Partitioning abstractions

MPRI — Cours 2.6 “Interprétation abstraite : application à la vérification et à l’analyse statique”

Xavier Rival

INRIA, ENS, CNRS

Dec, 11th. 2018
Towards disjunctive abstractions

Extending the expressiveness of abstract domains

- **disjunctions** are **often needed**...
- ... but **potentially costly**

In this lecture, we will discuss:

- **precision issues** that motivate the use of abstract domains able to **express disjunctions**
- **several techniques** to **express disjunctive properties** using **abstract domain combination methods** (construction of abstract domains from other abstract domains):
  - disjunctive completion
  - cardinal power
  - state partitioning
  - trace partitioning
Domain combinators (or combiners)

General combination of abstract domains

- takes one or more abstract domains as **inputs**
- produces a **new abstract domain**

Input and output abstract domains are **characterized by an “interface”**:
- concrete domain,
- abstraction relation,
- and abstract operations (post-conditions, widening...)

**Advantages:**
- **general definition**, formalized and proved once
- can be **implemented** in a separate way, e.g., in ML:
  - abstract domain: **module**
    
    ```
    module D = (struct ... end: I)
    ```
  - abstract domain combinator: **functor**
    
    ```
    module C = functor (D: I0) -> (struct ... end: I1)
    ```
Example: product abstraction

Set notations:
- $\mathbb{V}$: values
- $\mathbb{X}$: variables
- $\mathbb{M}$: stores
  $\mathbb{M} = \mathbb{X} \rightarrow \mathbb{V}$

Assumptions:
- concrete domain $(\mathcal{P}(\mathbb{M}), \subseteq)$ with $\mathbb{M} = \mathbb{X} \rightarrow \mathbb{V}$
- we assume an abstract domain $\mathbb{D}^\#$ that provides
  - concretization function $\gamma: \mathbb{D}^\# \rightarrow \mathcal{P}(\mathbb{M})$
  - element $\bot$ with empty concretization $\gamma(\bot) = \emptyset$

Product combinator (implemented as a functor)

Given abstract domains $(\mathbb{D}^\#_0, \gamma_0, \bot_0)$ and $(\mathbb{D}^\#_1, \gamma_1, \bot_1)$, the product abstraction is $(\mathbb{D}^\#_\times, \gamma_\times, \bot_\times)$ where:
- $\mathbb{D}^\#_\times = \mathbb{D}^\#_0 \times \mathbb{D}^\#_1$
- $\gamma_\times(x^\#_0, x^\#_1) = \gamma_0(x^\#_0) \cap \gamma_1(x^\#_1)$
- $\bot_\times = (\bot_0, \bot_1)$

This amounts to expressing conjunctions of elements of $\mathbb{D}^\#_0$ and $\mathbb{D}^\#_1$
Example: product abstraction, coalescent product

The product abstraction is not very precise and needs a reduction:

\[ \forall x_0 \in D_0, x_1 \in D_1, \gamma_x(\bot_0, x_1) = \gamma_x(x_0, \bot_1) = \emptyset = \gamma_x(\bot_\times) \]

Coalescent product

Given abstract domains \((D_0, \gamma_0, \bot_0)\) and \((D_1, \gamma_1, \bot_1)\), the coalescent product abstraction is \((D_\times, \gamma_\times, \bot_\times)\) where:

- \(D_\times = \{\bot_\times\} \cup \{(x_0, x_1) \in D_0 \times D_1 | x_0 \neq \bot_0 \land x_1 \neq \bot_1\}\)
- \(\gamma_\times(\bot_\times) = \emptyset, \gamma_\times(x_0, x_1) = \gamma_0(x_0) \cap \gamma_1(x_1)\)

In many cases, this is not enough to achieve reduction:

- let \(D_0\) be the interval abstraction, \(D_1\) be the congruences abstraction
- \(\gamma_\times(\{x \in [3, 4]\}, \{x \equiv 0 \mod 5\}) = \emptyset\)

- how to define abstract domain combinators to add disjunctions?
Outline

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2 Imprecisions in convex abstractions

3 Disjunctive completion

4 Cardinal power and partitioning abstractions

5 State partitioning

6 Trace partitioning

7 Conclusion
Imprecisions in convex abstractions

Convex abstractions

Many numerical abstractions describe convex sets of points

Imprecisions inherent in the convexity, and when computing abstract join (over-approximation of concrete union):

Such imprecisions may make analyses fail

Similar issues also arise in non-numerical static analyses
Non convex abstractions

We consider abstractions of $\mathbb{D} = \mathcal{P}(\mathbb{Z})$

**Congruences:**

- $\mathbb{D}^\# = \mathbb{Z} \times \mathbb{N}$
- $\gamma(n, k) = \{n + k \cdot p \mid p \in \mathbb{Z}\}$
- $-2 \in \gamma(1, 2)$ and $1 \in \gamma(1, 2)$ but $0 \notin \gamma(1, 2)$

**Signs:**

- $0 \notin \gamma([\neq 0])$ so $[\neq 0]$ describes a non convex set
- other abstract elements describe convex sets
Example 1: verification problem

```c
bool b0, b1;
int x, y;  // (uninitialized)
b0 = x ≥ 0;
b1 = x ≤ 0;
if(b0 && b1){
    y = 0;
} else {
    y = 100/x;
}
```

- if \(\neg b_0\), then \(x < 0\)
- if \(\neg b_1\), then \(x > 0\)
- if either \(b_0\) or \(b_1\) is false, then \(x \neq 0\)
- thus, if point ① is reached the division is safe

How to verify the division operation?

- Non relational abstraction (e.g., intervals), at point ①:
  \[
  \{ \begin{array}{l}
    b_0 \in \{\text{FALSE, TRUE}\} \land b_1 \in \{\text{FALSE, TRUE}\} \\
    x : \top
  \end{array} \]

- Signs, congruences do not help:
  in the concrete, \(x\) may take any value but 0
Example 1: program annotated with local invariants

```c
bool b0, b1;
int x, y;  // (uninitialized)
b0 = x >= 0;
    (b0 ∧ x ≥ 0) ∨ (¬b0 ∧ x < 0)
b1 = x <= 0;
    (b0 ∧ b1 ∧ x = 0) ∨ (b0 ∧ ¬b1 ∧ x > 0) ∨ (¬b0 ∧ b1 ∧ x < 0)
if(b0 && b1){
    (b0 ∧ b1 ∧ x = 0)
y = 0;
    (b0 ∧ b1 ∧ x = 0 ∧ y = 0)
} else {
    (b0 ∧ ¬b1 ∧ x > 0) ∨ (¬b0 ∧ b1 ∧ x < 0)
y = 100/x;
    (b0 ∧ ¬b1 ∧ x > 0) ∨ (¬b0 ∧ b1 ∧ x < 0)
}
```

The obvious way to sucessfully analyzing this program consists in adding symbolic disjunctions to our abstract domain
Example 2: verification problem

```c
int x ∈ ℤ;
int s;
int y;
if(x ≥ 0){
    s = 1;
} else {
    s = -1;
}
① y = x/s;
② assert(y ≥ 0);
```

- s is either 1 or -1
- thus, the division at ① should not fail
- moreover s has the same sign as x
- thus, the value stored in y should always be positive at ②

**How to verify the division operation?**

- In the concrete, s is always non null: convex abstractions cannot establish this; congruences can
- Moreover, s has always the same sign as x
  expressing this would require a non trivial numerical abstraction
Example 2: program annotated with local invariants

```c
int x \in \mathbb{Z};
int s;
int y;
if(x \geq 0) {
  (x \geq 0)
  s = 1;
  (x \geq 0 \land s = 1)
} else {
  (x < 0)
  s = -1;
  (x < 0 \land s = -1)
}
(x \geq 0 \land s = 1) \lor (x < 0 \land s = -1)

y = x/s;
(x \geq 0 \land s = 1 \land y \geq 0) \lor (x < 0 \land s = -1 \land y > 0)

assert(y \geq 0);
```

Again, the obvious solution consists in adding disjunctions to our abstract domain.
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1. Introduction
2. Imprecisions in convex abstractions
3. Disjunctive completion
4. Cardinal power and partitioning abstractions
5. State partitioning
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Disjunctive completion

Distributive abstract domain

**Principle:**
1. consider concrete domain \((\mathbb{D}, \subseteq)\), with least upper bound operator \(\sqcup\)
2. assume an abstract domain \((\mathbb{D}^\#, \subseteq^\#)\) with concretization \(\gamma : \mathbb{D}^\# \rightarrow \mathbb{D}\)
3. build a domain containing all the disjunctions of elements of \(\mathbb{D}^\#\)

**Definition: distributive abstract domain**

Abstract domain \((\mathbb{D}^\#, \subseteq^\#)\) with concretization function \(\gamma : \mathbb{D}^\# \rightarrow \mathbb{D}\) is **distributive** (or **disjunctive**, or **complete for disjunction**) if and only if:

\[\forall \mathcal{E} \subseteq \mathbb{D}^\#, \exists x^\# \in \mathbb{D}^\#, \gamma(x^\#) = \bigcup_{y^\# \in \mathcal{E}} \gamma(y^\#)\]

**Examples:**
- the lattice \(\{\bot, < 0, = 0, > 0, \leq 0, \neq 0, \geq 0, \top\}\) is distributive
- the lattice of intervals is not distributive: there is no interval with concretization \(\gamma([0, 10]) \cup \gamma([12, 20])\)
Definition: disjunctive completion

The disjunctive completion of abstract domain \((\mathbb{D}^\#, \sqsubseteq^\#)\) with concretization function \(\gamma : \mathbb{D}^\# \rightarrow \mathbb{D}\) is the smallest abstract domain \((\mathbb{D}_{\text{disj}}^\#, \sqsubseteq_{\text{disj}}^\#)\) with concretization function \(\gamma_{\text{disj}} : \mathbb{D}_{\text{disj}}^\# \rightarrow \mathbb{D}\) such that:

1. \(\mathbb{D}^\# \subseteq \mathbb{D}_{\text{disj}}^\#\)
2. \(\forall x^\# \in \mathbb{D}^\#, \gamma_{\text{disj}}(x^\#) = \gamma(x^\#)\)
3. \((\mathbb{D}_{\text{disj}}^\#, \sqsubseteq_{\text{disj}}^\#)\) with concretization \(\gamma_{\text{disj}}\) is distributive

Building a disjunctive completion domain:

1. start with \(\mathbb{D}_{\text{disj}}^\# = \mathbb{D}^\#\)
2. for all set \(\mathcal{E} \subseteq \mathbb{D}^\#\) such that there is no \(x^\# \in \mathbb{D}^\#,\) such that \(\gamma(x^\#) = \bigcup_{y^\# \in \mathcal{E}} \gamma(y^\#)\), add \([\sqcup \mathcal{E}]\) to \(\mathbb{D}_{\text{disj}}^\#\), and extend \(\gamma_{\text{disj}}\) by

\[
\gamma_{\text{disj}}([\sqcup \mathcal{E}]) = \bigcup_{y^\# \in \mathcal{E}} \gamma(y^\#)
\]

Theorem: this process constructs a disjunctive abstraction
Example 1: completion of signs

We consider **concrete lattice** $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $\subseteq \subseteq$ and $(\mathbb{D}^\#, \subseteq^\#)$ defined by:

\[
\begin{align*}
\top & \quad \downarrow \quad \downarrow \\
[-] & \quad [0] & \quad [+] \\
\downarrow & \quad \downarrow \quad \downarrow \\
\bot
\end{align*}
\]

\[
\gamma : \begin{array}{c}
\bot \\
[< 0] \\
[= 0] \\
[> 0] \\
\top
\end{array} \quad \mapsto \begin{array}{c}
\emptyset \\
\{ k \in \mathbb{Z} \mid k < 0 \} \\
\{ k \in \mathbb{Z} \mid k = 0 \} \\
\{ k \in \mathbb{Z} \mid k > 0 \} \\
\mathbb{Z}
\end{array}
\]

Then, the disjunctive completion is defined by adding elements corresponding to:

- $\cup\{[-], [0]\}$
- $\cup\{[-], [+]\}$
- $\cup\{[0], [+]\}$
Example 2: completion of constants

We consider **concrete lattice** \( \mathbb{D} = \mathcal{P}(\mathbb{Z}) \), with \( \sqsubseteq = \subseteq \) and \( (\mathbb{D}^\#, \sqsubseteq^\#) \) defined by:

\[
\begin{align*}
\vdots & \quad \{-2\} & \quad \{-1\} & \quad \{0\} & \quad \{1\} & \quad \{2\} & \quad \vdots \\
\bot & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow & \quad \bot
\end{align*}
\]

\[
\gamma : \quad \bot & \quad \longmapsto \quad \emptyset \\
\{k\} & \quad \longmapsto \quad \{k\} \\
\top & \quad \longmapsto \quad \mathbb{Z}
\]

Then, the disjunctive completion coincides with the **power-set**:

- \( \mathbb{D}^\#_{\text{disj}} \equiv \mathcal{P}(\mathbb{Z}) \)
- **this abstraction loses no information**: \( \gamma_{\text{disj}} \) is the **identity function**!
- obviously, this lattice contains **infinite sets which are not representable**

**Middle ground solution**: \( k \)-**bounded disjunctive completion**

- only add disjunctions of **at most** \( k \) **elements**
- \( e.g. \), if \( k = 2 \), pairs are represented precisely, other sets abstracted to \( \top \)
Example 3: completion of intervals

We consider concrete lattice $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $\subseteq \subseteq$ and let $(\mathbb{D}^\#, \sqsubseteq^\#)$ the domain of intervals

- $\mathbb{D}^\# = \{\bot, \top\} \uplus \{[a, b] \mid a \leq b\}$
- $\gamma([a, b]) = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$

Then, the disjunctive completion is the set of unions of intervals :

- $\mathbb{D}^\#_{\text{disj}}$ collects all the families of disjoint intervals
- this lattice contains infinite sets which are not representable
- as expressive as the completion of constants, but more efficient representation

The disjunctive completion of $(\mathbb{D}^\#)^n$ is not equivalent to $(\mathbb{D}^\#_{\text{disj}})^n$

- which is more expressive ?
- show it on an example !
We use the disjunctive completion of $(\mathbb{D}^\#)^3$. The invariants below can be expressed in the disjunctive completion:

```plaintext
int x ∈ \mathbb{Z};
int s;
in y;
if(x ≥ 0){
    (x ≥ 0)
    s = 1;
    (x ≥ 0 ∧ s = 1)
} else {
    (x < 0)
    s = -1;
    (x < 0 ∧ s = -1)
}
(x ≥ 0 ∧ s = 1) ∨ (x < 0 ∧ s = -1)
y = x/s;
(x ≥ 0 ∧ s = 1 ∧ y ≥ 0) ∨ (x < 0 ∧ s = -1 ∧ y > 0)
assert(y ≥ 0);
```
To carry out the analysis of a basic imperative language, we will define:

- **Operations for the computation of post-conditions:**
  - sound over-approximation for basic program steps
  - concrete $\text{post} : \mathcal{P}(\mathcal{S}) \rightarrow \mathcal{P}(\mathcal{S})$ (where $\mathcal{S}$ is the set of states);
  - the abstract $\text{post}^\# : \mathcal{D}^\# \rightarrow \mathcal{D}^\#$ should be such that
    \[ \text{post} \circ \gamma \subseteq \gamma \circ \text{post}^\# \]
  - case where $\text{post}$ is an assignment: $\text{post}^\# = \text{assign}$
    inputs a variable, an expression, an abstract pre-condition, outputs an abstract post-condition
  - case where $\text{post}$ is a condition test: $\text{post}^\# = \text{test}$ inputs a boolean expression, an abstract pre-condition, outputs an abstract post-condition

- An operator $\text{join}$ for **over-approximation of for concrete unions**

- A **widening operator** $\triangledown$ for the analysis of loops

- A **conservative inclusion checking operator**
Disjunctive completion

Static analysis with disjunctive completion

**Transfer functions** for the computation of **abstract post-conditions**:

- we assume a monotone concrete post-condition operation \( \text{post} : \mathbb{D} \rightarrow \mathbb{D} \), and an abstract \( \text{post}^\# : \mathbb{D}^\# \rightarrow \mathbb{D}^\# \) such that \( \text{post} \circ \gamma \subseteq \gamma \circ \text{post}^\# \)

- convention: if \( \gamma(y^\#) = \bigsqcup \{ \gamma(z^\#) \mid z^\# \in \mathcal{E} \} \), we note \( y^\# = [\bigsqcup \mathcal{E}] \)

- then, we can simply use, for the **disjunctive completion domain**:

  \[
  \text{post}^\#_{\text{disj}}([\bigsqcup \mathcal{E}]) = [\bigsqcup \{\text{post}^\#(x^\#) \mid x^\# \in \mathcal{E}\}]
  \]

  (note it may be an element of the initial domain)

- the proof is left as **exercise**

- this works for assignment, condition tests...

**Abstract join**:

- disjunctive completion provides **an exact join** (exercise !)

**Inclusion check**: exercise !

**Widening**: **no general definition**
Limitations of disjunctive completion

**Combinatorial explosion:**
- if $D$ is infinite, $D_{\text{disj}}$ may have elements that cannot be represented e.g., completion of constants or intervals
- even when $D$ is finite, $D_{\text{disj}}$ may be huge
  - in the worst case, if $D$ has $n$ elements, $D_{\text{disj}}$ may have $2^n$ elements

Many elements useless in practice:
disjunctive completion of intervals: may express any set of integers...

**No general definition of a widening operator**
- most common approach to achieve that: $k$-limiting
  - bound the numbers of disjuncts
    - i.e., the size of the sets added to the base domain
- remaining issue: the join operator should “select” which disjoints to merge
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**Principle**

**Observation**

Disjuncts that are required for static analysis can usually be characterized by some semantic property.

**Examples:** each disjunct is characterized by
- the sign of a variable
- the value of a boolean variable
- the execution path, e.g., side of a condition that was visited

**Solution:** perform a kind of indexing of disjuncts

1. introduce a new abstraction to describe labels, e.g., the sign of a variable, the value of a boolean, or another trace property...
2. apply the store abstraction (or another abstraction) to the set of states associated to each label
Disjuncts indexing: example

```c
int x ∈ ℤ;
int s;
int y;
if(x ≥ 0){
    (x ≥ 0)
    s = 1;
    (x ≥ 0 ∧ s = 1)
} else {
    (x < 0)
    s = -1;
    (x < 0 ∧ s = -1)
}
(x ≥ 0 ∧ s = 1) ∨ (x < 0 ∧ s = -1)
y = x/s;
(x ≥ 0 ∧ s = 1 ∧ y ≥ 0) ∨ (x < 0 ∧ s = -1 ∧ y > 0)
assert(y ≥ 0);
```

- natural “indexing”: **sign of** `x`
- but we could also rely on the **sign of** `s`
Cardinal power abstraction

We assume \((\mathbb{D}, \subseteq) = (\mathcal{P}(\mathcal{E}), \subseteq)\), and two abstractions \((\mathbb{D}_0, \subseteq_0), (\mathbb{D}_1, \subseteq_1)\) given by their concretization functions:

\[
\gamma_0 : \mathbb{D}_0 \longrightarrow \mathbb{D} \quad \gamma_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}
\]

**Definition**

We let the **cardinal power abstract domain** be defined by:

- \(\mathbb{D}^{\#}_{cp} = \mathbb{D}_0 \overset{\mathcal{M}}{\longrightarrow} \mathbb{D}_1\) be the set of monotone functions from \(\mathbb{D}_0\) into \(\mathbb{D}_1\)
- \(\subseteq^{\#}_{cp}\) be the pointwise extension of \(\subseteq_1\)
- \(\gamma^{\#}_{cp}\) is defined by:

\[
\gamma^{\#}_{cp} : \mathbb{D}^{\#}_{cp} \longrightarrow \mathbb{D} \\
X^{\#} \longmapsto \{y \in \mathcal{E} \mid \forall z^{\#} \in \mathbb{D}_0, y \in \gamma_0(z^{\#}) \implies y \in \gamma_1(X^{\#}(z^{\#}))\}
\]

We sometimes denote it by \(\mathbb{D}^{\#}_0 \Rightarrow \mathbb{D}^{\#}_1, \gamma^{\#}_{\mathbb{D}^{\#}_0 \Rightarrow \mathbb{D}^{\#}_1}\) to make it more explicit.
Use of cardinal power abstractions

Intuition: cardinal power expresses properties of the form

\[
\begin{align*}
  p_0 & \implies p'_0 \\
  \land p_1 & \implies p'_1 \\
  \vdots & \implies \vdots \\
  \land p_n & \implies p'_n
\end{align*}
\]

Two independent choices:
1. \( \mathcal{D}_0 \): set of partitions (the “labels”), represents \( p_0, \ldots, p_n \)
2. \( \mathcal{D}_1 \): abstraction of sets of states, e.g., a numerical abstraction, represents \( p'_0, \ldots, p'_n \)

Application \((x \geq 0 \land s = 1 \land y \geq 0) \lor (x < 0 \land s = -1 \land y > 0)\)
- \( \mathcal{D}_0 \): sign of \( s \)
- \( \mathcal{D}_1 \): other constraints
- we get: \( s > 0 \implies (x \geq 0 \land s = 1 \land y \geq 0) \land s \leq 0 \implies (\ldots) \)
Another example, with a single variable

**Assumptions:**

- concrete lattice $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $(\subseteq) = (\subseteq)$
- $(\mathbb{D}_0, \sqsubseteq_0)$ be the **lattice of signs**
  (strict inequalities only)
- $(\mathbb{D}_1, \sqsubseteq_1)$ be the **lattice of intervals**

**Example abstract values:**

- $[0, 8]$ is expressed by:
  $$\begin{align*}
  \bot &\mapsto \bot_1 \\
  [-] &\mapsto \bot_1 \\
  [0] &\mapsto [0, 0] \\
  [+ ] &\mapsto [1, 8] \\
  \top &\mapsto [0, 8]
  \end{align*}$$

- $[-10, -3] \cup [7, 10]$ is expressed by:
  $$\begin{align*}
  \bot &\mapsto \bot_1 \\
  [-] &\mapsto [-10, -3] \\
  [0] &\mapsto \bot_1 \\
  [+ ] &\mapsto [7, 10] \\
  \top &\mapsto [-10, 10]
  \end{align*}$$
Cardinal power: why monotone functions?

We have seen the reduced cardinal power intuitively denotes a conjunction of implications, thus, assuming that $D_0$ has two comparable elements $p_0, p_1$ and:

$$\begin{align*}
    p_0 &\iff p_0' \\
    \land p_1 &\iff p_1'
\end{align*}$$

Then:

- $p_0, p_1$ are comparable, so let us fix $p_0 \subseteq_0 p_1$
- logically, this means $p_0 \implies p_1$
- thus the abstract element represents states where $p_0 \implies p_1 \implies p_1'$
- as a conclusion, if $p_0'$ is not as strong as $p_1'$, it is possible to reinforce it!
- new abstract state:

$$\begin{align*}
    p_0 &\implies p_0 \land p_1' \\
    \land p_1 &\implies p_1'
\end{align*}$$

This is a reduction operation.

Non monotone functions can be reduced into monotone functions.
Example reduction (1): relation between the two domains

- **concrete lattice** $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $\subseteq \subseteq$
- $(\mathbb{D}_0^\#, \subseteq_0^\#)$ be the **lattice of signs**
- $(\mathbb{D}_1^\#, \subseteq_1^\#)$ be the **lattice of intervals**

We let:

$$X^\# = \begin{cases} \bot & \mapsto \bot_1 \\ [-] & \mapsto [1, 8] \\ [0] & \mapsto [1, 8] \\ [+ ] & \mapsto \bot_1 \\ T & \mapsto [1, 8] \end{cases}$$

$$Y^\# = \begin{cases} \bot & \mapsto \bot_1 \\ [-] & \mapsto [2, 45] \\ [0] & \mapsto [-5, -2] \\ [+ ] & \mapsto [-5, -2] \\ T & \mapsto T_1 \end{cases}$$

$$Z^\# = \begin{cases} \bot & \mapsto \bot_1 \\ [-] & \mapsto \bot_1 \\ [0] & \mapsto \bot_1 \\ [+ ] & \mapsto \bot_1 \\ T & \mapsto \bot_1 \end{cases}$$

Then,

$$\gamma_{cp}(X^\#) = \gamma_{cp}(Y^\#) = \gamma_{cp}(Z^\#) = \emptyset$$
Example reduction (2): tightening relations

- concrete lattice $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $\subseteq \subseteq$
- $(\mathbb{D}_0^\#, \sqsubseteq_0^\#)$ be the lattice of signs
- $(\mathbb{D}_1^\#, \sqsubseteq_1^\#)$ be the lattice of intervals

We let:

$$X^\# = \begin{cases} \perp & \mapsto \perp_1 \\ [-] & \mapsto [-5, -1] \\ [0] & \mapsto [0, 0] \\ [+ ] & \mapsto [1, 5] \\ \top & \mapsto [-10, 10] \end{cases}$$

$$Y^\# = \begin{cases} \perp & \mapsto \perp_1 \\ [-] & \mapsto [-5, -1] \\ [0] & \mapsto [0, 0] \\ [+ ] & \mapsto [1, 5] \\ \top & \mapsto [-5, 5] \end{cases}$$

Then, $\gamma_{cp}(X^\#) = \gamma_{cp}(Y^\#)$

$\gamma_0([-]) \cup \gamma_0([0]) \cup \gamma([+]) = \gamma(\top)$

but

$$\gamma_0(X^\#([-])) \cup \gamma_0(X^\#([0])) \cup \gamma(X^\#([+])) \subset \gamma(X^\#(\top))$$

In fact, we can improve the image of $\top$ into $[-5, 5]$
Reduction, and improving precision in the cardinal power

In general, the cardinal power construction requires reduction

**Strengthening using both sides of ⇒**

Tightening of $y_0^\# \Rightarrow y_1^\#$ when:
- $\exists z_1^\# \neq y_1^\#$, $\gamma(y_1^\#) \cap \gamma(y_0^\#) \subseteq \gamma(z_1^\#)$

- in the example, $z_1^\# = \perp_1$...

**Strengthening of one relation using other relations**

Tightening of relation $(\cup\{z^\# | z^\# \in \mathcal{E}\}) \Rightarrow x_1^\#$ when:
- $\cup\{\gamma_0(z^\#) | z^\# \in \mathcal{E}\} = \gamma_0(\cup\{z^\# | z^\# \in \mathcal{E}\})$
- $\exists y^\#$, $\cup\{\gamma_1(X^\#(z^\#)) | z^\# \in \mathcal{E}\} \subseteq \gamma_1(y^\#) \subseteq \gamma_1(X^\#(\cup\{z^\# | z^\# \in \mathcal{E}\}))$

- in the example, we use a set of elements that cover $\top$...
Representation of the cardinal power

**Basic ML representation:**

- using **functions**, i.e. type \( \text{cp} = \text{d0} \rightarrow \text{d1} \)
  \( \Rightarrow \) usually a bad choice, as it makes it hard to operate in the \( \mathbb{D}_0 \) side
- using **some kind of dictionnaries** type \( \text{cp} = (\text{d0}, \text{d1}) \text{ map} \)
  \( \Rightarrow \) better, but not straightforward...

**Even the latter is not a very efficient representation:**

- if \( \mathbb{D}_0 \) has \( N \) elements, then an abstract value in \( \mathbb{D}_{\text{cp}} \) requires \( N \) elements of \( \mathbb{D}_1 \)
- if \( \mathbb{D}_0 \) is infinite, and \( \mathbb{D}_1 \) is non trivial, then \( \mathbb{D}_{\text{cp}} \) **has elements that cannot be represented**
- the 1st reduction shows it is **unnecessary to represent bindings for all elements of \( \mathbb{D}_0 \)**

example: this is the case of \( \bot_0 \)
More compact representation of the cardinal power

**Principle:**
- use a **dictionary data-type** (most likely functional arrays)
- avoid representing information attached to redundant elements

A compact representation should be just sufficient to “represent” all elements of $\mathbb{D}_0^\#$.

**Compact representation**

Reduced cardinal power of $\mathbb{D}_0^\#$ and $\mathbb{D}_1^\#$ can be represented by considering only a subset $C \subseteq \mathbb{D}_0^\#$ where

$$\forall x^\# \in \mathbb{D}_0^\#, \ \exists \mathcal{E} \subseteq C, \ \gamma_0(x^\#) = \bigcup \{\gamma_0(y^\#) \mid y^\# \in \mathcal{E}\}$$

In particular:
- if possible, $C$ should be **minimal**
- in any case, $\bot_0 \notin C$
- also, when $\top_0$ can be generated by a union of a set of elements, it can be removed
Example: compact cardinal power over signs

- concrete lattice $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $\subseteq = \subseteq$
- $(\mathbb{D}_0^\#, \sqsubseteq_0^\#)$ be the lattice of signs
- $(\mathbb{D}_1^\#, \sqsubseteq_1^\#)$ be the lattice of intervals

**Observations**

- $\bot$ does not need be considered (obvious right hand side: $\bot_1$)
- $\gamma_0([< 0]) \cup \gamma_0([\leq 0]) \cup \gamma([> 0]) = \gamma(\top)$ thus $\top$ does not need be considered

**Thus, we let** $C = \{[-], [0], [+]\}$

- $[0, 8]$ is expressed by: $\begin{cases} 
[-] & \mapsto \bot_1 \\
[0] & \mapsto [0, 0] \\
[+] & \mapsto [1, 8]
\end{cases}$

- $[-10, -3] \cup [7, 10]$ is expressed by: $\begin{cases} 
[-] & \mapsto [-10, -3] \\
[0] & \mapsto \bot_1 \\
[+] & \mapsto [7, 10]
\end{cases}$
Lattice operations

**Infimum:**
- If \( \bot_1 \) is the infimum of \( D_1^# \), \( \bot_{cp} = \lambda(z^# \in D_0^#) \cdot \bot_1 \) is the infimum of \( D_{cp}^# \)

**Abstract post-conditions:** no easy general definition, will be discussed later, based on specific instances of \( D_0^# \)

**Ordering test** (sound, not necessarily optimal):
- We define \( \subseteq_{cp} \) as the pointwise ordering:
  \[
  X_0^# \subseteq_{cp} X_1^# \quad \text{def} \quad \forall z^# \in D_0^#, \ X_0^#(z^#) \subseteq_1 X_1^#(z^#)
  \]
- Then, \( X_0^# \subseteq_{cp} X_1^# \implies \gamma_{cp}(X_0^#) \subseteq \gamma_{cp}(X_1^#) \)

**Join operation:**
- We assume that \( \sqcup_1 \) is a sound upper bound operator in \( D_1^# \)
- Then, \( \sqcup_{cp} \) defined below is a sound upper bound operator in \( D_{cp}^# \):
  \[
  X_0^# \sqcup_{cp} X_1^# \quad \text{def} \quad \lambda(z^# \in D_0^#) \cdot (X_0^#(z^#) \sqcup_1 X_1^#(z^#))
  \]
- The same construction applies to widening, if \( D_0^# \) is finite
Composition with another abstraction

We assume three abstractions
- \((D_0^\#, \subseteq_0^\#)\), with concretization \(\gamma_0 : D_0^\# \rightarrow D\)
- \((D_1^\#, \subseteq_1^\#)\), with concretization \(\gamma_1 : D_1^\# \rightarrow D\)
- \((D_2^\#, \subseteq_2^\#)\), with concretization \(\gamma_2 : D_2^\# \rightarrow D_1^\#\)

Cardinal power abstract domains \(D_0^\# \Rightarrow D_1^\#\) and \(D_0^\# \Rightarrow D_2^\#\) can be bound by an abstraction relation defined by concretization function \(\gamma\):

\[
\gamma : (D_0^\# \Rightarrow D_2^\#) \rightarrow (D_0^\# \Rightarrow D_1^\#) \quad \lambda(z^\# \in D_0^\#) \cdot \gamma(X^\#(z^\#))
\]

Applications:
- start with \(D_1^\#, \gamma_1\) defined as the identity abstraction
- compose an abstraction for right hand side of relations
- compose several cardinal power abstractions (or partitioning abstractions)
Composition with another abstraction

- Concrete lattice $\mathbb{D} = \mathcal{P}(\mathbb{Z})$, with $\subseteq \subseteq$
- $(\mathbb{D}_0^\#, \subseteq_0^\#)$ be the lattice of signs
- $(\mathbb{D}_1^\#, \subseteq_1^\#)$ be the identity abstraction
  $\mathbb{D}_1^\# = \mathcal{P}(\mathbb{Z})$, $\gamma_1 = \text{Id}$
- $(\mathbb{D}_2^\#, \subseteq_2^\#)$ be the lattice of intervals

Then, $[-10, -3] \uplus [7, 10]$ is abstracted in two steps:

- in $\mathbb{D}_0^\# \Rightarrow \mathbb{D}_1^\#$, \[
\begin{align*}
[-] & \mapsto \{-10, -9, -8, -7, -6, -5, -4, -3\} \\
[0] & \mapsto \emptyset \\
[+] & \mapsto \{7, 8, 9, 10\}
\end{align*}
\]  
  (note that, at this stage, the right hand sides are simply sets of values)

- in $\mathbb{D}_0^\# \Rightarrow \mathbb{D}_2^\#$, \[
\begin{align*}
[-] & \mapsto [-10, -3] \\
[0] & \mapsto \bot_1 \\
[+] & \mapsto [7, 10]
\end{align*}
\]
Outline

1 Introduction
2 Imprecisions in convex abstractions
3 Disjunctive completion
4 Cardinal power and partitioning abstractions
5 State partitioning
   - Definition and examples
   - Abstract interpretation with boolean partitioning
6 Trace partitioning
7 Conclusion
Definition

We consider **concrete domain** $\mathbb{D} = \mathcal{P}(\mathbb{S})$ where

- $\mathbb{S} = \mathbb{L} \times \mathbb{M}$ where $\mathbb{L}$ denotes the set of control states
- $\mathbb{M} = \mathbb{X} \rightarrow \mathbb{V}$

**State partitioning**

A **state partitioning** abstraction is defined as the cardinal power of two abstractions $(\mathbb{D}_0^\#, \sqsubseteq_0, \gamma_0)$ and $(\mathbb{D}_1^\#, \sqsubseteq_1, \gamma_1)$ of the domain of sets of states $(\mathcal{P}(\mathbb{S}), \subseteq)$:

- $(\mathbb{D}_0^\#, \sqsubseteq_0, \gamma_0)$ defines the **partitions**
- $(\mathbb{D}_1^\#, \sqsubseteq_1, \gamma_1)$ defines the **abstraction of each element of partitions**

**Typical instances:**

- either $\mathbb{D}_1^\# = \mathcal{P}(\mathbb{S}) = \mathbb{D}$
- or an abstraction of sets of memory states: numerical abstraction can be obtained by composing another abstraction on top of $(\mathcal{P}(\mathbb{S}), \subseteq)$
Use of a partition: intuition

We fix a partition $\mathcal{U}$ of $\mathcal{P}(S)$:

1. $\forall E, E' \in \mathcal{U}, E \neq E' \implies E \cap E' = \emptyset$
2. $S = \bigcup \mathcal{U}$

We can apply the **cardinal power construction**: 

**State partitioning abstraction**

We let $\mathbb{D}_0^\# = \mathcal{U} \cup \{ \bot, \top \}$ and $\gamma_0 : E \mapsto E$. Thus, $\mathbb{D}_{cp}^\# = \mathcal{U} \rightarrow \mathbb{D}_1^\#$ and:

$$
\gamma_{cp} : \mathbb{D}_{cp}^\# \rightarrow \mathbb{D} \\
\mathbb{X}^\# \mapsto \{ s \in S \mid \forall E \in \mathcal{U}, s \in E \implies s \in \gamma_0(\mathbb{X}^\#(E)) \}
$$

- each $E \in \mathcal{U}$ is attached to a piece of information in $\mathbb{D}_1^\#$
- exercise: what happens if we use only a covering, i.e., if we drop property 1?
- we will often focus on $\mathcal{U}$ and drop $\bot, \top$
Application 1: flow sensitive abstraction

**Principle**: abstract separately the states at distinct control states

This is *what we have been often doing already*, without formalizing it for instance, using the *the interval abstract domain*:

\[
\begin{align*}
\ell_0 & : \quad // \text{ assume } x \geq 0 \\
\ell_1 & : \quad \textbf{if}(x < 10)\{ \\
\ell_2 & : \quad y = x - 2; \\
\ell_3 & : \quad }\textbf{else}\{ \\
\ell_4 & : \quad y = 2 - x; \\
\ell_5 & : \quad } \\
\ell_6 & : \quad ...
\end{align*}
\]

\[
\begin{align*}
\ell_0 & \mapsto x : \top \land y : \top \\
\ell_1 & \mapsto x : [0, +\infty[ \land y : \top \\
\ell_2 & \mapsto x : [0, 9] \land y : \top \\
\ell_3 & \mapsto x : [0, 9] \land y : [-2, 7] \\
\ell_4 & \mapsto x : [10, +\infty[ \land y : \top \\
\ell_5 & \mapsto x : [10, +\infty[ \land y : ]-\infty, -8] \\
\ell_6 & \mapsto x : [0, +\infty[ \land y : ]-\infty, 7]
\end{align*}
\]
Application 1: flow sensitive abstraction

**Principle**: abstract separately the states at distinct control states

**Flow sensitive abstraction**

We apply the cardinal power based partitioning abstraction with:

- $\mathcal{U} = \mathbb{L}$
- $\gamma_0 : \mathcal{L} \mapsto \{\mathcal{L}\} \times \mathbb{M}$

It is induced by partition $\{\{\mathcal{L}\} \times \mathbb{M} | \mathcal{L} \in \mathbb{L}\}$

Then, if $X^\#$ is an element of the reduced cardinal power,

$$
\gamma_{cp}(X^\#) = \{s \in \mathbb{S} | \forall x \in \mathbb{D}_0^\#, \ s \in \gamma_0(x) \implies s \in \gamma_1(X^\#(x))\}
\]

$$
= \{(l, m) \in \mathbb{S} | m \in \gamma_1(X^\#(l))\}
$$

- after this abstraction step, $\mathbb{D}_1^\#$ only needs to represent sets of memory states (numeric abstractions...)
- this abstraction step is *very common* as part of the design of abstract interpreters
Application 1: flow insensitive abstraction

Flow sensitive abstraction is **sometimes too costly**:

- *e.g., ultra fast pointer analyses* (a few seconds for 1 MLOC) for compilation and program transformation
- **context insensitive** abstraction simply **collapses all control states**

**Flow insensitive abstraction**

We apply the cardinal power based partitioning abstraction with:

- $D^0 = \emptyset$
- $\gamma_0 : \cdot \mapsto S$
- $D^1 = P(M)$
- $\gamma_1 : M \mapsto \{(l, m) \mid l \in L, m \in M\}$

It is induced by a trivial partition of $\mathcal{P}(S)$
Application 1: flow insensitive abstraction

We compare with flow sensitive abstraction:

\begin{align*}
    l_0 & : \text{// assume } x \geq 0 \\
    l_1 & : \text{if}(x < 10) \{
        y = x - 2; \\
    \} \text{else} \\
    l_3 & : \}
    l_4 & : y = 2 - x; \\
    l_5 & : \\
    l_6 & : \ldots \\
    l_0 & \mapsto x : \top \land y : \top \\
    l_1 & \mapsto x : [0, +\infty[ \land y : \top \\
    l_2 & \mapsto x : [0, 9] \land y : \top \\
    l_3 & \mapsto x : [0, 9] \land y : [-2, 7] \\
    l_4 & \mapsto x : [10, +\infty[ \land y : \top \\
    l_5 & \mapsto x : [10, +\infty[ \land y : ] - \infty, -8] \\
    l_6 & \mapsto x : [0, +\infty[ \land y : ] - \infty, 7]
\end{align*}

- the best global information is $x : \top \land y : \top$ (very imprecise)
- even if we exclude the entry point before the assumption point, we get $x : [0, +\infty[ \land y : \top$ (still very imprecise)

For a few specific applications flow insensitive is ok

In most cases (e.g., numeric properties), flow sensitive is absolutely needed
Application 2: context sensitive abstraction

We consider programs with procedures

**Example:**

```plaintext
void main(){... l_0 : f(); ... l_1 : f(); ... l_2 : g() ...}
void f(){...}
void g(){if(...) {l_3 : g()} else {l_4 : f()}}
```

- assumption: **flow sensitive abstraction** used inside each function
- we need to also describe the **call stack state**

**Call string**

Thus, $S = K \times L \times M$, where $K$ is the set of **call strings**

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\in$</th>
<th>$K$</th>
<th>calling contexts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>::=</td>
<td>$\epsilon$</td>
<td>empty call stack</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(f, \ell) \cdot \kappa$</td>
<td>call to $f$ from stack $\kappa$ at point $\ell$</td>
</tr>
</tbody>
</table>
Application 2: context sensitive abstraction, $\infty$-CFA

Full context sensitive abstraction ($\infty$-CFA)

- $D_0^f = K \times L$
- $\gamma_0 : (\kappa, \ell) \mapsto \{ (\kappa, \ell, m) \mid m \in M \}$

```c
void main(){... l_0 : f();... l_1 : f();... l_2 : g() ...}
void f(){...}
void g(){if(...){l_3 : g()} else{l_4 : f()}}
```

Abstract contexts in function $f$:

- $(l_0, f) \cdot \epsilon$, $(l_1, f) \cdot \epsilon$, $(l_4, f) \cdot (l_2, g) \cdot \epsilon$,
- $(l_4, f) \cdot (l_3, g) \cdot (l_2, g) \cdot \epsilon$, $(l_4, f) \cdot (l_3, g) \cdot (l_3, g) \cdot (l_2, g) \cdot \epsilon$, ...

- one invariant per calling context, very precise
- infinite in presence of recursion (i.e., not practical in this case)
Application 2: context insensitive abstraction, 0-CFA

Context insensitive abstraction (0-CFA)

- $D^0_0 = L$
- $\gamma_0 : l \mapsto \{(\kappa, l, m) \mid \kappa \in \mathbb{K}, m \in \mathbb{M}\}$

```c
void main(){...\ell_0 : f();...\ell_1 : f();... \ell_2 : g()...}
void f(){...}
void g(){if(...){\ell_3 : g()}else{\ell_4 : f()}}
```

Abstract contexts in function $f$ are of the form $(?, f) \cdot \ldots$, 
- 0-CFA merges all calling contexts to a same procedure, very coarse abstraction
- but is usually quite efficient to compute
Application 2: context sensitive abstraction, \( k \)-CFA

### Partially context sensitive abstraction (k-CFA)

- \( D_0^k = \{ \kappa \in K \mid \text{length}(\kappa) \leq k \} \times L \)
- \( \gamma_0 : (\kappa, \ell) \mapsto \{ (\kappa \cdot \kappa', \ell, m) \mid \kappa' \in K, m \in M \} \)

```c
void main()
  { ... l_0 : f(); ... l_1 : f(); ... l_2 : g() ... }
void f()
void g()
  if(...) { l_3 : g() } else { l_4 : f() }
```

**Abstract contexts in function \( f \), in 2-CFA:**

\((l_0, f) \cdot \epsilon, (l_1, f) \cdot \epsilon, (l_4, f) \cdot (l_3, g) \cdot (? , g) \cdot \ldots, (l_4, f) \cdot (l_2, g) \cdot (?) , \text{main}) \ldots\)

- usually **intermediate** level of precision and efficiency
- can be applied to programs with **recursive procedures**
Application 3: partitioning by a boolean condition

- so far, we only used abstractions of the control states to partition
- we now consider abstractions of memory states properties

Function guided memory states partitioning

We let:
- $D_0 = A$ where $A$ finite set is a finite set of values / properties
- $\phi : M \rightarrow A$ maps each store to its property
- $\gamma_0$ is of the form $(a \in A) \mapsto \{(l, m) \in S \mid \phi(m) = a\}$

Common choice for $A$: the set of boolean values $\mathbb{B}$
(or another finite set of values —convenient for enum types!)

Many choices for function $\phi$ are possible:
- value of one or several variables (boolean or scalar)
- sign of a variable
- ...
Application 3: partitioning by a boolean condition

We assume:

- \( \mathbb{X} = \mathbb{X}_{\text{bool}} \uplus \mathbb{X}_{\text{int}} \), where \( \mathbb{X}_{\text{bool}} \) (resp., \( \mathbb{X}_{\text{int}} \)) collects boolean (resp., integer) variables

- \( \mathbb{X}_{\text{bool}} = \{b_0, \ldots, b_{k-1}\} \)

- \( \mathbb{X}_{\text{int}} = \{x_0, \ldots, x_{l-1}\} \)

Thus, \( \mathbb{M} = \mathbb{X} \rightarrow \mathbb{V} \equiv (\mathbb{X}_{\text{bool}} \rightarrow \mathbb{V}_{\text{bool}}) \times (\mathbb{X}_{\text{int}} \rightarrow \mathbb{V}_{\text{int}}) \equiv \mathbb{V}^k_{\text{bool}} \times \mathbb{V}^l_{\text{int}} \)

Boolean partitioning abstract domain

We apply the cardinal power abstraction, with a domain of partitions defined by a function, with:

- \( A = \mathbb{B}^k \)

- \( \phi(m) = (m(b_0), \ldots, m(b_{k-1})) \)

- we let \( (\mathbb{D}_1, \sqsubseteq_1, \gamma_1) \) be any \textbf{numerical abstract domain} for \( \mathcal{P}(\mathbb{V}^l_{\text{int}}) \)
Application 3: example

With $\mathbb{X}_{\text{bool}} = \{b_0, b_1\}$, $\mathbb{X}_{\text{int}} = \{x, y\}$, we can express:

\[
\begin{align*}
(b_0 \land b_1) & \quad \implies \quad x \in [-3, 0] \land y \in [-2, 0] \\
(b_0 \land \neg b_1) & \quad \implies \quad x \in [-3, 0] \land y \in [-2, 0] \\
\neg b_0 \land b_1 & \quad \implies \quad x \in [0, 3] \land y \in [0, 2] \\
\neg b_0 \land \neg b_1 & \quad \implies \quad x \in [0, 3] \land y \in [0, 2]
\end{align*}
\]

- this abstract value expresses a relation between $b_0$ and $x, y$ (which induces a relation between $x$ and $y$)
- alternative: partition with respect to only some variables e.g., here $b_0$ only since $b_1$ is irrelevant
- typical representation of abstract values: based on some kind of decision trees (variants of BDDs)
Application 3: example

- Left side abstraction shown in blue: boolean partitioning for $b_0, b_1$
- Right side abstraction shown in green: interval abstraction
- We omit the cases of the form $P \implies \bot$...

```c
bool b0, b1;
int x, y;  // (uninitialized)
b0 = x >= 0;
    (b0 \implies x \geq 0) \land (\lnot b0 \implies x < 0)
b1 = x <= 0;
    (b0 \land b1 \implies x = 0) \land (b0 \land \lnot b1 \implies x > 0) \land (\lnot b0 \land b1 \implies x < 0)
if(b0 && b1){
    (b0 \land b1 \implies x = 0)
y = 0;
    (b0 \land b1 \implies x = 0 \land y = 0)
} else{
    (b0 \land \lnot b1 \implies x > 0) \land (\lnot b0 \land b1 \implies x < 0)
y = 100/x;
    (b0 \land \lnot b1 \implies x > 0 \land y \geq 0) \land (\lnot b0 \land b1 \implies x < 0 \land y \leq 0)
}
```
Application 3: partitioning by the sign of a variable

We now consider a **semantic property**: the **sign of a variable**

We assume:

- $X = X_{\text{int}}$, i.e., all variables have integer type
- $X_{\text{int}} = \{x_0, \ldots, x_{l-1}\}$

Thus, $M = X \rightarrow V \equiv V'_{\text{int}}$

### Sign partitioning abstract domain

We apply the cardinal power abstraction, with a domain of partitions defined by a function, with:

- $A = \{[< 0], [= 0], [> 0]\}$
- $\phi(m) = \begin{cases} [< 0] & \text{if } m(x_0) < 0 \\ [= 0] & \text{if } m(x_0) = 0 \\ [> 0] & \text{if } m(x_0) > 0 \end{cases}$
- $(\mathbb{D}^h_1, \sqsubseteq_1, \gamma_1)$ an abstraction of $P(V'_{\text{int}}^{-1})$ (no need to abstract $x_0$ twice)
Application 3: example

- Sign abstraction fixing partitions shown in blue
- States abstraction shown in green: interval abstraction
- We omit the cases of the form $P \implies \bot$...

```plaintext
int x ∈ Z;
int s;
int y;
if(x ≥ 0) {
  (x < 0 ⇒ ⊥) ∧ (x = 0 ⇒ T) ∧ (x > 0 ⇒ T)
  s = 1;
  (x < 0 ⇒ ⊥) ∧ (x = 0 ⇒ s = 1) ∧ (x > 0 ⇒ s = 1)
} else {
  (x < 0 ⇒ T) ∧ (x = 0 ⇒ ⊥) ∧ (x > 0 ⇒ ⊥)
  s = -1;
  (x < 0 ⇒ s = -1) ∧ (x = 0 ⇒ ⊥) ∧ (x > 0 ⇒ ⊥)
}
(x < 0 ⇒ s = -1) ∧ (x = 0 ⇒ s = 1) ∧ (x > 0 ⇒ s = 1)
1. y = x/s;
    (x < 0 ⇒ s = -1 ∧ y > 0) ∧ (x = 0 ⇒ s = 1 ∧ y = 0) ∧ (x > 0 ⇒ s = 1 ∧ y > 0)
2. assert(y ≥ 0);
```
Outline

1 Introduction

2 Imprecisions in convex abstractions

3 Disjunctive completion

4 Cardinal power and partitioning abstractions

5 State partitioning
   - Definition and examples
   - Abstract interpretation with boolean partitioning

6 Trace partitioning

7 Conclusion
Computation of abstract semantics and partitioning

We present abstract operations in the context of an analysis that combines two forms of partitioning:

- by control states (as previously), using a chaotic iteration strategy
- by the values of the boolean variables

Intuitively, the abstract values are of the form:

\[ f^\# : (\mathbb{I} \times \mathbb{V}^k_{\text{bool}}) \rightarrow \mathbb{D}_1^\# \]

Yet, this is not a very good representation:

- program transition from one control state to another are known before the analysis: they correspond to the program transitions
- program transition from one boolean configuration to another are not known before the analysis: we need to know information about the values of the boolean variables, which the analysis is supposed to compute
A combination of two cardinal powers

Sequence of abstractions:

1. **concrete states:** \( \mathcal{P}(L \times M) \equiv \mathcal{P}(L \times (V^k_{\text{bool}} \times V^k_{\text{int}})) \)

2. **partitioning of states by the control state:**
   \[
   L \rightarrow \mathcal{P}(M) \equiv L \rightarrow \mathcal{P}((V^k_{\text{bool}} \times V^l_{\text{int}}))
   \]

3. **partitioning by the boolean configuration:**
   \[
   L \rightarrow (V^k_{\text{bool}} \rightarrow \mathcal{P}(V^l_{\text{int}}))
   \]

4. **numerical abstraction of numerical stores:**
   \[
   L \rightarrow (V^k_{\text{bool}} \rightarrow D^1_{\text{int}})
   \]

**Computer representation:**

```plaintext
type abs1 = ... (* abstract elements of \(D^1_{\text{int}}\) *)
type abs_state = ... (* boolean trees with elements of type abs1 at the leaves *)
type abs_cp = (labels, abs_state) Map.t
```
Abstract operations

Abstract post-conditions

- concrete \( post : \mathcal{P}(\mathbb{S}) \rightarrow \mathcal{P}(\mathbb{S}) \) (where \( \mathbb{S} \) is the set of states);
- the abstract \( post^\# : \mathbb{D}^\# \rightarrow \mathbb{D}^\# \) should be such that

\[
post \circ \gamma \subseteq \gamma \circ post^\#
\]

In the next part, we seek for abstract post-conditions for the following operations, in the cardinal power domain, assuming similar functions are defined in the underlying domain (numeric abstract domain, cf previous course):

- assignment to scalar, e.g., \( x = 1 - x \);
- assignment to boolean, e.g., \( b_0 = x \leq 7 \)
- scalar test, e.g., \( \text{if}(x \geq 8) \ldots \)
- boolean test, e.g., \( \text{if}(\neg b_1) \ldots \)

Other lattice operations (inclusion check, join, widening) are left as exercise
Transfer functions: assignment to scalar (1/2)

Computation of an abstract post-condition

\[ x_k = e; \]

Example:

- **statement** \( x = 1 - x; \)
- **abstract pre-condition:**

\[
\left\{ \begin{array}{c}
    b \Rightarrow x \geq 0 \\
    \land \neg b \Rightarrow x \leq 0
\end{array} \right. 
\]

Intuition:

- the values of the boolean variables do not change
- the values of the numeric values can be updated separately for each partition
Transfer functions: assignment to scalar (2/2)

Definition of the abstract post-condition

\[ \text{assign}_{cp}(x, e, X^\#) = \lambda(z^\# \in \bigvee_{\text{bool}}^k) \cdot \text{assign}_1(x, e, X^\#(z^\#)) \]

This post-condition is sound:

Soundness

If \( \text{assign}_1 \) is sound, so is \( \text{assign}_{cp} \), in the sense that:

\[ \forall X^\# \in \mathbb{D}_{cp}^\#, \forall m \in \gamma_{cp}(X^\#), \ m[x \leftarrow \llbracket e \rrbracket(m)] \in \gamma_{cp}(\text{assign}_{cp}(x, e, X^\#)) \]

- proof by case analysis over the value of the boolean variables

Example:

\[ \text{assign}_{cp} \left(x, 1 - x, \left\{ \begin{array}{l} b \Rightarrow x \geq 0 \\ \neg b \Rightarrow x \leq 0 \end{array} \right\} \right) = \left\{ \begin{array}{l} b \Rightarrow x \leq 1 \\ \neg b \Rightarrow x \geq 1 \end{array} \right\} \]
Transfer functions: scalar test (1/2)

Computation of an abstract post-condition

$$\text{if}(e)$$

where e only refers to numeric variables
(analysis of a condition test, of a loop test, of an assertion)

Example:

- **statement**: if($x \geq 8$){...}
- **abstract pre-condition**:

$$\{ b \Rightarrow x \geq 0 \ \wedge \ \neg b \Rightarrow x \leq 0 \}$$

Intuition:
- the values of the variables do not change, no relations between boolean and numeric variables can be inferred
- new conditions on the numeric variables can be inferred, separately for each partition (possibly leading to empty abstract states)
Transfer functions: scalar test (2/2)

Definition of the abstract post-condition

\[
test_{cp}(c, X^\#) = \lambda(z^\# \in \bigvee^k_{bool}) \cdot test_1(c, X^\#(z^\#))
\]

This post-condition is sound:

Soundness

If \(test_1\) is sound, so is \(test_{cp}\), in the sense that:

\[
\forall X^\# \in D^\#_{cp}, \forall m \in \gamma_{cp}(X^\#), \llbracket c \rrbracket(m) = \text{TRUE} \implies m \in \gamma_{cp}(test_{cp}(x, e, X^\#))
\]

- proof by case analysis over the value of the boolean variables

Example:

\[
test_{cp} \left( x \geq 8, \left\{ \begin{array}{c}
  b \ \Rightarrow \ x \geq 0 \\
  \neg b \ \Rightarrow \ x \leq 0
\end{array} \right\} \right) = \left\{ \begin{array}{c}
  b \ \Rightarrow \ x \geq 8 \\
  \neg b \ \Rightarrow \ \bot
\end{array} \right\}
\]
Transfer functions: boolean condition test (1/3)

Computation of an abstract post-condition

\[ \text{if(e)} \]

where \( e \) only refers to boolean variables

(analysis of a condition test, of a loop test, of an assertion)

Example:

- **statement**: \( \text{if}(\neg b_1) \ldots \)
- **abstract pre-condition**:
  \[
  \begin{align*}
  b_0 \land b_1 & \Rightarrow 15 \leq x \\
  \land b_0 \land \neg b_1 & \Rightarrow 9 \leq x \leq 14 \\
  \land \neg b_0 \land b_1 & \Rightarrow 6 \leq x \leq 8 \\
  \land \neg b_0 \land \neg b_1 & \Rightarrow x \leq 5
  \end{align*}
  \]

Intuition:

- the values of the variables do not change, no new relations between boolean and numeric variables can be inferred
- certain boolean configurations get discarded or refined
Transfer functions: boolean condition test (2/3)

**Definition of the abstract post-condition**

\[
\text{test}_{cp}(c, X^\#) = \lambda (z^\# \in \forall_{bool}^k). \begin{cases} 
X^\#(z^\#) & \text{if test}_0(c, X^\#(z^\#)) \neq \bot_0 \\
\bot_1 & \text{otherwise}
\end{cases}
\]

This post-condition is sound:

**Soundness**

If test\(_0\) is sound, so is test\(_{cp}\), in the sense that:

\[
\forall X^\# \in D^\#_{cp}, \forall m \in \gamma_{cp}(X^\#), \llbracket c \rrbracket(m) = \text{TRUE} \implies m \in \gamma_{cp}(\text{test}_{cp}(x, e, X^\#))
\]

Proof:

- case analysis over the boolean configurations
- in each situation, two cases depending on whether or not the condition test evaluates to TRUE or to FALSE
Transfer functions: boolean condition test (2/3)

Example abstract post-condition:

\[
\text{test}_{cp}(\neg b_1, \begin{cases}
  b_0 \land b_1 & \Rightarrow 15 \leq x \\
  \land b_0 \land \neg b_1 & \Rightarrow 9 \leq x \leq 14 \\
  \land \neg b_0 \land b_1 & \Rightarrow 6 \leq x \leq 8 \\
  \land \neg b_0 \land \neg b_1 & \Rightarrow x \leq 5
\end{cases})
\]

\[
= \begin{cases}
  b_0 \land b_1 & \Rightarrow \bot_1 \\
  \land b_0 \land \neg b_1 & \Rightarrow 9 \leq x \leq 14 \\
  \land \neg b_0 \land b_1 & \Rightarrow \bot_1 \\
  \land \neg b_0 \land \neg b_1 & \Rightarrow x \leq 5
\end{cases}
\]
Transfer functions: assignment to boolean (1/3)

Computation of an abstract post-condition

\[ b_j = e; \]

where e only refers to numeric variables

Example:
- **statement:** \( b_0 = x \leq 7 \)
- **abstract pre-condition:**
  \[
  \begin{align*}
  & b_0 \land b_1 \quad \Rightarrow \quad 15 \leq x \\
  & \land \quad b_0 \land \neg b_1 \quad \Rightarrow \quad 9 \leq x \leq 14 \\
  & \land \quad \neg b_0 \land b_1 \quad \Rightarrow \quad 6 \leq x \leq 8 \\
  & \land \quad \neg b_0 \land \neg b_1 \quad \Rightarrow \quad x \leq 5
  \end{align*}
  \]

Intuition:
- the value of the boolean variable in the left hand side changes, thus partitions need to be recomputed
- new relations between boolean variables and numeric variables emerge (old relations get discarded)
**Transfer functions: assignment to boolean (2/3)**

### Definition of the abstract post-condition

\[
\text{assign}_{\text{cp}}(b, e, X^\#)(z^#[b \leftarrow \text{TRUE}]) = \begin{cases} 
    \text{test}_1(e, X^#(z^#[b \leftarrow \text{TRUE}])) \\
    \sqcup_1 \text{test}_1(e, X^#(z^#[b \leftarrow \text{FALSE}]))
\end{cases}
\]
\[
\text{assign}_{\text{cp}}(b, e, X^#)(z^#[b \leftarrow \text{FALSE}]) = \begin{cases} 
    \text{test}_1(\neg e, X^#(z^#[b \leftarrow \text{TRUE}])) \\
    \sqcup_1 \text{test}_1(\neg e, X^#(z^#[b \leftarrow \text{FALSE}]))
\end{cases}
\]

### Soundness

\[
\forall X^\# \in \mathbb{D}_{\text{cp}}^#, \forall m \in \gamma_{\text{cp}}(X^#), \ m[b \leftarrow [e](m)] \in \gamma_{\text{cp}}(\text{assign}_{\text{cp}}(b, e, X^#))
\]

**Proof:** if \( z^\# \in \mathbb{D}_0^# \) and \( z^#(b) = \text{TRUE} \), then, \( \text{assign}_{\text{cp}}(b, e[x_0, \ldots, x_i], X^#)(z^#) \) should account for all states where \( b \) becomes true, whatever the previous value, other boolean variables remaining unchanged; the case where \( z^#(b) = \text{FALSE} \) is symmetric.

*The partitions get modified* (this is a *costly step*, involving join)
Transfer functions: assignment to boolean (3/3)

Example abstract post-condition:

$$assign_{cp} \left( b_0, x \leq 7, \begin{cases} b_0 \land b_1 \Rightarrow 15 \leq x \\ \land b_0 \land \neg b_1 \Rightarrow 9 \leq x \leq 14 \\ \land \neg b_0 \land b_1 \Rightarrow 6 \leq x \leq 8 \\ \land \neg b_0 \land \neg b_1 \Rightarrow x \leq 5 \end{cases} \right)$$

$$= \begin{cases} b_0 \land b_1 \Rightarrow 6 \leq x \leq 7 \\ \land b_0 \land \neg b_1 \Rightarrow x \leq 5 \\ \land \neg b_0 \land b_1 \Rightarrow 8 \leq x \\ \land \neg b_0 \land \neg b_1 \Rightarrow 9 \leq x \leq 14 \end{cases}$$

The partitions get modified (this is a costly step, involving join)
Choice of boolean partitions

Boolean partitioning allows to express relations between boolean and scalar variables, but these relations are expensive to maintain:

1. partitioning with respect to $N$ boolean variables translates into a $2^N$ space cost factor
2. after assignments, partitions need be recomputed (use of join)

Packing addresses the first issue

- select groups of variables for which relations would be useful
- can be based on syntactic or semantic criteria

Whatever the packs, the transfer functions will produce a sound result (but possibly not the most precise one)

In the last part of this course, we present another form of partitioning that can sometimes alleviate these issues
Outline

1 Introduction

2 Imprecisions in convex abstractions

3 Disjunctive completion

4 Cardinal power and partitioning abstractions

5 State partitioning

6 Trace partitioning
   - Principles and examples
   - Abstract interpretation with trace partitioning

7 Conclusion
Definition of trace partitioning

**Principle**

We start from a **trace semantics** and rely on an abstraction of execution history for partitioning.

- **Concrete domain:** $\mathcal{D} = \mathcal{P}(\mathcal{S}^*)$

- **Left side abstraction** $\gamma_0 : \mathcal{D}_0^\# \to \mathcal{D}$: a trace abstraction to be defined precisely later

- **Right side abstraction**, as a composition of two abstractions:
  - the final state abstraction defined by $(\mathcal{D}_1^\#, \subseteq_1^\#) = (\mathcal{P}(\mathcal{S}), \subseteq)$ and:
    $$\gamma_1 : M \mapsto \{ (s_0, \ldots, s_k, (l, m)) \mid m \in M, l \in \mathcal{L}, s_0, \ldots, s_k \in \mathcal{S} \}$$
  - a store abstraction applied to the traces final memory state:
    $$\gamma_2 : \mathcal{D}_2^\# \to \mathcal{D}_1^\#$$

**Trace partitioning**

**Cardinal power abstraction** defined by abstractions $\gamma_0$ and $\gamma_1 \circ \gamma_2$
Application 1: partitioning by control states

Flow sensitive abstraction

- We let $D_0^\# = L \cup \{T\}$
- Concretization is defined by:

$$\gamma_0 : D_0^\# \longrightarrow P(S^*)$$

This produces the same flow sensitive abstraction as with state partitioning; in the following we always compose context sensitive abstraction with other abstractions...

Trace partitioning is more general than state partitioning

Any state partitioning abstraction is also a trace partitioning abstraction:

- context-sensitivity, partial context sensitivity
- partitioning guided by a boolean condition...
Application 2: partitioning guided by a condition

We consider a program with a **conditional statement**:

\[
\begin{align*}
\ell_0 : & \text{ if}(c) \{ \\
\ell_1 : & \ldots \\
\ell_2 : & \} \text{else} \{ \\
\ell_3 : & \ldots \\
\ell_4 : & \} \\
\ell_5 : & \ldots 
\end{align*}
\]

**Domain of partitions**

The partitions are defined by \( \mathbb{D}_0^\# = \{ \tau_{\text{if:t}}, \tau_{\text{if:f}}, T \} \) and:

\[
\begin{align*}
\gamma_0 : \tau_{\text{if:t}} & \mapsto \{ \langle (\ell_0, m), (\ell_1, m'), \ldots \rangle \mid m \in \mathbb{M}, m' \in \mathbb{M} \} \\
\tau_{\text{if:f}} & \mapsto \{ \langle (\ell_0, m), (\ell_3, m'), \ldots \rangle \mid m \in \mathbb{M}, m' \in \mathbb{M} \} \\
T & \mapsto \mathbb{S}^*
\end{align*}
\]

**Application:**

discriminate the executions depending on the branch they visited
Application 2: partitioning guided by a condition

This partitioning resolves the second example:

```c
int x ∈ ℤ;
int s;
int y;
if(x ≥ 0){
    \( \tau_{\text{if:t}} \Rightarrow (0 ≤ x) \land \tau_{\text{if:f}} \Rightarrow \perp \)
    s = 1;
    \( \tau_{\text{if:t}} \Rightarrow (0 ≤ x \land s = 1) \land \tau_{\text{if:f}} \Rightarrow \perp \)
} else {
    \( \tau_{\text{if:f}} \Rightarrow (x < 0) \land \tau_{\text{if:t}} \Rightarrow \perp \)
    s = -1;
    \( \tau_{\text{if:f}} \Rightarrow (x < 0 \land s = -1) \land \tau_{\text{if:t}} \Rightarrow \perp \)
}
\( \begin{cases} 
    \tau_{\text{if:t}} & \Rightarrow (0 ≤ x \land s = 1) \\
    \land \tau_{\text{if:f}} & \Rightarrow (x < 0 \land s = -1) 
\end{cases} \)
y = x/s;
```

\( \begin{cases} 
    \tau_{\text{if:t}} & \Rightarrow (0 ≤ x \land s = 1 \land 0 ≤ y) \\
    \land \tau_{\text{if:f}} & \Rightarrow (x < 0 \land s = -1 \land 0 < y) 
\end{cases} \)
Application 3: partitioning guided by a loop

We consider a program with a **loop statement**:

\[
\begin{align*}
\ell_0 : & \quad \text{while}(c)\{ \\
\ell_1 : & \quad \ldots \\
\ell_2 : & \quad \} \\
\ell_3 : & \quad \ldots
\end{align*}
\]

**Domain of partitions**

For a given \( k \in \mathbb{N} \), the partitions are defined by

\[
\mathcal{D}_0^k = \{ \tau_{\text{loop}:0}, \tau_{\text{loop}:1}, \ldots, \tau_{\text{loop}:k}, \top \}
\]

and:

\[
\begin{align*}
\gamma_0 : \quad & \tau_{\text{loop}:i} & \quad \mapsto \quad \text{traces that visit } \ell_1 \text{ } i \text{ times} \\
& \top & \quad \mapsto \quad \mathcal{S}^\ast
\end{align*}
\]

**Application:**

discriminate executions depending on the number of iterations in a loop
Application 3: partitioning guided by a loop

An interpolation function:

\[
y = \begin{cases} 
-1 & \text{if } x \leq -1 \\
-\frac{1}{2} + \frac{x}{2} & \text{if } x \in [-1, 1] \\
-1 + x & \text{if } x \in [1, 3] \\
2 & \text{if } 3 \leq x
\end{cases}
\]

Typical implementation:

- use tables of coefficients and loops to search for the range of \( x \)
- here we assume the entrance is positive:

```c
int i = 0;
while (i < 4 && x > t_x[i + 1]){
    i + +;
}
```

\[
\begin{align*}
\tau_{\text{loop}:0} & \Rightarrow \quad \bot & \text{(case } x \leq -1) \\
\tau_{\text{loop}:1} & \Rightarrow \quad 0 \leq x \leq 1 \land i = 1 & \text{(case } -1 \leq x \leq 1) \\
\tau_{\text{loop}:2} & \Rightarrow \quad 1 \leq x \leq 3 \land i = 2 \\
\tau_{\text{loop}:3} & \Rightarrow \quad 3 \leq x \land i = 3
\end{align*}
\]

\[
y = t_c[i] \times (x - t_x[i]) + t_y[i]
\]
Application 4: partitioning guided by the value of a variable

We consider a program with an integer variable $x$, and a program point $\ell$:

$$\text{int } x; \ldots; \ell : \ldots$$

Domain of partitions: partitioning by the value of a variable

For a given $E \subseteq \mathbb{V}_{\text{int}}$ finite set of integer values, the partitions are defined by

$$D_0^\# = \{\tau_{\text{val}:i} \mid i \in E\} \cup \{\top\}$$

and:

$$\gamma_0 : \tau_{\text{val}:k} \mapsto \{\langle \ldots, (\ell, m), \ldots \rangle \mid m(x) = k\}$$

$$\top \mapsto S^*$$

Domain of partitions: partitioning by the property of a variable

For a given abstraction $\gamma : (\mathbb{V}^\#, \subseteq^\#) \to (\mathcal{P}(\mathbb{V}_{\text{int}}), \subseteq)$, the partitions are defined by

$$D_0^\# = \{\tau_{\text{var}:v^\#} \mid v^\# \in \mathbb{V}^\#\}$$

and:

$$\gamma_0 : \tau_{\text{val}:v^\#} \mapsto \{\langle \ldots, (\ell, m), \ldots \rangle \mid m(x) \in \tau_{\text{var}:v^\#}\}$$
Application 4: partitioning guided by the value of a variable

- Left side abstraction shown in blue: **sign of x at entry**
- Right side abstraction shown in green:
  non relational abstraction (we omit the information about x)
- **Same precision** and **similar results** as boolean partitioning, but **very different abstraction**, fewer partitions, no re-partitioning

```plaintext
bool b0, b1;
int x, y;      // (uninitialized)
1 (x < 0@1 ⇒ T) ∧ (x = 0@1 ⇒ T) ∧ (x > 0@1 ⇒ T)
b0 = x ≥ 0;
   (x < 0@1 ⇒ ¬b0) ∧ (x = 0@1 ⇒ b0) ∧ (x > 0@1 ⇒ b0)
b1 = x ≤ 0;
   (x < 0@1 ⇒ ¬b0 ∧ b1) ∧ (x = 0@1 ⇒ b0 ∧ b1) ∧ (x > 0@1 ⇒ b0 ∧ ¬b1)
if(b0 && b1){
   (x < 0@1 ⇒ ⊥) ∧ (x = 0@1 ⇒ b0 ∧ b1) ∧ (x > 0@1 ⇒ ⊥)
y = 0;
   (x < 0@1 ⇒ ⊥) ∧ (x = 0@1 ⇒ b0 ∧ b1 ∧ y = 0) ∧ (x > 0@1 ⇒ ⊥)
} else {
   (x < 0@1 ⇒ ¬b0 ∧ b1) ∧ (x = 0@1 ⇒ ⊥) ∧ (x > 0@1 ⇒ b0 ∧ ¬b1)
y = 100/x;
   (x < 0@1 ⇒ ¬b0 ∧ b1 ∧ y ≤ 0) ∧ (x = 0@1 ⇒ ⊥) ∧ (x > 0@1 ⇒ b0 ∧ ¬b1 ∧ y ≥ 0)
}
```
## Outline

1. Introduction
2. Imprecisions in convex abstractions
3. Disjunctive completion
4. Cardinal power and partitioning abstractions
5. State partitioning
6. **Trace partitioning**
   - Principles and examples
   - Abstract interpretation with trace partitioning
7. Conclusion
Trace partitioning induced by a refined transition system

We consider the **partitions for a condition, and formalize the analysis**:

- **$P_0$**: the analysis does merge them *right after the condition*, at $l_5$ (this amounts to doing no partitioning at all)
- **$P_1$**: the analysis may merge them *at a further point* $lab_6$ (more precise, but more expensive)
- **$P_2$**: the analysis may *never* merge traces from both branches (very precise, but very expensive)

**Intuition**: we can view this form of trace partitioning as *the use of a refined control flow graph*
Trace partitioning induced by a refined transition system

We now **formalize this intuition**:

- we **augment** control states with partitioning tokens: \( \mathcal{L}' = \mathcal{L} \times \mathbb{D}_0 \)
  and let \( \mathcal{S}' = \mathcal{L}' \times \mathcal{M} \)
- let \( \rightarrow' \subseteq \mathcal{S}' \times \mathcal{S}' \) be an **extended transition relation**

**Definition: partitioning transition system**

We say that system \( \mathcal{S}' = (\mathcal{S}', \rightarrow', \mathcal{S}'_I) \) is a **partition** of the transition system \( \mathcal{S} = (\mathcal{S}, \rightarrow, \mathcal{S}_I) \) if and only if:

- (initial states) \( \forall (\ell, m) \in \mathcal{S}_I, \exists \tau \in \mathbb{D}_0, ((\ell, \tau), m) \in \mathcal{S}'_I \)
- (transitions) \( \forall (\ell, m), (\ell', m') \in \mathcal{S}, \forall \tau \in \mathbb{D}_0, \) if \( ((\ell, \tau), m) \in \llbracket \mathcal{S} \rrbracket_{\mathcal{R}} \) then,
  \( (\ell, m) \rightarrow (\ell', m') \Longrightarrow \exists \tau' \in \mathbb{D}_0, ((\ell, \tau), m) \rightarrow ((\ell', \tau'), m') \)

In that case, we write:

\[ \mathcal{S}' \prec \mathcal{S} \]

**Meaning**: system \( \mathcal{S}' \) refines system \( \mathcal{S} \) with additional execution history information
Partitionned transition system and semantics

The partitioned transition system over-approximates the behaviors of the initial system:

**Partitioned system and semantic approximation**

Let us assume that $S' \preceq S$. We let $\llbracket S \rrbracket_T$ (resp., $\llbracket S' \rrbracket_T$) denote the trace semantics of $S$ (resp., $S'$). Then:

$$\forall \langle (l_0, m_0), \ldots, (l_n, m_n) \rangle \in \llbracket S \rrbracket_T,$$

$$\exists \tau_0, \ldots, \tau_n \in D^\#_0, \langle ((l_0, \tau_0), m_0), \ldots, ((l_n, \tau_n), m_n) \rangle \in \llbracket S' \rrbracket_T,$$

**Proof:** by induction over the length of executions (exercise).

**Properties of $S' \preceq S$**

- all traces of $S$ have a counterpart in $S'$ (up to token addition)
- a trace in $S'$ embeds more information than a trace in $S$
- moreover, if we reason up to isomorphisms (e.g., $l \equiv (l, \bullet)$, or $((l, \tau), \tau') \equiv (l, (\tau, \tau'))$), $\preceq$ extends into a pre-order
Trace partitioning induced by a refined transition system

Assumptions:

- **refined control system** \((S', \rightarrow', S'_I) < (S, \rightarrow, S_I)\)
- **erasure function**: \(\Psi : (S')^* \rightarrow S^*\) removes the tokens

Definition of a trace partitioning

The abstraction defining partitions is defined by:

\[
\begin{align*}
\gamma_0 : & \quad \mathcal{D}_0^T \quad \rightarrow \quad \mathcal{P}(S^*) \\
\tau & \quad \mapsto \quad \{\sigma \in S^* \mid \exists \sigma' = \langle \ldots, ((l, \tau), m) \rangle \in (S')^*, \Psi(\sigma') = \sigma\}
\end{align*}
\]

Not all instances of trace partitionings can be expressed that way but **many interesting instances can**:

- **control states** and **call stack** partitioning
- partitioning guided by **conditions** and **loops**
- partitioning **guided by the value of a variable**
Trace partitioning induced by a refined transition system

Example of the partitioning guided by a condition:

- each system induces a partitioning, with different merging points:
  \[ P_1 \prec P_0 \quad P_2 \prec P_1 \]
- these systems induce hierarchy of refining control structures
  \[ P_2 \prec P_1 \prec P_0 \quad \text{thus,} \quad [P_0]_\tau \subseteq [P_1]_\tau \subseteq [P_2]_\tau \]
- this approach also applies to:
  - partitioning induced by a loop
  - partitioning induced by the value of a variable at a given point...
Transfer functions: example

\begin{verbatim}
int x \in \mathbb{Z};
int s;
int y;
if(x \geq 0) {
    \tau_{if:t} \Rightarrow (0 \leq x) \land \tau_{if:f} \Rightarrow \bot
    s = 1;
    \tau_{if:t} \Rightarrow (0 \leq x \land s = 1) \land \tau_{if:f} \Rightarrow \bot
}\}
else {
    \tau_{if:f} \Rightarrow (x < 0) \land \tau_{if:t} \Rightarrow \bot
    s = -1;
    \tau_{if:f} \Rightarrow (x < 0 \land s = -1) \land \tau_{if:t} \Rightarrow \bot
}\}
\{ \begin{array}{c}
\tau_{if:t} \Rightarrow (0 \leq x \land s = 1) \\
\tau_{if:f} \Rightarrow (x < 0 \land s = -1)
\end{array} \}
\Rightarrow y = x/s;
\{ \begin{array}{c}
\tau_{if:t} \Rightarrow (0 \leq x \land s = 1 \land 0 \leq y) \\
\tau_{if:f} \Rightarrow (x < 0 \land s = -1 \land 0 < y)
\end{array} \}
\Rightarrow s \in [-1, 1] \land 0 \leq y
\end{verbatim}

Partitions are rarely modified, and only some (branching) points

partition creation: $\tau_{if:t}$

no modification of partitions

partition creation: $\tau_{if:f}$

no modification of partitions

no modification of partitions

fusion of partitions
Transfer functions: partition creation

Analysis of an if statement, with partitioning

\[ l_0 : \textbf{if}(c)\{ \]
\[ l_1 : \hspace{1cm} \ldots \]
\[ l_2 : \} \textbf{else}\{ \]
\[ l_3 : \hspace{1cm} \ldots \]
\[ l_4 : \} \]
\[ l_5 : \hspace{1cm} \ldots \]

\[ \delta^\#_{l_0,l_1}(X^\#) = [\tau_{\text{if}:t} \mapsto \text{test}(c, \sqcup X^\#(\tau)), \tau_{\text{if}:f} \mapsto \bot] \]

\[ \delta^\#_{l_0,l_3}(X^\#) = [\tau_{\text{if}:t} \mapsto \bot, \tau_{\text{if}:f} \mapsto \text{test}(\neg c, \sqcup X^\#(\tau))] \]

\[ \delta^\#_{l_2,l_5}(X^\#) = X^\# \]

\[ \delta^\#_{l_4,l_5}(X^\#) = X^\# \]

Observations:

- in the body of the condition: either \( \tau_{\text{if}:t} \) or \( \tau_{\text{if}:f} \)
  \( i.e., \) no partition modification there

- effect at point \( l_5 \): both \( \tau_{\text{if}:t} \) and \( \tau_{\text{if}:f} \) exist

- partitions are modified only at the condition point, that is only by \( \delta^\#_{l_0,l_1}(X^\#) \) and \( \delta^\#_{l_0,l_2}(X^\#) \)
Transfer functions: partition fusion

When partitions are not useful anymore, they can be merged

$$\delta^\#_{\ell_0, \ell_1}(X^\#) = \left[ \_ \mapsto \bigcup_{\tau} X^\#(\ell_0)(\tau) \right]$$

Remarks:

- at this point, all partitions are **effectively collapsed** into just one set
- **example**: fusion of the partition of a condition when not useful
- **choice of fusion point**:
  - **precision**: merge point should not occur as long as partitions are useful
  - **efficiency**: merge point should occur as early as partitions are not needed anymore
Choice of partitions

How are the partitions chosen?

Static partitioning [always the case in this lecture]

- a fixed partitioning abstraction $D_0^\#, \gamma_0$ is **fixed before the analysis**
- usually $D_0^\#, \gamma_0$ are chosen by a pre-analysis

- static partitioning is rather easy to formalize and implement
- but it might be limiting, when choosing partitions beforehand is hard

Dynamic partitioning

- the partitioning abstraction $D_0^\#, \gamma_0$ is **not fixed before the analysis**
- instead, it is **computed as part of the analysis**
- *i.e.*, the analysis uses on a lattice of partitioning abstractions $D^\#$ and computes $(D_0^\#, \gamma_0)$ as an element of this lattice
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Adding disjunctions in static analyses

Disjunctive completion: brutally adds disjunctions too expensive in practice

\[ P_0 \lor \ldots \lor P_n \]

Cardinal power abstraction expresses collections of implications between abstract facts in two abstract domains

\[ (P_0 \Rightarrow Q_0) \land \ldots \land (P_n \Rightarrow Q_n) \]

Two major cases:

- **State partitioning** is easier to use when the criteria for partitioning can be easily expressed at the state level

- **Trace partitioning** is more expressive in general
  it can also allow the use of simpler partitioning criteria, with less “re-partitioning”
Conclusion

Assignment: proofs and paper reading

Proof 1:
prove the disjunctive completion algorithm (Slide 15)

Proof 2:
what happens in the case we use coverings instead of partitions (Slide 41)

Refining static analyses by trace-partitioning using control flow
Maria Handjieva and Stanislas Tzolovski,
Static Analysis Symposium, 1998,
http://link.springer.com/chapter/10.1007/3-540-49727-7_12

Abstract interpretation by dynamic partitioning,
François Bourdoncle,
Extended report available at: