Correction
MPRI 2-6

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Problem 1

1. The concrete evaluation gives:

$$\begin{array}{rl} & wrap[-128,127](wrap[0,255](\{-1,0,1\}) + wrap[0,255](\{-1,0,1\})) \\ = & wrap[-128,127](\{0,1,255\} + \{0,1,255\}) \\ = & wrap[-128,127](\{0,1,2,255,256,510\}) \\ = & \{-2,-1,0,1,2\} \end{array}$$

2. We define the optimal $wrap[\ell,h]^\sharp_i$ as:

$$\begin{array}{rl} & wrap[\ell,h]^\sharp_i([a,b]) \\ = & \alpha_i(wrap[\ell,h]^\sharp_i(\gamma_i([a,b]))) \\ = & [\min\{\, wrap[\ell,h](v) \mid v \in [a,b]\,\}, \max\{\, wrap[\ell,h](v) \mid v \in [a,b]\,\}] \end{array}$$

   where α_i and γ_i are the interval abstraction and the interval concretization.

   We then have two cases:

   - either a and b are contained in a single interval of the form $[\ell+h]+k(h-\ell+1)$, i.e., if $\exists k : \ell + k(h-\ell+1) \le a \le b \le h + k(h-\ell+1)$. In that case, $wrap[\ell,h]^\sharp_i([a,b]) = [a-k(h-\ell+1), b-k(h-\ell+1)] = [wrap[\ell,h](a), wrap[\ell,h](b)]$;
   - otherwise, $wrap[\ell,h]^\sharp_i([a,b]) = [\ell,h]$, as the interval $[a,b]$ contains both a point x such that $wrap[\ell,h](x) = \ell$ and a point y such that $wrap[\ell,h](y) = h$.

   The operator is exact if and only if:

   - either we are in the first case: $\exists k : \ell + k(h-\ell+1) \le a \le b \le h + k(h-\ell+1)$;
   - or $b - a \ge h - \ell$, which implies $\{\, wrap[\ell,h](v) \mid v \in [a,b]\,\} = [\ell,h]$ in the concrete anyway.

   An example of non-exact application of the operator is $wrap[0,255]^\sharp([-1,0]) = [0,255]$ as, in the concrete, we would get the set $\{0,255\}$.

3. We get:

$$\begin{array}{rl} & wrap[-128,127]^\sharp_i(wrap[0,255]^\sharp_i(x^\sharp) +^\sharp_i wrap[0,255]^\sharp_i(y^\sharp)) \\ = & wrap[-128,127]^\sharp_i(wrap[0,255]^\sharp_i([-1,1]) +^\sharp_i wrap[0,255]^\sharp_i(y[-1,1])) \\ = & wrap[-128,127]^\sharp_i([0,255] +^\sharp_i [0,255]) \\ = & wrap[-128,127]^\sharp_i([0,510]) \\ = & [-128,127] \end{array}$$

The concrete is, by question 1, $\{-2,-1,0,1,2\}$. Note that it can be exactly represented as an interval $[-2,2]$, yet, the evaluation of the expression in the interval domain gives a much coarser result: $[-128,127]$. Hence, the abstract result is neither exact nor optimal.

This imprecision is caused by the accumulated loss of precision due to applying several optimal but non-exact operators in sequence (in general, the composition of optimal but non-exact operators is not an optimal operator). In particular, the first applications of $\mathit{wrap}[0,255]^\sharp_i$ results in a non-recoverable loss of precision.

4. The set of values $V \stackrel{\text{def}}{=} \{0,1,4\}$ can be abstracted both as $x^\sharp \stackrel{\text{def}}{=} [0,1]+3\mathbb{Z}$ and as $y^\sharp \stackrel{\text{def}}{=} [0,1]+4\mathbb{Z}$. Moreover, both abstract values are minimal in $\mathcal{D}_m$, i.e., no $z^\sharp$ such that $\gamma_m(z^\sharp) \subsetneq \gamma_m(x^\sharp)$ or $\gamma_m(z^\sharp) \subsetneq \gamma_m(y^\sharp)$ can satisfy $V \subseteq \gamma_m(z^\sharp)$. If it existed, α_m would allow constructing a *unique* minimal element $\alpha_m(V)$ overapproximating V.

5. To design an abstraction $+^\sharp_m$ of $+$ in $\mathcal{D}_m$, we add separately the interval component and the modular component:

$$([a_1,b_1]+k_1\mathbb{Z}) +^\sharp_m ([a_2,b_2]+k_2\mathbb{Z}) \stackrel{\text{def}}{=} [a_1+a_2, b_1+b_2] + \gcd(k_1,k_1)\mathbb{Z}$$

The operator is sound because, given $x_1 = c_1 + k_1 n_1$, $x_2 = c_2 + k_2 n_2$ where $c_1 \in [a_1,b_1]$ and $c_2 \in [a_2,b_2]$, we have $x_1+x_2 = (c_1+c_2)+(k_1n_1+k_2n_2)$, where $c_1+c_2 \in [a_1+a_2, b_1+b_2] = [a_1,b_1]+[a_2+b_2]$ and $k_1n_1+k_2n_2 \in k_1\mathbb{Z}+k_2\mathbb{Z} = \gcd(k_1,k_2)\mathbb{Z}$. Note that, in this definition, gcd is extended to $\mathbb{N}$ by defining $\forall x : \gcd(0,x) = \gcd(x,0) = x$ (similarly to the simple congruence domain seen in the course).

For $\mathit{wrap}[\ell,h]^\sharp_m([a,b]+k\mathbb{Z})$ we consider two different cases:

(a) when the result, in the concrete, can be exactly represented as an interval, we return this interval; this can be checked by ensuring that $[a,b]+k\mathbb{Z}$ does not cross any boundary in $\ell+(h-\ell+1)\mathbb{Z}$, i.e., that $[a,b]$ does not cross any boundary in $\ell+(h-\ell+1)\mathbb{Z}+k\mathbb{Z} = \ell+\gcd(k,h-\ell+1)\mathbb{Z}$;

(b) otherwise, we keep the interval component intact and adjust the modular component so that the result corresponds to the argument modulo $h-\ell+1$; i.e., we add $(h-\ell+1)\mathbb{Z}$ to $[a,b]+k\mathbb{Z}$ to get $[a,b]+\gcd(h-\ell+1,k)\mathbb{Z}$.

We get:

$$\begin{array}{l}
\mathit{wrap}[\ell,h]^\sharp_m([a,b]+k\mathbb{Z}) \stackrel{\text{def}}{=} \\
\quad \begin{cases} [\mathit{wrap}[\ell,h](a), \mathit{wrap}[\ell,h](b)] + 0\mathbb{Z} & \text{if } (\ell+k'\mathbb{Z}) \cap [a+1,b] = \emptyset \\ [a,b]+k'\mathbb{Z} & \text{otherwise} \end{cases} \\
\quad \text{where } k' \stackrel{\text{def}}{=} \gcd(k, h-\ell+1)
\end{array}$$

In our example, both applications of $\mathit{wrap}[0,255]^\sharp_m$ exercise the second case of the definition, while the application of $\mathit{wrap}[-128,127]^\sharp_m$ exercises the first case. We get:

$$\begin{array}{rl}
& \mathit{wrap}[-128,127]^\sharp_m(\mathit{wrap}[0,255]^\sharp_m(x^\sharp) +^\sharp_m \mathit{wrap}[0,255]^\sharp_m(y^\sharp)) \\
= & \mathit{wrap}[-128,127]^\sharp_m(\mathit{wrap}[0,255]^\sharp_m([-1,1]+0\mathbb{Z}) +^\sharp_m \mathit{wrap}[0,255]^\sharp_m(y[-1,1]+0\mathbb{Z})) \\
= & \mathit{wrap}[-128,127]^\sharp_m([-1,1]+256\mathbb{Z} +^\sharp_m [-1,1]+256\mathbb{Z}) \\
= & \mathit{wrap}[-128,127]^\sharp_m([-2,2]+256\mathbb{Z}) \\
= & [-2,2]
\end{array}$$

Hence, the result is optimal.

Problem 2

1. In the concrete, the set $X \subseteq \mathbb{R}$ of possible values for the variable `X` is given by the smallest solution of the equation:

$$X = \{0\} \cup \{\, \alpha x + b \mid x \in X,\, b \in [0, \beta] \,\}$$

which can be computed using Kleene iterations as:

$$X = \cup_i F^i(\emptyset) \text{ where } F(S) \stackrel{\text{def}}{=} \{0\} \cup \{\, \alpha x + b \mid x \in S,\, b \in [0, \beta] \,\}$$

We can prove by recurrence on i that $F^i(\emptyset) = [0, \sum_{k<i} \alpha^k \beta]$. The limit of this interval is the following interval, open at its upper bound: $\cup_i F^i = [0, \sum_k \alpha^k \beta[$. We have two cases:

   (a) if $0 \leq \alpha < 1$, then the limit is $[0, m[$ where $m \stackrel{\text{def}}{=} \beta/(1-\alpha)$;

   (b) if $\alpha \geq 1$, then the limit is $[0, +\infty[$.

In the following, we will consider only the first case.

2. An interval $[0, m']$ is an inductive invariant if and only if it is a post-fixpoint of F, i.e.: $F([0, m']) \subseteq [0, m']$. As $F([0, m']) = [0, \alpha m' + \beta]$, we deduce that $[0, m']$ is an inductive invariant if and only if $\alpha m' + \beta \leq m'$, i.e., $m' \geq \beta/(1-\alpha) = m$.

3. An analysis using the interval domain and the widening with threshold set T will find the smallest interval inductive invariant whose upper bound is in T. By the answer to the previous question, it will thus find an interval of the form $[0, m']$ where $m' \stackrel{\text{def}}{=} \min \{\, m' \in T \mid m' \geq \beta/(1-\alpha) \,\}$.

   In order to find a bounded interval invariant, it is necessary and sufficient to ensure that T contains a value greater than or equal to $\beta/(1-\alpha)$ and strictly smaller than $+\infty$.

   The most precise invariant representable in the interval domain is $[0, \beta/(1-\alpha)]$ (as we cannot represent open intervals). In order to find the most precise interval invariant, it is necessary and sufficient to have $\beta/(1-\alpha) \in T$.

4. Assume that the result of an interval analysis is the interval $[0, a]$ where $a \neq +\infty$.

   A first decreasing iteration will give $F([0, a]) = [0, \alpha a + \beta]$. We know, by the previous question that $a \geq \beta/(1-\alpha)$; this implies $a(1-\alpha) \geq \beta$ and then $a \geq a\alpha + \beta$. We thus get $F([0, a]) \subseteq [0, a]$. When the invariant is not optimal, i.e., $a > \beta/(1-\alpha)$ the inclusion is strict. By using decreasing iterations, we can compute a sequence $F^i([0, a])$ that converges to the optimal invariant $[0, \beta/(1-\alpha)]$. The decreasing sequence of intervals is infinite, so, a narrowing must be used to converge in finite time (possibly to an interval between the optimal $[0, \beta/(1-\alpha)]$ and the original invariant found $[0, a]$).

5. The first increasing iterates in the interval domain are:

$$\begin{aligned} F^0(\emptyset) &= \emptyset \\ F^1(\emptyset) &= [0, 0] \\ F^2(\emptyset) &= [0, \beta] \\ F^3(\emptyset) &= [0, \alpha\beta + \beta] \end{aligned}$$

Denoting x_i the upper bound of $F^i(\emptyset)$, we get that $\beta = x_2$ and $\alpha = (x_3 - \beta)/\beta = x_3/x_2 - 1$. The exact concrete bound is then $\beta/(1-\alpha) = (x_2)^2/(2x_2 - x_3)$.

We can modify the classic interval widening to check, after iteration 3, the stability of $(x_2)^2/(2x_2 - x_3)$. The new widening takes, as parameter, in addition to the two last iterates, the iteration count i. More precisely, the increasing sequence of intervals computed will now be $X_{i+1} = X_i \triangledown_i F(X_i)$ where, at iteration i, the widening is defined as:

$$[a,b] \triangledown_i [c,d] \stackrel{\text{def}}{=} \begin{cases} [c,d] & \text{if } c \leq a = b \leq d \\ [0, b^2/(2b-d)] & \text{if } a = c = 0 \wedge b^2/(2b-d) \geq b, d \wedge i = 2 \\ [a,b] \triangledown [c,d] & \text{otherwise} \end{cases}$$

where $\triangledown$ is the classic interval widening:

$$[a,b] \triangledown [c,d] \stackrel{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ -\infty & \text{otherwise} \end{cases}, \begin{cases} b & \text{if } b \geq d \\ +\infty & \text{otherwise} \end{cases} \right]$$

The first case $c \leq a = b \leq d$ ensures that, at iteration 1, when the upper bound goes from 0 to β, it is not immediately widened to $+\infty$. The second case ensures that, at iteration 2, the limit $\beta/(1-\alpha) = b^2/(2b-d)$ is chosen as upper bound, if it is sound (test $a = c = 0 \wedge b^2/(2b-d) \geq b, d$). The soundness of $\triangledown$ completes the soundness proof of $\triangledown_i$. To prove the termination, it is sufficient to remark that a strictly increasing sequence will keep applying $\triangledown$ after a certain iterate, and so, the sequence terminates by the termination property of $\triangledown$.