Order Theory

MPRI 2–6: Abstract Interpretation,
application to verification and static analysis

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Partial orders
Given a set $X$, a relation $\sqsubseteq \in X \times X$ is a partial order if it is:

1. reflexive: $\forall x \in X, x \sqsubseteq x$
2. antisymmetric: $\forall x, y \in X, x \sqsubseteq y \land y \sqsubseteq x \implies x = y$
3. transitive: $\forall x, y, z \in X, x \sqsubseteq y \land y \sqsubseteq z \implies x \sqsubseteq z$.

$(X, \sqsubseteq)$ is a poset (partially ordered set).

If we drop antisymmetry, we have a preorder instead.
Partial orders:

- \((\mathbb{Z}, \leq)\)  
  (completely ordered)

- \((\mathcal{P}(X), \subseteq)\)
  (not completely ordered: \(\{1\} \not\subseteq \{2\}, \{2\} \not\subseteq \{1\}\))

- \((S, =)\) is a poset for any \(S\)

- \((\mathbb{Z}^2, \sqsubseteq), \text{ where } (a, b) \sqsubseteq (a', b') \iff a \geq a' \land b \leq b'\)
  (ordering of interval bounds that implies inclusion)
Examples: preorders

Preorders:

- $(\mathcal{P}(X), \sqsubseteq)$, where $a \sqsubseteq b \iff |a| \leq |b|$
  
  (ordered by cardinal)

- $(\mathbb{Z}^2, \sqsubseteq)$, where
  
  $(a, b) \sqsubseteq (a', b') \iff \{x \mid a \leq x \leq b\} \subseteq \{x \mid a' \leq x \leq b'\}$
  
  (inclusion of intervals represented by pairs of bounds)

  not antisymmetric: $[1, 0] \not= [2, 0]$ but $[1, 0] \sqsubseteq [2, 0] \sqsubseteq [1, 0]$

Equivalence: $\equiv$

$X \equiv Y \iff (X \sqsubseteq Y) \land (Y \sqsubseteq X)$

We obtain a partial order by quotienting by $\equiv$. 
Given by a **Hasse diagram**, e.g.:

\[ a \sqsubseteq b \sqsubseteq c \sqsubseteq d \sqsubseteq e \sqsubseteq f \sqsubseteq g \]

\[ g \sqsubseteq g \]
\[ f \sqsubseteq f, g \]
\[ e \sqsubseteq e, g \]
\[ d \sqsubseteq d, f, g \]
\[ c \sqsubseteq c, e, f, g \]
\[ b \sqsubseteq b, c, d, e, f, g \]
\[ a \sqsubseteq a, b, c, d, e, f, g \]
Infinite Hasse diagram for \((\mathbb{N} \cup \{\infty\}, \leq)\):

\[
\begin{array}{c}
\infty \\
\vdots \\
3 \\
2 \\
1 \\
0
\end{array}
\begin{array}{c}
\infty \sqsubseteq \infty \\
\ldots \\
1 \sqsubseteq 1, 2, \ldots, \infty \\
0 \sqsubseteq 0, 1, 2, \ldots, \infty
\end{array}
\]
Use of posets (informally)

Posets are a very useful notion to discuss about:

- **logic**: ordered by implication \[ \implies \]

- **approximation**: \[ \sqsubseteq \] is an information order
  ("\( a \sqsubseteq b \)" means: "\( a \) caries more information than \( b \)"")

- **program verification**: program semantics \[ \sqsubseteq \] specification
  (e.g.: behaviors of program \[ \sqsubseteq \] accepted behaviors)

- **iteration**: fixpoint computation
  (e.g., a computation is directed, with a limit: \( X_1 \sqsubseteq X_2 \sqsubseteq \cdots \sqsubseteq X_n \))
(Least) Upper bounds

- $c$ is an upper bound of $a$ and $b$ if: $a \sqsubseteq c$ and $b \sqsubseteq c$
- $c$ is a least upper bound (lub or join) of $a$ and $b$ if:
  - $c$ is an upper bound of $a$ and $b$
  - for every upper bound $d$ of $a$ and $b$, $c \sqsubseteq d$
(Least) Upper bounds

The lub is **unique** and denoted \( a \sqcup b \).

(proof: assume that \( c \) and \( d \) are both lubs of \( a \) and \( b \); by definition of lubs, \( c \sqsubseteq d \) and \( d \sqsubseteq c \); by antisymmetry of \( \sqsubseteq \), \( c = d \))

Generalized to upper bounds of arbitrary (even infinite) sets \( \sqcup Y, Y \subseteq X \)

(well-defined, as \( \sqcup \) is commutative and associative).

Similarly, we define **greatest lower bounds** (glb, meet) \( a \sqcap b, \sqcap Y \).

(\( a \sqcap b \sqsubseteq a, b \) and \( \forall c, c \sqsubseteq a, b \implies c \sqsubseteq a \sqcap b \))

**Note:** not all posets have lubs, glbs

(e.g.: \( a \sqcup b \) not defined on \( \{a, b\}, = \))
$C \subseteq X$ is a chain in $(X, \sqsubseteq)$ if it is totally ordered by $\sqsubseteq$:

\[ \forall x, y \in C, \ x \sqsubseteq y \lor y \sqsubseteq x. \]
A poset \((X, \sqsubseteq)\) is a complete partial order (CPO) if every chain \(C\) (including \(\emptyset\)) has a least upper bound \(\sqcup C\).

A CPO has a least element \(\sqcup \emptyset\), denoted \(\bot\).

**Examples:**

- \((\mathbb{N}, \leq)\) is not complete, but \((\mathbb{N} \cup \{\infty\}, \leq)\) is complete.
- \((\{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}, \leq)\) is not complete, but \((\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}, \leq)\) is complete.
- \((\mathcal{P}(Y), \subseteq)\) is complete for any \(Y\).
- \((X, \sqsubseteq)\) is complete if \(X\) is finite.
Complete partial order examples

\((\mathbb{N}, \leq)\) non-complete

\((\mathbb{N} \cup \{\infty\}, \leq)\) complete
Lattices
A lattice \((X, \sqsubseteq, \sqcup, \sqcap)\) is a poset with

1. a lub \(a \sqcup b\) for every pair of elements \(a\) and \(b\);
2. a glb \(a \sqcap b\) for every pair of elements \(a\) and \(b\).

Examples:

- integers \((\mathbb{Z}, \leq, \max, \min)\)
- integer intervals \((\text{presenter later})\)
- divisibility \((\text{presenter later})\)

If we drop one condition, we have a (join or meet) semilattice.

Reference on lattices: Birkhoff [Birk76].
Example: the interval lattice

Integer intervals: \( \{ [a, b] \mid a, b \in \mathbb{Z}, a \leq b \} \cup \{ \emptyset \}, \subseteq, \cup, \cap \)

where \([a, b] \sqcup [a', b'] \overset{\text{def}}{=} [\min(a, a'), \max(b, b')]\).
Example: the divisibility lattice

Divisibility \((\mathbb{N}^*, |, \text{lcm}, \text{gcd})\) where \(x|y \iff \exists k \in \mathbb{N}, kx = y\)
Let $P \overset{\text{def}}{=} \{ p_1, p_2, \ldots \}$ be the (infinite) set of prime numbers.

We have a correspondence $\iota$ between $\mathbb{N}^*$ and $P \rightarrow \mathbb{N}$:

- $\alpha = \iota(x)$ is the (unique) decomposition of $x$ into prime factors
- $\iota^{-1}(\alpha) \overset{\text{def}}{=} \prod_{a \in P} a^{\alpha(a)} = x$
- $\iota$ is one-to-one on functions $P \rightarrow \mathbb{N}$ with finite support

$(\alpha(a) = 0$ except for finitely many factors $a)$

We have a correspondence between $(\mathbb{N}^*, |, \text{lcm}, \gcd)$ and $(\mathbb{N}, \leq, \text{max}, \text{min})$.

Assume that $\alpha = \iota(x)$ and $\beta = \iota(y)$ are the decompositions of $x$ and $y$, then:

- $\prod_{a \in P} a^{\max(\alpha(a), \beta(a))} = \text{lcm}(\prod_{a \in P} a^{\alpha(a)}, \prod_{a \in P} a^{\beta(a)}) = \text{lcm}(x, y)$
- $\prod_{a \in P} a^{\min(\alpha(a), \beta(a))} = \text{gcd}(\prod_{a \in P} a^{\alpha(a)}, \prod_{a \in P} a^{\beta(a)}) = \text{gcd}(x, y)$
- $(\forall a: \alpha(a) \leq \beta(a)) \iff (\prod_{a \in P} a^{\alpha(a)}) | (\prod_{a \in P} a^{\beta(a)}) \iff x | y$
A complete lattice \((X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)\) is a poset with

1. a lub \(\sqcup S\) for every set \(S \subseteq X\)
2. a glb \(\sqcap S\) for every set \(S \subseteq X\)
3. a least element \(\bot\)
4. a greatest element \(\top\)

Notes:

- 1 implies 2 as \(\sqcap S = \sqcup \{ y \mid \forall x \in S, y \sqsubseteq x \}\)
  (and 2 implies 1 as well),
- 1 and 2 imply 3 and 4: \(\bot = \sqcup \emptyset = \sqcap X\), \(\top = \sqcap \emptyset = \sqcup X\),
- a complete lattice is also a CPO.
Complete lattice examples

- **real segment** $[0, 1]$: $(\{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \}, \leq, \max, \min, 0, 1)$

- **powersets** $(\mathcal{P}(S), \subseteq, \cup, \cap, \emptyset, S)$

- **any finite lattice**
  $(\cup Y$ and $\cap Y$ for finite $Y \subseteq X$ are always defined)

- **integer intervals** with finite and **infinite** bounds:
  $(\{ [a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b \} \cup \{\emptyset\}$, $\subseteq, \cup, \cap, \emptyset, [-\infty, +\infty])$

  with $\bigcup_{i \in I} [a_i, b_i] \overset{\text{def}}{=} [\min_{i \in I} a_i, \max_{i \in I} b_i]$. 
Example: the powerset complete lattice

\[ (\mathcal{P}(\mathbb{Z}), \subseteq, \cup, \cap, \emptyset, \mathbb{Z}) \]
The integer intervals with finite and infinite bounds:

\[
\{ [a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, \ b \in \mathbb{Z} \cup \{+\infty\}, \ a \leq b \} \cup \{\emptyset\}, \ \subseteq, \ \cup, \ \cap, \ \emptyset, \ [-\infty, +\infty] \}
\]
Derivation

Given a (complete) lattice or partial order \((X, \sqsubseteq, \sqcup, \sqcap, \bot, \top)\) we can derive new (complete) lattices or partial orders by:

- **duality**
  \((X, \sqsubseteq', \sqcap', \sqcup', \bot', \top')\)
  - \(\sqsubseteq\) is reversed
  - \(\sqcup\) and \(\sqcap\) are switched
  - \(\bot\) and \(\top\) are switched

- **lifting** (adding a smallest element)
  \((X \cup \{\bot'\}, \sqsubseteq', \sqcup', \sqcap', \bot', \top')\)
  
  \begin{align*}
  a \sqsubseteq' b & \iff a = \bot' \lor a \sqsubseteq b \\
  \bot' \sqcup' a & = a \sqcup' \bot' = a, \text{ and } a \sqcup' b = a \sqcup b \text{ if } a, b \neq \bot' \\
  \bot' \sqcap' a & = a \sqcap' \bot' = \bot', \text{ and } a \sqcap' b = a \sqcap b \text{ if } a, b \neq \bot' \\
  \bot' & \text{ replaces } \bot \\
  \top & \text{ is unchanged}
  \end{align*}
Given (complete) lattices or partial orders:
\((X_1, \sqsubseteq_1, \sqcup_1, \sqcap_1, \bot_1, \top_1)\) and \((X_2, \sqsubseteq_2, \sqcup_2, \sqcap_2, \bot_2, \top_2)\)

We can combine them by:

- **product**
  \((X_1 \times X_2, \sqsubseteq, \sqcup, \sqcap, \bot, \top)\) where
  - \((x, y) \sqsubseteq (x', y') \iff x \sqsubseteq_1 x' \land y \sqsubseteq_2 y'\)
  - \((x, y) \sqcup (x', y') \defeq (x \sqcup_1 x', y \sqcup_2 y')\)
  - \((x, y) \sqcap (x', y') \defeq (x \sqcap_1 x', y \sqcap_2 y')\)
  - \(\bot \defeq (\bot_1, \bot_2)\)
  - \(\top \defeq (\top_1, \top_2)\)

- **smashed product** (coalescent product, merging \(\bot_1\) and \(\bot_2\))
  \(((((X_1 \setminus \{\bot_1\}) \times (X_2 \setminus \{\bot_2\}))) \cup \{\bot\}, \sqsubseteq, \sqcup, \sqcap, \bot, \top)\)
  (as \(X_1 \times X_2\), but all elements of the form \((\bot_1, y)\) and \((x, \bot_2)\) are identified to a unique \(\bot\) element)
Derivation (cont.)

Given a (complete) lattice or partial order \((X, \sqsubseteq, \cup, \cap, \perp, \top)\) and a set \(S\):

- **point-wise lifting** (functions from \(S\) to \(X\))

\[(S \rightarrow X, \sqsubseteq', \cup', \cap', \perp', \top')\] where

- \(x \sqsubseteq' y \iff \forall s \in S: x(s) \sqsubseteq y(s)\)
- \(\forall s \in S: (x \cup' y)(s) \overset{\text{def}}{=} x(s) \cup y(s)\)
- \(\forall s \in S: (x \cap' y)(s) \overset{\text{def}}{=} x(s) \cap y(s)\)
- \(\forall s \in S: \perp'(s) = \perp\)
- \(\forall s \in S: \top'(s) = \top\)
A lattice \((X, \subseteq, \cup, \cap)\) is **distributive** if:

- \(a \cup (b \cap c) = (a \cup b) \cap (a \cup c)\) and
- \(a \cap (b \cup c) = (a \cap b) \cup (a \cap c)\)

**Examples:**

- \((\mathcal{P}(X), \subseteq, \cup, \cap)\) is distributive

- Intervals are **not** distributive
  \[
  ([0, 0] \cup [2, 2]) \cap [1, 1] = [0, 2] \cap [1, 1] = [1, 1]
  \]
  but
  \[
  ([0, 0] \cap [1, 1]) \cup ([2, 2] \cap [1, 1]) = \emptyset \cup \emptyset = \emptyset
  \]
  (common cause of precision loss in static analyses)
Sublattice

Given a lattice \((X, \subseteq, \cup, \cap)\) and \(X' \subseteq X\)
\((X', \subseteq, \cup, \cap)\) is a sublattice of \(X\) if \(X'\) is closed under \(\cup\) and \(\cap\)

Examples:

- if \(Y \subseteq X\), \((\mathcal{P}(Y), \subseteq, \cup, \cap, \emptyset, Y)\) is a sublattice of \((\mathcal{P}(X), \subseteq, \cup, \cap, \emptyset, X)\)

- integer intervals are not a sublattice of \((\mathcal{P}(\mathbb{Z}), \subseteq, \cup, \cap, \emptyset, \mathbb{Z})\)
  
  \([\min(a, a'), \max(b, b')] \neq [a, b] \cup [a', b']\)

  (another common cause of precision loss in static analyses)
Fixpoints
Fixpoints

Functions

A function \( f : (X_1, \sqsubseteq_1, \sqcup_1, \bot_1) \rightarrow (X_2, \sqsubseteq_2, \sqcup_2, \bot_2) \) is

- **monotonic** if
  \[ \forall x, x', x \sqsubseteq_1 x' \implies f(x) \sqsubseteq_2 f(x') \]
  (aka: increasing, isotone, order-preserving, morphism)

- **strict** if \( f(\bot_1) = \bot_2 \)

- **continuous** between CPO if
  \[ \forall C \text{ chain } \subseteq X_1, \{ f(c) \mid c \in C \} \text{ is a chain in } X_2 \]
  and \( f(\sqcup_1 C) = \sqcup_2 \{ f(c) \mid c \in C \} \)

- a (complete) \( \sqcup \)-morphism between (complete) lattices
  if \( \forall S \subseteq X_1, f(\sqcup_1 S) = \sqcup_2 \{ f(s) \mid s \in S \} \)

- **extensive** if \( X_1 = X_2 \) and \( \forall x, x \sqsubseteq_1 f(x) \)

- **reductive** if \( X_1 = X_2 \) and \( \forall x, f(x) \sqsubseteq_1 x \)
Fixpoints

Given \( f : (X, \sqsubseteq) \to (X, \sqsubseteq) \)

- \( x \) is a fixpoint of \( f \) if \( f(x) = x \)
- \( x \) is a pre-fixpoint of \( f \) if \( x \sqsubseteq f(x) \)
- \( x \) is a post-fixpoint of \( f \) if \( f(x) \sqsubseteq x \)

We may have several fixpoints (or none)

- \( \text{fp}(f) \overset{\text{def}}{=} \{ x \in X \mid f(x) = x \} \)
- \( \text{lfp}_x f \overset{\text{def}}{=} \min_{\sqsubseteq} \{ y \in \text{fp}(f) \mid x \sqsubseteq y \} \) if it exists
  (least fixpoint greater than \( x \))
- \( \text{lfp} f \overset{\text{def}}{=} \text{lfp}_\bot f \)
  (least fixpoint)
- dually: \( \text{gfp}_x f \overset{\text{def}}{=} \max_{\sqsubseteq} \{ y \in \text{fp}(f) \mid y \sqsubseteq x \} \), \( \text{gfp} f \overset{\text{def}}{=} \text{gfp}_\top f \)
  (greatest fixpoints)
Fixpoints: illustration

Graph showing the concepts of least fixpoint (lfp), fixed point (fp), and greatest fixpoint (gfp) in an order theory context.
Monotonic function with two distinct fixpoints
Monotonic function with a unique fixpoint
Non-monotonic function with no fixpoint
Uses of fixpoints: examples

Express solutions of mutually recursive equation systems

Example:

The solutions of \[
\begin{cases}
x_1 = f(x_1, x_2) \\
x_2 = g(x_1, x_2)
\end{cases}
\]
with $x_1, x_2$ in lattice $X$

are exactly the fixpoint of $\vec{F}$ in lattice $X \times X$, where

$$\vec{F} \left( \begin{array}{c} x_1, \\ x_2 \end{array} \right) = \left( \begin{array}{c} f(x_1, x_2), \\ g(x_1, x_2) \end{array} \right)$$

The least solution of the system is lfp $\vec{F}$. 
Uses of fixpoints: examples

- Close (complete) sets to satisfy a given property

Example:

$r \subseteq X \times X$ is transitive if:

$\forall (a, b), (b, c) \in r \quad (a, b) \in r \land (b, c) \in r \implies (a, c) \in r$

The transitive closure of $r$ is the smallest transitive relation containing $r$.

Let $f(s) = r \cup \{(a, c) \mid (a, b) \in s \land (b, c) \in s\}$, then $\text{lfp } f$:

- $\text{lfp } f$ contains $r$
- $\text{lfp } f$ is transitive
- $\text{lfp } f$ is minimal

$\implies \text{lfp } f$ is the transitive closure of $r$. 
Tarski’s fixpoint theorem

Tarski’s theorem

If $f : X \to X$ is monotonic in a complete lattice $X$ then $\text{fp}(f)$ is a complete lattice.

Proved by Knaster and Tarski [Tars55].
Tarski’s fixpoint theorem

**Tarski’s theorem**

If $f : X \rightarrow X$ is monotonic in a complete lattice $X$ then $\text{fp}(f)$ is a complete lattice.

**Proof:**
We prove $\text{lfp } f = \sqcap \{ x \mid f(x) \sqsubseteq x \}$ (meet of post-fixpoints).
Tarski’s fixpoint theorem

If \( f : X \to X \) is monotonic in a complete lattice \( X \) then \( \text{fp}(f) \) is a complete lattice.

**Proof:**

We prove \( \text{lfp } f = \sqcap \{ x \mid f(x) \sqsubseteq x \} \) (meet of post-fixpoints).

Let \( f^* = \{ x \mid f(x) \sqsubseteq x \} \) and \( a = \sqcap f^* \).

\[ \forall x \in f^*, \ a \sqsubseteq x \quad \text{(by definition of } \sqcap) \]

so \( f(a) \sqsubseteq f(x) \quad \text{(as } f \text{ is monotonic)} \)

so \( f(a) \sqsubseteq x \quad \text{(as } x \text{ is a post-fixpoint)} \).

We deduce that \( f(a) \sqsubseteq \sqcap f^* \), i.e. \( f(a) \sqsubseteq a \).
Fixpoints

Tarski’s fixpoint theorem

Tarski’s theorem

If \( f : X \rightarrow X \) is monotonic in a complete lattice \( X \) then \( \text{fp}(f) \) is a complete lattice.

Proof:

We prove \( \text{lfp} f = \bigcap \{ x \mid f(x) \sqsubseteq x \} \) (meet of post-fixpoints).

\[
f(a) \sqsubseteq a
\]

so \( f(f(a)) \sqsubseteq f(a) \) (as \( f \) is monotonic)

so \( f(a) \in f^* \) (by definition of \( f^* \))

so \( a \sqsubseteq f(a) \).

We deduce that \( f(a) = a \), so \( a \in \text{fp}(f) \).

Note that \( y \in \text{fp}(f) \) implies \( y \in f^* \).

As \( a = \bigcap f^* \), \( a \sqsubseteq y \), and we deduce \( a = \text{lfp} f \).
Tarski’s fixpoint theorem

If $f : X \rightarrow X$ is monotonic in a complete lattice $X$ then $\text{fp}(f)$ is a complete lattice.

**Proof:**
Given $S \subseteq \text{fp}(f)$, we prove that $\text{lfp}_{\uplus S} f$ exists.

Consider $X' = \{ x \in X | \uplus S \subseteq x \}$. $X'$ is a complete lattice. Moreover $\forall x' \in X', f(x') \in X'$. $f$ can be restricted to a monotonic function $f'$ on $X'$. We apply the preceding result, so that $\text{lfp} f' = \text{lfp}_{\uplus S} f$ exists. By definition, $\text{lfp}_{\uplus S} f \in \text{fp}(f)$ and is smaller than any fixpoint larger than all $s \in S$. 
Tarski’s fixpoint theorem

Tarski’s theorem
If \( f : X \rightarrow X \) is monotonic in a complete lattice \( X \) then \( \text{fp}(f) \) is a complete lattice.

Proof:
By duality, we construct \( \text{gfp } f \) and \( \text{gfp } \sqcap S f \).

The complete lattice of fixpoints is:
\( \langle \text{fp}(f), \sqsubseteq, \lambda S. \text{lfp } \sqcup S f, \lambda S. \text{gfp } \sqcap S f, \text{lfp } f, \text{gfp } f \rangle \).

Not necessarily a sublattice of \( (X, \sqsubseteq, \sqcup, \sqcap, \bot, \top) \)!
Tarski’s fixpoint theorem: example

Lattice: \((\{ \text{lfp}, \text{fp1}, \text{fp2}, \text{pre}, \text{gfp} \}, \sqcup, \sqcap, \text{lfp}, \text{gfp})\)

Fixpoint lattice: \((\{ \text{lfp}, \text{fp1}, \text{fp2}, \text{gfp} \}, \sqcup', \sqcap', \text{lfp}, \text{gfp})\)

(not a sublattice as \(\text{fp1} \sqcup' \text{fp2} = \text{gfp}\) while \(\text{fp1} \sqcup \text{fp2} = \text{pre}\), but \(\text{gfp}\) is the smallest fixpoint greater than \(\text{pre}\))
“Kleene” fixpoint theorem

If $f : X \to X$ is continuous in a CPO $X$ and $a \sqsubseteq f(a)$ then $\text{Lfp}_a f$ exists.

Inspired by Kleene [Klee52].
“Kleene” fixpoint theorem

If $f : X \rightarrow X$ is continuous in a CPO $X$ and $a \sqsubseteq f(a)$ then $\text{lfp}_a f$ exists.

We prove that $\{ f^n(a) \mid n \in \mathbb{N} \}$ is a chain and

$$\text{lfp}_a f = \bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \}.$$
“Kleene” fixpoint theorem

If \( f : X \to X \) is continuous in a CPO \( X \) and \( a \sqsubseteq f(a) \) then \( \text{lfp}_a f \) exists.

We prove that \( \{ f^n(a) \mid n \in \mathbb{N} \} \) is a chain and
\[
\text{lfp}_a f = \bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \}.
\]

\( a \sqsubseteq f(a) \) by hypothesis.

\( f(a) \sqsubseteq f(f(a)) \) by monotony of \( f \).

(Note that any continuous function is monotonic. Indeed, \( x \sqsubseteq y \implies x \sqcup y = y \implies f(x \sqcup y) = f(y) \); by continuity \( f(x) \sqcup f(y) = f(x \sqcup y) = f(y) \), which implies \( f(x) \sqsubseteq f(y) \).)

By recurrence \( \forall n, f^n(a) \sqsubseteq f^{n+1}(a) \).

Thus, \( \{ f^n(a) \mid n \in \mathbb{N} \} \) is a chain and \( \bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \} \) exists.
"Kleene" fixpoint theorem

If $f : X \to X$ is continuous in a CPO $X$ and $a \sqsubseteq f(a)$ then $\operatorname{lfp}_a f$ exists.

\[
f(\bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \}) \\
= \bigsqcup \{ f^{n+1}(a) \mid n \in \mathbb{N} \} \quad \text{(by continuity)} \\
= a \sqcup (\bigsqcup \{ f^{n+1}(a) \mid n \in \mathbb{N} \}) \quad \text{(as all $f^{n+1}(a)$ are greater than $a$)} \\
= \bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \}. \\
\]

So, $\bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \} \in \operatorname{fp}(f)$

Moreover, any fixpoint greater than $a$ must also be greater than all $f^n(a)$, $n \in \mathbb{N}$.

So, $\bigsqcup \{ f^n(a) \mid n \in \mathbb{N} \} = \operatorname{lfp}_a f$. 
Well-ordered sets

\((S, \sqsubseteq)\) is a **well-ordered set** if:

- \(\sqsubseteq\) is a **total order** on \(S\)
- every \(X \subseteq S\) such that \(X \neq \emptyset\) has a **least element** \(\sqcap X \in X\)

**Consequences:**

- any element \(x \in S\) has a **successor** \(x + 1 \overset{\text{def}}{=} \sqcap \{ y \mid x \sqsubseteq y \}\)
  (except the greatest element, if it exists)
- if \(\forall y, x = y + 1\), \(x\) is a **limit** and \(x = \sqcup \{ y \mid y \sqsubseteq x \}\)
  (every bounded subset \(X \subseteq S\) has a lub \(\sqcup X = \sqcap \{ y \mid \forall x \in X, x \sqsubseteq y \}\))

**Examples:**

- \((\mathbb{N}, \leq)\) and \((\mathbb{N} \cup \{ \infty \}, \leq)\) are well-ordered
- \((\mathbb{Z}, \leq), (\mathbb{R}, \leq), (\mathbb{R}^+, \leq)\) are **not** well-ordered
- ordinals \(0, 1, 2, \ldots, \omega, \omega + 1, \ldots\) are well-ordered (\(\omega\) is a limit)
  well-ordered sets are ordinals **up to order-isomorphism**
  (i.e., bijective functions \(f\) such that \(f\) and \(f^{-1}\) are monotonic)
Given a function $f : X \to X$ and $a \in X$, the transfinite iterates of $f$ from $a$ are:

$$
\begin{align*}
  x_0 &\overset{\text{def}}{=} a \\
  x_n &\overset{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\
  x_n &\overset{\text{def}}{=} \bigsqcup \{ x_m \mid m < n \} & \text{if } n \text{ is a limit ordinal}
\end{align*}
$$

**Constructive Tarski theorem**

If $f : X \to X$ is monotonic in a CPO $X$ and $a \sqsubseteq f(a)$, then $\text{lfp}_a f = x_\delta$ for some ordinal $\delta$.

Generalisation of “Kleene” fixpoint theorem, from [Cous79].
Proof

Let $f$ be monotonic in a CPO $X$,
\[
\begin{cases}
    x_0 \overset{\text{def}}{=} a \sqsubseteq f(a) \\
    x_n \overset{\text{def}}{=} f(x_{n-1}) & \text{if } n \text{ is a successor ordinal} \\
    x_n \overset{\text{def}}{=} \sqcup \{ x_m \mid m < n \} & \text{if } n \text{ is a limit ordinal}
\end{cases}
\]

Proof:
We prove that $\exists \delta, x_\delta = x_{\delta+1}$.

We note that $m \leq n \implies x_m \sqsubseteq x_n$.
Assume by contradiction that $\forall \delta, x_\delta \neq x_{\delta+1}$.
If $n$ is a successor ordinal, then $x_{n-1} \sqsubseteq x_n$.
If $n$ is a limit ordinal, then $\forall m < n, x_m \sqsubseteq x_n$.
Thus, all the $x_n$ are distinct.
By choosing $n > |X|$, we arrive at a contradiction.
Thus $\delta$ exists.
Proof

$f$ is monotonic in a CPO $X$,

$$
\begin{align*}
    x_0 & \overset{\text{def}}{=} a \sqsubseteq f(a) \\
    x_n & \overset{\text{def}}{=} f(x_{n-1}) \quad \text{if } n \text{ is a successor ordinal} \\
    x_n & \overset{\text{def}}{=} \bigsqcup \{ x_m \mid m < n \} \quad \text{if } n \text{ is a limit ordinal}
\end{align*}
$$

Proof:

Given $\delta$ such that $x_{\delta+1} = x_\delta$, we prove that $x_\delta = \text{lfp}_a f$.

$f(x_\delta) = x_{\delta+1} = x_\delta$, so $x_\delta \in \text{fp}(f)$.

Given any $y \in \text{fp}(f)$, $y \sqsubseteq a$, we prove by transfinite induction that $\forall n, x_n \sqsubseteq y$.

By definition $x_0 = a \sqsubseteq y$.

If $n$ is a successor ordinal, by monotony,

$x_{n-1} \sqsubseteq y \implies f(x_{n-1}) \sqsubseteq f(y)$, i.e., $x_n \sqsubseteq y$.

If $n$ is a limit ordinal, $\forall m < n, x_m \sqsubseteq y$ implies

$x_n = \bigsqcup \{ x_m \mid m < n \} \sqsubseteq y$.

Hence, $x_\delta \sqsubseteq y$ and $x_\delta = \text{lfp}_a f$. 
An ascending chain $C$ in $(X, \sqsubseteq)$ is a sequence $c_i \in X$ such that $i \leq j \implies c_i \sqsubseteq c_j$.

A poset $(X, \sqsubseteq)$ satisfies the ascending chain condition (ACC) iff for every ascending chain $C$, $\exists i \in \mathbb{N}$, $\forall j \geq i$, $c_i = c_j$.

Similarly, we can define the descending chain condition (DCC).

Examples:

- the powerset poset $(\mathcal{P}(X), \subseteq)$ is ACC when $X$ is finite
- the pointed integer poset $(\mathbb{Z} \cup \{\bot\}, \subseteq)$ where $x \subseteq y \iff x = \bot \lor x = y$ is ACC and DCC
- the divisibility poset $(\mathbb{N}^*, \mid)$ is DCC but not ACC.
“Kleene” finite fixpoint theorem

If $f : X \rightarrow X$ is monotonic in an ACC poset $X$ and $a \sqsubseteq f(a)$ then $\text{lfp}_a f$ exists.

Proof:

We prove $\exists n \in \mathbb{N}, \text{lfp}_a f = f^n(a)$.

By monotony of $f$, the sequence $x_n = f^n(a)$ is an increasing chain.
By definition of ACC, $\exists n \in \mathbb{N}, x_n = x_{n+1} = f(x_n)$.
Thus, $x_n \in \text{fp}(f)$.

Obviously, $a = x_0 \sqsubseteq f(x_n)$.

Moreover, if $y \in \text{fp}(f)$ and $y \sqsupseteq a$, then $\forall i, y \sqsupseteq f^i(a) = x_i$.

Hence, $y \sqsupseteq x_n$ and $x_n = \text{lfp}_a (f)$. 
## Comparison of fixpoint theorems

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Galois connections
Given two posets \((C, \leq)\) and \((A, \sqsubseteq)\), the pair \((\alpha : C \to A, \gamma : A \to C)\) is a **Galois connection** iff:

\[
\forall a \in A, c \in C, \quad \alpha(c) \sqsubseteq a \iff c \leq \gamma(a)
\]

which is noted \((C, \leq) \xrightarrow{\alpha} (A, \sqsubseteq)\).

\(\alpha\) is the **upper adjoint** or **abstraction**; \(A\) is the abstract domain.

\(\gamma\) is the **lower adjoint** or **concretization**; \(C\) is the concrete domain.
Properties of Galois connections

Assuming $\forall a, c, \alpha(c) \sqsubseteq a \iff c \leq \gamma(a)$, we have:

1. $\gamma \circ \alpha$ is extensive: $\forall c, c \leq \gamma(\alpha(c))$
   
   **Proof:** $c \leq \gamma(\alpha(c))$

2. $\alpha \circ \gamma$ is reductive: $\forall a, \alpha(\gamma(a)) \sqsubseteq a$

3. $\alpha$ is monotonic
   
   **Proof:** $c \leq c' \implies c \leq \gamma(\alpha(c')) \implies \alpha(c) \sqsubseteq \alpha(c')$

4. $\gamma$ is monotonic

5. $\gamma \circ \alpha \circ \gamma = \gamma$
   
   **Proof:** $\alpha(\gamma(a)) \sqsubseteq \alpha(\gamma(a)) \implies \gamma(a) \leq \gamma(\alpha(\gamma(a)))$ and $\gamma(a) \leq \gamma(\alpha(\gamma(a)))$

6. $\alpha \circ \gamma \circ \alpha = \alpha$

7. $\alpha \circ \gamma$ is idempotent: $\alpha \circ \gamma \circ \alpha \circ \gamma = \alpha \circ \gamma$

8. $\gamma \circ \alpha$ is idempotent
Alternate characterization

If the pair \((\alpha : C \rightarrow A, \gamma : A \rightarrow C)\) satisfies:

1. \(\gamma\) is monotonic,
2. \(\alpha\) is monotonic,
3. \(\gamma \circ \alpha\) is extensive
4. \(\alpha \circ \gamma\) is reductive

then \((\alpha, \gamma)\) is a Galois connection.

(proof left as exercise)
Given \((C, \leq) \xleftarrow{\gamma} (A, \sqsubseteq)\),
each adjoint can be \textit{uniquely defined} in term of the other:
\begin{enumerate}
\item \(\alpha(c) = \bigcap \{ a \mid c \leq \gamma(a) \}\)
\item \(\gamma(a) = \bigvee \{ c \mid \alpha(c) \sqsubseteq a \}\)
\end{enumerate}

\textbf{Proof:} of 1
\[\forall a, \ c \leq \gamma(a) \implies \alpha(c) \sqsubseteq a.\]
Hence, \(\alpha(c)\) is a lower bound of \(\{ a \mid c \leq \gamma(a) \}\).
Assume that \(a'\) is another lower bound.
Then, \(\forall a, \ c \leq \gamma(a) \implies a' \sqsubseteq a.\)
By Galois connection, we have then \(\forall a, \alpha(c) \sqsubseteq a \implies a' \sqsubseteq a.\)
This implies \(a' \sqsubseteq \alpha(c)\).
Hence, the greatest lower bound of \(\{ a \mid c \leq \gamma(a) \}\) exists,
and equals \(\alpha(c)\).

The proof of 2 is similar (by duality).
If \((\alpha : C \to A, \gamma : A \to C)\), then:

1. For all \(X \subseteq C\), if \(\lor X\) exists, then 
   \[\alpha(\lor X) = \bigcup \{ \alpha(x) \mid x \in X \} .\]

2. For all \(X \subseteq A\), if \(\land X\) exists, then 
   \[\gamma(\land X) = \bigwedge \{ \gamma(x) \mid x \in X \} .\]

Proof: of 1

By definition of lubs, \(\forall x \in X, x \leq \lor X\).
By monotony, \(\forall x \in X, \alpha(x) \subseteq \alpha(\lor X)\).
Hence, \(\alpha(\lor X)\) is an upper bound of \(\{ \alpha(x) \mid x \in X \}\).

Assume that \(y\) is another upper bound of \(\{ \alpha(x) \mid x \in X \}\).
Then, \(\forall x \in X, \alpha(x) \subseteq y\).
By Galois connection \(\forall x \in X, x \leq \gamma(y)\).
By definition of lubs, \(\lor X \leq \gamma(y)\).
By Galois connection, \(\alpha(\lor X) \subseteq y\).
Hence, \(\{ \alpha(x) \mid x \in X \}\) has a lub, which equals \(\alpha(\lor X)\).

The proof of 2 is similar (by duality).
Deriving Galois connections

Given \((C, \leq) \leftrightarrow_{\alpha} (A, \sqsubseteq)\), we have:

- **duality**: \((A, \sqsubseteq) \leftrightarrow_{\gamma} (C, \geq)\)
  
  \[(\alpha(c) \sqsubseteq a \iff c \leq \gamma(a) \text{ is exactly } \gamma(a) \geq c \iff a \sqsupseteq \alpha(c))\]

- **point-wise lifting** by some set \(S\):
  
  \[(S \to C, \leq) \leftrightarrow_{\hat{\gamma}} (S \to A, \sqsubseteq)\]  
  
  where
  
  \[f \leq f' \iff \forall s, f(s) \leq f'(s), \quad (\hat{\gamma}(f))(s) = \gamma(f(s)),\]
  
  \[f \sqsubseteq f' \iff \forall s, f(s) \sqsubseteq f'(s), \quad (\hat{\alpha}(f))(s) = \alpha(f(s)).\]

Given \((X_1, \sqsubseteq_1) \leftrightarrow_{\alpha_1} (X_2, \sqsubseteq_2) \leftrightarrow_{\gamma_2} (X_3, \sqsubseteq_3)\):

- **composition**: \((X_1, \sqsubseteq_1) \leftrightarrow_{\gamma_1 \circ \gamma_2} (X_3, \sqsubseteq_3)\)
  
  \[((\alpha_2 \circ \alpha_1)(c) \sqsubseteq_3 a \iff \alpha_1(c) \sqsubseteq_2 \gamma_2(a) \iff c \sqsubseteq_1 (\gamma_1 \circ \gamma_2)(a))\]
Galois connections

Galois connection example

Abstract domain of intervals of integers $\mathbb{Z}$ represented as pairs of bounds $(a, b)$.

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma} (I, \sqsubseteq)$

- $I \overset{\text{def}}{=} (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$
- $(a, b) \sqsubseteq (a', b') \iff a \geq a' \land b \leq b'$
- $\gamma(a, b) \overset{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X)$

proof:
Galois connections

Galois connection example

Abstract domain of intervals of integers $\mathbb{Z}$ represented as pairs of bounds $(a, b)$.

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \xleftarrow{\gamma} (I, \sqsubseteq) \xrightarrow{\alpha} (\mathcal{P}(\mathbb{Z}), \subseteq)$

- $I \overset{\text{def}}{=} (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$
- $(a, b) \subseteq (a', b') \iff a \geq a' \land b \leq b'$
- $\gamma(a, b) \overset{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X)$

proof:

\[
\alpha(X) \subseteq (a, b) \\
\iff \min X \geq a \land \max X \leq b \\
\iff \forall x \in X : a \leq x \leq b \\
\iff \forall x \in X : x \in \{y \mid a \leq y \leq b\} \\
\iff \forall x \in X : x \in \gamma(a, b) \\
\iff X \subseteq \gamma(a, b)
\]
Galois embeddings

If \((C, \leq) \xrightarrow{\gamma}_\alpha (A, \sqsubseteq)\), the following properties are equivalent:

1. \(\alpha\) is surjective \(\quad (\forall a \in A, \exists c \in C, \alpha(c) = a)\)
2. \(\gamma\) is injective \(\quad (\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')\)
3. \(\alpha \circ \gamma = id\) \(\quad (\forall a \in A, id(a) = a)\)

Such \((\alpha, \gamma)\) is called a **Galois embedding**, which is noted \((C, \leq) \xleftarrow{\gamma}_\alpha (A, \sqsubseteq)\)

**Proof:**
Galois embeddings

If \((C, \leq) \xleftarrow{\gamma} \alpha \rightarrow (A, \subseteq)\), the following properties are equivalent:

1. \(\alpha\) is surjective \((\forall a \in A, \exists c \in C, \alpha(c) = a)\)
2. \(\gamma\) is injective \((\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')\)
3. \(\alpha \circ \gamma = id\) \((\forall a \in A, id(a) = a)\)

Such \((\alpha, \gamma)\) is called a **Galois embedding**, which is noted \((C, \leq) \xleftarrow{\gamma} \alpha \rightarrow (A, \subseteq)\)

**Proof:** 1  \(\implies\) 2
Assume that \(\gamma(a) = \gamma(a')\).
By surjectivity, take \(c, c'\) such that \(a = \alpha(c), a' = \alpha(c')\).
Then \(\gamma(\alpha(c)) = \gamma(\alpha(c'))\).
And \(\alpha(\gamma(\alpha(c))) = \alpha(\gamma(\alpha(c')))\).
As \(\alpha \circ \gamma \circ \alpha = \alpha, \alpha(c) = \alpha(c')\).
Hence \(a = a'\).
Galois embeddings

If \((C, \leq) \xleftarrow{\gamma} \xrightarrow{\alpha} (A, \sqsubseteq)\), the following properties are equivalent:

1. \(\alpha\) is surjective
   \((\forall a \in A, \exists c \in C, \alpha(c) = a)\)

2. \(\gamma\) is injective
   \((\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')\)

3. \(\alpha \circ \gamma = id\)
   \((\forall a \in A, id(a) = a)\)

Such \((\alpha, \gamma)\) is called a **Galois embedding**, which is noted
\((C, \leq) \xleftarrow{\gamma} \xrightarrow{\alpha} (A, \sqsubseteq)\)

**Proof:** \(2 \implies 3\)

Given \(a \in A\), we know that \(\gamma(\alpha(\gamma(a))) = \gamma(a)\).
By injectivity of \(\gamma\), \(\alpha(\gamma(a)) = a\).
Galois connections

Galois embeddings

If \((C, \leq) \xrightarrow{\gamma} (A, \sqsubseteq)\), the following properties are equivalent:

1. \(\alpha\) is surjective
   \[(\forall a \in A, \exists c \in C, \alpha(c) = a)\]
2. \(\gamma\) is injective
   \[(\forall a, a' \in A, \gamma(a) = \gamma(a') \implies a = a')\]
3. \(\alpha \circ \gamma = id\)
   \[(\forall a \in A, id(a) = a)\]

Such \((\alpha, \gamma)\) is called a \textbf{Galois embedding}, which is noted
\[(C, \leq) \xleftarrow{\gamma} \xrightarrow{\alpha} (A, \sqsubseteq)\]

Proof: 3 \implies 1

Given \(a \in A\), we have \(\alpha(\gamma(a)) = a\).
Hence, \(\exists c \in C, \alpha(c) = a\), using \(c = \gamma(a)\).
A Galois connection can be made into an embedding by quotienting $A$ by the equivalence relation $a \equiv a' \iff \gamma(a) = \gamma(a')$. 
Galois connections

Galois embedding example

Abstract domain of intervals of integers $\mathbb{Z}$ represented as pairs of ordered bounds $(a, b)$ or ⊥.

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma} (I, \sqsubseteq)$

- $I \overset{\text{def}}{=} \{(a, b) \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b\} \cup \{\bot\}$
- $(a, b) \sqsubseteq (a', b') \iff a \geq a' \land b \leq b', \quad \forall x: \bot \subseteq x$
- $\gamma(a, b) \overset{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}, \quad \gamma(\bot) = \emptyset$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X), \text{ or } \bot \text{ if } X = \emptyset$

proof:
Abstract domain of **intervals of integers $\mathbb{Z}$** represented as **pairs of ordered bounds $(a, b)$ or $\perp$**.

We have: $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma} (I, \sqsubseteq)$

- $I \overset{\text{def}}{=} \{(a, b) \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b\} \cup \{\perp\}$
- $(a, b) \sqsubseteq (a', b') \iff a \geq a' \land b \leq b', \forall x: \perp \sqsubseteq x$
- $\gamma(a, b) \overset{\text{def}}{=} \{x \in \mathbb{Z} \mid a \leq x \leq b\}, \quad \gamma(\perp) = \emptyset$
- $\alpha(X) \overset{\text{def}}{=} (\min X, \max X), \text{ or } \perp \text{ if } X = \emptyset$

**proof:**

Quotient of the “pair of bounds” domain $(\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\})$ by the relation $(a, b) \equiv (a', b') \iff \gamma(a, b) = \gamma(a', b')$

i.e., $(a \leq b \land a = a' \land b = b') \lor (a > b \land a' > b')$. 
$\rho : X \to X$ is an upper closure in the poset $(X, \sqsubseteq)$ if it is:

1. **monotonic**: $x \sqsubseteq x' \implies \rho(x) \sqsubseteq \rho(x')$,
2. **extensive**: $x \sqsubseteq \rho(x)$, and
3. **idempotent**: $\rho \circ \rho = \rho$.
Given \((C, \leq) \xleftarrow{\gamma} (A, \sqsubseteq)\),
\(\gamma \circ \alpha\) is an upper closure on \((C, \leq)\).

Given an upper closure \(\rho\) on \((X, \sqsubseteq)\), we have a Galois embedding:
\((X, \sqsubseteq) \xleftarrow{id} (\rho(X), \sqsubseteq)\)

\(\implies\) we can rephrase abstract interpretation using upper closures instead of Galois connections, but we lose:

- the notion of **abstract representation**
  (a data-structure \(A\) representing elements in \(\rho(X)\))
- the ability to have **several distinct** abstract representations for a single concrete object
  (non-necessarily injective \(\gamma\) versus \(id\))
Operator approximations
Abstractions in the concretization framework

Given a concrete \((C, \leq)\) and an abstract \((A, \sqsubseteq)\) poset and a monotonic concretization \(\gamma : A \rightarrow C\)

\((\gamma(a)\) is the “meaning” of \(a\) in \(C\); we use intervals in our examples\)

- \(a \in A\) is a sound abstraction of \(c \in C\) if \(c \leq \gamma(a)\).
  (e.g.: \([0, 10]\) is a sound abstraction of \(\{0, 1, 2, 5\}\) in the integer interval domain)

- \(g : A \rightarrow A\) is a sound abstraction of \(f : C \rightarrow C\) if \(\forall a \in A: (f \circ \gamma)(a) \leq (\gamma \circ g)(a)\).
  (e.g.: \(\lambda([a, b].[−\infty, +\infty]\) is a sound abstraction of \(\lambda X.\{x + 1 | x \in X\}\) in the interval domain)

- \(g : A \rightarrow A\) is an exact abstraction of \(f : C \rightarrow C\) if \(f \circ \gamma = \gamma \circ g\).
  (e.g.: \(\lambda([a, b].[a + 1, b + 1]\) is an exact abstraction of \(\lambda X.\{x + 1 | x \in X\}\) in the interval domain)
Assume now that \((C, \leq) \xrightarrow{\gamma} (A, \sqsubseteq)\).

- **sound abstractions**
  - \(c \leq \gamma(a)\) is equivalent to \(\alpha(c) \sqsubseteq a\).
  - \((f \circ \gamma)(a) \leq (\gamma \circ g)(a)\) is equivalent to \((\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a)\).

- Given \(c \in C\), its **best abstraction** is \(\alpha(c)\).
  
    (proof: recall that \(\alpha(c) = \sqcap \{ a \mid c \leq \gamma(a) \}\), so, \(\alpha(c)\) is the smallest sound abstraction of \(c\))

    (e.g.: \(\alpha(\{0, 1, 2, 5\}) = [0, 5]\) in the interval domain)

- Given \(f : C \rightarrow C\), its **best abstraction** is \(\alpha \circ f \circ \gamma\)

  (proof: \(g\) sound \(\iff\) \(\forall a, (\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a),\) so \(\alpha \circ f \circ \gamma\) is the smallest sound abstraction of \(f\))

  (e.g.: \(g([a, b]) = [2a, 2b]\) is the best abstraction in the interval domain of \(f(X) = \{ 2x \mid x \in X \}\); it is not an exact abstraction as \(\gamma(g([0, 1])) = \{0, 1, 2\} \not\subset \{0, 2\} = f(\gamma([0, 1]))\)
If $g$ and $g'$ soundly abstract respectively $f$ and $f'$ then:

- if $f$ is monotonic, then $g \circ g'$ is a sound abstraction of $f \circ f'$,
  
  \[
  \text{(proof: } \forall a, (f \circ f' \circ \gamma)(a) \leq (f \circ \gamma \circ g')(a) \leq (\gamma \circ g \circ g')(a))
  \]

- if $g$, $g'$ are exact abstractions of $f$ and $f'$, then $g \circ g'$ is an exact abstraction,
  
  \[
  \text{(proof: } f \circ f' \circ \gamma = f \circ \gamma \circ g' = \gamma \circ g \circ g')
  \]

- if $g$ and $g'$ are the best abstractions of $f$ and $f'$, then $g \circ g'$ is not always the best abstraction!
  
  (e.g.: $g([a, b]) = [a, \min(b, 1)]$ and $g'([a, b]) = [2a, 2b]$ are the best abstractions of $f(X) = \{ x \in X \mid x \leq 1 \}$ and $f'(X) = \{ 2x \mid x \in X \}$ in the interval domain, but $g \circ g'$ is not the best abstraction of $f \circ f'$ as $(g \circ g')([0, 1]) = [0, 1]$ while $(\alpha \circ f \circ f' \circ \gamma)([0, 1]) = [0, 0]$)
Fixpoint approximations
Fixpoint transfer

If we have:

- a Galois connection \((C, \leq) \xleftrightarrow{\gamma}{\alpha} (A, \sqsubseteq)\) between CPOs
- monotonic concrete and abstract functions \(f : C \to C, f^\# : A \to A\)
- a commutation condition \(\alpha \circ f = f^\# \circ \alpha\)
- an element \(a\) and its abstraction \(a^\# = \alpha(a)\)

then \(\alpha(\text{lfp}_a f) = \text{lfp}_{a^\#} f^\#\).

(proof on next slide)
Fixpoint approximations

Fixpoint transfer (proof)

Proof:

By the constructive Tarski theorem, $\text{lfp}_a f$ is the limit of transfinite iterations:

$a_0 \overset{\text{def}}{=} a$, $a_{n+1} \overset{\text{def}}{=} f(a_n)$, and $a_n \overset{\text{def}}{=} \bigvee \{ a_m \mid m < n \}$ for limit ordinals $n$.

Likewise, $\text{lfp}_{a^\#} f^\#$ is the limit of a transfinite iteration $a_n^\#$.

We prove by transfinite induction that $a_n^\# = \alpha(a_n)$ for all ordinals $n$:

- $a_0^\# = \alpha(a_0)$, by definition;
- $a_{n+1}^\# = f^\#(a_n^\#) = f^\#(\alpha(a_n)) = \alpha(f(a_n)) = \alpha(a_{n+1})$ for successor ordinals, by commutation;
- $a_n^\# = \bigsqcup \{ a_m^\# \mid m < n \} = \bigsqcup \{ \alpha(a_m) \mid m < n \} = \alpha(\bigvee \{ a_m \mid m < n \}) = \alpha(a_n)$ for limit ordinals, because $\alpha$ is always continuous in Galois connections.

Hence, $\text{lfp}_{a^\#} f^\# = \alpha(\text{lfp}_a f)$. 
Fixpoint approximations

Fixpoint approximation

If we have:

- a complete lattice \((C, \leq, \lor, \land, \bot, \top)\)
- a monotonic concrete function \(f\)
- a sound abstraction \(f^\# : A \rightarrow A\) of \(f\)

\[(\forall x^\#: (f \circ \gamma)(x^\#) \leq (\gamma \circ f^\#)(x^\#))\]

- a post-fixpoint \(a^\#\) of \(f^\#\) \((f^\#(a^\#) \sqsubseteq a^\#)\)

then \(a^\#\) is a sound abstraction of \(\text{lfp } f\): \(\text{lfp } f \leq \gamma(a^\#)\).

Proof:
By definition, \(f^\#(a^\#) \sqsubseteq a^\#\).
By monotony, \(\gamma(f^\#(a^\#)) \leq \gamma(a^\#)\).
By soundness, \(f(\gamma(a^\#)) \leq \gamma(a^\#)\).
By Tarski's theorem \(\text{lfp } f = \land \{x \mid f(x) \leq x\}\).
Hence, \(\text{lfp } f \leq \gamma(a^\#)\).

Other fixpoint transfer / approximation theorems can be constructed...


