The Linear-Non-Linear Substitution Monad

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Christine Tasson (tasson@irif.fr)
Joint work with Martin Hyland at https://arxiv.org/abs/2005.09559

Institut de Recherche en Informatique Fondamentale

Differential Lambda Calculus

(Ehrhard-Regnier 2003, Ehrhard 2018)

Semantical observation: in quantitative models of Linear Logic, programs are interpreted by smooth functions, hence differentiation.

	Programs	Functions	
	M, N	f, g	
Variable	x	x	Variable
Abstraction	$\lambda x.M$	$f : x \mapsto f(x)$	Map
Application	$(\lambda x.M)N$	$f \circ g : x \mapsto f(\,g(x)\,)$	Composition
Differentiation	$D\lambda x.M \cdot N$	$u, x \mapsto Df_x(u)$	Derivation

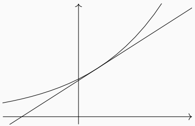
Linear and Non-Linear substitutions in Differential λ-calculus

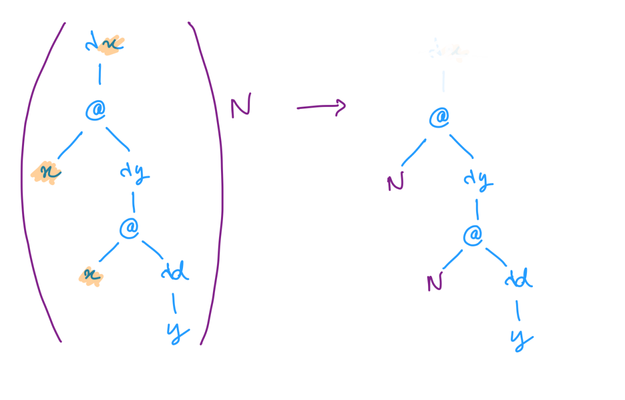
Substitution

$$(\lambda x.M)N \quad \to \quad M[x\backslash N]$$

$$D\lambda x.M \cdot N \quad \to \quad \lambda x.\left(\frac{\partial M}{\partial x} \cdot N\right)$$

Linear approximation





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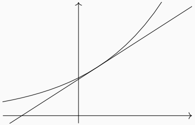

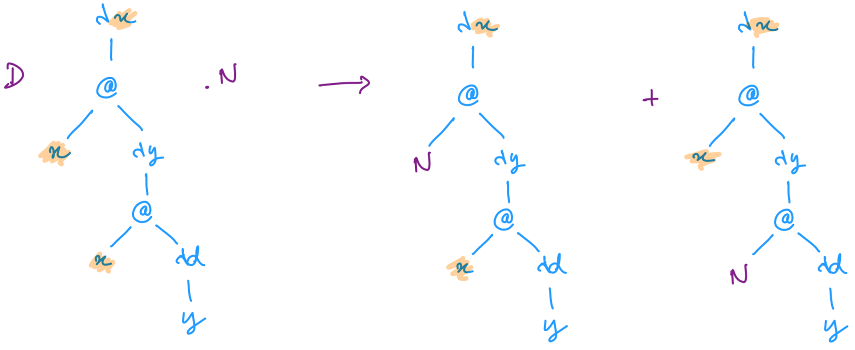

Categorical Model of Differential Lambda-Calculus

(Blute-Cockett-Seely 2010, Bucciarelli-Ehrhard-Manzonetto 2010)

Definition 4.2 A *Cartesian (closed) differential category* is a Cartesian (closed) left-additive category having an operator $D(-)$ that maps a morphism $f : A \to B$ into a morphism $D(f) : A \times A \to B$ and satisfies the following axioms:

D1. $D(f+g) = D(f) + D(g)$ and $D(0) = 0$

D2. $D(f) \circ \langle h+k, v\rangle = D(f) \circ \langle h, v\rangle + D(f) \circ \langle k, v\rangle$ and $D(f) \circ \langle 0, v\rangle = 0$

D3. $D(\mathrm{Id}) = \pi_1$, $D(\pi_1) = \pi_1 \circ \pi_1$ and $D(\pi_2) = \pi_2 \circ \pi_1$

D4. $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$

D5. $D(f \circ g) = D(f) \circ \langle D(g), g \circ \pi_2\rangle$

D6. $D(D(f)) \circ \langle\langle g, 0\rangle, \langle h, k\rangle\rangle = D(f) \circ \langle g, k\rangle$

D7. $D(D(f)) \circ \langle\langle 0, h\rangle, \langle g, k\rangle\rangle = D(D(f)) \circ \langle\langle 0, g\rangle, \langle h, k\rangle\rangle$

A differential operator such that if $f : A \longrightarrow B$, then $Df : A \times A \longrightarrow B$ corresponds to $u, x \mapsto Df_x(u)$ with axioms (D2) for linearity in 1st coord.

Categorical Model of Differential Linear Logic

(Blute-Cockett-Seely 2006, Blute-Cockett-Lemay-Seely 2019)

Idea: *Non-Linear* types are annotated with !.

Definition: A differential category is

- an additive symmetric monoidal closed category X with
- a coalgebra modality made of a (monoidal) comonad $! : X \to X$ together with digging $\delta : !A \multimap !!A$ and dereliction $\epsilon : !A \multimap A$ s.t.
- $!A$ is a cocommutative comonoid together with contraction $\Delta : !A \multimap !A \otimes !A$ and weakening $e : !A \multimap 1$
- a deriving transformation $d : A \otimes !A \multimap !A$ with commuting diagrams.

The deriving transformation maps a morphism $f : A \to B = !A \multimap B$ in the cartesian closed coKleisli category into a morphism

$$D(f) : A \otimes !A \overset{d}{\multimap} !A \overset{f}{\multimap} B.$$

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Problem: What is the nature of a morphism $!A \otimes (!(B \otimes !C) \otimes D) \multimap !A$

A term calculus for Linear-Non-Linear Logic

(Benton-Bierman-de Paiva-Hyland 1993, Barber 1996)

Dual Intuitionistic Linear Logic: $\overset{Linear}{\Gamma} \mid \overset{Non-Linear}{\Delta} \vdash t : c$

Linear rules:

$$\frac{}{x : a \mid \Delta \vdash x : a} \qquad \frac{\Gamma, x : a \mid \Delta \vdash t : b}{\Gamma \mid \Delta \vdash \lambda x^a.t : a \multimap b}$$

$$\frac{\Gamma \mid \Delta \vdash s : a \multimap b \qquad \Gamma' \mid \Delta \vdash t : a}{\Gamma, \Gamma' \mid \Delta \vdash \langle s \rangle t : b}$$

Non-Linear rules:

$$\frac{}{\Gamma \mid \Delta, x : \underline{b} \vdash x : \underline{b}} \qquad \frac{\Gamma \mid \Delta, x : \underline{a} \vdash t : b}{\Gamma \mid \Delta \vdash \lambda x^{\underline{a}}.t : \underline{a} \to b}$$

$$\frac{\Gamma \mid \Delta \vdash s : a \to b \qquad \cdot \mid \Delta \vdash t : a}{\Gamma \mid \Delta \vdash (s)t : b}$$

Linear-Non-Linear rule:

$$\frac{\Gamma , x : a \mid \Delta \vdash t : b}{\Gamma \mid \Delta , x : \underline{a} \vdash t : b}$$

If Δ is empty, we recover Multiplicative ILL and if Γ is empty, we recover simply typed λ-calculus

Axiomatic using Categories

In a category X, equipped with the right structure (SMC/ CC)	
Types	are interpreted as objects
Contexts	are interpreted as objects (products/tensors)
Terms	are interpreted as morphisms.
Substitution	is interpreted as composition.

In **Multiplicative Linear Logic,** a proof of is interpreted as a morphism

$$x_1 : a_1, \ldots, x_\ell : a_\ell \vdash t : c \qquad \text{as} \qquad a_1 \otimes \cdots \otimes a_\ell \multimap c.$$

In **λ-calculus,** a term is interpreted as a morphism

$$x_1 : \underline{b}_1, \ldots, x_n : \underline{b}_n \vdash t : c \qquad \text{as} \qquad \underline{b}_1 \times \cdots \times \underline{b}_n \to c.$$

Axiomatic using Multicategories

In a multicategory

Types	are interpreted as objects in X
Terms	are interpreted as multimorphisms
Substitution	is interpreted as multicomposition.

In **Multiplicative Linear Logic,** a term is interpreted as a multimorphism in a symmetric multicategory:

$$x_1 : a_1, \ldots, x_\ell : a_\ell \vdash t : c \qquad \text{as} \qquad a_1, \ldots, a_\ell \multimap c.$$

In **λ-calculus,** a term is interpreted as a multimorphism in a cartesian multicategory:

$$x_1 : \underline{b}_1, \ldots, x_n : \underline{b}_n \vdash t : c \qquad \text{as} \qquad \underline{b}_1, \ldots, \underline{b}_n \to c.$$

Multicategories can be seen as distributors combined with a monad.

(Fiore-Plotkin-Turi 1999, Tanaka-Power 2006, Hirschowitz-Maggesi 2010)

Distributors, Kleisli Bicategories and 2-Monads

The category **Rel** has objects sets and morphisms $F : X \nrightarrow Y$ relations $F : Y \times X \to 2$, composition $G \circ F = \{(a,c) \mid \exists b\ (b,c) \in G \wedge (a,b) \in F\}$

The bicategory **Prof** has objects categories and morphisms $F : X \nrightarrow Y$ distributors $F : Y^{op} \times X \to \mathbf{Set}$ and composition $G \circ F(a,c) = \int^{b \in B} G(b,c) \times F(a,b)$

A relation is $X \to \mathcal{P}\,Y$ with $\mathcal{P}$ is the powserset monad, so **Rel** is $\mathbf{Set}_{\mathcal{P}}$.

A distributor is a functor $X \to Psh\,Y$ where Psh is the presheaf pseudomonad. The bicategory **Prof** is the Kleisli bicategory $\mathbf{Cat}_{Psh}$

The commutative monoid monad $\mathcal{M}_{fin}$ extends from **Set** to **Rel**, thanks to a distributive law of $\mathcal{P}$ over $\mathcal{M}_{fin}$. (Beck 1969)

A 2-monad $\mathcal{T}$ extends from **Cat** to **Prof**, thanks to a pseudo distributive law of Psh over $\mathcal{T}$. (Tanaka 2005, Fiore-Gambino-Hyland-Winskel 2016)

Multicategories as Distributors with Context Monad

Let $\mathcal{T}$ be a 2-monad on **Cat** that extends to **Prof**. For instance,

- $\mathcal{L}$ the 2-monad for free symmetric monoidal categories
- $\mathcal{M}$ the 2-monad for free categories with products

A **multicategory** can be seen as a **distributor** in the Kleisli bicat of $\mathcal{T}$:

$$M : X \nrightarrow \mathcal{T}X \qquad M : \mathcal{T}X^{\mathsf{op}} \times X \to \mathbf{Set}$$

Together with a monadic structure that represents:

- identity: $1 \Rightarrow M$
- multicomposition: $M \circ M \Rightarrow M$

Axiomatization using Multicategories via Distributors

In a multicategory $M : X \nrightarrow \mathcal{T}X$, that is $M : \mathcal{T}X^{op} \times X \to \mathbf{Set}$

Types are interpreted as objects in X

Terms are interpreted as elements of M

Substitution is interpreted by the monadic structure $M \circ M \Rightarrow M$

In **Multiplicative Linear Logic,** $\mathcal{T} = \mathcal{L}$ with $\mathcal{L}X$ the free symmetric monoidal category over X (Fiore-Gambino-Hyland-Winskel 2007)

$x_1 : a_1, \ldots, x_\ell : a_\ell \vdash t : c$ as $a_1, \ldots, a_\ell \multimap c$ in $M(\langle a_1, \ldots, a_\ell \rangle; c)$

In **λ-calculus,** $\mathcal{T} = \mathcal{M}$ with $\mathcal{M}X$ the free category with product over X

(Tanaka-Power 2004, Hyland 2017)

$x'_1 : \underline{b}_1, \ldots, x'_n : \underline{b}_n \vdash t : c$ as $\underline{b}_1, \ldots, \underline{b}_n \to c$ in $M(\langle \underline{b}_1, \ldots, \underline{b}_n \rangle; c)$

What is $\mathcal{T}$ for a Mixed Linear-Non-Linear Calculus ? What is $\mathcal{T}X$?

$$x_1 : a_1, \ldots, x_\ell : a_\ell \mid x'_1 : \underline{b}_1, \ldots, x'_n : \underline{b}_n \vdash t : c$$

Mathematical Theory of Linear / Non-Linear Substitution

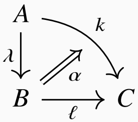
What construction to combine into a 2-monad lifting to profunctors ?

- $\mathcal{L}$ free symmetric monoidal category 2-monad
  $\mathcal{L}X$: objects are sequences $\langle a_1, \ldots, a_\ell \rangle$
  morphisms are bijections and sequence of morphisms.

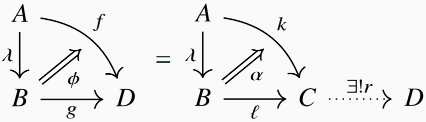
- $\mathcal{M}$ free cateogory with products 2-monad
  $\mathcal{M}X$: objects are sequences $\langle \underline{b}_1, \ldots, \underline{b}_n \rangle$
  morphisms are functions and sequence of morphisms.

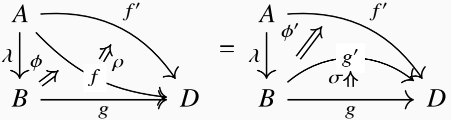
- $\mathcal{Q}$ Mixed linear / non linear 2-monad (Power-Tanaka 2005,Fiore 2006)
  $\mathcal{Q}X$: objects are mixed sequences $\langle a_1, \ldots, a_\ell \mid \underline{b}_1, \ldots, \underline{b}_n \rangle$
  morphisms combine functions, bijections and sequence of morphisms.

Colax colimits induced by a map in a 2-category $\mathcal{K}$

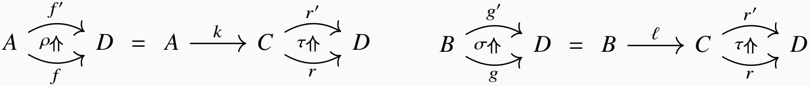
If $\lambda : A \to B$ is a map in $\mathcal{K}$, then the induced colax colimit is 

There are two universal aspects for 1-cells and 2-cells

- for any 

- for any 

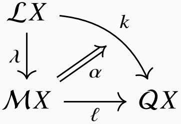
, $\exists!\ r \overset{\tau}{\Rightarrow} r'$ s.t.



Mixing Linear and Non-Linear monads via a colimit

A map of 2-monads: $\lambda : \mathcal{L} \to \mathcal{M}$ as every category with products is a symmetric monoidal category.

Colax Colimit over λ in the 2-category of SymmMonCat satisfies



$$
\begin{aligned}
\lambda : \langle a_1, \ldots, a_\ell \rangle &\mapsto \langle \underline{a}_1, \ldots, \underline{a}_\ell \rangle \\
k : \langle a_1, \ldots, a_\ell \rangle &\mapsto \langle a_1, \ldots, a_\ell \mid \cdot \rangle \\
\ell : \langle b_1, \ldots, b_n \rangle &\mapsto \langle \cdot \mid \underline{b}_1, \ldots, \underline{b}_n \rangle
\end{aligned}
$$

$$
\frac{x_1 : a_1, \ldots, x_\ell : a_\ell \mid \cdot \vdash t : b}{\cdot \mid x_1 : \underline{a}_1, \ldots, x_\ell : \underline{a}_\ell \vdash t : b}
\qquad
\alpha : \langle \cdot \mid \underline{a}_1, \ldots, \underline{a}_\ell \rangle \mapsto \langle a_1, \ldots, a_\ell \mid \cdot \rangle
$$

I can substitute a linear variable with a non-linear one.

Properties of the $\mathcal{Q}X$ from universality for 1-cell and 2-cell

$\mathcal{Q}X$ a category whose objects are mixed sequences $\langle a_1, \ldots, a_\ell \mid \underline{b}_1, \ldots, \underline{b}_n \rangle$ and morphisms combine functions, bijections and sequence of morphisms.

- $\mathcal{Q}X$ is a symmetric monoidal category
- $\mathcal{Q}X$ splits through the free category with $\mathcal{M}X$:

$$f : \mathcal{Q}X \to \mathcal{M}X \to \mathcal{Q}X$$

$$\langle a_1, \ldots, a_\ell \mid \underline{b}_1, \ldots, \underline{b}_n \rangle \mapsto \langle a_1, \ldots, a_\ell, b_1, \ldots b_n \rangle \mapsto \langle \cdot \mid \underline{a}_1, \ldots, \underline{a}_\ell, \underline{b}_1, \ldots, \underline{b}_n \rangle$$

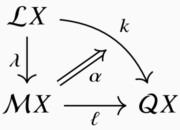
- f is a strictly idempotent comonad that is strictly monoidal:

$$\beta : f \Rightarrow 1 \qquad \frac{x_1 : a_1, \ldots, x_\ell : a_\ell \mid \Delta \vdash t : b}{\cdot \mid x_1 : \underline{a}_1, \ldots, x_\ell : \underline{a}_\ell, \Delta \vdash t : b}$$

$$\langle \cdot \mid \underline{a}_1, \ldots, \underline{a}_\ell, \underline{b}_1, \ldots, \underline{b}_n \rangle \to \langle a_1, \ldots, a_\ell \mid \underline{b}_1, \ldots, \underline{b}_n \rangle$$

Mixed Linear-Non-Linear 2-Monad

Compute a Colimit in the 2-category of Symmetric Monoidal Categories.



- $\mathcal{L}X$ the free symmetric monoidal category X
- $\mathcal{M}X$ the free category with products over X

Theorem

$\mathcal{Q}$ is a 2-monad on **Cat**.

The proof uses universality of the colimit.

Theorem

A $\mathcal{Q}$-algebra is a Symmetric Monoidal Category that splits through a Cartesian Category with coherences.

What is a model of Linear-Non-Linear Calculus

We are looking for a multicategorical axiomatisation: $M : X \nrightarrow QX$

DONE We have defined a 2 monad $\mathcal{Q}$ on **Cat** which describes Linear-Non-Linear contexts.

TODO To describe what is a $\mathcal{Q}$-multicategory, we need to extend $\mathcal{Q}$ to a pseudo-monad on the bicategory of distributors.

HOW Instead we prove that *Psh* lifts to pseudo $\mathcal{Q}$ algebras

PROBLEM The presheaf pseudo monad lifts from $\mathcal{L}$-algebra (symmetric strict monoidal category) to pseudo-$\mathcal{L}$-algebra, where there are NO COLIMITS.

IDEA Pseudo version of the characterisation of $\mathcal{Q}$-algebras, together with a strictification to recover colimits.

Back to differential λ-calculus

Mixed Linear-Non-Linear Calculus ?

- Closed structure to interpret abstractions

(Fiore-Plotkin-Turi 1999, Hyland 2017)

Differentiation $u, x \mapsto Df_x(u)$

- Derivation operator transforms a LNL-multimap of type $\langle \Gamma \mid \underline{b}, \Delta \rangle \to c$ to a LNL-multimap of type $\langle \Gamma, b \mid \underline{b}, \Delta \rangle \to c$ instead of $d : A \otimes !A \multimap !A$
- Chain rule will induce an additive structure