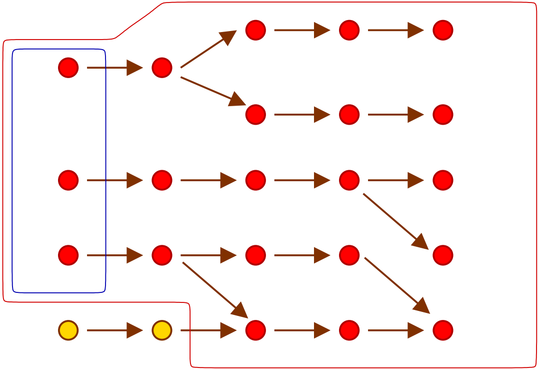
Iterate $F_{\mathcal{R}}^2(\mathcal{I})$.

Forward reachability: graphical illustration



Iterate $F_{\mathcal{R}}^3(\mathcal{I})$.

Forward reachability: graphical illustration



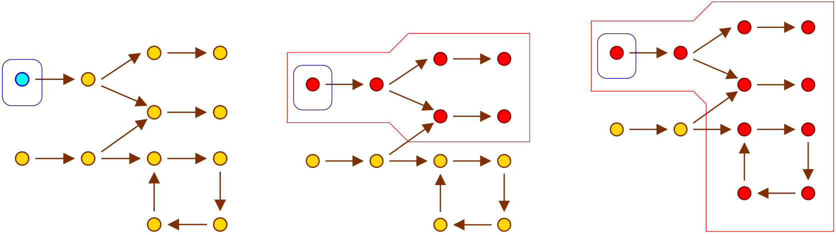
States reachable from $\mathcal{I}$: $\mathcal{R}(\mathcal{I}) = F_{\mathcal{R}}^5(\mathcal{I})$.

Multiple forward fixpoints

Recall: $\mathcal{R}(\mathcal{I}) = \text{lfp}\ F_\mathcal{R}$ where $F_\mathcal{R}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$.

Note that $F_\mathcal{R}$ may have several fixpoints.

Example:



Initial state $\mathcal{I}$ $\quad$ $\mathcal{R}(\mathcal{I}) = \text{lfp}\ F_\mathcal{R}$ $\quad$ $\text{gfp}\ F_\mathcal{R}$

Exercise:

Compute all the fixpoints of $G_\mathcal{R}(S) \stackrel{\text{def}}{=} S \cup \text{post}_\tau(S)$ on this example.

Forward reachability: applications

- Infer the set of possible states at program end: $\mathcal{R}(\mathcal{I}) \cap \mathcal{F}$.

example

$\bullet$ $i \leftarrow 0$;
while $i < 100$ **do**
  $i \leftarrow i + 1$;
  $j \leftarrow j + [0, 1]$
done $\bullet$

  - initial states $\mathcal{I}$: $j \in [0, 10]$ at control state $\bullet$,
  - final states $\mathcal{F}$: any memory state at control state $\bullet$,
  - $\Longrightarrow \mathcal{R}(\mathcal{I}) \cap \mathcal{F}$: control at $\bullet$, $i = 100$, and $j \in [0, 110]$.

- Prove the absence of run-time error: $\mathcal{R}(\mathcal{I}) \cap \mathcal{B} \subseteq \mathcal{F}$.

(never block except when reaching the end of the program)

Forward reachability equation system

By partitioning forward reachability wrt. control states,
we retrieve the equation system form of program semantics.

Control state partitioning

We assume $\Sigma \stackrel{\text{def}}{=} \mathcal{L} \times \mathcal{E}$; note that: $\mathcal{P}(\Sigma) \simeq \mathcal{L} \to \mathcal{P}(\mathcal{E})$.

We have a Galois isomorphism:

$$(\mathcal{P}(\Sigma), \subseteq) \underset{\alpha_{\mathcal{L}}}{\overset{\gamma_{\mathcal{L}}}{\leftleftarrows\!\!\rightrightarrows}} (\mathcal{L} \to \mathcal{P}(\mathcal{E}), \dot{\subseteq})$$

- $X \dot{\subseteq} Y \stackrel{\text{def}}{\iff} \forall \ell \in \mathcal{L}: X(\ell) \subseteq Y(\ell)$
- $\alpha_{\mathcal{L}}(S) \stackrel{\text{def}}{=} \lambda \ell. \{ \rho \,|\, (\ell, \rho) \in S \}$
- $\gamma_{\mathcal{L}}(X) \stackrel{\text{def}}{=} \{ (\ell, \rho) \,|\, \ell \in \mathcal{L}, \rho \in X(\ell) \}$
- simply reorganize the states by control location!

Note that: $\alpha_{\mathcal{L}} \circ \gamma_{\mathcal{L}} = \gamma_{\mathcal{L}} \circ \alpha_{\mathcal{L}} = id$. (no abstraction)

Forward reachability equation system: example

Idea:

From the global fixpoint equation $S = F_{\mathcal{R}}(S)$, $S \subseteq \mathcal{P}(\mathcal{L} \times \mathcal{E})$, consider its abstraction $F_{eq} \stackrel{\text{def}}{=} \alpha_{\mathcal{L}} \circ F_{\mathcal{R}} \circ \gamma_{\mathcal{L}}$ in $\mathcal{L} \to \mathcal{P}(\mathcal{E})$, we get $(\mathcal{X}_\ell)_{\ell \in \mathcal{L}} = F_{eq}((\mathcal{X}_\ell)_{\ell \in \mathcal{L}})$, i.e., an equation system with equation variables $\mathcal{X}_\ell \subseteq \mathcal{E}$.

Example:

$\ell 1$ $i \leftarrow 2;$
$\ell 2$ $n \leftarrow [-\infty, +\infty];$
$\ell 3$ **while** $\ell 4$ $i < n$ **do**
  $\ell 5$ **if** $[0,1] = 0$ **then**
    $\ell 6$ $i \leftarrow i + 1$
  $\ell 7$
$\ell 8$

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{I}_1 \\
\mathcal{X}_2 &= \mathsf{C}[\![\, i \leftarrow 2 \,]\!]\, \mathcal{X}_1 \\
\mathcal{X}_3 &= \mathsf{C}[\![\, n \leftarrow [-\infty, +\infty] \,]\!]\, \mathcal{X}_2 \\
\mathcal{X}_4 &= \mathcal{X}_3 \cup \mathcal{X}_7 \\
\mathcal{X}_5 &= \mathsf{C}[\![\, i < n \,]\!]\, \mathcal{X}_4 \\
\mathcal{X}_6 &= \mathcal{X}_5 \\
\mathcal{X}_7 &= \mathcal{X}_5 \cup \mathsf{C}[\![\, i \leftarrow i + 1 \,]\!]\, \mathcal{X}_6 \\
\mathcal{X}_8 &= \mathsf{C}[\![\, i \geq n \,]\!]\, \mathcal{X}_4
\end{aligned}$$

- initial states $\mathcal{I} \stackrel{\text{def}}{=} \{\, (\ell 1, \rho) \mid \rho \in \mathcal{I}_1 \,\}$ for some $\mathcal{I}_1 \subseteq \mathcal{E}$,
- $\mathsf{C}[\![\, \cdot \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$ model assignments and tests (see next slide).

Forward reachability equation system: construction

We derive the equation system $eq({}^{\ell}stat^{\ell'})$
from the program syntax ${}^{\ell}stat^{\ell'}$ by structural induction:

$$eq({}^{\ell 1}X \leftarrow e^{\ell 2}) \stackrel{\text{def}}{=} \{\, \mathcal{X}_{\ell 2} = \mathsf{C}[\![\, X \leftarrow e \,]\!]\, \mathcal{X}_{\ell 1} \,\}$$

$$eq({}^{\ell 1}\textbf{if } e \bowtie 0 \textbf{ then } {}^{\ell 2}s^{\ell 3}) \stackrel{\text{def}}{=}$$
$$\{\, \mathcal{X}_{\ell 2} = \mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X}_{\ell 1},\ \mathcal{X}_{\ell 3} = \mathcal{X}_{\ell 3'} \cup \mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\, \mathcal{X}_{\ell 1} \,\} \cup eq({}^{\ell 2}s^{\ell 3'})$$

$$eq({}^{\ell 1}\textbf{while } {}^{\ell 2}e \bowtie 0 \textbf{ do } {}^{\ell 3}s^{\ell 4}) \stackrel{\text{def}}{=}$$
$$\{\, \mathcal{X}_{\ell 2} = \mathcal{X}_{\ell 1} \cup \mathcal{X}_{\ell 4'},\ \mathcal{X}_{\ell 3} = \mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X}_{\ell 2},\ \mathcal{X}_{\ell 4} = \mathsf{C}[\![\, e \not\bowtie 0 \,]\!]\, \mathcal{X}_{\ell 2} \,\} \cup eq({}^{\ell 3}s^{\ell 4'})$$

$$eq({}^{\ell 1}s_1; {}^{\ell 2}s_2{}^{\ell 3}) \stackrel{\text{def}}{=} eq({}^{\ell 1}s_1{}^{\ell 2}) \cup ({}^{\ell 2}s_2{}^{\ell 3})$$

where:

- $\mathcal{X}^{\ell 3'}$, $\mathcal{X}^{\ell 4'}$ are fresh variables storing intermediate results
- $\mathsf{C}[\![\, X \leftarrow e \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho[X \mapsto v] \mid \rho \in \mathcal{X},\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\}$
  $\mathsf{C}[\![\, e \bowtie 0 \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho \in \mathcal{X} \mid \exists v \in \mathsf{E}[\![\, \rho \,]\!]\, \rho\colon v \bowtie 0 \,\}$

$\cup-$morphisms in a complete lattice $\implies$ a smallest solution exists

pre: co-reachability state semantics

Backward reachability

$\mathcal{C}(\mathcal{F})$: states co-reachable from $\mathcal{F}$ in the transition system:

$$\mathcal{C}(\mathcal{F}) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists n \geq 0, \sigma_0, \ldots, \sigma_n : \sigma = \sigma_0, \sigma_n \in \mathcal{F}, \forall i : \sigma_i \to \sigma_{i+1} \,\}$$
$$= \bigcup_{n \geq 0} \mathsf{pre}_\tau^n(\mathcal{F})$$

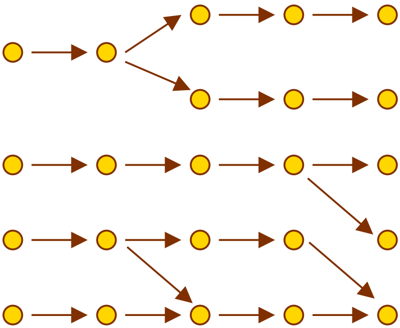
$\mathcal{C}(\mathcal{F})$ can also be expressed in fixpoint form:

$$\mathcal{C}(\mathcal{F}) = \mathsf{lfp}\, F_\mathcal{C} \text{ where } F_\mathcal{C}(S) \stackrel{\text{def}}{=} \mathcal{F} \cup \mathsf{pre}_\tau(S)$$

Alternate characterization: $\mathcal{C}(\mathcal{F}) = \mathsf{lfp}_\mathcal{F}\ G_\mathcal{C}$ where $G_\mathcal{C}(S) = S \cup \mathsf{pre}_\tau(S)$

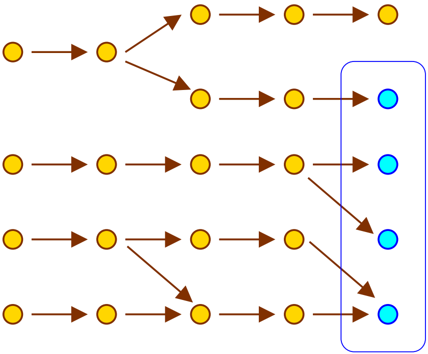
Justification: $\mathcal{C}(\mathcal{F})$ in τ is exactly $\mathcal{R}(\mathcal{F})$ in τ^{-1}.

Backward reachability: graphical illustration



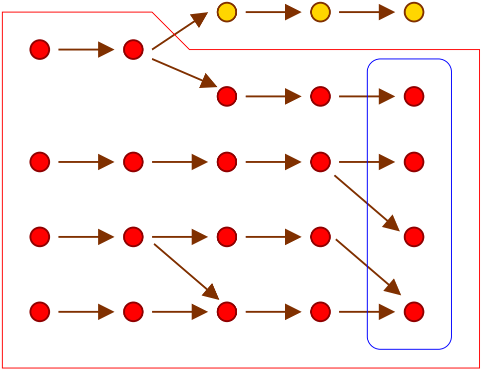
Transition system.

Backward reachability: graphical illustration



Final states $\mathcal{F}$.

Backward reachability: graphical illustration



States co-reachable from $\mathcal{F}$.

Backward reachability: applications

- $\mathcal{I} \cap \mathcal{C}(\mathcal{B} \setminus \mathcal{F})$
  Initial states that have at least one erroneous execution.

program

```
•  j ← 0;
   while i > 0 do
      i ← i − 1;
      j ← j + [0, 10]
   done •
```

- initial states $\mathcal{I}$: $i \in [0, 100]$ at •
- final states $\mathcal{F}$: any memory state at •
- blocking states $\mathcal{B}$: final, or $j > 200$ at any location
- $\mathcal{I} \cap \mathcal{C}(\mathcal{B} \setminus \mathcal{F})$: at •, $i > 20$

- $\mathcal{I} \cap (\Sigma \setminus \mathcal{C}(\mathcal{B}))$
  Initial states that necessarily cause the program to loop.

- Iterate forward and backward analyses interactively
  $\Longrightarrow$ abstract debugging [Bour93].

Backward reachability equation system: example

Principle:

As before, reorganize transitions by label $\ell \in \mathcal{L}$,
to get an equation system on $(\mathcal{X}_\ell)_\ell$, with $\mathcal{X}_\ell \subseteq \mathcal{E}$

Example:

$^{\ell 1}$ $i \leftarrow 2$;
$^{\ell 2}$ $n \leftarrow [-\infty, +\infty]$;
$^{\ell 3}$ **while** $^{\ell 4}$ $i < n$ **do**
  $^{\ell 5}$ **if** $[0, 1] = 0$ **then**
    $^{\ell 6}$ $i \leftarrow i + 1$
  $^{\ell 7}$
$^{\ell 8}$

$$
\begin{aligned}
\mathcal{X}_1 &= \mathsf{C}[\![\, i \rightarrow 2 \,]\!]\, \mathcal{X}_2 \\
\mathcal{X}_2 &= \mathsf{C}[\![\, n \rightarrow [-\infty, +\infty] \,]\!]\, \mathcal{X}_3 \\
\mathcal{X}_3 &= \mathcal{X}_4 \\
\mathcal{X}_4 &= \mathsf{C}[\![\, i < n \,]\!]\, \mathcal{X}_5 \cup \mathsf{C}[\![\, i \geq n \,]\!]\, \mathcal{X}_8 \\
\mathcal{X}_5 &= \mathcal{X}_6 \cup \mathcal{X}_7 \\
\mathcal{X}_6 &= \mathsf{C}[\![\, i \rightarrow i + 1 \,]\!]\, \mathcal{X}_7 \\
\mathcal{X}_7 &= \mathcal{X}_4 \\
\mathcal{X}_8 &= \mathcal{F}_8
\end{aligned}
$$

- final states $\mathcal{F} \stackrel{\text{def}}{=} \{\, (\ell 8, \rho) \mid \rho \in \mathcal{F}_8 \,\}$ for some $\mathcal{F}_8 \subseteq \mathcal{E}$,
- $\mathsf{C}[\![\, X \rightarrow e \,]\!]\, \mathcal{X} \stackrel{\text{def}}{=} \{\, \rho \mid \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon \rho[X \mapsto v] \in X \,\}$.

$\widetilde{\text{pre}}$: pre-condition state semantics

Sufficient preconditions

$\mathcal{S}(\mathcal{Y})$: states with executions staying in $\mathcal{Y}$.

$$\mathcal{S}(\mathcal{Y}) \stackrel{\text{def}}{=} \{ \sigma \mid \forall n \geq 0, \sigma_0, \ldots, \sigma_n : (\sigma = \sigma_0 \land \forall i : \sigma_i \to \sigma_{i+1}) \implies \sigma_n \in \mathcal{Y} \}$$
$$= \bigcap_{n \geq 0} \widetilde{\text{pre}}_\tau^n(\mathcal{Y})$$

$\mathcal{S}(\mathcal{Y})$ can be expressed in fixpoint form:

$$\mathcal{S}(\mathcal{Y}) = \text{gfp}\, F_\mathcal{S} \text{ where } F_\mathcal{S}(S) \stackrel{\text{def}}{=} \mathcal{Y} \cap \widetilde{\text{pre}}_\tau(S)$$

proof sketch: similar to that of $\mathcal{R}(\mathcal{I})$, in the dual.

$F_\mathcal{S}$ is continuous in the dual CPO $(\mathcal{P}(\Sigma), \supseteq)$, because $\widetilde{\text{pre}}_\tau$ is:
$F_\mathcal{S}(\cap_{i \in I} A_i) = \cap_{i \in I} F_\mathcal{S}(A_i)$.
By Kleene's theorem in the dual, $\text{gfp}\, F_\mathcal{S} = \cap_{n \in \mathbb{N}} F_\mathcal{S}^n(\Sigma)$.
We would prove by recurrence that $F_\mathcal{S}^n(\Sigma) = \cap_{i<n} \widetilde{\text{pre}}_\tau^i(\mathcal{Y})$.

Sufficient preconditions and reachability

Correspondence with reachability:

We have a Galois connection:

$$(\mathcal{P}(\Sigma), \subseteq) \underset{\mathcal{R}}{\overset{\mathcal{S}}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$$

- $\mathcal{R}(\mathcal{I}) \subseteq \mathcal{Y} \iff \mathcal{I} \subseteq \mathcal{S}(\mathcal{Y})$
  definition of a Galois connection
  all executions from $\mathcal{I}$ stay in $\mathcal{Y}$
  $\iff$ $\mathcal{I}$ includes only sufficient pre-conditions for $\mathcal{Y}$
- so $\mathcal{S}(\mathcal{Y}) = \bigcup \{\, X \mid \mathcal{R}(X) \subseteq \mathcal{Y} \,\}$
  by Galois connection property
  $\mathcal{S}(\mathcal{Y})$ is the largest initial set whose reachability is in $\mathcal{Y}$

We retrieve Dijkstra's weakest liberal preconditions.

(proof sketch on next slide)

Sufficient preconditions and reachability (proof)

proof sketch:

Recall that $\mathcal{R}(\mathcal{I}) = \text{lfp}_{\mathcal{I}}\, G_{\mathcal{R}}$ where $G_{\mathcal{R}}(S) = S \cup \text{post}_\tau(S)$.
Likewise, $\mathcal{S}(\mathcal{Y}) = \text{gfp}_{\mathcal{Y}}\, G_{\mathcal{S}}$ where $G_{\mathcal{S}}(S) = S \cap \widetilde{\text{pre}}_\tau(S)$.

Recall the Galois connection $(\mathcal{P}(\Sigma), \subseteq) \underset{\text{post}_\tau}{\overset{\widetilde{\text{pre}}_\tau}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$.

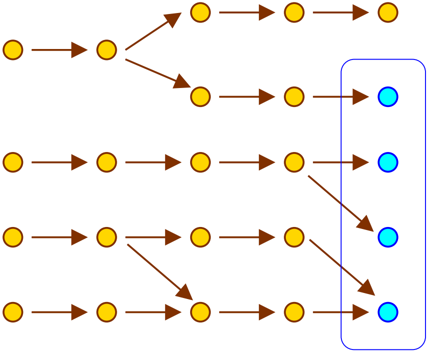
As a consequence $(\mathcal{P}(\Sigma), \subseteq) \underset{G_{\mathcal{R}}}{\overset{G_{\mathcal{S}}}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$.

The Galois connection can be lifted to fixpoint operators:

$(\mathcal{P}(\Sigma), \subseteq) \underset{x \mapsto \text{lfp}_x\, G_{\mathcal{R}}}{\overset{x \mapsto \text{gfp}_x\, G_{\mathcal{S}}}{\leftrightarrows}} (\mathcal{P}(\Sigma), \subseteq)$.

Exercise: complete the proof sketch.

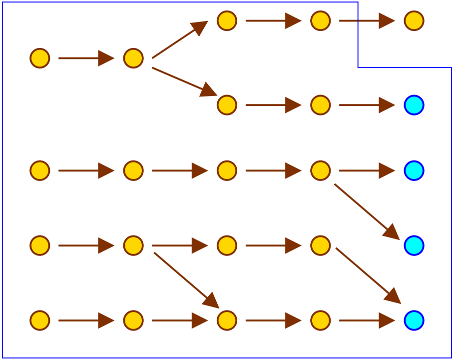
Sufficient preconditions: graphical illustration



Final states $\mathcal{F}$.

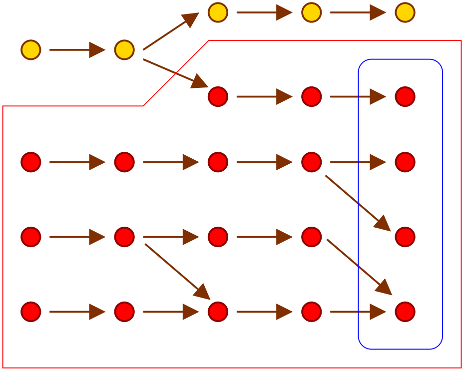
Goal: when stopping, stop in $\mathcal{F}$.

Sufficient preconditions: graphical illustration



Goal: avoid stopping in a non-final state (i.e., error state)
but passing through a non-blocking state is not (yet) an error
$\Longrightarrow$ consider $\mathcal{Y} = \mathcal{F} \cup (\Sigma \setminus \mathcal{B})$.

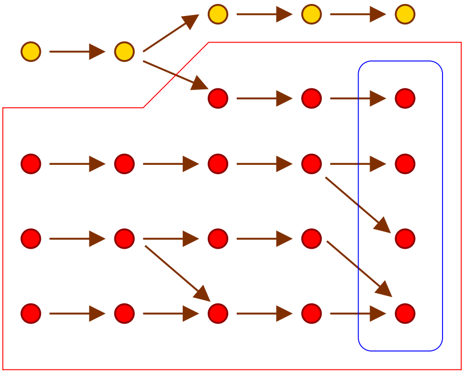
Sufficient preconditions: graphical illustration

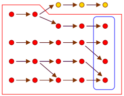


Sufficient preconditions $\mathcal{S}(\mathcal{Y})$ to stop in $\mathcal{F}$.

(without forcing the program to stop at all)

Sufficient preconditions: graphical illustration





Sufficient preconditions $\mathcal{S}(\mathcal{Y})$ to stop in $\mathcal{F}$.

(without forcing the program to stop at all)

$\mathcal{C}(\mathcal{F})$

Note: $\mathcal{S}(\mathcal{Y}) \subset \mathcal{C}(\mathcal{F})$

Sufficient preconditions: application

Initial states such that all executions are correct:
$\mathcal{I} \cap \mathcal{S}(\mathcal{F} \cup (\Sigma \setminus \mathcal{B}))$.

(the only blocking states reachable from initial states are final states)

program

```
• i ← 0;
  while i < 100 do
    i ← i + 1;
    j ← j + [0, 1]
  done •
```

- initial states $\mathcal{I}$: $j \in [0, 10]$ at •
- final states $\mathcal{F}$: any memory state at •
- blocking states $\mathcal{B}$: final, or $j > 105$ at any location
- $\mathcal{I} \cap \mathcal{S}(\mathcal{F} \cup (\Sigma \setminus \mathcal{B}))$: at •, $j \in [0, 5]$

  (note that $\mathcal{I} \cap \mathcal{C}(\mathcal{F} \cup (\Sigma \setminus \mathcal{B}))$ gives $\mathcal{I}$)

Applications: infer contracts;
optimize (hoist) tests;
infer counter-examples.

Trace semantics

Traces and trace operations

Sequences, traces

Trace: sequence of elements from Σ

- ϵ: empty trace (unique)
- σ: trace of length 1 (assimilated to a state)
- $\sigma_0, \ldots, \sigma_{n-1}$: trace of length n
- $\sigma_0, \ldots, \sigma_n, \ldots$: infinite trace (length ω)

Trace sets:

- Σ^n: the set of traces of length n
- $\Sigma^{\leq n} \stackrel{\text{def}}{=} \cup_{i \leq n} \Sigma^i$: the set of traces of length at most n
- $\Sigma^* \stackrel{\text{def}}{=} \cup_{i \in \mathbb{N}} \Sigma^i$: the set of finite traces
- Σ^ω: the set of infinite traces
- $\Sigma^\infty \stackrel{\text{def}}{=} \Sigma^* \cup \Sigma^\omega$: the set of all traces

Traces of a transition system

Execution traces:

Non-empty sequences of states linked by the transition relation τ.

- can be finite (in $\mathcal{P}(\Sigma^*)$) or infinite (in $\mathcal{P}(\Sigma^\omega)$)
- can be anchored at initial states, or final states, or none

Atomic traces:

- $\mathcal{I}$: initial states $\simeq$ set of traces of length 1
- $\mathcal{F}$: final states $\simeq$ set of traces of length 1
- τ: transition relation $\simeq$ set of traces of length 2

  $(\{\, \sigma, \sigma' \mid \sigma \to \sigma' \,\})$

(as $\Sigma \simeq \Sigma^1$ and $\Sigma \times \Sigma \simeq \Sigma^2$)

Trace operations

Operations on traces:

- length: $|t| \in \mathbb{N} \cup \{\omega\}$ of a trace $t \in \Sigma^\infty$
- concatenation $\cdot$
  - $(\sigma_0, \ldots, \sigma_n) \cdot (\sigma'_0, \ldots) \stackrel{\text{def}}{=} \sigma_0, \ldots, \sigma_n, \sigma'_0, \ldots$
    (append to a finite trace)
  - $t \cdot t' \stackrel{\text{def}}{=} t$ if $t \in \Sigma^\omega$ (append to an infinite trace does nothing)
  - $\epsilon \cdot t \stackrel{\text{def}}{=} t \cdot \epsilon \stackrel{\text{def}}{=} t$ (ϵ is neutral)
- junction $\frown$
  - $(\sigma_0, \ldots, \sigma_n) \frown (\sigma'_0, \sigma'_1 \ldots) \stackrel{\text{def}}{=} \sigma_0, \ldots, \sigma_n, \sigma'_1, \ldots$ when $\sigma_n = \sigma'_0$
    undefined if $\sigma_n \neq \sigma'_0$
  - $\epsilon \frown t$ and $t \frown \epsilon$ are undefined
  - $t \frown t' \stackrel{\text{def}}{=} t$, if $t \in \Sigma^\omega$

Trace operations (cont.)

Extension to sets of traces:

- $A \cdot B \stackrel{\text{def}}{=} \{\, a \cdot b \mid a \in A,\ b \in B \,\}$
  $\{\epsilon\}$ is the neutral element for $\cdot$
- $A ^\frown B \stackrel{\text{def}}{=} \{\, a ^\frown b \mid a \in A,\ b \in B,\ a ^\frown b \text{ defined} \,\}$
  Σ is the neutral element for $^\frown$

A^0	$\stackrel{\text{def}}{=}$	$\{\epsilon\}$	$A^{\frown 0}$	$\stackrel{\text{def}}{=}$	Σ
A^{n+1}	$\stackrel{\text{def}}{=}$	$A \cdot A^n$	$A^{\frown n+1}$	$\stackrel{\text{def}}{=}$	$A ^\frown A^{\frown n}$
A^ω	$\stackrel{\text{def}}{=}$	$A \cdot A \cdot \cdots$	$A^{\frown \omega}$	$\stackrel{\text{def}}{=}$	$A ^\frown A ^\frown \cdots$
A^*	$\stackrel{\text{def}}{=}$	$\cup_{n<\omega} A^n$	$A^{\frown *}$	$\stackrel{\text{def}}{=}$	$\cup_{n<\omega} A^{\frown n}$
A^∞	$\stackrel{\text{def}}{=}$	$\cup_{n\leq\omega} A^n$	$A^{\frown \infty}$	$\stackrel{\text{def}}{=}$	$\cup_{n\leq\omega} A^{\frown n}$

Note: $A^n \neq \{\, a^n \mid a \in A \,\}$, $A^{\frown n} \neq \{\, a^{\frown n} \mid a \in A \,\}$ when $|A| > 1$

Finite prefix trace semantics

Prefix trace semantics

$\mathcal{T}_p(\mathcal{I})$: partial, finite execution traces starting in $\mathcal{I}$.

$$\mathcal{T}_p(\mathcal{I}) \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \sigma_0 \in \mathcal{I}, \forall i : \sigma_i \to \sigma_{i+1} \,\}$$
$$= \bigcup_{n \geq 0} \mathcal{I}^\frown (\tau^{\frown n})$$

(traces of length n, for any n, starting in $\mathcal{I}$ and following τ)

$\mathcal{T}_p(\mathcal{I})$ can be expressed in fixpoint form:

$$\mathcal{T}_p(\mathcal{I}) = \text{lfp}\, F_p \text{ where } F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T^\frown \tau$$

(F_p appends a transition to each trace, and adds back $\mathcal{I}$)

(proof on next slide)

Prefix trace semantics: proof

proof of: $\mathcal{T}_p(\mathcal{I}) = \text{lfp}\, F_p$ where $F_p(T) = \mathcal{I} \cup T \frown \tau$

Similar to the proof of $\mathcal{R}(\mathcal{I}) = \text{lfp}\, F_{\mathcal{R}}$ where $F_{\mathcal{R}}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$.

F_p is continuous in a CPO $(\mathcal{P}(\Sigma^*), \subseteq)$:

$$\begin{array}{ll} & F_p(\cup_{i\in I} T_i) \\ = & \mathcal{I} \cup (\cup_{i\in I} T_i) \frown \tau \\ = & \mathcal{I} \cup (\cup_{i\in I} T_i \frown \tau) = \cup_{i\in I} (\mathcal{I} \cup T_i \frown \tau) \end{array}$$

hence (Kleene), $\text{lfp}\, F_p = \cup_{n\geq 0} F_p^i(\emptyset)$

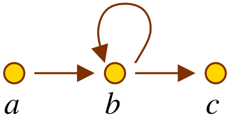
We prove by recurrence on n that $\forall n$: $F_p^n(\emptyset) = \cup_{i<n} \mathcal{I} \frown \tau^{\frown i}$:

- $F_p^0(\emptyset) = \emptyset$,
- $$\begin{array}{ll} & F_p^{n+1}(\emptyset) \\ = & \mathcal{I} \cup F_p^n(\emptyset) \frown \tau \\ = & \mathcal{I} \cup (\cup_{i<n} \mathcal{I} \frown \tau^{\frown i}) \frown \tau \\ = & \mathcal{I} \cup \cup_{i<n} (\mathcal{I} \frown \tau^{\frown i}) \frown \tau \\ = & \mathcal{I} \frown \tau^{\frown 0} \cup \cup_{i<n} (\mathcal{I} \frown \tau^{\frown i+1}) \\ = & \cup_{i<n+1} \mathcal{I} \frown \tau^{\frown i} \end{array}$$

Thus, $\text{lfp}\, F_p = \cup_{n\in\mathbb{N}} F_p^n(\emptyset) = \cup_{n\in\mathbb{N}} \cup_{i<n} \mathcal{I} \frown \tau^{\frown i} = \cup_{i\in\mathbb{N}} \mathcal{I} \frown \tau^{\frown i}$.

Note: we also have $\mathcal{T}_p(\mathcal{I}) = \text{lfp}_{\mathcal{I}}\, G_p$ where $G_p(T) = T \cup T \frown \tau$.

Prefix trace semantics: graphical illustration



$\mathcal{I} \stackrel{\text{def}}{=} \{a\}$

$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$

Iterates: $\mathcal{T}_p(\mathcal{I}) = \text{lfp } F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$.

- $F_p^0(\emptyset) = \emptyset$
- $F_p^1(\emptyset) = \mathcal{I} = \{a\}$
- $F_p^2(\emptyset) = \{a, ab\}$
- $F_p^3(\emptyset) = \{a, ab, abb, abc\}$
- $F_p^n(\emptyset) = \{\, a, ab^i, ab^j c \mid i \in [1, n-1], j \in [1, n-2] \,\}$
- $\mathcal{T}_p(\mathcal{I}) = \cup_{n \geq 0} F_p^n(\emptyset) = \{\, a, ab^i, ab^i c \mid i \geq 1 \,\}$

Prefix trace semantics: expressive power

The prefix trace semantics is the collection of finite observations of program executions.

$\Longrightarrow$ Semantics of testing.

Limitations:

- no information on infinite executions,
  (we will add infinite traces later)
- can bound maximal execution time: $\mathcal{T}_p(\mathcal{I}) \subseteq \Sigma^{\leq n}$
  but cannot bound minimal execution time.
  (we will consider maximal traces later)

Abstracting traces into states

Idea: view state semantics as abstractions of traces semantics.

We have a Galois embedding between finite traces and states:

$$(\mathcal{P}(\Sigma^*), \subseteq) \xrightleftharpoons[\alpha_p]{\gamma_p} (\mathcal{P}(\Sigma), \subseteq)$$

- $\alpha_p(T) \stackrel{\text{def}}{=} \{\, \sigma \in \Sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T \colon \sigma = \sigma_n \,\}$
  (last state in traces in T)
- $\gamma_p(S) \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \in \Sigma^* \mid \sigma_n \in S \,\}$
  (traces ending in a state in S)

(proof on next slide)

Abstracting traces into states (proof)

proof of: (α_p, γ_p) forms a Galois embedding.

Instead of the definition $\alpha(c) \subseteq a \iff c \subseteq \gamma(a)$, we use the alternate characterization of Galois connections: α and γ are monotonic, $\gamma \circ \alpha$ is extensive, and $\alpha \circ \gamma$ is reductive.
Embedding means that, additionally, $\alpha \circ \gamma = id$.

- α_p, γ_p are $\cup-$morphisms, hence monotonic
- $(\gamma_p \circ \alpha_p)(T)$
  $= \{ \sigma_0, \ldots, \sigma_n \mid \sigma_n \in \alpha_p(T) \}$
  $= \{ \sigma_0, \ldots, \sigma_n \mid \exists \sigma'_0, \ldots, \sigma'_m \in T : \sigma_n = \sigma'_m \}$
  $\supseteq T$
- $(\alpha_p \circ \gamma_p)(S)$
  $= \{ \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in \gamma_p(S) : \sigma = \sigma_n \}$
  $= \{ \sigma \mid \exists \sigma_0, \ldots, \sigma_n : \sigma_n \in S,\ \sigma = \sigma_n \}$
  $= S$

Abstracting prefix traces into reachability

Recall that:

- $\mathcal{T}_p(\mathcal{I}) = \operatorname{lfp} F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$,
- $\mathcal{R}(\mathcal{I}) = \operatorname{lfp} F_{\mathcal{R}}$ where $F_{\mathcal{R}}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \operatorname{post}_\tau(S)$,
- $(\mathcal{P}(\Sigma^*), \subseteq) \xleftarrow[\alpha_p]{\gamma_p} (\mathcal{P}(\Sigma), \subseteq)$.

We have: $\alpha_p \circ F_p = F_{\mathcal{R}} \circ \alpha_p$;

by fixpoint transfer, we get: $\alpha_p(\mathcal{T}_p(\mathcal{I})) = \mathcal{R}(\mathcal{I})$.

(proof on next slide)

Abstracting prefix traces into reachability (proof)

proof: of $\alpha_p \circ F_p = F_{\mathcal{R}} \circ \alpha_p$

$$
\begin{aligned}
& (\alpha_p \circ F_p)(T) \\
& = \alpha_p(\mathcal{I} \cup T^\frown \tau) \\
& = \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in \mathcal{I} \cup T^\frown \tau \colon \sigma = \sigma_n \,\} \\
& = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T^\frown \tau \colon \sigma = \sigma_n \,\} \\
& = \mathcal{I} \cup \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T \colon \sigma_n \to \sigma \,\} \\
& = \mathcal{I} \cup \mathrm{post}_\tau(\{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T \colon \sigma = \sigma_n \,\}) \\
& = \mathcal{I} \cup \mathrm{post}_\tau(\alpha_p(T)) \\
& = (F_{\mathcal{R}} \circ \alpha_p)(T)
\end{aligned}
$$

Abstracting traces into states (example)

program

$j \leftarrow 0;$
$i \leftarrow 0;$
while $i < 100$ **do**
  $i \leftarrow i + 1;$
  $j \leftarrow j + [0, 1]$
done

- prefix trace semantics:
  i and j are increasing and $0 \leq j \leq i \leq 100$
- forward reachable state semantics:
  $0 \leq j \leq i \leq 100$

$\Longrightarrow$ the abstraction forgets the ordering of states.

Prefix closure

Prefix partial order: $\preceq$ on Σ^∞

$$x \preceq y \overset{\text{def}}{\iff} \exists u \in \Sigma^\infty : x \cdot u = y$$

$(\Sigma^\infty, \preceq)$ is a CPO, while $(\Sigma^*, \preceq)$ is not complete.

Prefix closure: $\rho_p : \mathcal{P}(\Sigma^\infty) \to \mathcal{P}(\Sigma^\infty)$

$$\rho_p(T) \overset{\text{def}}{=} \{\, u \mid \exists t \in T : u \preceq t,\ u \neq \epsilon \,\}$$

ρ_p is an upper closure operator on $\mathcal{P}(\Sigma^\infty \setminus \{\epsilon\})$.
(monotonic, extensive $T \subseteq \rho_p(T)$, idempotent $\rho_p \circ \rho_p = \rho_p$)

The prefix trace semantics is closed by prefix:
$\rho_p(\mathcal{T}_p(\mathcal{I})) = \mathcal{T}_p(\mathcal{I})$.

(note that $\epsilon \notin \mathcal{T}_p(\mathcal{I})$, which is why we disallowed ϵ in ρ_p)

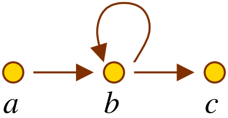
Finite suffix trace semantics

Suffix trace semantics

Similar results on the suffix trace semantics,
going backwards from $\mathcal{F}$:

- $\mathcal{T}_s(\mathcal{F}) \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \sigma_n \in \mathcal{F}, \forall i\colon \sigma_i \to \sigma_{i+1} \,\}$
  (traces following τ and ending in a state in $\mathcal{F}$)
- $\mathcal{T}_s(\mathcal{F}) = \bigcup_{n \geq 0} \tau^{\frown n \frown} \mathcal{F}$
- $\mathcal{T}_s(\mathcal{F}) = \text{lfp}\, F_s$ where $F_s(T) \stackrel{\text{def}}{=} \mathcal{F} \cup \tau \frown T$
  (F_s prepends a transition to each trace, and adds back $\mathcal{F}$)
- $\alpha_s(\mathcal{T}_s(\mathcal{F})) = \mathcal{C}(\mathcal{F})$
  where $\alpha_s(T) \stackrel{\text{def}}{=} \{\, \sigma \mid \exists \sigma_0, \ldots, \sigma_n \in T\colon \sigma = \sigma_0 \,\}$
- $\rho_s(\mathcal{T}_s(\mathcal{F})) = \mathcal{T}_s(\mathcal{F})$
  where $\rho_s(T) \stackrel{\text{def}}{=} \{\, u \mid \exists t \in \Sigma^\infty\colon t \cdot u \in T, u \neq \epsilon \,\}$
  (closed by suffix)

Suffix trace semantics: graphical illustration



$\mathcal{F} \stackrel{\text{def}}{=} \{c\}$

$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$

Iterates: $\mathcal{T}_s(\mathcal{F}) = \text{lfp } F_s$ where $F_s(T) \stackrel{\text{def}}{=} \mathcal{F} \cup \tau \frown T$.

- $F_s^0(\emptyset) = \emptyset$
- $F_s^1(\emptyset) = \mathcal{F} = \{c\}$
- $F_s^2(\emptyset) = \{c, bc\}$
- $F_s^3(\emptyset) = \{c, bc, bbc, abc\}$
- $F_s^n(\emptyset) = \{\, c, b^i c, ab^j c \mid i \in [1, n-1], j \in [1, n-2] \,\}$
- $\mathcal{T}_s(\mathcal{F}) = \cup_{n \geq 0} F_s^n(\emptyset) = \{\, c, b^i c, ab^i c \mid i \geq 1 \,\}$

Finite partial trace semantics

Finite partial trace semantics

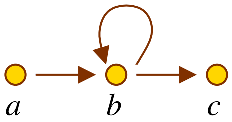
$\mathcal{T}$: all finite partial finite execution traces.

(not necessarily starting in $\mathcal{I}$ or ending in $\mathcal{F}$)

$$\begin{aligned}\mathcal{T} &\stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \mid n \geq 0, \forall i: \sigma_i \to \sigma_{i+1} \,\} \\ &= \bigcup_{n \geq 0} \Sigma^\frown \tau^{\frown n} \\ &= \bigcup_{n \geq 0} \tau^{\frown n \frown} \Sigma\end{aligned}$$

- $\mathcal{T} = \mathcal{T}_p(\Sigma)$, hence $\mathcal{T} = \text{lfp}\, F_{p*}$ where $F_{p*}(T) \stackrel{\text{def}}{=} \Sigma \cup T^\frown \tau$
  (prefix partial traces from any initial state)
- $\mathcal{T} = \mathcal{T}_s(\Sigma)$, hence $\mathcal{T} = \text{lfp}\, F_{s*}$ where $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup \tau^\frown T$
  (suffix partial traces to any final state)
- $F_{p*}^n(\emptyset) = F_{s*}^n(\emptyset) = \bigcup_{i<n} \Sigma^\frown \tau^{\frown i} = \bigcup_{i<n} \tau^{\frown i \frown} \Sigma = \mathcal{T} \cap \Sigma^{<n}$
- $\mathcal{T}_p(\mathcal{I}) = \mathcal{T} \cap (\mathcal{I} \cdot \Sigma^*)$ (restricted initial states)
- $\mathcal{T}_s(\mathcal{F}) = \mathcal{T} \cap (\Sigma^* \cdot \mathcal{F})$ (restricted final states)

Partial trace semantics: graphical illustration



$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$

Iterates: $\mathcal{T}(\Sigma) = \text{lfp } F_{p*}$ where $F_{p*}(T) \stackrel{\text{def}}{=} \Sigma \cup T \frown \tau$.

- $F^0_{p*}(\emptyset) = \emptyset$
- $F^1_{p*}(\emptyset) = \Sigma = \{a, b, c\}$
- $F^2_{p*}(\emptyset) = \{a, b, c, ab, bb, bc\}$
- $F^3_{p*}(\emptyset) = \{a, b, c, ab, bb, bc, abb, abc, bbb, bbc\}$
- $F^n_{p*}(\emptyset) = \{\, ab^i, ab^jc, b^ic, b^k \mid i \in [0, n-1], j \in [1, n-2], k \in [1, n] \,\}$
- $\mathcal{T} = \cup_{n\geq 0} F^n_{p*}(\emptyset) = \{\, ab^i, ab^jc, b^ic, b^j \mid i \geq 0, j > 1 \,\}$

(using $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup \tau \frown T$, we get the exact same iterates)

Abstracting partial traces to prefix traces

Idea: anchor partial traces at initial states $\mathcal{I}$.

We have a Galois connection:

$$(\mathcal{P}(\Sigma^*), \subseteq) \underset{\alpha_{\mathcal{I}}}{\overset{\gamma_{\mathcal{I}}}{\leftrightarrows}} (\mathcal{P}(\Sigma^*), \subseteq)$$

- $\alpha_{\mathcal{I}}(T) \stackrel{\text{def}}{=} T \cap (\mathcal{I} \cdot \Sigma^*)$ (keep only traces starting in $\mathcal{I}$)
- $\gamma_{\mathcal{I}}(T) \stackrel{\text{def}}{=} T \cup ((\Sigma \setminus \mathcal{I}) \cdot \Sigma^*)$ (add all traces not starting in $\mathcal{I}$)

We then have: $\mathcal{T}_p(\mathcal{I}) = \alpha_{\mathcal{I}}(\mathcal{T})$.

(similarly $\mathcal{T}_s(\mathcal{F}) = \alpha_{\mathcal{F}}(\mathcal{T})$ where $\alpha_{\mathcal{F}}(T) \stackrel{\text{def}}{=} \mathcal{T} \cap (\Sigma^* \cdot \mathcal{F})$)

(proof on next slide)

Abstracting partial traces to prefix traces (proof)

proof

$\alpha_{\mathcal{I}}$ and $\gamma_{\mathcal{I}}$ are monotonic.
$(\alpha_{\mathcal{I}} \circ \gamma_{\mathcal{I}})(T) = (T \cup (\Sigma \setminus \mathcal{I}) \cdot \Sigma^*) \cap \mathcal{I} \cdot \Sigma^*) = T \cap \mathcal{I} \cdot \Sigma^* \subseteq T$.
$(\gamma_{\mathcal{I}} \circ \alpha_{\mathcal{I}})(T) = (T \cap \mathcal{I} \cdot \Sigma^*) \cup (\Sigma \setminus \mathcal{I}) \cdot \Sigma^* = T \cup (\Sigma \setminus \mathcal{I}) \cdot \Sigma^* \supseteq T$.
So, we have a Galois connection.

A direct proof of $\mathcal{T}_p(\mathcal{I}) = \alpha_{\mathcal{I}}(\mathcal{T})$ is straightforward, by definition of $\mathcal{T}_p$, $\alpha_{\mathcal{I}}$, and $\mathcal{T}$.

We can also retrieve the result by fixpoint transfer.
$\mathcal{T} = \text{lfp}\, F_{p*}$ where $F_{p*}(T) \stackrel{\text{def}}{=} \Sigma \cup T \frown \tau$.
$\mathcal{T}_p = \text{lfp}\, F_p$ where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$.
We have: $(\alpha_{\mathcal{I}} \circ F_{p*})(T) = (\Sigma \cup T \frown \tau) \cap (\mathcal{I} \cdot \Sigma^*) =$
$\mathcal{I} \cup ((T \frown \tau) \cap (\mathcal{I} \cdot \Sigma^*) = \mathcal{I} \cup ((T \cap (\mathcal{I} \cdot \Sigma^*)) \frown \tau) = (F_p \circ \alpha_{\mathcal{I}})(T)$.

Maximal finite and infinite trace semantics

Maximal traces

Maximal traces: $\mathcal{M}_\infty \in \mathcal{P}(\Sigma^\infty)$

- sequences of states linked by the transition relation τ,
- start in any state ($\mathcal{I} = \Sigma$),
- either finite and stop in a blocking state ($\mathcal{F} = \mathcal{B}$),
- or infinite.

maximal traces cannot be "extended"
by adding a new transition in τ at their end

$$\mathcal{M}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \in \Sigma^* \,|\, \sigma_n \in \mathcal{B}, \forall i < n: \sigma_i \to \sigma_{i+1} \,\} \cup \{\, \sigma_0, \ldots, \sigma_n, \ldots \in \Sigma^\omega \,|\, \forall i < \omega: \sigma_i \to \sigma_{i+1} \,\}$$

(can be anchored at $\mathcal{I}$ and $\mathcal{F}$ as: $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \cap ((\Sigma^* \cdot \mathcal{F}) \cup \Sigma^\omega)$)

Partitioned fixpoint formulation of maximal traces

Goal: we look for a fixpoint characterization of $\mathcal{M}_\infty$.

We consider separately finite and infinite maximal traces.

- Finite traces:

  From the suffix partial trace semantics, recall:

  $\mathcal{M}_\infty \cap \Sigma^* = \mathcal{T}_s(\mathcal{B}) = \text{lfp } F_s$

  where $F_s(T) \stackrel{\text{def}}{=} \mathcal{B} \cup \tau ^\frown T$ in $(\mathcal{P}(\Sigma^*), \subseteq)$.

- Infinite traces:

  Additionally, we will prove: $\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp } G_s$

  where $G_s(T) \stackrel{\text{def}}{=} \tau ^\frown T$ in $(\mathcal{P}(\Sigma^\omega), \subseteq)$.

(proof on next slide)

Partitioned fixpoint formulation of maximal traces (proof)

$\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp } G_s$
where $G_s(T) \stackrel{\text{def}}{=} \tau^\frown T$ in $(\mathcal{P}(\Sigma^\omega), \subseteq)$

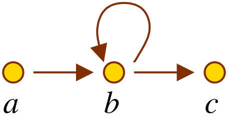
proof:

G_s is continuous in $(\mathcal{P}(\Sigma^\omega), \supseteq)$: $G_s(\cap_{i\in I} T_i) = \cap_{i\in I} G_s(T_i)$.
By Kleene's theorem in the dual: gfp $G_s = \cap_{n\in\mathbb{N}} G_s^n(\Sigma^\omega)$.
We prove by recurrence on n that $\forall n$: $G_s^n(\Sigma^\omega) = \tau^{\frown n \frown}\Sigma^\omega$:

- $G_s^0(\Sigma^\omega) = \Sigma^\omega = \tau^{\frown 0 \frown}\Sigma^\omega$,
- $G_s^{n+1}(\Sigma^\omega) = \tau^\frown G_s^n(\Sigma^\omega) = \tau^\frown(\tau^{\frown n \frown}\Sigma^\omega) = \tau^{\frown n+1 \frown}\Sigma^\omega$.

$$
\begin{aligned}
\text{gfp } G_s &= \cap_{n\in\mathbb{N}} \tau^{\frown n \frown}\Sigma^\omega \\
&= \{\, \sigma_0, \ldots \in \Sigma^\omega \mid \forall n \geq 0 \colon \sigma_0, \ldots, \sigma_{n-1} \in \tau^{\frown n} \,\} \\
&= \{\, \sigma_0, \ldots \in \Sigma^\omega \mid \forall n \geq 0 \colon \forall i < n \colon \sigma_i \to \sigma_{i+1} \,\} \\
&= \mathcal{M}_\infty \cap \Sigma^\omega
\end{aligned}
$$

Infinite trace semantics: graphical illustration



$\mathcal{B} \stackrel{\text{def}}{=} \{c\}$

$\tau \stackrel{\text{def}}{=} \{(a,b),(b,b),(b,c)\}$

Iterates: $\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp } G_s$ where $G_s(T) \stackrel{\text{def}}{=} \tau \frown T$.

- $G_s^0(\Sigma^\omega) = \Sigma^\omega$
- $G_s^1(\Sigma^\omega) = ab\Sigma^\omega \cup bb\Sigma^\omega \cup bc\Sigma^\omega$
- $G_s^2(\Sigma^\omega) = abb\Sigma^\omega \cup bbb\Sigma^\omega \cup abc\Sigma^\omega \cup bbc\Sigma^\omega$
- $G_s^3(\Sigma^\omega) = abbb\Sigma^\omega \cup bbbb\Sigma^\omega \cup abbc\Sigma^\omega \cup bbbc\Sigma^\omega$
- $G_s^n(\Sigma^\omega) = \{\, ab^n t,\ b^{n+1} t,\ ab^{n-1} ct,\ b^n ct \mid t \in \Sigma^\omega \,\}$
- $\mathcal{M}_\infty \cap \Sigma^\omega = \cap_{n \geq 0} G_s^n(\Sigma^\omega) = \{ab^\omega,\ b^\omega\}$

Least fixpoint formulation of maximal traces

Idea: To get a fixpoint formulation for whole $\mathcal{M}_\infty$,
merge finite and infinite maximal trace fixpoint forms.

Fixpoint fusion

$\mathcal{M}_\infty \cap \Sigma^*$ is best defined on $(\Sigma^*, \subseteq, \cup, \cap, \emptyset, \Sigma^*)$.
$\mathcal{M}_\infty \cap \Sigma^\omega$ is best defined on $(\Sigma^\omega, \supseteq, \cap, \cup, \Sigma^\omega, \emptyset)$.

We mix them into a new complete lattice $(\Sigma^\infty, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$:

- $A \sqsubseteq B \overset{\text{def}}{\iff} (A \cap \Sigma^*) \subseteq (B \cap \Sigma^*) \wedge (A \cap \Sigma^\omega) \supseteq (B \cap \Sigma^\omega)$
- $A \sqcup B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cup (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cap (B \cap \Sigma^\omega))$
- $A \sqcap B \overset{\text{def}}{=} ((A \cap \Sigma^*) \cap (B \cap \Sigma^*)) \cup ((A \cap \Sigma^\omega) \cup (B \cap \Sigma^\omega))$
- $\bot \overset{\text{def}}{=} \Sigma^\omega$
- $\top \overset{\text{def}}{=} \Sigma^*$

In this lattice, $\mathcal{M}_\infty = \text{lfp } F_s$ where $F_s(T) \overset{\text{def}}{=} \mathcal{B} \cup \tau \frown T$.

(proof on next slides)

Fixpoint fusion theorem

Theorem: fixpoint fusion

If $X_1 = \text{lfp}\, F_1$ in $(\mathcal{P}(\mathcal{D}_1), \sqsubseteq_1)$ and $X_2 = \text{lfp}\, F_2$ in $(\mathcal{P}(\mathcal{D}_2), \sqsubseteq_2)$
and $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$,
then $X_1 \cup X_2 = \text{lfp}\, F$ in $(\mathcal{P}(\mathcal{D}_1 \cup \mathcal{D}_2), \sqsubseteq)$ where:

- $F(X) \stackrel{\text{def}}{=} F_1(X \cap \mathcal{D}_1) \cup F_2(X \cap \mathcal{D}_2)$,
- $A \sqsubseteq B \stackrel{\text{def}}{\iff} (A \cap \mathcal{D}_1) \sqsubseteq_1 (B \cap \mathcal{D}_1) \wedge (A \cap \mathcal{D}_2) \sqsubseteq_2 (B \cap \mathcal{D}_2)$.

proof:

We have:
$F(X_1 \cup X_2) = F_1((X_1 \cup X_2) \cap \mathcal{D}_1) \cup F_2((X_1 \cup X_2) \cap \mathcal{D}_2) = F_1(X_1) \cup F_2(X_2) = X_1 \cup X_2$,
hence $X_1 \cup X_2$ is a fixpoint of F.

Let Y be a fixpoint. Then $Y = F(Y) = F_1(Y \cap \mathcal{D}_1) \cup F_2(Y \cap \mathcal{D}_2)$, hence, $Y \cap \mathcal{D}_1 = F_1(Y \cap \mathcal{D}_1)$ and $Y \cap \mathcal{D}_1$ is a fixpoint of F_1. Thus, $X_1 \sqsubseteq_1 Y \cap \mathcal{D}_1$. Likewise, $X_2 \sqsubseteq_2 Y \cap \mathcal{D}_2$. We deduce that $X = X_1 \cup X_2 \sqsubseteq (Y \cap \mathcal{D}_1) \cup (Y \cap \mathcal{D}_2) = Y$, and so, X is F's least fixpoint.

note: we also have $\text{gfp}\, F = \text{gfp}\, F_1 \cup \text{gfp}\, F_2$.

Least fixpoint formulation of maximal traces (proof)

We are now ready to finish the proof that $\mathcal{M}_\infty = \text{lfp } F_s$
where $F_s(T) \stackrel{\text{def}}{=} \mathcal{B} \cup \tau \frown T$

proof:

We have:

- $\mathcal{M}_\infty \cap \Sigma^* = \text{lfp } F_s$ in $(\mathcal{P}(\Sigma^*), \subseteq)$,
- $\mathcal{M}_\infty \cap \Sigma^\omega = \text{lfp } G_s$ in $(\mathcal{P}(\Sigma^\omega), \supseteq)$ where $G_s(T) \stackrel{\text{def}}{=} \tau \frown T$,
- in $\mathcal{P}(\Sigma^\infty)$, we have
  $F_s(A) = (F_s(A) \cap \Sigma^*) \cup (F_s(A) \cap \Sigma^\omega) = F_s(A \cap \Sigma^*) \cup G_s(A \cap \Sigma^\omega)$.

So, by fixpoint fusion in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$, we have:

$\mathcal{M}_\infty = (\mathcal{M}_\infty \cap \Sigma^*) \cup (\mathcal{M}_\infty \cap \Sigma^\omega) = \text{lfp } F_s$.

Abstracting maximal traces into partial traces

Finite and infinite partial trace semantics

Idea: complete the partial traces $\mathcal{T}$ with infinite traces.

$\mathcal{T}_\infty$: all finite and infinite sequences of states
linked by the transition relation τ:

$$\mathcal{T}_\infty \stackrel{\text{def}}{=} \{\, \sigma_0, \ldots, \sigma_n \in \Sigma^* \mid \forall i < n \colon \sigma_i \to \sigma_{i+1} \,\} \cup \{\, \sigma_0, \ldots, \sigma_n, \ldots \in \Sigma^\omega \mid \forall i < \omega \colon \sigma_i \to \sigma_{i+1} \,\}$$

(partial finite traces do not necessarily end in a blocking state)

Fixpoint form similar to $\mathcal{M}_\infty$:
$\mathcal{T}_\infty = \text{lfp}\, F_{s*}$ in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$ where $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup \tau \frown T$,

proof: similar to the proof of $\mathcal{M}_\infty = \text{lfp}\, F_s$.

Finite trace abstraction

Finite partial traces $\mathcal{T}$ are an abstraction of all partial traces $\mathcal{T}_\infty$.

We have a Galois embedding:

$$(\mathcal{P}(\Sigma^\infty), \sqsubseteq) \xrightleftharpoons[\alpha_*]{\gamma_*} (\mathcal{P}(\Sigma^*), \subseteq)$$

- $\sqsubseteq$ is the fused ordering on $\Sigma^* \cup \Sigma^\omega$:
  $A \sqsubseteq B \iff^{\text{def}} (A \cap \Sigma^*) \subseteq (B \cap \Sigma^*) \wedge (A \cap \Sigma^\omega) \supseteq (B \cap \Sigma^\omega)$
- $\alpha_*(T) \stackrel{\text{def}}{=} T \cap \Sigma^*$
  (remove infinite traces)
- $\gamma_*(T) \stackrel{\text{def}}{=} T$
  (embedding)
- $\mathcal{T} = \alpha_*(\mathcal{T}_\infty)$

(proof on next slide)

Finite trace abstraction (proof)

proof:

We have Galois embedding because:

- α_* and γ_* are monotonic,
- given $T \subseteq \Sigma^*$, we have $(\alpha_* \circ \gamma_*)(T) = T \cap \Sigma^* = T$,
- $(\gamma_* \circ \alpha_*)(T) = T \cap \Sigma^* \sqsupseteq T$, as we only remove infinite traces.

Recall that $\mathcal{T}_\infty = \operatorname{lfp} F_{s*}$ in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$ and $\mathcal{T} = \operatorname{lfp} F_{s*}$ in $(\mathcal{P}(\Sigma^*), \subseteq)$, where $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup T \frown \tau$.

As $\alpha_* \circ F_{s*} = F_{s*} \circ \alpha_*$ and $\alpha_*(\emptyset) = \emptyset$, we can apply the fixpoint transfer theorem to get $\alpha_*(\mathcal{T}_\infty) = \mathcal{T}$.

Prefix abstraction

Idea: complete maximal traces by adding (non-empty) prefixes.

We have a Galois connection:

$$(\mathcal{P}(\Sigma^\infty \setminus \{\epsilon\}), \subseteq) \underset{\alpha_\preceq}{\overset{\gamma_\preceq}{\leftrightarrows}} (\mathcal{P}(\Sigma^\infty \setminus \{\epsilon\}), \subseteq)$$

- $\alpha_\preceq(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^\infty \setminus \{\epsilon\} \mid \exists u \in T: t \preceq u \,\}$
  (set of all non-empty prefixes of traces in T)
- $\gamma_\preceq(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^\infty \setminus \{\epsilon\} \mid \forall u \in \Sigma^\infty \setminus \{\epsilon\}: u \preceq t \implies u \in T \,\}$
  (traces with non-empty prefixes in T)

proof:

$\alpha_\preceq$ and $\gamma_\preceq$ are monotonic.

$(\alpha_\preceq \circ \gamma_\preceq)(T) = \{\, t \in T \mid \rho_p(t) \subseteq T \,\} \subseteq T$ (prefix-closed trace sets).

$(\gamma_\preceq \circ \alpha_\preceq)(T) = \rho_p(T) \supseteq T$.

Abstraction from maximal traces to partial traces

Finite and infinite partial traces $\mathcal{T}_\infty$ are an abstraction of maximal traces $\mathcal{M}_\infty$: $\mathcal{T}_\infty = \alpha_\preceq(\mathcal{M}_\infty)$.

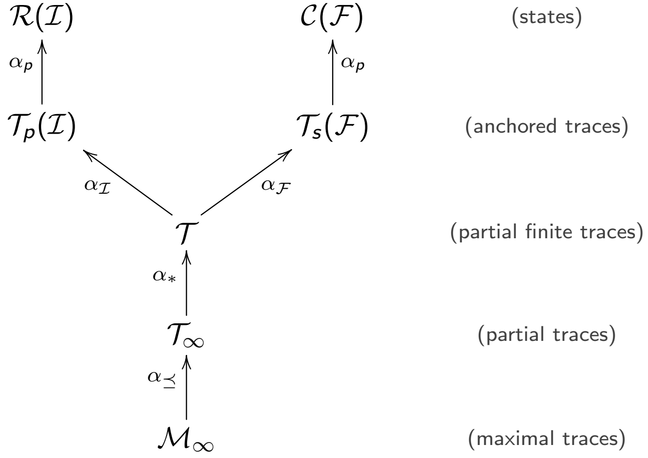
proof:

Firstly, $\mathcal{T}_\infty$ and $\alpha_\preceq(\mathcal{M}_\infty)$ coincide on infinite traces. Indeed, $\mathcal{T}_\infty \cap \Sigma^\omega = \mathcal{M}_\infty \cap \Sigma^\omega$ and $\alpha_\preceq$ does not add infinite traces, so: $\mathcal{T}_\infty \cap \Sigma^\omega = \alpha_\preceq(\mathcal{M}_\infty) \cap \Sigma^\omega$.

We now prove that they also coincide on finite traces. Assume $\sigma_0, \ldots, \sigma_n \in \alpha_\preceq(\mathcal{M}_\infty)$, then $\forall i < n: \sigma_i \to \sigma_{i+1}$, so, $\sigma_0, \ldots, \sigma_n \in \mathcal{T}_\infty$.
Assume $\sigma_0, \ldots, \sigma_n \in \mathcal{T}_\infty$, then it can be completed into a maximal trace, either finite or infinite, and so, $\sigma_0, \ldots, \sigma_n \in \alpha_\preceq(\mathcal{M}_\infty)$.

Note: no fixpoint transfer applies here.

(Partial) hierarchy of semantics



Relational semantics

Big-step semantics

Finite big-step semantics

Pairs of states linked by a sequence of transitions in τ.

$$\mathcal{BS} \stackrel{\text{def}}{=} \{\, (\sigma_0, \sigma_n) \in \Sigma \times \Sigma \mid n \geq 0, \exists \sigma_1, \ldots, \sigma_{n-1} : \forall i < n : \sigma_i \to \sigma_{i+1} \,\}$$

(symmetric and transitive closure of τ)

Fixpoint form:

$\mathcal{BS} = \text{lfp}\ F_B$
where $F_B(R) \stackrel{\text{def}}{=} id \cup \{\, (\sigma, \sigma'') \mid \exists \sigma' : (\sigma, \sigma') \in R, \sigma' \to \sigma'' \,\}$.

Relational abstraction

Relational abstraction: allows skipping intermediate steps.

We have a Galois embedding:

$$(\mathcal{P}(\Sigma^*), \subseteq) \underset{\alpha_{io}}{\overset{\gamma_{io}}{\leftarrows\!\!\twoheadrightarrow}} (\mathcal{P}(\Sigma \times \Sigma), \subseteq)$$

- $\alpha_{io}(T) \stackrel{\text{def}}{=} \{ (\sigma, \sigma') \mid \exists \sigma_0, \ldots, \sigma_n \in T : \sigma = \sigma_0, \sigma' = \sigma_n \}$
  (first and last state of a trace in T)
- $\gamma_{io}(R) \stackrel{\text{def}}{=} \{ \sigma_0, \ldots, \sigma_n \in \Sigma^* \mid \exists (\sigma, \sigma') \in R : \sigma = \sigma_0, \sigma' = \sigma_n \}$
  (traces respecting the first and last states from R)

proof sketch:

γ_{io} and α_{io} are monotonic.
$(\gamma_{io} \circ \alpha_{io})(T) = \{ \sigma_0, \ldots, \sigma_n \mid \exists \sigma'_0, \ldots, \sigma'_m \in T : \sigma_0 = \sigma'_0, \sigma_n = \sigma'_m \}$.
$(\alpha_{io} \circ \gamma_{io})(R) = R$.

Finite big-step semantics as an abstraction

The finite big-step semantics is an abstraction of the finite trace semantics: $\mathcal{BS} = \alpha_{io}(\mathcal{T})$.

proof sketch: by fixpoint transfer.

We have $\mathcal{T} = \operatorname{lfp} F_{p*}$ where $F_{p*}(T) \stackrel{\text{def}}{=} \Sigma \cup T \frown \tau$.
Moreover, $F_B(R) \stackrel{\text{def}}{=} id \cup \{ (\sigma, \sigma'') \mid \exists \sigma' : (\sigma, \sigma') \in R, \sigma' \to \sigma'' \}$.
Then, $\alpha_{io} \circ F_{p*} = F_B \circ \alpha_{io}$ because $\alpha_{io}(\Sigma) = id$ and $\alpha_{io}(T \frown \tau) = \{ (\sigma, \sigma'') \mid \exists \sigma' : (\sigma, \sigma') \in \alpha_{io}(T) \wedge \sigma' \to \sigma'' \}$.
By fixpoint transfer: $\alpha_{io}(\mathcal{T}) = \operatorname{lfp} F_B$.

We have a similar result using $F_{s*}(T) \stackrel{\text{def}}{=} \Sigma \cup \tau \frown T$ and $F'_B(R) \stackrel{\text{def}}{=} id \cup \{ (\sigma, \sigma'') \mid \exists \sigma' : (\sigma', \sigma'') \in R \wedge \sigma \to \sigma' \}$.

Finite big-step semantics (example)

program

$i \leftarrow [0, +\infty];$
while $i > 0$ **do**
  $i \leftarrow i - [0, 1];$
done

Finite big-step semantics $\mathcal{BS}$: $\{ (\rho, \rho') \mid 0 \leq \rho'(i) \leq \rho(i) \}$.

Relational denotational semantics

Denotational semantics (in relation form)

In the denotational semantics, we forget all the intermediate steps and only keep the input / output relation:

- $(\sigma, \sigma') \in \Sigma \times \mathcal{B}$: finite execution starting in σ, stopping in σ',
- $(\sigma, \circlearrowleft)$: non-terminating execution starting in σ.

($\neq$ big-step semantics: we no longer include (σ, σ') if σ' is not blocking!)

Construction by abstraction: of the maximal trace semantics $\mathcal{M}_\infty$.

$$(\mathcal{P}(\Sigma^\infty), \subseteq) \underset{\alpha_d}{\overset{\gamma_d}{\leftrightarrows}} (\mathcal{P}(\Sigma \times (\Sigma \cup \{\circlearrowleft\})), \subseteq)$$

- $\alpha_d(T) \stackrel{\text{def}}{=} \alpha_{io}(T \cap \Sigma^*) \cup \{\, (\sigma, \circlearrowleft) \,|\, \exists t \in \Sigma^\omega : \sigma \cdot t \in T \,\}$
- $\gamma_d(R) \stackrel{\text{def}}{=} \gamma_{io}(R \cap (\Sigma \times \Sigma)) \cup \{\, \sigma \cdot t \,|\, (\sigma, \circlearrowleft) \in R, t \in \Sigma^\omega \,\}$

  (extension of $(\alpha_{io}, \gamma_{io})$ to infinite traces)

The denotational semantics is $\mathcal{DS} \stackrel{\text{def}}{=} \alpha_d(\mathcal{M}_\infty)$.

Denotational fixpoint semantics

Idea: as $\mathcal{M}_\infty$, separate terminating and non-terminating behaviors, and use a fixpoint fusion theorem.

We have: $\mathcal{DS} = \text{lfp } F_d$

in $(\mathcal{P}(\Sigma \times (\Sigma \cup \{\circlearrowleft\})), \sqsubseteq^*, \sqcup^*, \sqcap^*, \bot^*, \top^*)$, where

- $\bot^* \stackrel{\text{def}}{=} \{\, (\sigma, \circlearrowleft) \mid \sigma \in \Sigma \,\}$
- $\top^* \stackrel{\text{def}}{=} \{\, (\sigma, \sigma') \mid \sigma, \sigma' \in \Sigma \,\}$
- $A \sqsubseteq^* B \iff ((A \cap \top^*) \subseteq (B \cap \top^*)) \wedge ((A \cap \bot^*) \supseteq (B \cap \bot^*))$
- $A \sqcup^* B \stackrel{\text{def}}{=} ((A \cap \top^*) \cup (B \cap \top^*)) \cup ((A \cap \bot^*) \cap (B \cap \bot^*))$
- $A \sqcap^* B \stackrel{\text{def}}{=} ((A \cap \top^*) \cap (B \cap \top^*)) \cup ((A \cap \bot^*) \cup (B \cap \bot^*))$
- $F_d(R) \stackrel{\text{def}}{=} \{\, (\sigma, \sigma) \mid \sigma \in \mathcal{B} \,\} \cup \{\, (\sigma, \sigma'') \mid \exists \sigma' : \sigma \to \sigma' \wedge (\sigma', \sigma'') \in R \,\}$

Denotational fixpoint semantics (proof)

<u>proof:</u>

We cannot use directly a fixpoint transfer on $\mathcal{M}_\infty = \text{lfp}\, F_s$ in $(\mathcal{P}(\Sigma^\infty), \sqsubseteq)$ because our Galois connection (α_d, γ_d) uses the $\subseteq$ order, not $\sqsubseteq$!
Instead, we use fixpoint transfer separately on finite and infinite executions, and then apply fixpoint fusion.

Recall that $\mathcal{M}_\infty \cap \Sigma^* = \text{lfp}\, F_s$ in $(\mathcal{P}(\Sigma^*), \subseteq)$ where $F_s(T) \stackrel{\text{def}}{=} \mathcal{B} \cup \tau \frown T$

and $\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp}\, G_s$ in $(\mathcal{P}(\Sigma^\omega), \subseteq)$ where $G_s(T) \stackrel{\text{def}}{=} \cup \tau \frown T$.

For finite execution, we have $\alpha_d \circ F_s = F_d \circ \alpha_d$ in $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma \times \Sigma)$.
We can apply directly fixpoint transfer and get that: $\mathcal{DS} \cap (\Sigma \times \Sigma) = \text{lfp}\, F_d$.

(proof continued on next slide)

Denotational fixpoint semantics (proof cont.)

proof (continued): proof sketch for infinite executions

We have $\alpha_d \circ G_s = G_d \circ \alpha_d$ in $\mathcal{P}(\Sigma^\omega) \to \mathcal{P}(\Sigma \times \{\circlearrowleft\})$, where
$G_d(R) \stackrel{\text{def}}{=} \{\, (\sigma, \sigma'') \,|\, \exists \sigma' \colon \sigma \to \sigma' \wedge (\sigma', \sigma'') \in R \,\}$.

A candidate proof would be to apply a fixpoint transfer theorem to $\mathcal{M}_\infty \cap \Sigma^\omega = \text{gfp}\, G_s$, in the dual, replacing lfp with gfp, and $\cup$ with $\cap$.
However, the proof of the theorem, which required α to be continous, would require α to be co-continuous in the dual, i.e., $\alpha_d(\cap_{i \in I} S_i) = \cap_{\in I} \alpha_d(S_i)$.
This does not hold. Consider for example: $I = \mathbb{N}$ and $S_i = \{\, a^n b^\omega \,|\, n > i \,\}$:
$\cap_{i \in \mathbb{N}} S_i = \emptyset$, but $\forall i \colon \alpha_d(S_i) = \{(a, \circlearrowleft)\}$.

We use instead a fixpoint transfer based on Tarksi's theorem.
We have $\text{gfp}\, G_s = \cup \{\, X \,|\, X \subseteq G_s(X) \,\}$.
Thus, $\alpha_d(\text{gfp}\, G_s) = \alpha_d(\cup \{\, X \,|\, X \subseteq G_s(X) \,\}) = \cup \{\, \alpha_d(X) \,|\, X \subseteq G_s(X) \,\}$ as α_d is a complete $\cup$ morphism. The proof is finished by noting that the commutation $\alpha_d \circ G_s = G_d \circ \alpha_d$ and the Galois embedding (α_d, γ_d) imply that
$\{\, \alpha_d(X) \,|\, X \subseteq G_s(X) \,\} = \{\, \alpha_d(X) \,|\, \alpha_d(X) \subseteq G_d(\alpha_d(X)) \,\} = \{\, Y \,|\, Y \subseteq G_d(Y) \,\}$.

(the complete proof can be found in [Cous02])

Denotational semantics (example)

program

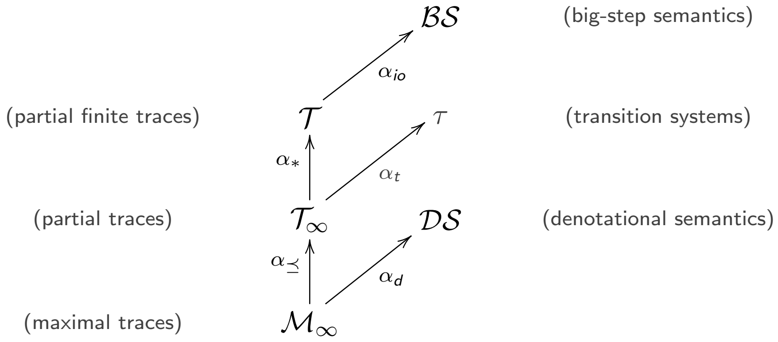
$i \leftarrow [0, +\infty];$
while $i > 0$ **do**
  $i \leftarrow i - [0, 1];$
done

Denotational semantics $\mathcal{DS}$:

$$\{\, (\rho, \rho') \mid \rho(i) \geq 0 \wedge \rho'(i) = 0 \,\} \cup \{\, (\rho, \circlearrowleft) \mid \rho(i) \geq 0 \,\}.$$

(quite different from the big-step semantics)

Another part of the hierarchy of semantics



See [Cou82] for more semantics in this diagram.

Note: we show transition systems as an abstraction of the partial trace semantics this is left as exercise (see assignment).

Properties and proofs

State properties

State properties

State property: $P \in \mathcal{P}(\Sigma)$.

Verification problem: $\mathcal{R}(\mathcal{I}) \subseteq P$.

(all the states reachable from $\mathcal{I}$ are in P)

Examples:

- absence of blocking: $P \stackrel{\text{def}}{=} \Sigma \setminus \mathcal{B}$,
- the variables remain in a safe range,
- dangerous program locations cannot be reached.

Invariance proof method

Invariance proof method: find an inductive invariant $I \subseteq \Sigma$

- $\mathcal{I} \subseteq I$

  (contains initial states)
- $\forall \sigma \in I: \sigma \to \sigma' \implies \sigma' \in I$

  (invariant by program transition)

that implies the desired property: $I \subseteq P$.

Link with the state semantics $\mathcal{R}(\mathcal{I})$:

Given $F_{\mathcal{R}}(S) \stackrel{\text{def}}{=} \mathcal{I} \cup \text{post}_\tau(S)$, we have $F_{\mathcal{R}}(I) \subseteq I$
$\implies$ I is a post-fixpoint of $F_{\mathcal{R}}$.

Recall that $\mathcal{R}(\mathcal{I}) = \text{lfp}\, F_{\mathcal{R}}$
$\implies$ $\mathcal{R}(\mathcal{I})$ is the tightest inductive invariant.

Link with Hoare logic

Hoare logic: proof method where we

- annotate program points with local sate invariants in $\mathcal{P}(\Sigma)$
- use logic rules to prove their correctness

$$\frac{}{\{P[e/X]\}\ X \leftarrow e\ \{P\}} \qquad \frac{\{P\}\ stat_1\ \{R\} \quad \{R\}\ stat_2\ \{Q\}}{\{P\}\ stat_1; stat_2\ \{Q\}}$$

$$\frac{\{P \wedge b\}\ stat\ \{Q\} \quad P \wedge \neg b \Rightarrow Q}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ stat\ \{Q\}} \qquad \frac{\{P \wedge b\}\ stat\ \{P\}}{\{P\}\ \textbf{while}\ b\ \textbf{do}\ stat\ \{P \wedge \neg b\}}$$

$$\frac{\{P\}\ stat\ \{Q\} \quad P' \Rightarrow P \quad Q \Rightarrow Q'}{\{P'\}\ stat\ \{Q'\}}$$

Link with the state semantics $\mathcal{R}(\mathcal{I})$:

Recall the equation system $(\mathcal{X}_\ell)_{\ell \in \mathcal{L}} = F_{eq}((\mathcal{X}_\ell)_{\ell \in \mathcal{L}})$ obtained by partitioning reachability $F_\mathcal{R}$ by control point $(\mathcal{P}(\Sigma), \subseteq) \xleftarrow[\alpha_\mathcal{L}]{\gamma_\mathcal{L}} (\mathcal{L} \rightarrow \mathcal{P}(\mathcal{E}), \dot{\subseteq})$.

- any post-fixpoint of F_{eq} gives valid Hoare triples
- lfp F_{eq} gives the tightest Hoare triples

Trace properties

Trace properties

Trace property: $P \in \mathcal{P}(\Sigma^\infty)$

Verification problem: $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$

(or, equivalently, as $\mathcal{M}_\infty \subseteq P'$ where $P' \stackrel{\text{def}}{=} P \cup ((\Sigma \setminus \mathcal{I}) \cdot \Sigma^\infty)$)

Examples:

- termination: $P \stackrel{\text{def}}{=} \Sigma^*$,
- non-termination: $P \stackrel{\text{def}}{=} \Sigma^\omega$,
- any state property $S \subseteq \Sigma$: $P \stackrel{\text{def}}{=} S^\infty$,
- maximal execution time: $P \stackrel{\text{def}}{=} \Sigma^{\leq k}$,
- minimal execution time: $P \stackrel{\text{def}}{=} \Sigma^{\geq k}$,
- ordering, e.g.: $P \stackrel{\text{def}}{=} (\Sigma \setminus \{b\})^* \cdot a \cdot \Sigma^* \cdot b \cdot \Sigma^\infty$.

  (a and b occur, and a occurs before b)

Safety properties

Idea: a safety property P models that "nothing bad ever occurs"

- P is provable by exhaustive testing;
  (observe the prefix trace semantics: $\mathcal{T}_p(\mathcal{I}) \subseteq P$)
- P is disprovable by finding a single finite execution not in P.

Examples:

- any state property: $P \stackrel{\text{def}}{=} S^\infty$ for $S \subseteq \Sigma$,
- ordering: $P \stackrel{\text{def}}{=} \Sigma^\infty \setminus ((\Sigma \setminus \{a\})^* \cdot b \cdot \Sigma^\infty)$,
  (no b can appear without an a before,
  but we can have only a, or neither a nor b)
  (not a state property)
- but termination $P \stackrel{\text{def}}{=} \Sigma^*$ is not a safety property.
  (disproving requires exhibiting an *infinite* execution)

Definition of safety properties

Reminder: finite prefix abstraction (simplified to allow ϵ)

$$(\mathcal{P}(\Sigma^\infty), \subseteq) \xleftarrow[\alpha_{*\preceq}]{\gamma_{*\preceq}} (\mathcal{P}(\Sigma^*), \subseteq)$$

- $\alpha_{*\preceq}(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^* \mid \exists u \in T \colon t \preceq u \,\}$
- $\gamma_{*\preceq}(T) \stackrel{\text{def}}{=} \{\, t \in \Sigma^\infty \mid \forall u \in \Sigma^* \colon u \preceq t \implies u \in T \,\}$

The associated upper closure $\rho_{*\preceq} \stackrel{\text{def}}{=} \gamma_\preceq \circ \alpha_\preceq$ is:
$\rho_{*\preceq} = \lim \circ \rho_p$ where:

- $\rho_p(T) \stackrel{\text{def}}{=} \{\, u \in \Sigma^\infty \mid \exists t \in T \colon u \preceq t \,\}$,
- $\lim(T) \stackrel{\text{def}}{=} T \cup \{\, t \in \Sigma^\omega \mid \forall u \in \Sigma^* \colon u \preceq t \implies u \in T \,\}$.

Definition: $P \in \mathcal{P}(\Sigma^\infty)$ is a safety property if $P = \rho_{*\preceq}(P)$.

Definition of safety properties (examples)

Definition: $P \subseteq \mathcal{P}(\Sigma^\infty)$ is a safety property if $P = \rho_{*\preceq}(P)$.

Examples and counter-examples:

- state property $P \stackrel{\text{def}}{=} S^\infty$ for $S \subseteq \Sigma$:

  $\rho_p(S^\infty) = \lim(S^\infty) = S^\infty \implies$ safety;

- termination $P \stackrel{\text{def}}{=} \Sigma^*$:

  $\rho_p(\Sigma^*) = \Sigma^*$, but $\lim(\Sigma^*) = \Sigma^\infty \neq \Sigma^* \implies$ not safety;

- even number of steps $P \stackrel{\text{def}}{=} (\Sigma^2)^\infty$:

  $\rho_p((\Sigma^2)^\infty) = \Sigma^\infty \neq (\Sigma^2)^\infty \implies$ not safety.

Proving safety properties

Invariance proof method: find an inductive invariant I

- set of finite traces $I \subseteq \Sigma^*$
- $\mathcal{I} \subseteq I$

  (contains traces reduced to an initial state)
- $\forall \sigma_0, \ldots, \sigma_n \in I$: $\sigma_n \to \sigma_{n+1} \implies \sigma_0, \ldots, \sigma_n, \sigma_{n+1} \in I$

  (invariant by program transition)

and implies the desired property: $I \subseteq P$.

Link with the finite prefix trace semantics $\mathcal{T}_p(\mathcal{I})$:

An inductive invariant is a post-fixpoint of F_p: $F_p(I) \subseteq I$
where $F_p(T) \stackrel{\text{def}}{=} \mathcal{I} \cup T \frown \tau$.
$\mathcal{T}_p(\mathcal{I}) = \operatorname{lfp} F_p$ is the tightest inductive invariant.

Correctness of the invariant method for safety

Soundness:

if P is a safety property and an inductive invariant I exists
then: $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$

proof:

Using the Galois connection between $\mathcal{M}_\infty$ and $\mathcal{T}$, we get:
$\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq \rho_{*\preceq}(\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty)) = \gamma_{*\preceq}(\alpha_{*\preceq}(\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty))) =$
$\gamma_{*\preceq}(\alpha_{*\preceq}(\mathcal{M}_\infty) \cap (\mathcal{I} \cdot \Sigma^*)) = \gamma_{*\preceq}(\mathcal{T} \cap (\mathcal{I} \cdot \Sigma^*)) = \gamma_{*\preceq}(\mathcal{T}_p(\mathcal{I}))$.
Using the link between invariants and the finite prefix trace semantics, we have:
$\mathcal{T}_p(\mathcal{I}) \subseteq I \subseteq P$.

As P is a safety property, $P = \gamma_{*\preceq}(P)$, so, $\gamma_{*\preceq}(\mathcal{T}_p(\mathcal{I})) \subseteq \gamma_{*\preceq}(P) = P$, and so,
$\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$.

Completeness: an inductive invariant always exists

proof: $\mathcal{T}_p(\mathcal{I})$ provides an inductive invariant.

Disproving safety properties

Proof method:

A safety property P can be disproved by constructing a finite prefix of execution that does not satisfy the property:

$$\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \not\subseteq P \implies \exists t \in \mathcal{T}_p(\mathcal{I}) : t \notin P$$

proof:

By contradiction, assume that no such trace exists, i.e., $\mathcal{T}_p(\mathcal{I}) \subseteq P$.

We proved in the previous slide that this implies $\mathcal{M}_\infty \cap (\mathcal{I} \cdot \Sigma^\infty) \subseteq P$.

Examples:

- disproving a state property $P \stackrel{\text{def}}{=} S^\infty$:
  $\Rightarrow$ find a partial execution containing a state in $\Sigma \setminus S$;
- disproving an order property $P \stackrel{\text{def}}{=} \Sigma^\infty \setminus ((\Sigma \setminus \{a\})^* \cdot b \cdot \Sigma^\infty)$
  $\Rightarrow$ find a partial execution where b appears and not a.

Liveness properties

Idea: liveness property $P \in \mathcal{P}(\Sigma^\infty)$

Liveness properties model that "something good eventually occurs"

- P cannot be proved by testing
  (if nothing good happens in a prefix execution,
  it can still happen in the rest of the execution)
- disproving P requires exhibiting an infinite execution not in P

Examples:

- termination: $P \stackrel{\text{def}}{=} \Sigma^*$,
- inevitability: $P \stackrel{\text{def}}{=} \Sigma^* \cdot a \cdot \Sigma^\infty$,
  (a eventually occurs in all executions)
- state properties are not liveness properties.

Definition of liveness properties

Definition: $P \in \mathcal{P}(\Sigma^\infty)$ is a liveness property if $\rho_{*\preceq}(P) = \Sigma^\infty$.

Examples and counter-examples:

- termination $P \stackrel{\text{def}}{=} \Sigma^*$:

  $\rho_p(\Sigma^*) = \Sigma^*$ and $\lim(\Sigma^*) = \Sigma^\infty \implies$ liveness;

- inevitability: $P \stackrel{\text{def}}{=} \Sigma^* \cdot a \cdot \Sigma^\infty$

  $\rho_p(P) = P \cup \Sigma^*$ and $\lim(P \cup \Sigma^*) = \Sigma^\infty \implies$ liveness;

- state property $P \stackrel{\text{def}}{=} S^\infty$ for $S \subseteq \Sigma$:

  $\rho_p(S^\infty) = \lim(S^\infty) = S^\infty \neq \Sigma^\infty$ if $S \neq \Sigma \implies$ not liveness;

- maximal execution time $P \stackrel{\text{def}}{=} \Sigma^{\leq k}$:

  $\rho_p(\Sigma^{\leq k}) = \lim(\Sigma^{\leq k}) = \Sigma^{\leq k} \neq \Sigma^\infty \implies$ not liveness;

- the only property which is both safety and liveness is Σ^∞.

Proving liveness properties

Variance proof method: (informal definition)

Find a decreasing quantity until something good happens.

Example: termination proof

- find $f : \Sigma \to \mathcal{S}$ where $(\mathcal{S}, \sqsubseteq)$ is well-ordered;

  (f is called a "ranking function")

- $\sigma \in \mathcal{B} \implies f = \min \mathcal{S}$;
- $\sigma \to \sigma' \implies f(\sigma') \sqsubset f(\sigma)$.

(f counts the number of steps remaining before termination)

Disproving liveness properties

Property:

If P is a liveness property, then $\forall t \in \Sigma^*: \exists u \in P: t \preceq u$.

proof:

By definition of liveness, $\rho_{*\preceq}(P) = \Sigma^\infty$, so $t \in \rho_{*\preceq}(P) = \lim(\alpha_p(P))$.
As $t \in \Sigma^*$ and lim only adds infinite traces, $t \in \alpha_p(P)$.

By definition of α_p, $\exists u \in P: t \preceq u$.

Consequence:

- liveness cannot be disproved by testing.

Trace topology

A topology on a set can be defined as:
– either a family of open sets (closed under union)
– or family of closed sets (closed under intersection)

Trace topology: on sets of traces in Σ^∞

- the closed sets are: $\mathcal{C} \stackrel{\text{def}}{=} \{\, P \in \mathcal{P}(\Sigma^\infty) \,|\, P \text{ is a safety property} \,\}$
- the open sets can be derived as $\mathcal{O} \stackrel{\text{def}}{=} \{\, \Sigma^\infty \setminus c \,|\, c \in \mathcal{C} \,\}$

Topological closure: $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$

- $\rho(x) \stackrel{\text{def}}{=} \cap \{\, c \in \mathcal{C} \,|\, x \subseteq c \,\}$ (upper closure operator in $(\mathcal{P}(X), \subseteq)$)
- on our trace topology, $\rho = \rho_{*\preceq}$.

Dense sets:

- $x \subseteq X$ is dense if $\rho(x) = X$;
- on our trace topology, dense sets are liveness properties.

Decomposition theorem

Theorem: decomposition on a topological space

Any set $x \subseteq X$ is the intersection of a closed set and a dense set.

proof:

We have $x = \rho(x) \cap (x \cup (X \setminus \rho(x)))$. Indeed:
$\rho(x) \cap (x \cup (X \setminus \rho(x))) = (\rho(x) \cap x) \cup (\rho(x) \cap (X \setminus \rho(x))) = \rho(x) \cap x = x$ as $x \subseteq \rho(x)$.

- $\rho(x)$ is closed
- $x \cup (X \setminus \rho(x))$ is dense because: $\rho(x \cup (X \setminus \rho(x))) \supseteq \rho(x) \cup \rho(X \setminus \rho(x))$ $\supseteq \rho(x) \cup (X \setminus \rho(x))$ $= X$

Consequence: on trace properties

Every trace property is the conjunction of a safety property and a liveness property.

proving a trace property can be decomposed into a soundness proof and a liveness proof

Beyond trace properties

Some verification problems cannot be expressed as $\mathcal{M}_\infty \subseteq P$

Examples:

- Program equivalence

  Do two programs (Σ, τ_1) and (Σ, τ_2) have the exact same executions?
  i.e., $\mathcal{M}_\infty[\tau_1] = \mathcal{M}_\infty[\tau_2]$

- Non-interference

  Does changing the initial value of X change its final value?
  $\forall \sigma_0, \ldots, \sigma_n \in \mathcal{M}_\infty : \forall \sigma'_0 : \sigma_0 \equiv \sigma'_0 \implies$
  $\exists \sigma'_0, \ldots, \sigma'_m \in \mathcal{M}_\infty : \sigma'_m \equiv \sigma_m$
  where $(\ell, \rho) \equiv (\ell', \rho') \iff \ell = \ell' \land \forall V \neq X : \rho(V) = \rho'(V)$

New verification problem: $\mathcal{M}_\infty \in H$ where $H \in \mathcal{P}(\mathcal{P}(\Sigma^\infty))$

- generalizes trace properties: $\mathcal{M}_\infty \subseteq P$ reduces to $\mathcal{M}_\infty \in \mathcal{P}(P)$;
- program equivalence is $\mathcal{M}_\infty[\tau_1] \in \{\mathcal{M}_\infty[\tau_2]\}$; etc.

Reading assignment: hyperproperties.

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