Recap on Betti numbers

1 - Notion of p-chain

Formal sum of p-simplices with modulo 2 coefficients

ex $c = \underbrace{e_0 + e_1}_{} + e_1 = e_0$

$d = c + e_1$
$\quad = e_0 + e_1 + e_1 + e_1$

$d = e_0 + e_1$

2 - Boundary operator

Formal sum modulo 2 of (p-1) faces

ex $\partial(d) = \partial(e_0 + e_1)$
$\qquad = v_0 + v_1 + v_0 + v_2$

$\partial(d) = v_1 + v_2$

3 - p-cycle

p-chain c such that $\boxed{\partial c = 0}$

ex $\partial(e_0 + e_1 + e_2)$
$\qquad = \underbrace{v_0 + v_1 + v_0 + v_2 + v_1 + v_2}_{=0}$

group → $\boxed{Z_p}$

4 - p-boundary

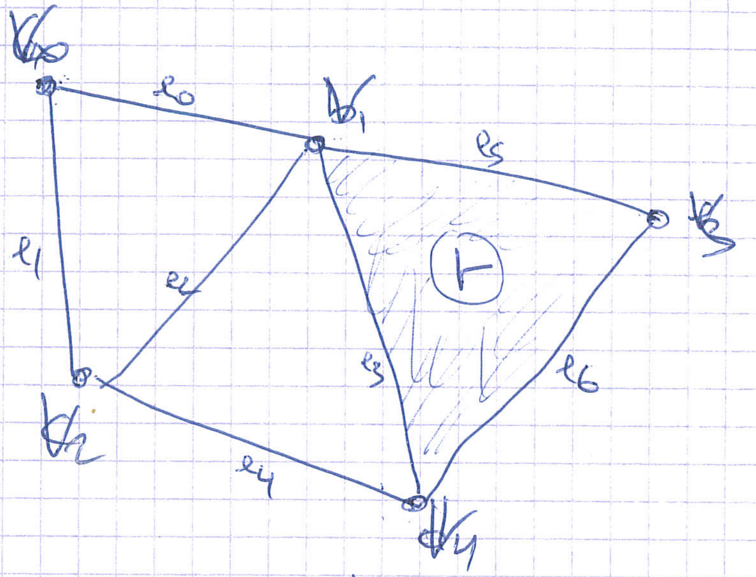
boundary of a (p+1)-chain

$b = \partial c$

ex $\partial T = e_3 + e_5 + e_6$

↳ only 1-boundary. more triangles → more boundaries

group : $\boxed{B_p}$



↳ 5 vertices, 7 edges, 1 triangle

5 - Fundamental lemma of homology

$\partial(\partial c) = 0$

$\Rightarrow \boxed{B_p \subset Z_p}$

all cycles not necessary boundaries
⇒ that's the idea of homology

6 - Homology group

$\boxed{H_p = Z_p / B_p}$ can be seen as quotient space by the equivalence relation

$c \sim c' \iff c \in Z_p$
$\qquad\qquad c' \in Z_p$

$\boxed{c' = c + b} \quad \exists\, b \in B_p$

ex: $c' = \underbrace{c}_{} + b$

$e_2 + e_3 + e_4 = e_2 + e_4 + e_6 + e_5$
$\qquad + e_3 + e_5 + e_6$
$\qquad = e_2 + e_3 + e_4$

↳ all cycles after having filled all possible holes with (p+1) chains vc' homologous cycles

↳ can be continuously transformed into each other by contracting $(p+1)$ chains

↳ "fill holes with triangles"

equivalence class → is called "homology class"

: order of H_p (its cardinality) number of homology classes

number can be seen as

rank(H_p) → number of linearly independent classes

$\log_2 |H_p|$

$$\underbrace{e_0 + e_3 + e_4 + e_1}_{a} = \underbrace{e_0 + e_1 + e_2}_{b} + \underbrace{\cancel{e_2} + e_3 + e_4}_{c}$$

a is linearly dependent to both b and c

$\beta_1 = 2$ only b and c are linearly independent